Nihonkai Math. J.
Vol.29(2018), 29–43

RATIONAL UNICUSPIDAL CURVES ON $\mathbb{Q}$-HOMOLOGY PROJECTIVE PLANES WHOSE COMPLEMENTS HAVE LOGARITHMIC KODAIRA DIMENSION $-\infty$

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Abstract. Let S be a $\mathbb{Q}$-homology projective plane, C a rational unicuspidal curve on $S^0 = S - \operatorname{Sing} S$ and C' the proper transform of C with respect to the minimal embedded resolution of C. We prove that $S^0 - C$ is affine ruled if and only if $C'^2 \geq -1$ and determine the pairs (S, C) when $\overline{\kappa}(S^0 - C) = -\infty$ and $C'^2 \leq -2$.

1. Introduction

We work over the complex number field $\mathbb{C}$. In this paper, a cuspidal curve means a projective curve whose singular points are locally irreducible. A $\mathbb{Q}$-homology projective plane is, by definition, a normal projective surface with at worst quotient singular points having the same rational homology as $\mathbb{P}^2$. For a normal algebraic surface S, let $\pi : \tilde{S} \to S$ be a resolution of the singularities on S. Then we denote by $\overline{\kappa}(S)$ the logarithmic Kodaira dimension of $\tilde{S}$ (see [17] for the definition), here we note that $\overline{\kappa}(S)$ does not depend on the choise of π.

Many mathematicians have been interested in rational cuspidal projective plane curves and we have many results on the subject. Recently, M. Koras, K. Palka and T. Pełka obtained significant results on rational cuspidal projective plane curves whose complements have logarithmic Kodaira dimension two. See, e.g., [23], [24] and [14]. In particular, Koras and Palka [14] proved that any rational cuspial projective plane curve can be mapped onto a line by some birational transformation of $\mathbb{P}^2$. Furthermore, they announce that every rational cuspidal plane curve has at most four singular points. For other results, see, e.g., [2] and its references.

Recently, the study of rational cuspidal curves on $\mathbb{Q}$-homology projective planes have been motivated by the problem posed by Kollár [12, Problem 33] asking the classification of pairs (S, C) such that S is a $\mathbb{Q}$-homology projective plane and C

2010 *Mathematics Subject Classification.* Primary 14J26; Secondary 14J17.
Key words and phrases. Rational cuspidal curve, $\mathbb{Q}$-homology projective plane.
This work was supported by JSPS KAKENHI Grant Number JP17K05198.

is a rational cuspidal curve on S. Let S be a $\mathbb{Q}$-homology projective plane and set $S^0 = S - \operatorname{Sing} S$. Let C be a rational cuspidal curve in S^0. Here we note that the surface $S - C$ is then a $\mathbb{Q}$-homology plane, i.e., a normal affine surface with at worst quotient singular points having the same rational homology as $\mathbb{C}^2$. Then S is a rational surface by [25], [7] and [6].

In [5], Gurjar, Hwang and Kolte proved the following results.

(I) If $\overline{\kappa}(S - C) = -\infty$, then $\overline{\kappa}(S^0 - C) = -\infty$ and $\# \operatorname{Sing} C \leq 1$.
(II) If $\overline{\kappa}(S^0 - C) \leq 1$, then $\# \operatorname{Sing} C \leq 2$.
(III) $\overline{\kappa}(S^0 - C) \neq 0$.

The results (I) and (II) generalize results of Wakabayashi [27, Theorem], and the result (III) generalizes Orevkov [21, Theorem B(c)]. (See also Yoshihara [28] and the author [11] for results related to them.)

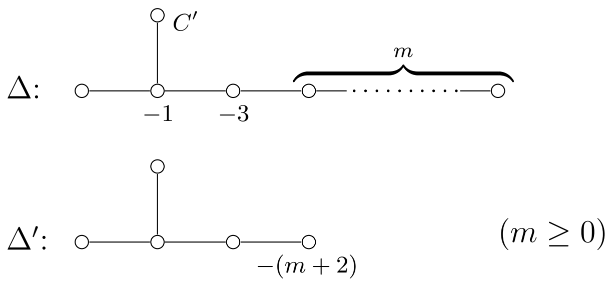
In [13], Kolte studied the case where $\overline{\kappa}(S^0 - C) = 1$ and $\overline{\kappa}(S - C) \neq -\infty$ more precisely and proved that there exists a smooth rational curve $\theta \subset S^0$ passing through the cusps of C. As a consequence of this result, we know that $S - \theta$ is a $\mathbb{Z}$-homology plane (a normal affine surface with at worst quotient singular points having the same homology as $\mathbb{C}^2$) and $\# \operatorname{Sing} S \leq 1$. Here, we note that if a projective curve T in S^0 satisfies $\overline{\kappa}(S - T) = -\infty$, then S is a rational surface and T is a rational cuspidal curve with $\# \operatorname{Sing} T \leq 1$. (See Proposition 4.1.)

