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Nagoya Math. J.
Vol. 160 (2000), 143–159

QUOTIENTS OF *L*-FUNCTIONS

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Abstract. In this paper a certain type of Dirichlet series, attached to a pair of Jacobi forms and Siegel modular forms is studied. It is shown that this series can be analyzed by a new variant of the Rankin-Selberg method. We prove that for eigenforms the Dirichlet series have an Euler product and we calculate all the local L-factors. Globally this Euler product is essentially the quotient of the standard L-functions of the involved Jacobi- and Siegel modular form.

Introduction

Let $F, G \in S_2^k$ be two Siegel cusp forms of weight k and degree 2. It had been discovered by Kohnen and Skoruppa [K-S89] that the Dirichlet series

$$D_{F,G}^{\mathrm{KS}}(s) = \zeta(2s-2k+4) \sum_{N=1}^{\infty} \langle \Phi_N^F, \Phi_N^G \rangle_{\mathcal{J}} \, N^{-s}, \tag{1}$$

where Φ_N^F, Φ_N^G are the N^{th} coefficients of the Fourier-Jacobi expansion of F and G, respectively and $\langle \cdot\,, \cdot \rangle_{\mathcal{J}}$ denotes the Petersson scalar product on Jacobi cusp forms, can be studied by the Rankin-Selberg method. Moreover they proved, that if F is a Hecke eigenform and G in the Maass space, then $D_{F,G}^{\mathrm{KS}}(s)$ is proportional to the spinor zeta function of F, i.e.,

$$D_{F,G}^{\mathrm{KS}}(s) = \langle \Phi_1^F, \Phi_1^G \rangle_{\mathcal{J}} \, Z_F(s). \tag{2}$$

In this paper we study Dirichlet series $D_{\Phi,F}^{\diamond}(s)$ attached to Jacobi cusp forms Φ on $\mathbb{H} \times \mathbb{C}$ and Siegel cusp forms $F \in S_2^k$ of degree 2 and even weight k of formally similar type, but of surprisingly different properties. Let U_λ be the operator $\Phi(\tau, z) \mapsto \Phi(\tau, \lambda z)$. Then by definition

$$D_{\Phi,F}^{\diamond}(s) = \sum_{\lambda=1}^{\infty} \langle \Phi | U_\lambda, \Phi_{t\lambda^2}^F \rangle_{\mathcal{J}} \, \lambda^{-(2s+2k-4)}, \tag{3}$$

Received February 28, 1999.
2000 Mathematics Subject Classification: 11F27, 11F55, 11F67.

where $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$, i.e., Φ is a Jacobi cusp form of weight k and index t. We employ the method introduced in [He97], to obtain a Rankin type integral representation of $D_{\Phi,F}^{\diamond}(s)$. Thus analytic and arithmetic properties can be deduced from a certain kind of Jacobi Eisenstein series. In contrast to other generalizations of Kohnen and Skoruppa's work e.g. Yamazaki [Ya90] which do not lead to an Euler product, we can prove the following: Let $F \in S_2^k$ be a Hecke eigenform and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ a Hecke-Jacobi newform (cf. Section 5.1), then $D_{\Phi,F}^{\diamond}(s)$ has an Euler product. More precisely, let $D_F(s)$ and $L(s,\Phi)$ be the standard L-functions attached to F and Φ. Then

THEOREM. *Let $k,t \in \mathbb{N}$ and let k be even. Let $F \in S_2^k$ be a Hecke eigenform and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi newform. Then*

$$\begin{aligned} D_{\Phi,F}^{\diamond}(s) = \langle \Phi, \Phi_t^F \rangle_{\mathcal{J}}\, &\zeta(4s+2k-4)^{-1} \\ &\times D_F(2s+k-2) L(2s+2k-3,\Phi)^{-1}. \end{aligned}$$

Hence in contrast to Kohnen and Skoruppa's Dirichlet series we do not need the existence of a Maass space to get an Euler product, which gives some hope for generalization to higher degrees. Moreover the Euler product in the theorem involves L-functions of Φ and F. This is not the case for $D_{F,G}^{\text{KS}}(s)$. Let the index of the Jacobi form be one, then our results have some direct relation to the work of Murase and Sugano [M-S91]. Let $X = \{s \in \mathbb{C} \mid 2\,\mathrm{Re}(s)+k > 5\}$ and $\mathcal{H}(X)$ the vector space of all holomorphic functions on X. Then the construction of the Dirichlet series $D_{\Phi,F}^{\diamond}(s)$ can be interpreted as a bilinear map

$$\mathcal{J}_{k,t}^{\text{cusp}} \times S_2^k \longrightarrow \mathcal{H}(X),$$

which can be continued to $\mathcal{J}_k^{\text{cusp}} = \bigoplus_{t=1}^{\infty} \mathcal{J}_{k,t}^{\text{cusp}}$, the *Jacobi-Siegel pairing*. This pairing can be used to study either standard L-functions of Siegel modular forms of degree 2 or Jacobi forms of arbitrary index. It follows from the work of [He98], that every analytic Klingen-Jacobi Eisenstein series attached to a Hecke-Jacobi eigenform $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ has a meromorphic continuation on the whole complex plane. Thus the image of $\Phi \times S_2^k$ has a meromorphic continuation. This means for example, that the image of $\mathcal{J}_{k,t}^{\text{cusp}} \times S_2^k$ for t square free has the same property.

Finally we would like to mention, that we believe that our results can be generalized to Jacobi forms on $\mathbb{H}_n \times \mathbb{C}^{n,1}$ and Siegel modular forms of degree n. But for this more knowledge on the involved Hecke-Jacobi theory has to be obtained.

Acknowledgements. The author would like to thank the Max-Planck Institut für Mathematik in Bonn for hospitality and the DFG for financial support.

