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MALGRANGE'S VANISHING THEOREM IN 1-CONCAVE CR MANIFOLDS

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Abstract. We prove a vanishing theorem for the $\overline{\partial}_b$-cohomology in top degree on 1-concave CR generic manifolds.

The aim of this paper is an analogous in the CR setting of Malgrange's theorem [13] for the vanishing of the $\overline{\partial}$-cohomology in top degree in connected, non compact complex manifolds. We prove the following theorem

THEOREM 0.1. *If M is a connected, $\mathcal{C}^{2+\ell}$-smooth, $\ell \in \mathbb{N}$, non compact, 1-concave, CR generic manifold of real codimension k in a complex manifold of complex dimension n, $n \geq 3$, then for all p, $0 \leq p \leq n$,*

$$H_\ell^{p,n-k}(M) = 0,$$

where $H_\ell^{p,n-k}(M)$, $0 \leq p \leq n$, denote the $\overline{\partial}_M$-cohomology groups of top degree on M with coefficients of class $\mathcal{C}^\ell$.

If moreover M is assumed to be $\mathcal{C}^\infty$-smooth, then

$$H_\infty^{p,n-k}(M) = 0 \ .$$

We point out that this theorem holds without any global condition on M (1-concavity is a local condition, cf. Sect. 1). If, additional, certain global convexity condition is fulfilled then the vanishing of $H_\ell^{p,n-k}(M)$ is well-known. The first result of this type can be found in the paper [1] (Th. 7.2.4) of Airapetjan and Henkin, where the vanishing of $H_\infty^{p,n-k}(M)$ is obtained under the hypothesis that M is a closed submanifold of a Stein manifold. Generalizations of this result can be found in [9] and [12].

Note that in view of the lack of the Dolbeault isomorphism in top degree on 1-concave, CR-generic manifolds, one cannot deduce the vanishing of the groups $H_\ell^{p,n-k}(M)$, $0 \leq \ell \leq \infty$, from the vanishing of one of them.

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The proof of the theorem is based on some local results on the solvability of the tangential Cauchy-Riemann equation in top degree and the approximation of $\overline{\partial}_M$-closed $\mathcal{C}^\ell$-forms of top degree minus one by $\mathcal{C}^{\ell+1}$-smooth, $\overline{\partial}_M$-closed forms in 1-concave, CR generic manifolds, on the unique continuation of CR functions and on the Grauert bumping method.

We may notice by looking precisely to the proof that the manifold M needs not to be a 1-concave CR-generic manifold embedded into a complex manifold but that Theorem 0.1 still holds under the following assumptions:

(i) The CR-manifold M is either locally embeddable and minimal in the sense of Tumanov [14] or abstract and 1-concave (this ensures in both cases the unique continuation of CR functions, see [14], [3]).

(ii) One can solve locally the tangential Cauchy-Riemann equation in top degree in the $\mathcal{C}^\ell$-class with an arbitrary small gain of regularity and approximate locally $\overline{\partial}_M$-closed $\mathcal{C}^\ell$-forms of top degree minus one by $\mathcal{C}^{\ell+1}$-smooth, $\overline{\partial}_M$-closed forms.

Note, moreover, that if E is a vector bundle over M, which locally extends as an holomorphic vector bundle, then Theorem 0.1 still holds for $H_\ell^{p,n-k}(M,E)$.

As a consequence of Theorem 0.1, we get a global approximation theorem.

THEOREM 0.2. *If M is a connected, $\mathcal{C}^\infty$-smooth, non compact, 1-concave, CR-generic manifold of real codimension k in a complex manifold X of complex dimension n, $n \geq 3$, and p an integer, $0 \leq p \leq n$, then each continuous, $\overline{\partial}_M$-closed, $(p, n-k-1)$-form in M can be approximated uniformly on compact subsets of M by $\overline{\partial}_M$-closed, $(p, n-k-1)$-forms of class $\mathcal{C}^\infty$ in M.*

Again this theorem holds without any global condition on M. In the case when M is a closed submanifold of a Stein manifold, it was proved by Airapetjan and Henkin (cf. [1], Th. 7.2.3).

§1. Notations and definitions

Let X be a complex manifold of complex dimension n. If M is a $\mathcal{C}^{2+\ell}$-smooth real submanifold of real codimension k in X, we denote by $T_\tau^{\mathbb{C}}(M)$ the complex tangent space to M at $\tau \in M$.

Such a manifold M can be represented locally in the form

$$M = \{z \in \Omega | \rho_1(z) = \cdots = \rho_k(z) = 0\} \tag{1}$$

where the ρ_ν's, $1 \leq \nu \leq k$, are real $\mathcal{C}^{2+\ell}$ functions in an open subset Ω of X. If M is $\mathcal{C}^\infty$ smooth the functions ρ_ν can be chosen of class $\mathcal{C}^\infty$.

In this representation we have

$$T_\tau^{\mathbb{C}}(M) = \left\{ \zeta \in \mathbb{C}^n \mid \sum_{j=1}^{n} \frac{\partial \rho_\nu}{\partial z_j}(\tau)\zeta_j = 0, \quad \nu = 1, \ldots, k \right\} \tag{2}$$

and $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M) \geq n-k$, for $\tau \in M \cap \Omega$, where $(z_1, \ldots, z_n)$ are local holomorphic coordinates in a neighborhood of τ.

Definition 1.1. The submanifold M is called CR if the number $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M)$ is independent of the point $\tau \in M$. If moreover $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M) = n-k$ for every $\tau \in M$, then M is called CR *generic*.