Let S and C be the same as above. In this paper, we study the case where $\overline{\kappa}(S^0 - C) = -\infty$. Let $\pi : V \to S$ be a resolution of singularities of both S and C such that $\Delta = \pi^{-1}(C)$ is a simple normal crossing divisor. Here $\Delta' = \pi^{-1}(\operatorname{Sing} S)$ is a simple normal crossing divisor since S has at worst quotient singular points ([1]). Set $D = \Delta + \Delta'$. We assume further that the map π is minimal, that is, the Picard number of V is the least possible. Then such a map π is determined uniquely. We call the map π the minimal SNC-map for (S, C). The main result of this article is the following.

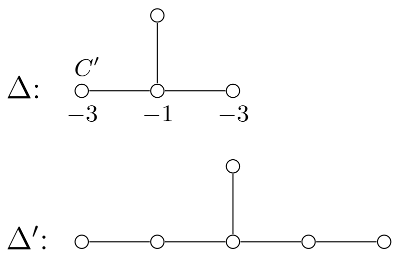
Theorem 1.1. *Let S be a $\mathbb{Q}$-homology projective plane and C a rational cuspidal curve with $\# \operatorname{Sing} C \leq 1$ in $S^0 = S - \operatorname{Sing} S$. Let $\pi : V \to S$ be the minimal SNC-map for the pair (S, C) and set $\Delta = \pi^{-1}(C)$, $\Delta' = \pi^{-1}(\operatorname{Sing} S)$ and $D = \Delta + \Delta'$. Let C' be the proper transform of C on V. Then the following assertions hold.*

(1) *$S^0 - C$ is affine ruled (see Proposition* 2.1 *for the definition) if and only if $C'^2 \geq -1$.*
(2) *Assume that $C'^2 \leq -2$ and $\overline{\kappa}(S^0 - C) = -\infty$. Then $-5 \leq C'^2 \leq -2$ and the weighted dual graph of $D = \Delta + \Delta'$ is one of* (a)–(g)*, where we omit the weight corresponding to a (-2)-curve. All the cases* (a)–(g) *can be realized.*

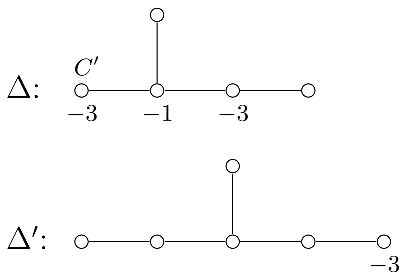
(a)


C'
m
Δ:
-1
-3
Δ':
$-(m+2)$
$(m \geq 0)$

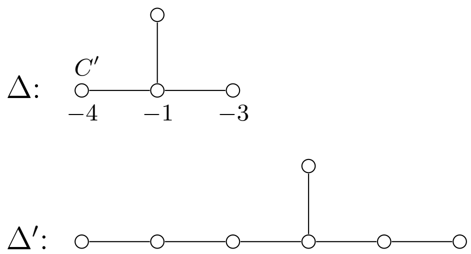
(b)


C'
Δ:
-3
-1
-3
Δ':

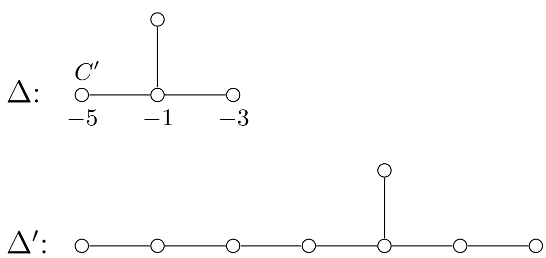
(c)


C'
Δ:
-3
-1
-3
Δ':
-3

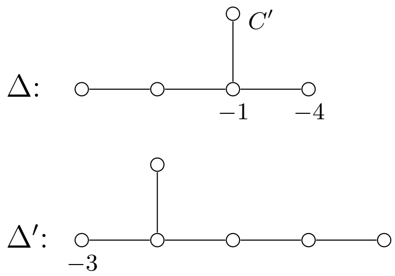
(d)


C'
Δ:
-4
-1
-3
Δ':