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Notation: For an associative ring R with identity element, we denote by $R^\times$ the group of all invertible elements of R. If M is a matrix, M^t, $\det(M)$, and $\mathrm{tr}(M)$ stand for its transpose, determinant, and trace. We put $M_n(R) = R^{n,n}$, $Gl_n(R) = M_n(R)^\times$. The identity and zero elements of $M_n(R)$ are denoted by 1_n and 0_n respectively (when n needs to be stressed). Let $J_n = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then the symplectic group of degree n is defined by $Sp_n(R) = \{M \in Gl_{2n}(R) \mid M^t J_n M = J_n\}$.

$$P_{n,r}(R) = \left\{ \begin{pmatrix} \alpha & * \\ 0_{n+r,n-r} & * \end{pmatrix} \in Sp_n(R) \right\},$$
$$C_{n,r}(R) = \left\{ \begin{pmatrix} * & * \\ 0_{n-r,n+r} & \alpha \end{pmatrix} \in Sp_n(R) \right\}.$$

Let $P^J_{n,r}$, $C^J_{n,r}$ be the subgroups of $P_{n,r}$ and $C_{n,r}$ respectively, where $\alpha = 1_{n-r}$. Let R be a subring of $\mathbb{R}$. Then $R^+ = \{r \in R \mid r > 0\}$ and

$$G^+Sp_n(R) = \{M \in Gl_{2n}(R) \mid M^t J_n M = \mu(M) J_n \text{ with } \mu(M) \in R^+\}.$$

For real symmetric matrices A and B, we put $A[B] = B^t AB$ if A, B are suitable. If $A_1, A_2, \ldots, A_n$ are square matrices, $[A_1, A_2, \ldots, A_n]$ denotes the matrix with $A_1, A_2, \ldots, A_n$ in the diagonal blocks and 0 in all other blocks. Let $Z \in \mathbb{C}^{n,n}$, then we put $e\{Z\} = e^{2\pi i\,\mathrm{tr}(Z)}$ and $\mathrm{Re}(Z)$, $\mathrm{Im}(Z)$ for the real and imaginary part of Z. Further let $\delta(Z) = \det(\mathrm{Im}(Z))$.

§1. Automorphic forms

1.1. Review of Siegel modular forms

The group of positive symplectic similitudes $G^+Sp_n(\mathbb{R})$ acts on Siegel's half space $\mathbb{H}_n = \{Z = Z^t \in M_n(\mathbb{C}) \mid \mathrm{Im}(Z) > 0\}$ of degree n as a group of biholomorphic automorphisms by

$$(M, Z) \longmapsto M(Z) = (AZ + B)(CZ + D)^{-1},$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A, B, C, D \in M_n(\mathbb{R})$. We denote the factor of automorphy by $j(M, Z) = \det(CZ + D)$. For $F : \mathbb{H}_n \to \mathbb{C}$ and $k \in \mathbb{Z}$ we define the Petersson operator

$$(F|_k M)(Z) := \mu(M)^{nk-n(n+1)/2} j(M, Z)^{-k} F(M(Z)). \tag{4}$$

Let us denote by M_n^k the space of Siegel modular forms and by S_n^k its subspace of cupsforms of degree n and weight k for $\Gamma_n = Sp_n(\mathbb{Z})$. Let $\langle \cdot, \cdot \rangle$ denote the Petersson scalar product, i.e., for arbitrary complex valued functions F and G on $\mathbb{H}_n$, which satisfy the same transformation law as modular forms the Petersson integral, convergence assumed, it is given by

$$\langle F, G \rangle = \int_{\Gamma_n \backslash \mathbb{H}_n} F(Z)\overline{G(Z)} \det(\mathrm{Im}\, Z)^k \, d^*Z. \tag{5}$$

Here $d^*Z = \det(Y)^{-(n+1)} \, dXdY$ denotes the symplectic volume element. For more details the reader is referred to Klingen [Kl90].

Let $F \in S_n^k$ be a Hecke eigenform with Satake parameter $(\alpha_{0,p}; \alpha_{1,p}; \cdots; \alpha_{n,p})$. Then the standard zeta function $D_F^n(s)$ of F is given by

$$D_F^n(s) = \prod_p \left\{D_{p,F}(p^{-s})\right\}^{-1}, \tag{6}$$

where the Rankin polynomial is

$$D_{p,F}(X) = (1 - X) \prod_{j=1}^{n} (1 - \alpha_{j,p}X)(1 - \alpha_{j,p}^{-1}X). \tag{7}$$

1.2. Jacobi forms

Let $k, n, t \in \mathbb{N}$. Then we denote by $\mathcal{J}_{k,t}^n$ and $\mathcal{J}_{k,t}^{n,\mathrm{cusp}}$ the space of Jacobi forms and Jacobi cusp forms, respectively, on $\mathcal{D}_{n,1} = \mathbb{H}_n \times \mathbb{C}^{1,n}$. We shall write $\langle \cdot, \cdot \rangle_{\mathcal{J}}$ for the Petersson scalar product of Jacobi forms. Let

$((\lambda,\mu,\rho),M)$ be the parameterization of $g\in P^J_{n+1,n}(\mathbb{Z})$ as given in [A-H98, Section 1.1]. Then $\Phi\in\mathcal{J}^n_{k,t}$ satisfies

$$\Phi(\tau,z)=j_{k,t}(g,(\tau,z))^{-1}\Phi(g(\tau,z)). \tag{8}$$

Here $g(\tau,z)=\big(M(\tau),zJ(M,\tau)^{-1}+\lambda M(\tau)+\mu\big)$, where $J(M,\tau)=c\tau+d$ and

$$\begin{aligned} j_{k,t}(g,(\tau,z)) &= j(M,\tau)^k e\{-t\rho\} \\ &\quad\times e\{-t[\lambda]M(\tau)-2\lambda^t tzJ(M,\tau)^{-1}+t[z]J(M,\tau)^{-1}c\}. \end{aligned} \tag{9}$$

At the same time we could also consider Φ as a Jacobi form with respect to $C^J_{n+1,n}(\mathbb{Z})$. Moreover let us introduce the projection map

$$*_{J,r}:\begin{cases} \mathcal{D}_{n,1} \longrightarrow \mathcal{D}_{r,1} \\ (\tau,z)\longmapsto \left(\tau\left[\begin{pmatrix}0\\1_r\end{pmatrix}\right],z\begin{pmatrix}0\\1_r\end{pmatrix}\right) \end{cases}, \tag{10}$$

which is a generalization of the projection $I\!H_n\to I\!H_r$, where $\tau\mapsto\tau_*=\tau\left[\begin{smallmatrix}0\\1_r\end{smallmatrix}\right]$. The groups

$$G^J_{n,1,r}(\mathbb{R})=\left\{((0\lambda_2,\mu,\rho),M)\;\middle|\;\begin{array}{l}\lambda_2\in\mathbb{R}^{1,r},\mu\in\mathbb{R}^{1,n},\rho\in\mathbb{R}^{1,1}\\ \text{and } M\in P_{n,r}(\mathbb{R}).\end{array}\right\}, \tag{11}$$

for $r\le n$, are involved in the definition of Eisenstein series.