In the local representation, M is CR generic if and only if

$$\overline{\partial}\rho_1 \wedge \cdots \wedge \overline{\partial}\rho_k \neq 0 \text{ on } M.$$

Definition 1.2. Let M be a $\mathcal{C}^{2+\ell}$-smooth CR generic submanifold of X. M is 1-*concave*, if for each $\tau \in M$, each local representation of M of type (1) in a neighborhood of τ in X and each $x \in \mathbb{R}^k \setminus \{0\}$, the quadratic form on $T_\tau^{\mathbb{C}}(M)$ defined by $\sum\limits_{\alpha,\beta} \frac{\partial^2 \rho_x}{\partial z_\alpha \partial \overline{z}_\beta}(\tau)\zeta_\alpha \overline{\zeta}_\beta$, where $\rho_x = x_1\rho_1 + \cdots + x_k\rho_k$ and $\zeta \in T_\tau^{\mathbb{C}}(M)$, has at least one negative eigenvalue.

The bundle of (p,q)-forms on M, denoted by $\Lambda^{p,q}{}_{|M}$, is, by definition, the restriction of the bundle $\Lambda^{p,q}$ of (p,q)-forms in X to the submanifold M. Thus a section f of $\Lambda^{p,q}\big|_M$ is obtained locally from an ambient form by restriction of the coefficients of the (p,q)-form to M. We denote by $\mathcal{C}^\ell_{p,q}(M)$ (resp. $\mathcal{C}^\infty_{p,q}(M)$, if M is $\mathcal{C}^\infty$-smooth) the $\mathcal{C}^\ell$ (resp. $\mathcal{C}^\infty$) sections of the bundle $\Lambda^{p,q}{}_{|M}$.

Following Kohn and Rossi [10], two forms $f, g \in \mathcal{C}^\ell_{p,q}(M)$ (resp. $\mathcal{C}^\infty_{p,q}(M)$) are said to be equal if and only if $\int_M f \wedge \varphi = \int_M g \wedge \varphi$ for every form $\varphi \in \mathcal{C}^\infty_{n-p,n-k-q}(X)$ with compact support.

We set on $\mathcal{C}^\ell_{p,q}(M)$ the topology of uniform convergence of the coefficients and all their derivatives up to order ℓ on compact subsets of M. This topology will be called the $\mathcal{C}^\ell$-topology on M. The dual space of $\mathcal{C}^\ell_{p,q}(M)$ is denoted by $\mathcal{E}'^\ell_{n-p,n-k-q}(M)$, it is the space of $(n-p, n-k-q)$-currents of order ℓ with compact support on M. If M is of class $\mathcal{C}^\infty$, then the space $\mathcal{C}^\infty_{p,q}(M)$ is provided with the topology of uniform convergence of the coefficients and all their derivatives on compact subsets of M. Its dual $\mathcal{E}'_{n-p,n-k-q}(M)$ is the space of $(n-p, n-k-q)$-currents with compact support on M.

We denote by $\mathcal{D}'^\ell_{p,q}(M)$ the space of (p,q)-currents of order l on M, this space is the dual of the space $\mathcal{D}^\ell_{n-p,n-k-q}(M)$ of $\mathcal{C}^\ell$-smooth $(n-p, n-k-q)$-forms with compact support on M provided with its usual inductive limit topology. If M is of class $\mathcal{C}^\infty$, $\mathcal{D}'_{p,q}(M)$ denotes the space of (p,q)-currents on M, this space is the dual of the space $\mathcal{D}_{n-p,n-k-q}(M)$ of $\mathcal{C}^\infty$-smooth $(n-p, n-k-q)$-forms with compact support on M provided with its usual inductive limit topology.

We denote by $\overline{\partial}_M$ the tangential Cauchy-Riemann operator on M.

A current $f \in \mathcal{D}'^\ell_{p,q}(M)$ is called CR if and only if $\overline{\partial}_M f = 0$.

If U is an open subset of M, then for $\ell \in \mathbb{N} \cup \{\infty\}$,

$Z^\ell_{p,q}(U)$ is the Fréchet space of CR (p,q)-forms of class $\mathcal{C}^\ell$ on U;

$E^\ell_{p,q}(U)$ is the subspace of $Z^\ell_{p,q}(U)$ of the forms f such that $f = \overline{\partial}_M g$ with $g \in \mathcal{C}^\ell_{p,q-1}(U)$;

$H^{p,q}_\ell(U)$ denotes the quotient space $Z^\ell_{p,q}(U)/E^\ell_{p,q}(U)$.

If Ω is a relatively compact open subset in M, we denote by $\mathcal{C}^\ell_{p,q-1}(\overline{\Omega})$ the Banach space of (p,q)-forms of class $\mathcal{C}^\ell$ on $\overline{\Omega}$ and by $\mathcal{C}^{\ell+\alpha}_{p,q-1}(\overline{\Omega})$ the Banach space of (p,q)-forms whose coefficients are of class $\mathcal{C}^{\ell+\alpha}$, $0<\alpha<1$, on $\overline{\Omega}$.

If D is a relatively compact open subset in M, we denote by germ $\mathcal{C}^\ell_{p,q}(\overline{D})$ the space of germs of (p,q)-forms of class $\mathcal{C}^\ell$ in neighborhoods of $\overline{D}$. Then germ $Z^\ell_{p,q}(\overline{D})$ is the space of germs of CR (p,q)-forms of class $\mathcal{C}^\ell$ in neighborhoods of $\overline{D}$, germ $E^\ell_{p,q}(\overline{D}) = \text{germ}\, Z^\ell_{p,q}(\overline{D}) \cap \overline{\partial}_M \,\text{germ}\, C^\ell_{p,q-1}(\overline{D})$ and germ $H^{p,q}_\ell(\overline{D}) = \text{germ}\, Z^\ell_{p,q}(\overline{D})/\,\text{germ}\, E^\ell_{p,q}(\overline{D})$.