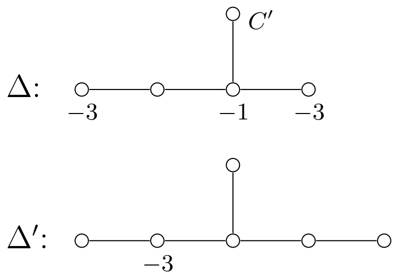
(e)


C'
Δ:
-5
-1
-3
Δ':

(f)



(g)



Let C be a cuspidal rational projective plane curve with $\#\operatorname{Sing} C \leq 1$ and let C' be the proper transform of C with respect to the minimal embedded resolution of C, which is the same as the miminal SNC-map for $(\mathbb{P}^2, C)$. Yoshihara [29] proved that $\overline{\kappa}(\mathbb{P}^2 - C) = -\infty$ if and only if $C'^2 \geq -1$. So our Theorem 1.1 generalizes this result of Yoshihara. Tono [26] determined the curves C with $C'^2 = -2$ and $\overline{\kappa}(\mathbb{P}^2 - C) = 2$.

2. Preliminary results

We employ the following notations.

K_V: the canonical divisor on V.
$\rho(V)$: the Picard number of V.
$\#D$: the number of irreducible components of a reduced effective divisor D.
$D_1 \equiv D_2$: D_1 and D_2 are numerically equivalent.
$\mu^*(D)$: the total transform of D by μ.
$\mu'(D)$: the proper transform of D by μ.

We give some notions on weighted graphs. As for the notions on weighted graphs, the reader may consult [4].

Definition 2.1. Let A be a graph and $v_1, \ldots, v_r$ the vertices of A. Then A is a *twig* if A is a connected linear graph together with a total ordering $v_1 > v_2 > \cdots > v_r$ among its vertices such that v_j and v_{j-1} are connected by a segment for each j $(2 \leq j \leq r)$. Such a twig is denoted by $[a_1, \ldots, a_r]$, where a_j is the weight of v_j. A twig $A = [a_1, \ldots, a_r]$ is said to be *admissible* if $a_j \leq -2$ for every j. For an admissible twig A, we denoted the determinant of A by $d(A)$ (cf. [4, (3.3)]). For an integer a and a positive integer s, we use the abbreviation $[a_s] = [a, a, \dots, a]$ that is a twig consisting of s vertices of weight a.

Definition 2.2. For an admissible twig $A = [a_1, \dots, a_r]$, the twig $[a_r, a_{r-1}, \dots, a_1]$ is called the *transposal* of A and denoted by ${}^tA$. We define also $\overline{A} = [a_2, \dots, a_r]$. If $r = 1$, we put $\overline{A} = \emptyset$ (the empty set). We call $e(A) = d(\overline{A})/d(A)$ the *inductance* of A. By [4, Corollary (3.8)], e defines a one-to-one correspondence from the set of all admissible twigs to the set of rational numbers in the interval $(0, 1)$. Hence there exists uniquely an admissible twig A^* whose inductance equals $1 - e({}^tA)$. We call the admissible twig A^* the *adjoint* of A.

We recall some basic notions in the theory of peeling. For more details, see [17, Chapter 2] or [19, Chapter 1]. A reduced effective divisor D on an algebraic variety is called an SNC-divisor if it has only simple normal crossings. Let X be a smooth projective surface and B an SNC-divisor on X. We call such a pair (X, B) an SNC-pair. A connected curve consisting only of irreducible components of B is called a connected curve in B for shortness. A connected curve T in B is *admissible* (resp. *rational*) if there are no (-1)-curves in $\operatorname{Supp} T$ and the intersection matrix of T is negative definite (resp. it consists only of rational curves). A connected curve T in B is a *twig* if its dual graph is a twig (see Definition 2.1) and T meets $B - T$ in a single point at one of the end components of T. An admissible rational twig T in B is *maximal* if it is not extended to an admissible rational twig with more irreducible components of B. A connected curve R (resp. F) in B is a *rational rod* (resp. *rational fork*) if it is a connected component of B and its weighted dual graph is the dual graph of the exceptional divisor of the minimal resolution of a cyclic quotient singular point (resp. a non-cyclic quotient singular point).

Let $\{T_\lambda\}$ (resp. $\{R_\mu\}$, $\{F_\nu\}$) be the set of all admissible rational maximal twigs (resp. all admissible rational rods, all admissible rational forks) in B, where no irreducible components of T_λ's belong to R_μ's or F_ν's. Then there exists a unique decomposition of B as a sum of effective $\mathbb{Q}$-divisors $B = B^{\#} + \operatorname{Bk} B$ such that the following conditions are satisfied:

(a) $\operatorname{Supp}(\operatorname{Bk} B) = (\cup_\lambda T_\lambda) \cup (\cup_\mu R_\mu) \cup (\cup_\nu F_\nu)$.