To simplify our notation we put $\mathcal{J}^{\mathrm{cusp}}_{k,t}=\mathcal{J}^{1,\mathrm{cusp}}_{k,t}$, $\mathbb{H}=\mathbb{H}_1$, $\Gamma^J_{n,1,r}=G^J_{n,1,r}(\mathbb{Z})$ and $\Gamma^J_{n,1}=G^J_{n,1,n}(\mathbb{Z})$. Jacobi cusp forms $\Phi\in\mathcal{J}^{\mathrm{cusp}}_{k,t}$ of index t on $\mathbb{H}\times\mathbb{C}$ can be completed to functions $\widehat{\Phi}$ on $\mathbb{H}_2$ via

$$\Phi(\tau,z)\longmapsto\widehat{\Phi}\begin{pmatrix}\tau' & z\\ z & \tau\end{pmatrix}=\Phi(\tau,z)e\{t\tau'\}. \tag{12}$$

Then $\widehat{\Phi}|_k\,g=\widehat{\Phi}$ for $g\in P^J_{2,1}(\mathbb{Z})$. Further if we would put $\widehat{\Phi}\left(\begin{smallmatrix}\tau & z\\ z & \tau'\end{smallmatrix}\right)=\Phi(\tau,z)e\{t\tau'\}$, then $\widehat{\Phi}|_k\,g=\widehat{\Phi}$ for $g\in C^J_{2,1}(\mathbb{Z})$.

We denote the space of completed Jacobi cusp forms of index t and weight k by $\widehat{\mathcal{J}}^{\mathrm{cusp}}_{k,t}$. On this space the Petersson scalar product is simulated by

$$\big\langle\widehat{\Phi},\widehat{\Psi}\big\rangle_{\mathcal{A}}=\int_{P^J_{2,1}(\mathbb{Z})\backslash\mathbb{H}_2}\widehat{\Phi}(Z)\overline{\widehat{\Psi}(Z)}\det(\operatorname{Im}Z)^k\,d^*Z, \tag{13}$$

where $\Phi, \Psi \in \mathcal{J}_{k,t}^{\text{cusp}}$. We have $\langle \Phi, \Psi \rangle_{\mathcal{J}} = \beta_k t^{k-2} \left\langle \widehat{\Phi}, \widehat{\Psi} \right\rangle_{\mathcal{A}}$. Here $\beta_k = (4\pi)^{k-2}\Gamma(k-2)^{-1}$. It is convenient to alternate frequently between the two notations Φ and $\widehat{\Phi}$ of a Jacobi form.

Next we state the definition of an analytic Jacobi Eisenstein series.

DEFINITION 1.1. Let $k, t, n \in \mathbb{N}$ with k even, $0 \leq r \leq n$. To $\Phi \in \mathcal{J}_{k,t}^{\text{r,cusp}}$ we attach an analytic Jacobi Eisenstein series of Klingen type on $\mathcal{D}_{n,1} = \mathbb{H}_n \times \mathbb{C}^{1,n}$ defined by:

$$
\begin{aligned}
(14) \qquad & E_{n,r}^{k,t}((\tau, z), \Phi; s) \\
&= \sum_{\gamma \in \Gamma_{n,1,r}^J \backslash \Gamma_{n,1}^J} \Phi\big(\gamma(\tau, z)_{*J}\big)\, j_{k,t}(\gamma, (\tau, z))^{-1} \left(\frac{\delta(M(\tau))}{\delta(M(\tau)_*)} \right)^s,
\end{aligned}
$$

here $\gamma = (h, M)$ and $s \in \mathbb{C}$ with $\text{Re}(s)$ sufficiently large. If $r = 0$ we denote by $E_n^{k,t}((\tau, z); s) = E_{n,0}^{k,t}((\tau, z), 1; s)$ and $E_n^{k,t}((\tau, z)) = E_n^{k,t}((\tau, z); 0)$ the (analytic) Siegel-Jacobi Eisenstein series.

The Eisenstein series is absolutely convergent for $k + 2\,\text{Re}(s) > n + r + 2$. We have proven in [He97], that under certain conditions the Klingen Jacobi Eisenstein series has a meromorphic continuation on the whole complex plane. For example, the conditions are satisfied for Hecke-Jacobi eigenforms $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$. This has been proven recently in [He98]. If $k > n + r + 2$, then $E_{n,r}^{k,t}((\tau, z), \Phi) = E_{n,r}^{k,t}((\tau, z), \Phi; 0) \in \mathcal{J}_{k,t}^n$.

§2. Jacobi-Siegel pairing

We introduce a bilinear map from $\mathcal{J}_{k,t}^{\text{cusp}} \times S_2^k$ to the space $\mathcal{H}(X)$ of holomorphic complex-valued functions on $\{s \in \mathbb{C} \mid 2\,\text{Re}(s) + k > 5\}$. It will turn out that these functions will have a meromorphic continuation on the whole complex plane, if we restrict ourselves to the subspace generated by Hecke-Jacobi eigenforms. Moreover these complex-valued functions can be described essentially as the quotient of L-series.

Let $\Phi, \Psi \in \mathcal{J}_{k,t}^{\text{cusp}}$ and $\tau' \in \mathbb{H}$, then we put $\widehat{\Phi}(Z) = \widehat{\Phi}\left(\begin{smallmatrix} \tau & z \\ z^t & \tau' \end{smallmatrix}\right) = \Phi(\tau', z) e\{t\tau\}$. In (13) and [He99, Definition 3.10], we have introduced a Petersson scalar product $\langle\, \cdot\,, \cdot\, \rangle_{\mathcal{A}}$ for the so called P-forms $\widehat{\Phi}$ on $\mathbb{H}_2$. We have

$$
\langle \Phi, \Psi \rangle_J = \beta_k\, t^{k-2} \left\langle \widehat{\Phi}, \widehat{\Psi} \right\rangle_{\mathcal{A}},
$$

where $\beta_k = (4\pi)^{k-2}\Gamma(k-2)^{-1}$ (see also Section 1.2). Moreover if A is a linear operator on the graded algebra of Jacobi cusp forms, then we denote by $\widetilde{A}$ the corresponding operator on the graded space of P-forms. Let us denote the Fourier-Jacobi expansion of F by

$$F\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} = \sum_{m=1}^{\infty} \Phi_m^F(\tau', z) e\{m\tau\}.$$

Moreover let U_λ be the operator $\Phi(\tau, z) \mapsto \Phi(\tau, \lambda z)$ and $\widetilde{U}_\lambda$ the corresponding one on the space of P-forms.