§2. Proof of Malgrange's theorem in the $\mathcal{C}^\ell$-case

Let X be a complex manifold of complex dimension n, $n\geq 3$, M a connected, $\mathcal{C}^{2+\ell}$-smooth, $\ell \in \mathbb{N}$, non compact, 1-concave, CR generic submanifold of real codimension k in X, and p an integer, $0 \leq p \leq n$.

Local results

We need first a result on the local solvability of the tangential Cauchy-Riemann equation in top degree on M.

PROPOSITION 2.1. *For every point z_0 in M, one can find a neighborhood M_0 of z_0 in M such that for each open subset $\Omega \subset\subset M_0$, there exists a continuous linear operator K_Ω from $\mathcal{C}^\ell_{p,n-k}(\overline{\Omega})$ into $\mathcal{C}^{\ell+\frac{1}{2}}_{p,n-k-1}(\overline{\Omega})$ which satisfies $\overline{\partial}_M K_\Omega f = f$ for all differential forms f in $\mathcal{C}^\ell_{p,n-k}(\overline{\Omega})$.*

Proof. This result can be easily deduced from Theorem 0.1 in [2]. Under the hypothesis $\ell > 0$, a slightly weaker result, also sufficient for our application, is given in Theorem 7.1.2 of [1].

We shall use also some approximation theorem for $\overline{\partial}_M$-closed $(p, n-k-1)$-differential forms.

DEFINITION 2.2. Let U and V be two open subsets of M such that $U \subset V$. We shall say that U *has no hole with respect to* V if for each compact subset K of U there exists a compact subset $\widetilde{K}$ of U such that $K \subset \widetilde{K}$ and $V \setminus \widetilde{K}$ has no connected component which is relatively compact in V.

PROPOSITION 2.3. *For every point z_0 in M, there exists a neighborhood M_0 of z_0 in M such that for each open subset $\Omega \subset\subset M_0$ without hole with respect to M_0 the image of the restriction map*

$$Z^\ell_{p,n-k-1}(M_0) \longrightarrow Z^\ell_{p,n-k-1}(\Omega)$$

is dense with respect to the uniform convergence of the coefficients and all their derivatives up to order ℓ on compact subsets of Ω.

Proof. Let z_0 be a fixed point in M. By the Hahn-Banach theorem, it is sufficient to prove that there exists a neighborhood M_0 of z_0 in M such that for each open subset $\Omega \subset\subset M_0$ without hole with respect to M_0, if L is a continuous linear form on $\mathcal{C}^\ell_{p,n-k-1}(\Omega)$, whose restriction to $Z^\ell_{p,n-k-1}(M_0)$ vanishes, then the restriction of L to $Z^\ell_{p,n-k-1}(\Omega)$ is identically equal to zero. Note that such a linear form L is a $\overline{\partial}_M$-closed $(n-p,1)$-current of order ℓ on M_0, with compact support in Ω. By Theorem 1' in [7] (see also Theorem 2.4 in [11]) in the case $\ell = 0$ and their direct generalization, using Proposition 2.1, to the case $\ell > 0$, we can find a neighborhood M_0 of z_0 in M on which we can solve the $\overline{\partial}_M$-equation with compact support in M_0 in bidegree $(n-p,1)$ for currents of order ℓ. We choose such an M_0 and $\Omega \subset\subset M_0$, then for $L \in \mathcal{E}'^\ell_{p,n-k-1}(\Omega)$ with $L_{|Z^\ell_{p,n-k-1}(M_0)} \equiv 0$, there exists a $(p,0)$-form T with compact support in M_0 such that $\overline{\partial}_M T = L$.

The $(p,0)$-form T is CR on $M_0 \setminus \operatorname{supp} L$ and vanishes on an open subset of $M_0 \setminus \operatorname{supp} L$. Since M is 1-concave, if Ω has no hole with respect to M_0, then T vanishes on a neighborhood of $M_0 \setminus \Omega$ by analytic extension (cf. [6]). Consequently the support of T is contained in Ω. Let $f \in Z^\ell_{p,n-k-1}(\Omega)$, then by the Airapetjan-Henkin Theorem 7.2.1 in [1], f can be approximated locally by $\mathcal{C}^{\ell+1}$-smooth $\overline{\partial}_M$-closed $(p,n-k-1)$-differential forms. Let $(U_i)_{i\in I}$ be a finite open covering of the support of T by open subsets satisfying the Airapetjan-Henkin approximation theorem and for each $i \in I$, $(f^i_\nu)_{\nu\in\mathbb{N}}$ a sequence of $\mathcal{C}^\infty$-smooth $\overline{\partial}_M$-closed $(p,n-k-1)$-differential forms in U_i, which converges to f on U_i in the $\mathcal{C}^\ell$-topology. If $(\chi_i)_{i\in I}$ denotes a partition of unity subordinated to the covering $(U_i)_{i\in I}$, then setting $f_\nu = \sum_{i\in I} \chi_i f^i_\nu$ we get a sequence $(f_\nu)_{\nu\in\mathbb{N}}$ of $\mathcal{C}^{\ell+1}$-smooth $(p,n-k-1)$-differential forms which converges to f on Ω in the $\mathcal{C}^\ell$-topology and such that the sequence $(\overline{\partial}_M f_\nu)_{\nu\in\mathbb{N}}$ converges to zero on Ω in the $\mathcal{C}^\ell$-topology. We obtain

$$L(f) = \lim_{\nu\to\infty} L(f_\nu) = \lim_{\nu\to\infty} \langle \overline{\partial}_M T, f_\nu\rangle = \lim_{\nu\to\infty} \langle T, \overline{\partial}_M f_\nu\rangle = 0.$$