(b) $(B^{\#} + K_X)Z = 0$ for every irreducible component Z of $\operatorname{Supp}(\operatorname{Bk} B)$.

Definition 2.3. An SNC-pair (X, B) is said to be *almost minimal* if, for every irreducible curve C on X, either $(B^{\#} + K_X)C \geq 0$ or $(B^{\#} + K_X)C < 0$ and the intersection matrix of $C + \operatorname{Bk} B$ is not negative definite.

For an SNC-pair (X, B), we set $r(X, B) = \rho(X) - \#B$. We note that, if $f : X \to X'$ is a birational morphism from X onto a smooth projective surface X' such that

$B' = f_*(B)$ is an SNC-divisor, then $r(X, B) \geq r(X', B')$ and the equality holds if and only if f contracts only curves in $\operatorname{Supp} B$. Here we give the following elementary result on open algebraic surfaces of $\overline{\kappa} = -\infty$.

Proposition 2.1. *Let (X, B) be an SNC-pair. If $\overline{\kappa}(X - B) = -\infty$ and $r(X, B) \leq -1$, then $X - B$ is affine ruled, i.e., it contains a Zariski open subset U isomorphic to $\mathbb{A}^1 \times U_0$, where U_0 is a smooth curve.*

Proof. By [17, Theorem 2.3.11.1 (p. 107)] (that is the same as [19, Theorem 1.11]), there exists a birational morphism $f : X \to V$ onto a smooth projective surface V such that $D = f_*(B)$ is an SNC-divisor, that the SNC-pair (V, D) is almost minimal and that $\overline{\kappa}(V - D) = \overline{\kappa}(X - B) = -\infty$. Since $r(X, B) \geq r(V, D)$, it suffices to show that $V - D$ is affine ruled.

We use the structure theorem for open algebraic surfaces of $\overline{\kappa} = -\infty$. For more details, see [17, Chapter 2] or [19]. Note that $D - \operatorname{Bk} D \geq 0$ and each connected component of $\operatorname{Supp}(\operatorname{Bk} D)$ can be contracted to a quotient singular point. Let $p : V \to \overline{V}$ be the contraction of $\operatorname{Supp}(\operatorname{Bk} D)$ to quotient singular points and set $\overline{D} = p_*(D)$. It then follows from [17, Lemmas 2.3.14.3 and 2.3.14.4 (pp. 113–114)] that one of the following cases takes place.

(A) There exists a $\mathbb{P}^1$-fibration $\overline{h} : \overline{V} \to T$ onto a smooth projective curve T such that every fiber of $\overline{h}$ is irreducible and $\overline{D}F \leq 1$ for a general fiber F of $\overline{h}$.

(B) $\rho(\overline{V}) = 1$ and $-(\overline{D} + K_{\overline{V}})$ is an ample $\mathbb{Q}$-Cartier divisor.

It is clear that, in Case (A), the surface $V - D$ is affine ruled; so is $X - B$. We consider Case (B). We set as $\operatorname{Supp}(\operatorname{Bk} D) = \cup_{i=1}^r D_i$ and $D' = D - \sum_{i=1}^r D_i$. Each irreducible component of D' has coefficient one in $D^\#$ and $D^\# - D' = \sum_{i=1}^r \alpha_i D_i$, where $0 \leq \alpha_i < 1$ for $i = 1, \ldots, r$.

Since $-(\overline{D} + K_{\overline{V}})$ is an ample $\mathbb{Q}$-Cartier divisor and $p^*(\overline{D} + K_{\overline{V}}) \equiv D^\# + K_V$, we see that $-(D^\# + K_V)$ is nef and big and that, for an irreducible curve E on V, $-(D^\# + K_V)E = 0$ if and only if $E \subset \operatorname{Supp}(\operatorname{Bk} D)$, i.e., $E = D_i$ for some $i \in \{1, \ldots, r\}$. We note that if $D' \neq 0$ then it is connected since $\rho(\overline{V}) = 1$. Since $r(V, D) \leq r(X, B) \leq -1$, $\#D' \geq 2$. If $\#D' \geq 3$, then D' has an irreducible component Y such that $(D' - Y)Y \geq 2$. Then we have

$$0 > (D^\# + K_V)Y = (Y + K_V)Y + (D' - Y)Y + \sum_{i=1}^r \alpha_i D_i Y \geq 0,$$

which is a contradiction. Thus, $\#D' = 2$.