THEOREM 2.1. *Let $k, t \in \mathbb{N}$ and $k > 5$ be even and $2\operatorname{Re}(s) + k > 5$. Let $\Phi \in \mathcal{J}_{k,t}^{\mathrm{cusp}}$ and $F \in S_2^k$. Then*

$$\begin{aligned} JS(\Phi, F; s) &:= \left\langle E_{2,1}^{k,t}\big((*, 0), \Phi; s\big), F(*)\right\rangle \\ &= (4\pi)^{-(s+k-2)}\Gamma(s+k-2) \sum_{\lambda=1}^{\infty} \left\langle \Phi|U_\lambda, \Phi_{t\lambda^2}^F \right\rangle_{\mathcal{J}} (t\lambda^2)^{-(s+k-2)}. \end{aligned}$$

Proof. Before we start to compute $JS(\Phi, F; s)$ explicitly, it is convenient to simplify the (restricted) Eisenstein series

$$E_{2,1}^{k,t}((Z,0), \Phi; s) = \sum_{\gamma \in \Gamma_{2,1,1}^J \backslash \Gamma_{2,1}^J} \Phi\big(\gamma(Z,0)_{*_J}\big)\, j_{k,t}(\gamma, (Z,0))^{-1} \left(\frac{\delta(M(Z))}{\delta(M(Z)_*)}\right)^s.$$

Here $\gamma = (h, M)$. We choose the complete representative system

$$\bigcup_{\lambda \in \mathbb{Z}} (\lambda\, 0, 00; 0) P_{2,1} \backslash \Gamma_2$$

of $\Gamma_{2,1,1}^J \backslash \Gamma_{2,1}^J$. Let $h = (\lambda\, 0, 00; 0)$ be in the Heisenberg group, then it is easy to see, that

$$\begin{aligned} \Phi\big(h(Z,0)_{*_J}\big) &= \Phi(\tau', \lambda z), \\ j_{k,t}(h, (Z,0))^{-1} &= e\{\lambda^2 t\tau\}. \end{aligned}$$

Now we are ready to compute $JS(\Phi, F; s)$. After *unwinding* we reach to

$$JS(\Phi, F; s) = \sum_{\lambda=-\infty}^{\infty} \int_{P_{2,1} \backslash \mathbb{H}_2} \Phi(\tau', \lambda z) e\{\lambda^2 t\tau\} \overline{F(Z)} \delta(Z)^{s+k-3} \delta(\tau')^{-s}\, dZ$$

To use the formalism and reduction theory given in Heim [He99, §3.4], it is convenient to exchange $P_{2,1}$ by $C_{2,1}$. Moreover let $w \in \mathbb{C}$, then x_w and y_w denote the real and imaginary part of w. This leads to

$$\begin{aligned} JS(\Phi, F; s) &= 2 \sum_{\lambda=1}^{\infty} \int_{C_{2,1} \backslash \mathbb{H}_2} \Phi(\tau, \lambda z) e\{\lambda^2 t \tau'\} \, \overline{\Phi^F_{t\lambda^2}(\tau, z) e\{t\lambda^2 \tau'\}} \\ &\quad \times \delta \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}^{k+s-3} \delta(\tau)^{-s} \, d\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \\ &= 2 \sum_{\lambda=1}^{\infty} \int_{\mathcal{B}_{1,1}} d\tau dz \, \Phi(\tau, \lambda z) \, \overline{\Phi^F_{t\lambda^2}(\tau, z)} \\ &\quad \times \int_{y_{\tau'} > y_\tau^{-1}[y_z]} dy_{\tau'} \int_0^1 dx_{\tau'} \, e^{-4\pi t \lambda^2 y_{\tau'}} y_\tau^{-s} \, \delta \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}^{k+s-3}. \end{aligned}$$

Let $\mathcal{B}_{1,1}$ be a fundamental domain of the action of $(C^J_{2,1}(\mathbb{Z})/\text{center})$ on $\mathbb{H} \times \mathbb{C}$. Then

$$\mathcal{Q}_{1,1} = \left\{ \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathbb{H}_2 \;\middle|\; (\tau, z) \in \mathcal{B}_{1,1} \text{ and } |x_{\tau'}| \leq 1/2 \right\} \tag{15}$$

is a fundamental domain of the action of $C^J_{2,1}(\mathbb{Z})$ on $\mathbb{H}_2$.

Next we substitute $y_{\tau'}$ by $y + y_\tau^{-1}[y_z]$. Hence we get

$$\begin{aligned} JS(\Phi, F; s) = 2 \sum_{\lambda=1}^{\infty} \int_{\mathcal{B}_{1,1}} & d\tau dz \, \Phi(\tau, \lambda z) \, \overline{\Phi^F_{t\lambda^2}(\tau, z)} \\ & \times y_\tau^{k-3} \, e^{-4\pi t \lambda^2 y_\tau^{-1}[y_z]} \int_0^\infty \frac{dy}{y} \, y^{s+k-2} \, e^{-4\pi t \lambda^2 y}. \end{aligned}$$

After some obvious simplifications we get the desired result. $\square$

Putting everything together shows that the Jacobi-Siegel pairing

$$\begin{array}{ccc} \mathcal{J}^{\text{cusp}}_{k,t} \times S^k_2 & \longrightarrow & \mathcal{H}(X) \\ (\Phi, F) & \longmapsto & JS(\Phi, F; s) \end{array}$$

leads to a Dirichlet series (which has a meromorphic continuation on the whole complex plane, if we assume t to be square free of if we restrict ourselves to $\mathcal{J}^{\text{cusp,new}}_{k,t}$).

§3. Hecke-Jacobi theory

Let $\mathcal{R} = \mathbb{Z}$, $\mathbb{Z}[\frac{1}{p}]$ or $\mathbb{Q}$. We put $G^J_{\mathcal{R}} = \left\{\left(\begin{smallmatrix} & & * & \\ 0 & 0 & 0 & \beta\end{smallmatrix}\right) \in C_{2,1}(\mathcal{R}) \mid \beta > 0\right\}$, $H_{\mathcal{R}} = \{(X, \kappa) \mid X \in \mathcal{R} \times \mathcal{R},\, \kappa \in \mathcal{R}\}$, $\Gamma^J_{\mathcal{R}} = G^+Sp_1(\mathcal{R}) \times \mathcal{R}^+$ and $\Gamma^J = C^J_{2,1}(\mathbb{Z})$. Then the exact sequence

$$1 \longrightarrow H_{\mathcal{R}} \xrightarrow{\varphi} G^J_{\mathcal{R}} \xrightarrow{p} \Gamma^J_{\mathcal{R}} \longrightarrow 1 \tag{16}$$

with

$$\varphi(\lambda, \mu, \kappa) = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad p \begin{pmatrix} a & 0 & b & \mu' \\ \lambda & \alpha & \mu & \kappa \\ c & 0 & d & -\lambda' \\ 0 & 0 & 0 & \beta \end{pmatrix} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \beta\right)$$