A first global consequence of the local results

By standard arguments (see e.g. the proofs of Lemma 2.3.1 in [8] and Proposition 3 in Appendix 2 of [8]), it follows from Proposition 2.1 that, if D is a relatively compact open subset of M, $E^\ell_{p,n-k}(\overline{D})$ is closed and finite codimensional in $Z^\ell_{p,n-k}(\overline{D})$. Moreover we have

PROPOSITION 2.4. *Let D be a relatively compact open subset of M. There exists a continuous linear operator $A : Z^\ell_{p,n-k}(\overline{D}) \to \mathcal{C}^\ell_{p,n-k-1}(\overline{D})$ such that $\overline{\partial}_M Af = f$ for all $f \in E^\ell_{p,n-k}(\overline{D})$.*

The bumping method

DEFINITION 2.5. A *bump* in M is an ordered collection $[M_0, \Omega_1, \Omega_2]$, where M_0, Ω_1 and Ω_2 are open subsets of M such that

(i) M_0 is as in Propositions 2.1 and 2.3.

(ii) Ω_1 and Ω_2 have $\mathcal{C}^2$-smooth boundary and $\Omega_1 \subset \Omega_2 \subset\subset M_0$.

(iii) $\overline{\Omega}_1$ admits a basis of neighborhoods without hole with respect to M_0.

Note that $\Omega_1 = \emptyset$ is allowed in this definition.

Definition 2.6. An *extension element* in M is an ordered pair $[D_1, D_2]$, where $D_1 \subset D_2$ are open subsets with $\mathcal{C}^2$-boundary in M such that there exists a bump $[M_0, \Omega_1, \Omega_2]$ in M with the following properties:

$$D_2 = D_1 \cup \Omega_2, \quad \Omega_1 = D_1 \cap \Omega_2 \quad \text{and} \quad \overline{(D_1 \smallsetminus \Omega_2)} \cap \overline{(\Omega_2 \smallsetminus \Omega_1)} = \emptyset.$$

Proposition 2.7. *Let $[D_1, D_2]$ be an extension element in M, then the restriction map*

$$\operatorname{germ} H_\ell^{p,n-k}(\overline{D}_2) \longrightarrow \operatorname{germ} H_\ell^{p,n-k}(\overline{D}_1)$$

is injective.

Proof. Let $U_1 \subset U_2$ be open neighborhoods of $\overline{D}_1$ and $\overline{D}_2$ in M respectively and let $f \in Z^\ell_{p,n-k}(U_2)$ and $u_1 \in \mathcal{C}^\ell_{p,n-k-1}(U_1)$ be given such that $\overline{\partial}_M u_1 = f$ on U_1. We have to prove the existence of a neighborhood $W_2 \subset U_2$ of $\overline{D}_2$ in M and of a differential form $u_2 \in \mathcal{C}^\ell_{p,n-k-1}(W_2)$ with $\overline{\partial}_M u_2 = f$ on W_2.

Let $[M_0, \Omega_1, \Omega_2]$ be the bump associated to the extension element $[D_1, D_2]$ and $V_2 \subset\subset U_2 \cap M_0$ a neighborhood of $\overline{\Omega}_2$ in M. By Proposition 2.1, there exists $u \in \mathcal{C}^\ell_{p,n-k-1}(V_2)$ such that $\overline{\partial}_M u = f$ on V_2. Hence we get $\overline{\partial}_M(u_1 - u) = 0$ on $U_1 \cap V_2$. We choose a neighborhood $W_1 \subset U_1 \cap V_2$ of $\overline{\Omega}_1$ without hole with respect to M_0, then by Proposition 2.3, we can find a sequence $(\omega_\nu)_{\nu\in\mathbb{N}} \subset Z^\ell_{p,n-k-1}(M_0)$ which converges to $u_1 - u$ in the $\mathcal{C}^\ell$-topology on W_1. Let V be a neighborhood of $\overline{\Omega_2 \smallsetminus \Omega_1}$ such that $V \subset V_2 \cap M_0$ and $V \cap \overline{(D_1 \smallsetminus \Omega_2)} = \emptyset$, and χ a $\mathcal{C}^{\ell+1}$-smooth function with compact support in V equal to 1 on a neighborhood $\widetilde{V}$ of $\overline{\Omega_2 \smallsetminus \Omega_1}$. Setting $v_\nu = (1-\chi)u_1 + \chi(u + w_\nu)$, we define a sequence $(v_\nu)_{\nu\in\mathbb{N}}$ in $\mathcal{C}^\ell_{p,n-k-1}(U_1 \cup V)$ such that the sequence $\overline{\partial}_M v_\nu = f - \overline{\partial}_M \chi \wedge (u_1 - u - w_\nu)$ converges to f in the $\mathcal{C}^\ell$-topology on the neighborhood $\widetilde{U}_1 \cup \widetilde{V}$ of $\overline{D}_2$ in M, where $\widetilde{U}_1$ is a neighborhood of $\overline{D}_1$ such that $\widetilde{U}_1 \subset U_1$ and $\widetilde{U}_1 \cap V = W_1 \cap V$. Let $W_2 \subset\subset \widetilde{U}_1 \cup \widetilde{V}$ be a neighborhood of $\overline{D}_2$. Then, using Proposition 2.4, we get a $(p, n-k-1)$-differential form u_2 of class $\mathcal{C}^\ell$ on W_2 such that $\overline{\partial}_M u_2 = f$ on W_2.