Let $D' = Y_1 + Y_2$ be the decomposition of D' into irreducible components. Then

$$0 > (D^\# + K_V)Y_1 = (Y_1 + K_V)Y_1 + Y_2Y_1 + \sum_{i=1}^r \alpha_i D_i Y_1.$$

Since $Y_1Y_2 = 1$, we know that Y_1 is a smooth rational curve and $\sum_{i=1}^{r} \alpha_i D_i Y_1 < 1$. Similarly, Y_2 is a smooth rational curve and $\sum_{i=1}^{r} \alpha_i D_i Y_2 < 1$.

Here we note that any connected component T of $\mathrm{Supp}(\mathrm{Bk}\, D)$ meeting Y_1 (or Y_2) is an admissible maximal rational twig in D and the irreducible component of T meeting Y_1 (or Y_2) has the coefficient $1 - \frac{1}{d(T)}$ in $D^{\#}$, where $d(T)$ is the absolute value of the determinant of the intersection matrix of T. Since $d(T) \geq 2$ and $\sum_{i=1}^{r} \alpha_i D_i Y_j < 1$ for $j = 1, 2$, we see that there exists at most one twig meeting Y_i ($i = 1,\ 2$). So the dual graph of the connected component of D containing $D' = Y_1 + Y_2$ is linear. If $Y_i^2 \leq -2$ ($i = 1$ or 2), then Y_i becomes a component of $\mathrm{Supp}(\mathrm{Bk}\, D)$, which is a contradiction. Hence $Y_1^2, Y_2^2 \geq -1$. It then follows from [17, Corollary 2.2.11.1 (p.82)] that $V - D$ is affine ruled. □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Let S, C, $\pi : V \to S$, Δ, Δ', D and C' be the same as in Theorem 1.1. Since S has only quotient singular points, Δ' is an SNC-divisor. If C is smooth, then $C' = D$ has positive self-intersection number since C is ample. So, $V - D$ is affine ruled by [17, Corollary 2.2.11.1 (p.82)]. From now on, we assume that C is not smooth. Then C is a rational unicuspidal curve (a rational cuspidal curve with unique singular point).

Since C is unicuspidal, there exists a unique (-1)-curve H in $\mathrm{Supp}(D - C')$. Then $H(\Delta - H) = 3$. Moreover, $\Delta - H$ consists of three connected components and C' becomes one of the connected components. Let T_1 and T_2 be the other connected components of $\Delta - H$. Since $H + T_1 + T_2$ is contracted to a smooth point, we may assume that T_1 is linear and H meets one of the terminal components of T_1. Then T_1 is an admissible maximal rational twig in D.

Lemma 3.1. *With the same notations and assumptions as above, $C'^2 \geq -1$ if and only if $V - D$ is affine ruled.*

Proof. The "only if" part follows from [17, Corollary 2.2.11.1 (p. 82)]. We prove the "if" part.

Suppose that $C'^2 \leq -2$ and $V - D$ is affine ruled. Then H is a unique irreducible component of D with self-intersection number ≥ -1. Since $V - D$ is affine ruled, it contains a smooth surface $U \cong \mathbb{C} \times U_0$, where U_0 is a smooth curve, as a Zariski open subset. Here we note that $\overline{\kappa}(V - D) = -\infty$ and S is a rational surface (see Section 1). We may assume that $U_0 \subset \mathbb{C}$ as a Zariski open subset. Since $V - D$ contains no complete algebraic curves, we know that the second projection $U \cong \mathbb{C} \times U_0 \to U_0$ gives rise to an irreducible pencil Λ of rational curves on V such that

$\mathrm{Bs}\,\Lambda \cap (V-D) = \emptyset$ and $\Phi_\Lambda|_{V-D}$ is an $\mathbb{C}$-fibration from $V-D$ onto a smooth rational curve containing U_0. We consider the following cases separately.