splits. Hence $G^J_{\mathcal{R}}$ can be viewed as the semi-direct product of $H_{\mathcal{R}}$ with $\Gamma^J_{\mathcal{R}}$. We consider $H_{\mathcal{R}}$, $\Gamma^J_{\mathcal{R}}$ as subgroups of $G^J_{\mathcal{R}}$. Further we denote by $\mathcal{H}^n$, $\mathcal{H}^J$, $\mathcal{H}^J_p$ the Hecke algebras of the Hecke pairs $(\Gamma_n, G^+Sp_n(\mathbb{Q}))$, $(\Gamma^J, G^J_{\mathbb{Q}})$ and $\left(\Gamma^J, G^J_{\mathbb{Z}[\frac{1}{p}]}\right)$. It is known that $\mathcal{H}^n$ is commutative and has no zero divisors in contrast to $\mathcal{H}^J$. Several maps $*$, j_-, j_+ will be used to study $\mathcal{H}^J$. We start with the important map $*$. It is an anti-automorphism of $\mathcal{H}^J$ given by

$$\Gamma^J(h; (M, \beta))\Gamma^J \longmapsto \left(\Gamma^J(h; (M, \beta))\Gamma^J\right)^* = \Gamma^J \mu(M)(h; (M, \beta))^{-1}\Gamma^J, \tag{17}$$

where $(h; (M, \beta))$ is the parametrization of $g \in G^J_{\mathbb{Q}}$ via the splitting of (16). This map somehow simulates the rule how to construct the adjoint operator of a Hecke operator with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$. Let us put $\Gamma = \Gamma_1$. We have two algebra monomorphism

$$j_- : \begin{cases} \mathcal{H}^1 & \longrightarrow \quad \mathcal{H}^J \\ \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma & \longmapsto \quad \Gamma^J \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, 1\right) \Gamma^J \end{cases}, \tag{18}$$

$$j_+ : \begin{cases} \mathcal{H}^1 & \longrightarrow \quad \mathcal{H}^J \\ \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma & \longmapsto \quad \Gamma^J \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, ad\right) \Gamma^J \end{cases}. \tag{19}$$

We have the relation $j_+(X) = j_-(X)^*$ for $X \in \mathcal{H}^1$. This means that $j_+(X)$ is the adjoint operator of $j_-(X)$.

Let $T(n,n) = \Gamma[n,n]\Gamma$. We introduce some elements of $\mathcal{H}^J$. Let $r \in \mathbb{Q}$ and $n \in \mathbb{N}$:

$$\begin{aligned}
&T^g_-(n) := j_-(T(n)), && T^g_+(n) := j_+(T(n)),\\
&\Lambda^g_-(n) := j_-(T(n,n)), && \Lambda^g_+(n) := j_+(T(n,n)),\\
&\nabla(r) := \Gamma^J\left((0;r);(1_2,1)\right)\Gamma^J, && \Delta_n := \Gamma^J[n,n,n,n]\Gamma^J,\\
&\nabla^g_n := \Delta_n \sum_{b \bmod n} \nabla\left(\tfrac{b}{n}\right), && \Xi^g_n := \Delta_n \sum_{\lambda,\mu,\kappa \bmod n} \Gamma^J\left(\tfrac{\lambda}{n},\tfrac{\mu}{n},\tfrac{\kappa}{n}\right),\\
&T^{J,g}(n) := \Gamma^J[1,n,n^2,n]\Gamma^J.
\end{aligned}$$

We also put $\nabla^r_n = (\Delta_n)^{-1}\nabla^g_n$, $\Xi^r_n = (\Delta_n)^{-1}\Xi^g_n$, $\Lambda^r_\pm(n) = (\Delta_n)^{-1}\Lambda^g_\pm(n)$ and $T^{J,r}(n) = (\Delta_n)^{-1}T^{J,g}(n)$ (as a rule, we do this only when it is convenient).

Proposition 3.4 in [He99] states, that the elements T^g_-, T^g_+, $T^{J,g}$, ∇^g, Ξ^g, Λ^g_-, Λ^g_+ and Δ in $\mathcal{H}^J$ commute with each other when we only allow coprime arguments. Moreover the *functions* $\Lambda^g_-(n)$, $\Lambda^g_+(n)$ and Δ_n are strong multiplicative.

The subalgebra $\widetilde{\mathcal{H}}^J$ of $\mathcal{H}^J$ generated by T^g_-, T^g_+, $T^{J,g}$, ∇^g, Ξ^g, Λ^g_-, Λ^g_+, Δ is called Hecke-Jacobi algebra. The local Hecke-Jacobi algebra is given by $\widetilde{\mathcal{H}}^J_p = \mathcal{H}^J_p \cap \widetilde{\mathcal{H}}^J$. We have

$$\widetilde{\mathcal{H}}^J = \bigotimes_p \widetilde{\mathcal{H}}^J_p. \tag{20}$$

Moreover let $\widetilde{\mathcal{H}}^J_0$ be the subalgebra generated by $T^{J,g}$, ∇^g, Ξ^g and Δ_n and $\widetilde{\mathcal{H}}^J_{0,p} = \widetilde{\mathcal{H}}^J_0 \cap \widetilde{\mathcal{H}}^J_p$.

Remark 3.1. The Hecke-Jacobi algebra $\widetilde{\mathcal{H}}^J$ is not commutative and has zero divisors, because $\Lambda^g_-(p)\cdot(\nabla^r_p - p) = 0$ and $\Lambda^g_-(p)T^g_+(p) \neq T^g_+(p)\Lambda^g_-(p)$.

The heart of our considerations is the following result proven in [He99, Section 3.3].

THEOREM 3.2. *The Rankin polynomial $d^2_p(X)$ has the following factorization in $\widetilde{\mathcal{H}}^J_p[X]$:*

$$d^2_p(X) = (1-X)(1-p^{-2}\Lambda^r_-(p)X)S^{(2)}(X)(1-p^{-2}\Lambda^r_+(p)X) \tag{21}$$

with $S^{(2)}(X) = \sum_{j=0}^3 (-1)^j S^{(2)}_j X^j$. Here

$$S^{(2)}_0 = 1,$$

$$S_1^{(2)} = p^{-2}\big(T^{J,r}(p) + \nabla_p^r - p^2\big),$$
$$S_2^{(2)} = p^{-3}\big(T^{J,r}(p)(\nabla_p^r - p) + \Xi_p^r - p\nabla_p^r + p^2\big),$$
$$S_3^{(2)} = p^{-2}\big(\nabla_p^r - p\big).$$

§4. Euler products

We call $\Phi \in \mathcal{J}_{k,t}^{\mathrm{cusp}}$ a weak Hecke-Jacobi eigenform, if Φ is an eigenform for all $T^{J,r}(p)$, where $(p,t)=1$. It is known that $\mathcal{J}_{k,t}^{\mathrm{cusp}}$ has a basis of weak Hecke-Jacobi eigenforms. At this point we are satisfied with this definition, but later on we have to assume stronger conditions on Φ.