Proposition 2.8. *Let $[D_1, D_2]$ be an extension element in M such that $D_1 \subset\subset M$, then the restriction map*

$$\operatorname{germ} Z^\ell_{p,n-k-1}(\overline{D}_2) \longrightarrow \operatorname{germ} Z^\ell_{p,n-k-1}(\overline{D}_1)$$

has dense image with respect to uniform convergence of the coefficients and their derivatives up to order ℓ on $\overline{D}_1$.

Proof. Let U_1 be an open neighborhood of $\overline{D}_1$ in M and $[M_0, \Omega_1, \Omega_2]$ the bump associated to the extension element $[D_1,D_2]$. Let $f \in Z^\ell_{p,n-k-1}(U_1)$ be given and $W_1 \subset U_1$ a neighborhood of $\overline{\Omega}_1$ without hole with respect to M_0. By Proposition 2.3, there exists a sequence $(g_\nu)_{\nu\in\mathbb{N}} \subset Z^\ell_{p,n-k-1}(M_0)$ which converges to f in the $\mathcal{C}^\ell$-topology on W_1. Let V be a neighborhood of $\overline{\Omega_2 \smallsetminus \Omega_1}$ such that $V \subset M_0$ and $V \cap \overline{(D_1 \smallsetminus \Omega_2)} = \emptyset$, and χ a $\mathcal{C}^{\ell+1}$-smooth function with compact support in V equal to 1 on a neighborhood $\widetilde{V}$ of $\overline{\Omega_2 \smallsetminus \Omega_1}$. Setting $\tilde{f}_\nu = (1-\chi)f + \chi g_\nu$, we define a sequence $(\tilde{f}_\nu)_{\nu\in\mathbb{N}}$ of forms of class $\mathcal{C}^\ell$ on the neighborhood $U_1 \cup V$ of $\overline{D}_2$, which converges to f in the $\mathcal{C}^\ell$-topology on $\overline{D}_1$. Moreover, since $\overline{\partial}_M \tilde{f}_\nu = \overline{\partial}_M \chi \wedge (f - g_\nu)$ the sequence $(\overline{\partial}_M \tilde{f}_\nu)_{\nu\in\mathbb{N}}$ converges to zero in the $\mathcal{C}^\ell$-topology on $U_2 = \widetilde{U}_1 \cup \widetilde{V}$, where $\widetilde{U}_1$ is a neighborhood of $\overline{D}_1$ such that $\widetilde{U}_1 \subset U_1$ and $\widetilde{U}_1 \cap V = W_1 \cap V$. As $D_1 \subset\subset M$, we can choose a relatively compact neighborhood W_2 of $\overline{D}_2$ in M and apply Proposition 2.4. Therefore, there exists a sequence $(u_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{C}^\ell_{p,n-k-1}(\overline{W}_2)$ which converges to zero in the $\mathcal{C}^\ell$-topology on $\overline{W}_2$ and satisfies $\overline{\partial}_M u_\nu = \overline{\partial}_M \tilde{f}_\nu$. If $f_\nu = \tilde{f}_\nu - u_\nu$, we get a sequence $(f_\nu)_{\nu\in\mathbb{N}} \subset Z^\ell_{p,n-k}(W_2)$ which converges to f in the $\mathcal{C}^\ell$-topology on $\overline{D}_1$.

We need now two technical lemmas about the existence of extension elements to jump from one level of an exhausting function on M to another level.

LEMMA 2.9. *Let φ be a function of class $\mathcal{C}^2$ on M and z_0 a non degenerate critical point for φ. Suppose $\varphi(z_0) = 0$, $\varphi^{-1}(0)$ is compact and z_0 is the only critical point on $\varphi^{-1}(0)$. Then there exists a neighborhood V_0 of z_0 in M such that for all neighborhood $V \subset\subset V_0$ of z_0 in M, we can find an extension element $[D_1, D_2]$ in M with the following properties:*

(i) $D_1 \supset \varphi^{-1}((-\infty, 0[) \smallsetminus V$;

(ii) $z_0 \in D_2 \smallsetminus \overline{D}_1 \subset V$.

Proof. If z_0 is a point of local minimum, we choose V_0 so small that $V_0 \cap \varphi^{-1}((-\infty, 0[) = \emptyset$ and $M_0 \subset V \subset V_0$ a neighborhood of z_0 satisfying Propositions 2.1 and 2.3. Taking $\Omega_1 = \emptyset$, $\Omega_2 \subset\subset M_0$ a neighborhood of z_0 and setting $D_1 = \varphi^{-1}((-\infty, 0[)$ and $D_2 = D_1 \cup \Omega_2$, we get the required extension element.