Case 1: $\mathrm{Bs}\,\Lambda \neq \emptyset$. Since $\Phi_\Lambda|_{V-D}$ is a $\mathbb{C}$-fibration, $\#\,\mathrm{Bs}\,\Lambda = 1$. We set $P = \mathrm{Bs}\,\Lambda$. Let $\nu : \tilde{V} \to V$ be the minimal resolution of the base points of Λ and let $\tilde{\Lambda}$ be the proper transform of Λ on $\tilde{V}$. Then $\tilde{\Lambda}$ defines a $\mathbb{P}^1$-fibration $\tilde{\Phi} := \Phi_{\tilde{\Lambda}} : \tilde{V} \to \mathbb{P}^1$ from $\tilde{V}$ onto $\mathbb{P}^1$ and the last exceptional curve, say $\tilde{E}$, in the process of ν becomes a section of $\tilde{\Phi}$. We note that $\mu^{-1}(\Delta)$ is a big divisor because so is Δ. If $P \notin \mathrm{Supp}\,\Delta$, then $\mu^{-1}(\Delta)$ is contained in a fiber of $\tilde{\Phi}$. This contradicts the bigness of $\mu^{-1}(\Delta)$. Hence $P \in \mathrm{Supp}\,\Delta$. Then $\nu^{-1}(\Delta')$ is contained in fibers of $\tilde{\Phi}$ and $f := \tilde{\Phi} \circ \nu^{-1} \circ \pi^{-1}|_{S-C}$ gives a $\mathbb{C}$-fibration from $S-C$ onto a smooth rational curve T'. Since $S-C$ is a $\mathbb{Q}$-homology plane with only quotient singular points, we infer from [18, Theorems 2.7 and 2.8] that $T' \cong \mathbb{C}$. Then $\tilde{\Phi}^{-1}(\mathbb{P}^1 - T') \subset \mathrm{Supp}\,\nu^{-1}(D)$.

Suppose that $P \notin H$. Then $\nu'(H)$ is a fiber component of $\tilde{\Phi}$. Let $\tilde{F}$ be the fiber of $\tilde{\Phi}$ containing $\nu'(H)$. Since $\nu'(H)$ is a (-1)-curve and $\nu'(H)(\nu^*(D)_{\mathrm{red}} - \nu'(H)) = 3$, at least one of the three adjacent components of $\nu'(H)$ in $\nu^*(D)_{\mathrm{red}}$ must be $\tilde{E}$. So the multiplicity of $\nu'(H)$ in $\tilde{F}$ equals one since $\tilde{E}$ is a section of $\tilde{\Phi}$ and $\tilde{E}\nu'(H) = 1$. However, the two other adjacent components of $\nu'(H)$ in $\nu^*(D)_{\mathrm{red}}$ are components of $\tilde{F}$. This contradicts the fact that the multiplicity of $\nu'(H)$ in $\tilde{F}$ equals one. Hence $P \in H$. This implies that $\tilde{E}$ is the unique curve in $\mathrm{Supp}\,\nu^*(D)_{\mathrm{red}}$ with self-intersection number ≥ -1. So $\mathrm{Supp}\,\nu^*(D)_{\mathrm{red}}$ contains no full fibers of $\tilde{\Phi}$. This is a contradiction because $\tilde{\Phi}^{-1}(\mathbb{P}^1 - T') \subset \mathrm{Supp}\,\nu^{-1}(D)$. Therefore, this case does not take place.

Case 2. $\mathrm{Bs}\,\Lambda = \emptyset$. In this case, $\Phi := \Phi_\Lambda : V \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration and $f := \Phi \circ \pi^{-1}|_{S-C}$ gives a $\mathbb{C}$-fibration onto $T' = \mathbb{C}$ by [18, Theorems 2.7 and 2.8]. In particular, $\Phi^{-1}(\mathbb{P}^1 - T') \subset \mathrm{Supp}\,D$. Since $\Phi|_{V-D}$ is a $\mathbb{C}$-fibration, there exists a unique component H' of D such that H' is a section of Φ. By using the same argument as in the previous paragraph, we know that $H' = H$ and derive a contradiction. Therefore, this case does not take place, neither.

Therefore, we know that $C'^2 \leq -1$ if $V-D$ is affine ruled. □

In the following two lemmas, we consider the case $C'^2 \leq -2$.

Lemma 3.2. *If $C'^2 \leq -2$ and $\overline{\kappa}(V-D) = -\infty$, then (V,D) is almost minimal.*

Proof. Suppose that (V,D) is not almost minimal. Then there exists an irreducible curve E on V such that $E(D^\# + K_V) < 0$ and the intersection matrix of $E + \mathrm{Bk}\,D$ is negative definite. Then E is a (-1)-curve, $E+D$ is an SNC-divisor and $E \not\subset \mathrm{Supp}\,D$. Moreover, $\overline{\kappa}(V-(E+D)) = \overline{\kappa}(V-D) = -\infty$. Since $\rho(V) = \#D$, $r(V, E+D) = \rho(V) - \#(E+D) = -1$. It then follows from Proposition 2.1 that $V-(E+D)$ is affine ruled; so is $V-D$. This contradicts Lemma 3.1. □

Lemma 3.3. *Assume that* $C'^2 \leq -2$ *and* $\overline{\kappa}(V - D) = -\infty$. *Then the weighted dual graph of* $D = \Delta + \Delta'$ *is one of* (a)–(h) *in Theorem* 1.1. *All the cases* (a)–(h) *can take place.*