In this section we show that the Dirichlet series obtained in Theorem 2.1

$$D_{\Phi,F}(s) = \sum_{\lambda=1}^{\infty} \big\langle \widehat{\Phi}|_k \widetilde{U}_\lambda, \widehat{\Phi}^F_{t\lambda^2} \big\rangle_{\mathcal{A}} \, (t\lambda^2)^{-s} \tag{22}$$

can be written essentially as an Euler product times a function which only depends on the 'ramified part of $D_{\Phi,F}(s)$', if F is a Hecke eigenform and Φ a weak Hecke-Jacobi eigenform. The proof depends on the Hecke-Jacobi theory developed in [He99, §3]. Moreover, the following results can also be considered as an extension and application of the Hecke-Jacobi theory summarized in the last section.

We start with a representation of the Hecke-Jacobi algebra. Let $F : \mathbb{H}_2 \to \mathbb{C}$ be invariant under the $|_k$-action of Γ^J, then for $X = \sum a_j \Gamma^J g_j \in \widetilde{\mathcal{H}}^J$ we put

$$F|_k \mathcal{D}(X) := \sum_j a_j (F|_k \, g_j). \tag{23}$$

Definition 4.1. Let $p \nmid t$ and let $\Phi \in \mathcal{J}_{k,t}^{\mathrm{cusp}}$ be a weak Hecke-Jacobi eigenform. Let $\widehat{\Phi}|_k \mathcal{D}(T^{J,r}(p)) = \lambda \widehat{\Phi}$ and $\lambda^{\mathrm{EZ}} := p^{k-3}\lambda$. We define

$$\widetilde{L}_p^{\mathrm{EZ}}(s,\Phi) = 1 - \lambda^{\mathrm{EZ}} p^{-s} + p^{2k-3} p^{-2s}.$$

The following observations will turn out to provide a transparent interpretation of how these L-factors occur in a natural way.

Remark 4.2. In [He99, Sections 3.3 and 3.4], certain operators $S^{(2)}(X)^{\mathrm{factor}}$ and $S^{(2)}(X)^{\mathrm{n.prim}}$ have been introduced. They are closely related to $S^{(2)}(X)$. For instance we have

$$S^{(2)}(X)^{\mathrm{factor}} = 1 - p^{-2} T^{J,r}(p) X + p^{-1} X^2.$$

Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ and $(p,t)=1$. Then the action of $S^{(2)}(X)$ is given by

$$\begin{aligned}(24)\quad \widehat{\Phi}|_k\mathcal{D}\big(S^{(2)}(p^{-s+k-1})\big) &= (1+p^{-s+k-1})\,\widehat{\Phi}|_k\mathcal{D}\big(S^{(2)}(p^{-s+k-1})^{\text{factor}}\big)\\ &= (1+p^{-s+k-1})\,\widetilde{L}_p^{\text{EZ}}(s,\Phi)\,\widehat{\Phi}.\end{aligned}$$

Let $\Phi \in \mathcal{J}_{k,1}^{\text{cusp}}$ be a weak Hecke-Jacobi eigenform and let $\varphi \in S_1^{2k-2}$ be the corresponding elliptic cusp form by the Saito-Kurokawa lift, then

$$(25)\qquad \widehat{\Phi}|_k\mathcal{D}\big(S^{(2)}(p^{-2s-k+2})\big) = (1+p^{-2s-k+2})\,L_p(2s+2k-3,\varphi)\,\widehat{\Phi},$$

where $L_p(s,\varphi)$ denotes the local L-factor of the Hecke L-series of φ.

THEOREM 4.3. *Let $k,t\in\mathbb{N}$ and $k>5$ and $2\operatorname{Re}(s)+k>5$. Let $F\in S_2^k$ be a Hecke eigenform and $\Phi\in\mathcal{J}_{k,t}^{\text{cusp}}$ be a weak Hecke-Jacobi eigenform. Let $t=p_1^{a_1}\cdot p_2^{a_2}\cdots p_d^{a_d}$ be the prime number decomposition of t, then*

$$\begin{aligned}(26)\quad D_{\Phi,F}(s) &= \zeta(2s+k-2)^{-1}D_F(2s+k-2)\\ &\times\Big(\prod_{p\nmid t}(1+p^{-2s-k+2})\,\widetilde{L}_p^{\text{EZ}}(2s+2k-3,\Phi)\Big)\\ &\times\sum_{\delta_1,\dots,\delta_d=0}^{\infty}\Big\langle\widehat{\Phi},\,\widehat{\Phi}^F_{tp_1^{2\delta_1}\cdots p_d^{2\delta_d}}\big|_k\mathcal{D}\big(\Lambda_+^g(p_1^{\delta_1}\cdots p_d^{\delta_d})\big)\Big\rangle_{\mathcal{A}}\\ &\times\big(p_1^{\delta_1}\cdots p_d^{\delta_d}\big)^{-2s+6-3k}.\end{aligned}$$

COROLLARY 4.4. *Let $k\in\mathbb{N}$ be even. Let $F\in S_2^k$ be a Hecke eigenform and $\Phi\in\mathcal{J}_{k,1}^{\text{cusp}}$ be a weak Hecke-Jacobi eigenform. Then*

$$\begin{aligned}(27)\qquad D_{\Phi,F}(s) &= \big\langle\widehat{\Phi},\widehat{\Phi}_1^F\big\rangle_{\mathcal{A}}\,\zeta(4s+2k-4)^{-1}D_F(2s+k-2)\\ &\times L(2s+2k-3,\Phi)^{-1}.\end{aligned}$$