Assume now that z_0 is not a point of local minimum. By the Morse lemma, there exist local real coordinates $(x_1, \ldots, x_{2n})$ around z_0 in X such that $\varphi = x_1^2 + \ldots + x_s^2 - x_{s+1}^2 - \ldots - x_{2n-k}^2$. Let V_0 be a neighborhood of z_0 on which we are in the above situation and $M_0 \subset\subset V \subset V_0$ the intersection of M with a small ball centered in z_0 in holomorphic coordinates around z_0 as in Propositions 2.1 and 2.3. Let B be a ball centered in z_0 with respect to the Morse coordinates $(x_1, \ldots, x_{2n-k})$ such that $B \subset M_0$, and U a small neighborhood of z_0 relatively compact in B. Let ε be equal to $\frac{1}{2} \min\limits_{z \in \overline{U}} |\varphi(z)|$. We choose $\theta \in \mathcal{D}(U)$ such that $0 < \theta(z) < \varepsilon$, if $z \in U$, and we set $\Omega_1 = \{z \in B \mid \varphi(z) + \theta(z) < 0\}$ and $\Omega_2 = \{z \in B \mid \varphi(z) - \theta(z) < 0\}$. Then it is clear that Ω_1 has no hole with respect to B (it is sufficient to look at the picture in the Morse coordinates) and as the boundary of B is connected and M_0 has no compact connected component then Ω_1 has also no hole with respect to M_0. Smoothing the boundary of Ω_1 and Ω_2 we get a bump $[M_0, \Omega_1, \Omega_2]$ in M such that $D_2 = \varphi^{-1}((-\infty, 0[) \cup \Omega_2$ and $D_1 = D_2 \setminus (\overline{\Omega}_2 \setminus \Omega_1)$ have the required properties.

From Lemma 2.9, one easily obtains the following lemma (cp. the proof of Theorem 7.10 in [12]).

Lemma 2.10. *Let φ be a function of class $\mathcal{C}^2$ on M all critical points of which are non degenerate such that the following conditions are fulfilled:*

(i) *no critical point of φ lies on $\varphi^{-1}(\{0,1\})$;*

(ii) *$\varphi^{-1}([0,1])$ is compact;*

(iii) *φ has no point of local maximum in $\varphi^{-1}(]0,1[)$.*

Then there exists a finite number of extension elements $[D_j, D_{j+1}]$, $j = 0, \ldots, N$, such that $D_0 = \varphi^{-1}((-\infty, 0[)$ and $D_{N+1} = \varphi^{-1}((-\infty, 1[)$.

As an easy consequence of Propositions 2.7 and 2.8 and Lemma 2.10, we obtain the following result:

Proposition 2.11. *Let φ be a real exhausting function of class $\mathcal{C}^2$ on M without local maximum and such that all critical points of φ are non degenerate. Let $\alpha, \beta \in \varphi(M)$ with $\alpha < \beta$ and such that no critical point of φ lies on $\varphi^{-1}(\{\alpha, \beta\})$ and set $D_\alpha = \varphi^{-1}((-\infty, \alpha[)$ and $D_\beta = \varphi^{-1}((-\infty, \beta[)$.*

(i) *The restriction map*

$$\operatorname{germ} H^{p,n-k}_{\ell}(\overline{D}_{\beta}) \longrightarrow \operatorname{germ} H^{p,n-k}_{\ell}(\overline{D}_{\alpha})$$

is injective

(ii) *The restriction map*

$$\operatorname{germ} Z^{\ell}_{p,n-k-1}(\overline{D}_{\beta}) \longrightarrow \operatorname{germ} Z^{\ell}_{p,n-k-1}(\overline{D}_{\alpha})$$

has dense image with respect to uniform convergence of the coefficients and their derivatives up to order ℓ on $\overline{D}_{\alpha}$.

Proof of the first assertion of Theorem 0.1

We may now conclude the proof of our Malgrange type theorem in non compact, 1-concave CR manifolds in the $\mathcal{C}^{\ell}$ case, $\ell \in \mathbb{N}$.

Since M is connected and not compact, by a theorem of Green and Wu [4], M admits a real exhausting function φ of class $\mathcal{C}^2$ without local maximum and we may assume that all critical points of φ are non degenerate (cp. e.g. [5]). Let z_0 be a point where φ takes its minimum value. By Proposition 2.1, there exists a neighborhood Ω_0 of z_0 such that $H^{p,n-k}_{\ell}(D) = 0$ for all $D \subset\subset \Omega_0$. As φ is an exhausting function on M, it admits only a finite number of points where φ takes its minimum value. We denote by Ω the union of the previous neighborhoods associated to these points and we choose $\alpha_0 \in \varphi(M)$ such that $\varphi^{-1}((-\infty, \alpha_0[)$ is not empty and contained in Ω and $(\alpha_j)_{j\geq 1} \subset \varphi(M)$ such that no critical point of φ lies on $\varphi^{-1}(\alpha_j)$, $j\geq 0$, and if $D_j = \varphi^{-1}((-\infty, \alpha_j[)$, $D_j \subset D_{j+1}$ for $j\geq 0$ and $M = \bigcup\limits_{j\geq 0} D_j$. We deduce from Proposition 2.11 (i) and from the choice of D_0 that, for all $j\geq 0$,

$$\operatorname{germ} H^{p,n-k}_{\ell}(\overline{D}_j) = 0.$$

Let $f \in Z^{\ell}_{p,n-k}(M)$ be given. Then from Proposition 2.11 (ii) we obtain a sequence $(u_j)_{j\in\mathbb{N}}$ such that $u_j \in \operatorname{germ} \mathcal{C}^{\ell}_{p,n-k}(\overline{D}_j)$, $\overline{\partial}_M u_j = f$ on a neighborhood of $\overline{D}_j$ and $\|u_{j+1} - u_j\|_{\ell,\overline{D}_j} \leq \frac{1}{2^j}$. Hence $u = \lim\limits_{j\to\infty} u_j$ exists, belongs to $\mathcal{C}^{\ell}_{p,n-k-1}(M)$, and solves the equation $\overline{\partial}_M u = f$ on M. □

§3. Proof of Malgrange's theorem in the $\mathcal{C}^{\infty}$-case

We shall first prove an approximation theorem in 1-concave CR manifolds, which is a direct consequence of Malgrange's theorem in the $\mathcal{C}^{\ell}$-case. Then we shall use this theorem to get Malgrange's theorem in the $\mathcal{C}^{\infty}$-case.