Proof. Lemmas 3.1 and 3.2 imply that the pair (V, D) is almost minimal and $V - D$ is not affine ruled. Set $U = V - D$ $(= S^0 - C)$. Since $\Delta = \pi^{-1}(C)$ is a big divisor, we infer from [17, Theorem 2.5.1.2 (p. 143)] (that is the same as [20, Main Theorem]) that U has a structure of Platonic $\mathbb{C}^*$-fiber space over $\mathbb{P}^1$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$. More precisely, there exists a surjective morphism $f : U \to \mathbb{P}^1$ from U onto $\mathbb{P}^1$ such that the following conditions are satisfied:

(i) f has no singular fibers except for three multiple fibers $\Gamma_i = \mu_i \Delta_i$, $i = 1, 2, 3$, such that $\Delta_i \cong \mathbb{C}^*$ and that $\{\mu_1, \mu_2, \mu_3\} = \{2, 2, m\}$ $(m \geq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$.

(ii) There exist an SNC-pair (X, B) and a $\mathbb{P}^1$-fibration $\overline{f} : X \to \mathbb{P}^1$ from X onto $\mathbb{P}^1$ such that:
   - (a) $X - B = U$.
   - (b) B contains two irreducible components B_1 and B_2 that are sections of $\overline{f}$ with $B_1 \cap B_2 = \emptyset$, and the other irreducible components of B are contaned in fibers of $\overline{f}$.
   - (c) Every fiber of $\overline{f}$ has linear chain as its weighted dual graph and contains a unique (-1)-curve if the fiber is reducible.

Recalling [16, Section 2], we give a description of the dual graph of B. B consists of two connected components, say $\tilde{B}_1$ and $\tilde{B}_2$, containing B_1 and B_2, respectively. We may assume that $B_1^2 = -b \leq -2$. Then $\tilde{B}_1$ has the weighted dual graph in Figure 1, where the subgraph $A^{(i)} = [-a_1^{(i)}, -a_2^{(i)}, \ldots, -a_{r_i}^{(i)}]$ $(i = 1, 2, 3)$ is an admissible twig and $\{d(A^{(1)}), d(A^{(2)}), d(A^{(3)})\} = \{\mu_1, \mu_2, \mu_3\}$ in (i). Since $-b \leq -2$, $\tilde{B}_1$ is contracted to a non-cyclic quotient singular point. It follows from the arguments in [16, pp. 40–41] that the weighted dual graph of $\tilde{B}_2$ is given as in Figure 2, where the subgraph $B^{(i)} = [-b_1^{(i)}, -b_2^{(i)}, \ldots, -b_{s_i}^{(i)}]$ $(i = 1, 2, 3)$ is the adjoint of ${}^t A^{(i)}$ (see Definition 2.2).

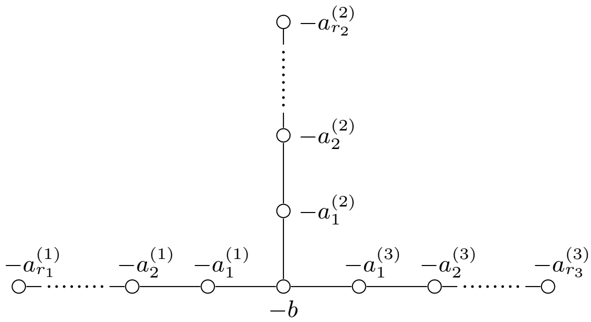


Figure 1

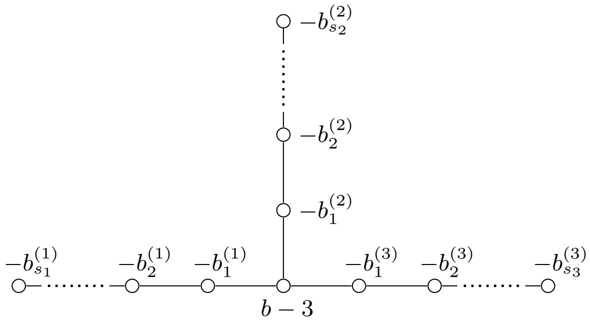


Figure 2

It follows that B contains no irreducible components B' such that $B'^2 \geq -1$ and $B'(B-B') \leq 2$. Hence the pair (X, B) is isomorphic to (V, D). Namely, there exists an isomorphism $f : X \to V$ whose restriction on B gives rise to an isomorphism between B and D. Since Δ is a big divisor and Δ' can be contracted to quotient singular points, the weighted dual graph of Δ (resp. Δ') is the same as that of $\tilde{B}_2$ (resp. $\tilde{B}_1$). Since H is a branch component of Δ and is a (-1)-curve, $b = 2$. Since one of $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ must be the dual graph of C', we may assume that the dual graph of C' (resp. T_1, T_2) is $B^{(1)}$ (resp. $B^{(2)}$, $B^{(3)}$). Then $s_1 = 1$ and T_2 is linear.