Here $L(s,\Phi)=\prod_p\widetilde{L}_p^{\text{EZ}}(s,\Phi)^{-1}$. Let $\varphi\in S_1^{2k-2}$ correspond to Φ with respect to the Saito-Kurokawa lift, then the Hecke L-function of φ is equal to $L(s,\Phi)$, i.e.,

$$L(s,\varphi)=L(s,\Phi).$$

Proof. We would like to apply [He99, Proposition 3.13], to analyze the Dirichlet series $D_{\Phi,F}(s)$. Hence we have to introduce the adjoint operator of $\widetilde{U}_\lambda$. Therefore it is advisable to use the rescaling rule

$$\widetilde{U}_\lambda=\lambda^{3(2-k)}\lambda^{2k-6}\Lambda_-^r(\lambda)$$

and the fact that the adjoint operator of $\Lambda_-^r(\lambda)$ is $\Lambda_+^r(\lambda)$, see [He99, Section 3.4], for more details. Let $\big(\widetilde{U}_\lambda\big)^{\mathrm{ad}}$ be the adjoint operator of $\widetilde{U}_\lambda$. Then putting $X = p^{-2s-k+2}$ gives

$$\widehat{\Phi}^F_{tp^{2\delta}}|_k\big(\widetilde{U}_{p^\delta}\big)^{\mathrm{ad}}\, p^{-2\delta s} = \widehat{\Phi}^F_{tp^{2\delta}}|_k\mathcal{D}\big(\Lambda_+^r(p^\delta)\big)\,(p^{-2}X)^\delta. \tag{28}$$

In [He99, Proposition 3.13], we have proven the following: Let $F \in S_2^k$ be a Hecke eigenform and $l \in \mathbb{N}$, then

$$\begin{aligned} D^2_{p,F}(\overline{X}) &\sum_{\delta=0}^{\infty} \widehat{\Phi}^F_{tp^{2\delta}}|_k\mathcal{D}\big(\Lambda_+^r(p^\delta)(p^{-2}\overline{X})^\delta\big) \\ &= (1-\overline{X})\Big(\widehat{\Phi}^F_t|_k\,\mathcal{D}\big(S^{(2)}(\overline{X})\big) - \widehat{\Phi}^F_{t/p^2}|_k\mathcal{D}\big(\Lambda_-^r(p)S^{(2)}(\overline{X})p^{-2}\overline{X}\big)\Big). \end{aligned} \tag{29}$$

Here $S^{(2)}(X)$ is defined in Theorem 3.2, a polynomial of degree 3 in $\widetilde{\mathcal{H}}^J_{0,p}[X]$. At this point we would like to mention that our argument for getting an Euler product only works because of [He99, Proposition 3.4], which gives a sufficient condition for the commutativity of certain Hecke-Jacobi operators.

Now we assume that p does not divide the index of Φ. Then $S^{(2)}(X)$ simplifies to $(1+X)S^{(2)}(X)^{\mathrm{factor}}$, i.e.,

$$\begin{aligned} \big\langle\widehat{\Phi},\widehat{\Phi}^F_t|\mathcal{D}\big(S^{(2)}(\overline{X})\big)\big\rangle_{\mathcal{A}} &= (1+X)\,\big\langle\widehat{\Phi}|\mathcal{D}\big(S^{(2)}(X)^{\mathrm{factor}}\big),\widehat{\Phi}^F_t\big\rangle_{\mathcal{A}} \\ &= \big\langle\widehat{\Phi},\widehat{\Phi}^F_t\big\rangle_{\mathcal{A}}\,(1+X)(1-p^{-2}\lambda X+p^{-1}X^2) \\ &= \big\langle\widehat{\Phi},\widehat{\Phi}^F_t\big\rangle_{\mathcal{A}}\,(1+p^{-2s-k+2})\widetilde{L}^{\mathrm{EZ}}_p(2s+2k-3,\Phi). \end{aligned}$$

Finally we would like to remark that $\overline{D_F(\overline{s})} = D_F(s)$. Hence the theorem is proven. □

§5. Quotients of L-functions

We begin with the concept of (strong) Hecke-Jacobi eigenforms and newforms in the context of Jacobi forms. Then we compute the local factors of $D_{\Phi,F}(s)$ at the bad primes and define a global L-function attached to a Jacobi form. Let $\Phi \in \mathcal{J}^{\mathrm{cusp}}_{k,t}$. Then Φ is a Hecke-Jacobi eigenform, if Φ is eigenform with respect to $T^{J,r}(n)$ for all $n \in \mathbb{N}$. We have seen in [He98], that this implies also, that Φ is automatically an eigenform with respect to Ξ_p^r.

5.1. Jacobi newforms

We would like to recall the definition of newforms in the setting of Jacobi forms introduced by Skoruppa and Zagier [S-Z88]. The space of Jacobi newforms is defined as always as the orthogonal complement of oldforms and is equal to

$$(30)\quad \widehat{\mathcal{J}}_{k,t}^{\text{cusp,new}} = \bigcap_{p|t} \operatorname{Ker}\left(\widehat{\mathcal{J}}_{k,t}^{\text{cusp}}|_k\mathcal{D}(T_+^r(p))\right) \cap \operatorname{Ker}\left(\widehat{\mathcal{J}}_{k,t}^{\text{cusp}}|_k\mathcal{D}(\Lambda_+^r(p))\right).$$

The space $\mathcal{J}_{k,t}^{\text{cusp,new}}$ has a basis of Hecke Jacobi eigenforms and

$$(31)\qquad \widehat{\mathcal{J}}_{k,t}^{\text{cusp}} = \widehat{\mathcal{J}}_{k,t}^{\text{cusp,new}} \oplus \bigoplus_{\substack{l,d>0 \\ ld^2|t,\, ld^2>1}} \widehat{\mathcal{J}}_{k,t/(ld^2)}^{\text{cusp,new}}|_k\mathcal{D}\big(\Lambda_-^r(d)T_-^r(l)\big).$$

Hecke-Jacobi eigenforms which are newforms are called Hecke-Jacobi newforms. (see [Gr95, page 80], and [He98] for more details).

Remark 5.1. Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi newform and $F \in S_2^k$ be a Hecke eigenform. Then Φ is an eigenform with respect to the operator $S^{(2)}(X)$. Let us denote the eigenvalue by $S_\Phi^{(2)}(X)$. In particular we have $\widehat{\Phi}|_k\mathcal{D}(\Lambda_+^r(p)) = 0$ and

$$\begin{aligned}
&\overline{D_{p,F}^2(\overline{X})}\left\langle \widehat{\Phi}, \sum_{\delta=0}^{\infty} \widehat{\Phi}_{tp^{2\delta}}^F|_k\mathcal{D}\big(\Lambda_+^r(p^\delta)(p^{-2}\overline{X})^\delta\big)\right\rangle_{\mathcal{A}} \\
&\quad = (1-X)\left\langle \widehat{\Phi}|_k\mathcal{D}\big(S^{(2)}(X)\big), \widehat{\Phi}_t^F\right\rangle_{\mathcal{A}} \\
&\quad = (1-X)S_\Phi^{(2)}(X)\left\langle \widehat{\Phi}, \widehat{\Phi}_t^F\right\rangle_{\mathcal{A}}.
\end{aligned}$$

5.2. Local L-factors for newforms

In this section we restrict our attention on the computation of the local L-factors of $D_{\Phi,F}(s)$ at the bad primes. We give a complete solution for Hecke-Jacobi newforms $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ of weight k and arbitrary index t.