THEOREM 3.1. *Let X be a complex manifold of complex dimension n, $n \geq 3$, M a connected, $\mathcal{C}^{3+\ell}$-smooth, $\ell \in \mathbb{N}$, non compact, 1-concave, CR generic submanifold of real codimension k in X and p an integer, $0 \leq p \leq n$. Then the space $Z^{\ell+1}_{p,n-k-1}(M)$ is dense in the space $Z^{\ell}_{p,n-k-1}(M)$ for the topology of uniform convergence of the coefficients and their derivatives up to order ℓ on each compact subset of M.*

Proof. By the Hahn-Banach theorem, it is sufficient to prove that for any $T \in \mathcal{E}'^{\ell}_{n-p,1}(M)$ such that $\langle T, \varphi\rangle = 0$ for all $\varphi \in Z^{\ell+1}_{p,n-k-1}(M)$ we have $\langle T, \psi\rangle = 0$ for all $\psi \in Z^{\ell}_{p,n-k-1}(M)$. Note that the hypothesis on T implies that T is $\overline{\partial}_M$-closed. We shall prove that T is $\overline{\partial}_M$-exact on M.

We define a linear form L on $\mathcal{C}^{\ell+1}_{p,n-k}(M)$ by setting $L(\varphi) = \langle T, \psi\rangle$ for $\varphi \in \mathcal{C}^{\ell+1}_{p,n-k}(M)$, where $\overline{\partial}_M \Psi = \varphi$. The application L is well defined since first $H^{p,n-k}_{\ell+1}(M) = 0$ and consequently all $\varphi \in \mathcal{C}^{\ell+1}_{p,n-k}(M)$ can be written in the form $\varphi = \overline{\partial}_M \psi$ with $\psi \in \mathcal{C}^{\ell+1}_{p,n-k-1}(M)$ and second $\langle T, \psi\rangle$ is independent of the choice of ψ satisfying $\overline{\partial}_M \psi = \varphi$ because $T_{|Z^{\ell+1}_{p,n-k-1}(M)} = 0$.

Moreover $\overline{\partial}_M$ is a closed operator between $\mathcal{C}^{\ell+1}_{p,n-k-1}(M)$ and $\mathcal{C}^{\ell+1}_{p,n-k}(M)$ which is surjective since $H^{p,n-k}_{\ell+1}(M) = 0$, consequently by the open mapping theorem this implies the continuity of L. It follows that L can be represented by a current $S \in \mathcal{E}'^{\ell+1}_{n-p,0}$ which satisfies

$$\langle \overline{\partial}_M S, \varphi\rangle = \langle S, \overline{\partial}_M \varphi\rangle = \langle T, \varphi\rangle$$

for all $\varphi \in \mathcal{C}^{\infty}_{p,n-k-1}(M)$, *i.e.* $\overline{\partial}_M S = T$. By regularity of $\overline{\partial}_M$ in bidegree $(n-p, 1)$, the $(n-p, 0)$-current S is of order ℓ since T is of order ℓ.

It remains to prove that $\langle T, \psi\rangle = 0$ for all $\psi \in Z^{\ell}_{p,n-k-1}(M)$. Let $\psi \in Z^{\ell}_{p,n-k-1}(M)$. In the same way as at the end of the proof of Proposition 2.3, we can construct a sequence $(\psi_\nu)_{\nu\in\mathbb{N}}$ of $\mathcal{C}^{\ell+1}$-smooth $(p, n-k-1)$-differential forms which converges to ψ on M in the $\mathcal{C}^{\ell}$-topology and such that the sequence $(\overline{\partial}_M \psi_\nu)_{\nu\in\mathbb{N}}$ converges to zero on M in the $\mathcal{C}^{\ell}$-topology. It follows that

$$\langle T, \psi\rangle = \lim_{\nu\to\infty} \langle T, \psi_\nu\rangle = \lim_{\nu\to\infty} \langle \overline{\partial}_M S, \psi_\nu\rangle = \lim_{\nu\to\infty} \langle S, \overline{\partial}_M \psi_\nu\rangle = 0.$$

Assume now that M is $\mathcal{C}^{\infty}$-smooth, we shall prove that $H^{p,n-k}_{\infty}(M) = 0$.

Proof of the second assertion of Theorem 0.1

Since M is connected and not compact, by a theorem of Green and Wu [4], M admits a real exhausting function φ of class $\mathcal{C}^\infty$ without local maximum and we may assume that all critical points of φ are non degenerate. Following the proof of the $\mathcal{C}^\ell$-case we can construct a sequence $(D_j)_{j\in\mathbb{N}}$ of open subsets of M such that $D_j \subset D_{j+1}$ and $M = \bigcup\limits_{j\geq 0} D_j$ and satisfying the following two conditions:

(i) $\operatorname{germ} H_j^{p,n-k}(\overline{D}_j) = 0$.

(ii) The restriction map

$$\operatorname{germ} Z^j_{p,n-k-1}(\overline{D}_{j+1}) \longrightarrow \operatorname{germ} Z^j_{p,n-k-1}(\overline{D}_j)$$

has dense image with respect to the $\mathcal{C}^j$-topology.

Let $f \in Z^\infty_{p,n-k}(M)$ and $\varepsilon > 0$ be given. Then we can construct a sequence $(u_j)_{j\in\mathbb{N}}$ such that $u_j \in \operatorname{germ} \mathcal{C}^j_{p,n-k-1}(\overline{D}_j)$, $\overline{\partial}_M u_j = f$ on a neighborhood of $\overline{D}_j$ and $\|u_{j+1} - u_j\|_{\overline{D}_j,j} < \frac{\varepsilon}{2^j}$. By (i) there exists $u_0 \in \operatorname{germ} \mathcal{C}^0_{p,n-k-1}(\overline{D}_0)$ such that $\overline{\partial}_M u_0 = f$ on a neighborhood of $\overline{D}_0$. Assume now that we have already constructed $(u_j)_{0\leq j\leq j_0}$. By (i) there exists $\tilde{u}_{j_0+1} \in \operatorname{germ} \mathcal{C}^{j_0+1}_{p,n-k-1}(\overline{D}_{j_0+1})$ such that $\overline{\partial}_M \tilde{u}_{j_0+1} = f$ on a neighborhood of $\overline{D}_{j_0+1}$. Then $\tilde{u}_{j_0+1} - u_{j_0} \in \operatorname{germ} Z^{j_0}_{p,n-k-1}(\overline{D}_{j_0+1})$ and by (ii) we can find $v_{j_0+1} \in \operatorname{germ} Z^{j_0}_{p,n-k-1}(\overline{D}_{j_0+1})$ such that $\|\tilde{u}_{j_0+1} - u_{j_0} - v_{j_0+1}\|_{\overline{D}_{j_0},j_0} < \frac{1}{2}\frac{\varepsilon}{2^{j_0}}$. Moreover by Theorem 3.1, we choose $\tilde{v}_{j_0+1} \in \operatorname{germ} Z^{j_0+1}_{p,n-k-1}(\overline{D}_{j_0+1})$ with $\|\tilde{v}_{j_0+1} - v_{j_0+1}\|_{\overline{D}_{j_0+1},j_0} < \frac{1}{2}\frac{\varepsilon}{2^{j_0}}$. Setting $u_{j_0+1} = \tilde{u}_{j_0+1} - \tilde{v}_{j_0+1}$, then u_{j_0+1} has the required properties. It follows from the properties of the forms u_j that the sequence $(u_j)_{j\in\mathbb{N}}$ converges to a form u uniformly on each compact subset of M and moreover $u \in \mathcal{C}^\infty_{p,n-k-1}(M)$ and $\overline{\partial}_M u = f$ on M. ∎

Some important consequences of vanishing theorems are approximation theorems. Using the first assertion in Theorem 0.1, we have proved Theorem 3.1. In the same way Theorem 0.2 follows from the second assertion in Theorem 0.1; it is sufficient to use that $H^{p,n-k}_\infty(M)$ vanishes instead of $H^{p,n-k}_{\ell+1}(M)$ and replace ℓ by zero and $\ell+1$ by ∞ in the proof of Theorem 3.1.

References

[1] R.A. Airapetjan, G.M. Henkin, *Integral representations of differential forms on Cauchy-Riemann manifolds and the theory of CR-functions*, Russian Math. Survey, **39** (1984), 41–118.

[2] M.Y. Barkatou, *Optimal regularity for $\overline{\partial}_b$ on CR manifolds*, Prépublication de l'Institut Fourier, **374**, 1997, to appear in Journal of Geometric Analysis 2.

[3] L. De Carli, M. Nacinovich, *Unique continuation in abstract pseudoconcave CR-manifolds*, Preprint Dipartimento di Matematica, Pisa 1. 177. 1028, April 1997.

[4] R.E. Green, H. Wu, *Embedding of open Riemannian manifolds by harmonic functions*, Ann. Inst. Fourier (Grenoble), **25** (1975), 215–235.

[5] V. Guillemin, A. Pollack, Differential Topology, Prentice-Hall, 1974.

[6] G.M. Henkin, *Solution des équations de Cauchy-Riemann tangentielles sur des variétés de Cauchy-Riemann q-convexes*, C. R. Acad. Sci. Paris Sér. I Math., **292** (1981), 27–30.

[7] ———, *The Hartogs-Bochner effect on CR manifolds*, Soviet. Math. Dokl., **29** (1984), 78–82.

[8] G.M. Henkin, J. Leiterer, Theory of functions on complex manifolds, Birkhäuser Verlag, 1984.

[9] C.D. Hill, M. Nacinovich, *Pseudoconcave CR manifolds*, Complex Analysis and Geometry, Lecture Notes in Pure and Appl. Math., **173**, Marcel Dekker, New York (1996), 275–297.

[10] J.-J. Kohn, H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. of Math., **81** (1965), 451–472.

[11] Ch. Laurent-Thiébaut, *Résolution du $\overline{\partial}_b$ à support compact et phénomène de Hartogs-Bochner dans les variétés CR*, Proc. of Symp. in Pure Math., **52** (1991), 239–249.

[12] Ch. Laurent-Thiébaut, J. Leiterer, Andreotti-Grauert theory on real hypersurfaces, Quaderni della Scuola Normale Superiore di Pisa, 1995.

[13] B. Malgrange, *Faisceaux sur des variétés analytiques réelles*, Bull. Soc. Math. de France, **85** (1957), 231–237.

[14] A.E. Tumanov, *Extension of CR functions into a wedge from a manifold of finite type*, Math. USSR - Sb -, **64** (1989), 129–140.

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