Let us first fix some notation. Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi eigenform. We denote the eigenvalues of Φ with respect to $T^{J,r}(p)$ and $p^{-2}\Xi_p^r$ by λ and ε, respectively. The operator $p^{-2}\Xi_p^r$ has some connection with the well known involution W_p in the theory of Jacobi forms, when $p\|t$, i.e., $p|t$ and $p^2 \nmid t$, see [He98, Remark 3.2]. Let $S^{(2)}(X)$ be as in Section 3, Theorem 3.2. Let $X = p^{-2s-k+2}$. For $p|t$ we have the following simplification

$$(32)\quad \widehat{\Phi}|_k\mathcal{D}(S^{(2)}(X)) = \widehat{\Phi}|_k\mathcal{D}\big(1-(p^{-2}T^{J,r}(p)+p^{-1}-1)X + p^{-3}\Xi_p^r X^2\big).$$

From now on, we assume Φ to be a Hecke-Jacobi newform. Thus Φ lies in the kernel of $T_+^r(p)$. This can be used to find a certain hidden relation between the operators $T^{J,r}(p)$ and $p^{-2}\Xi_p^r$ on the space $\mathcal{J}_{k,t}^{\text{cusp,new}}$. We have (cf. [He98, Section 3.2]): $0 = \widehat{\Phi}|_k\mathcal{D}\big(pT^{J,r}(p) + p^2 + \Xi_p^r\big)$, which allows us to identify $T^{J,r}(p)$ with $-(p + p^{-1}\Xi_p^r)$. Thus we have

$$\begin{aligned}\widehat{\Phi}|_k\mathcal{D}(S^{(2)}(X)) &= (1+X)\,\widehat{\Phi}|_k\mathcal{D}(1 + p^{-3}\Xi_p^r X)\\ &= (1+X)(1+p^{-1}\varepsilon X)\,\widehat{\Phi}.\end{aligned}$$

We know that $\varepsilon = \pm 1$ if $p|t$ with $p^2 \nmid t$.

Let $p^2|t$. Then similar as in [Gr95] and [He98], we obtain $\widehat{\Phi}|_k\mathcal{D}(\Xi_p^r) = 0$. In other words let $p^2|t$ and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp,new}}$, then we have

$$\widehat{\Phi}|_k\mathcal{D}\big(S^{(2)}(X)\big) = (1+X)\,\widehat{\Phi}.$$

Actually in this case we have to calculate

$$\Big\langle \widehat{\Phi},\, \Big(\widehat{\Phi}_t^F|_k\mathcal{D}\big(S^{(2)}(\overline{X})\big) - \widehat{\Phi}_{t/p^2}^F|_k\mathcal{D}\big(\Lambda_-^r(p)S^{(2)}(\overline{X})p^{-2}\,\overline{X}\,\big)\Big)\Big\rangle_{\mathcal{A}}. \tag{33}$$

But this leads after some simplifications essentially to the computation of the two expressions $\big\langle \widehat{\Phi}|_k\mathcal{D}(S^{(2)}(X)), \widehat{\Phi}_t^F\big\rangle_{\mathcal{A}}$ and $\big\langle \widehat{\Phi}|_k\mathcal{D}(\Lambda_+^r(p)), \widehat{\Phi}_{t/p^2}^F\big\rangle_{\mathcal{A}}$. Hence it is obvious that the term related to $\mathcal{D}(\Lambda_-^r(p))$ does not contribute to our formula.

The standard L-function $L(s,\Phi)$ attached to a Hecke-Jacobi newform $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ is defined in the following way: Let $L_p(s,\Phi) = \widetilde{L}^{\text{EZ}}(s,\Phi)$ if $p \nmid t$ and $L_p(s,\Phi) = 1 + \varepsilon_p p^{k-2}p^{-s}$ otherwise. Here ε_p is the eigenvalue of the operator $p^{-2}\Xi_p^r$. We know that $\varepsilon = \pm 1$ in the case $p|t$ with $p^2 \nmid t$ and 0 if $p^2|t$. Then

$$L(s,\Phi) = \prod_p L_p(s,\Phi)^{-1}.$$

The definition of $L(s,\Phi)$ is compatible with the one given in Corollary 4.4. Let $X = p^{-2s-k+2}$ and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp,new}}$ be a Hecke-Jacobi newform. For convenience we put $\zeta_p(s) = 1 - p^{-s}$. Then we have

$$(1-X)\,\widehat{\Phi}|_k\mathcal{D}(S^{(2)}(X)) = \zeta_p(4s+2k-4)L_p(2s+2k-3,\Phi)\,\widehat{\Phi}. \tag{34}$$

5.3. Main results

Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ and $F \in S_2^k$, where $k, t \in \mathbb{N}$ and k be even. In Section 2 we obtained an integral representation of the Dirichlet series

$$D_{\Phi,F}(s) = \sum_{\lambda=1}^{\infty} \left\langle \widehat{\Phi}|_k \widetilde{U}_\lambda, \widehat{\Phi}_{t\lambda^2}^F \right\rangle_{\mathcal{A}} (t\lambda^2)^{-s}. \tag{35}$$

More precisely we proved that

$$\left\langle E_{2,1}^{k,t}((*,0),\Phi;s), F(*)\right\rangle = \beta_k (4\pi)^{-(s+k-2)}\Gamma(s+k-2) D_{\Phi,F}(s).$$

Let F be a Hecke eigenform and Φ be a Hecke-Jacobi newform, then we showed that $D_{\Phi,F}(s)$ has an Euler product. We obtained the following result

Theorem 5.2. *Let $k, t \in \mathbb{N}$ and let k be even. Let $F \in S_2^k$ be a Hecke eigenform and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi newform. Let $s \in \mathbb{C}$. Then we have*

$$\begin{aligned} D_{\Phi,F}(s) = t^{-s} \left\langle \widehat{\Phi}, \widehat{\Phi}_t^F \right\rangle_{\mathcal{A}} \zeta(4s+2k-4)^{-1} \\ \times D_F(2s+k-2) L(2s+2k-3,\Phi)^{-1}. \end{aligned} \tag{36}$$

The formula presented in the theorem is well-defined, because $D_{\Phi,F}(s)$ possesses a meromorphic continuation on the whole complex plane. This follows from the integral representation (35). The reader familiar with the methods and results of our recent paper [He98] should be able to formulate Theorem 5.2 without the assumption newform, when we only assume t to be square-free.

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