Therefore, the weighted dual graph of $T_1 + H + T_2$ is linear and $\{d(T_1), d(T_2)\} = \{2, m\}$ $(m \geq 2)$, $\{3, 3\}$, $\{3, 4\}$ or $\{3, 5\}$. Since $T_1 + H + T_2$ is contracted to a smooth point, we can determine the weighted dual graph of $D = \Delta + \Delta'$. For example, we consider the case $d(T_1) = 3$ and $d(T_2) = 5$ (i.e., $\mu_2 = 3$ and $\mu_3 = 5$); the other cases can be treated similarly. Since $\mu_1 = 2$, $C'^2 = -2$. Since $d(T_1) = d(B^{(2)}) = 3$ (resp. $d(T_2) = d(B^{(3)}) = 5$), $B^{(2)} = [-3]$ or $[(-2)_2]$ (resp. $B^{(3)} = [-5]$, $[-2, -3]$, $[-3, -2]$ or $[(-2)_4]$). Since $T_1 + H + T_2$ is linear and contracted to a smooth point, we know that $B^{(2)} = [-3]$ and $B^{(3)} = [-2, -3]$. Then $A^{(1)} = [-2]$, $A^{(2)} = [(-2)_2]$ and $A^{(3)} = [-3, -2]$. Hence the weighted dual graph of D is (g) in Theorem 1.1.

The last assertion can be verified easily. $\square$

The proof of Theorem 1.1 is thus completed.

4. Some results on S

In this section, we remark several elementary results on some $\mathbb{Q}$-homology projective planes.

Proposition 4.1. *Let S be a $\mathbb{Q}$-homology projective plane and C an irreducible curve in $S^0 = S - \operatorname{Sing} S$. Assume that $\overline{\kappa}(S - C) = -\infty$. Then the following assertions hold true.*

(1) $\overline{\kappa}(S^0 - C) = -\infty$ *and* C *is a rational cuspidal curve with* $\#\operatorname{Sing} C \leq 1$.
(2) S *is a log del Pezzo surface of rank one, namely,* S *is a normal projective surface of Picard number one with only quotient singular points and with ample anticanonical divisor.*

Proof. (1) Since $\overline{\kappa}(S - C) = -\infty$, it follows from [10, Theorem 3] that C is a rational cuspidal curve with $\#\operatorname{Sing} C \leq 1$. So $\overline{\kappa}(S^0 - C) = -\infty$ by (I) in Introduction.

(2) Since $\overline{\kappa}(S^0) = -\infty$, we infer from [30, Remark 1.2 (2)] that S is a log del Pezzo surface of rank one. □

Proposition 4.2. *Let* S *be a* $\mathbb{Q}$*-homology projective plane. Assume that* $S^0 = S - \operatorname{Sing} S$ *contains a rational cuspidal curve. Then* $\#\operatorname{Sing} S \leq 2$*. Moreover, if* $\#\operatorname{Sing} S = 2$*, then the both singular points on* S *are cyclic quotient singular points.*

Proof. Take a cuspidal rational curve C on S^0, which exists by the assumption. Then $S - C$ is a $\mathbb{Q}$-homology plane. If $\overline{\kappa}(S^0 - C) \geq 0$, then the assertions follow from [22, Proposition 1.3]. If $\overline{\kappa}(S^0 - C) = -\infty$ and $S^0 - C$ is not affine ruled, then $\#\operatorname{Sing} S = 1$ by Theorem 1.1 (2). So we assume further that $S^0 - C$ is affine ruled. It is well known that every singular point on S is a cyclic quotient singular point (see [15]). From now on, we use the same notations as in Theorem 1.1.

Case 1: C **is smooth.** We may set $C = C'$. Then $\Delta = C$ and $m = C^2 \geq 1$. Let $\mu : \tilde{V} \to V$ be the composite of blowing-ups over a point $P \in C$ such that $\mu'(C)^2 = 0$ and $\mu^{-1}(P) = E_1 + \cdots + E_m$ is a linear chain of $\mathbb{P}^1$'s with $E_i^2 = -2$, $E_i E_{i+1} = 1$ $(i = 1, \ldots, m-1)$ and $E_m^2 = -1$. Here we identify $\mu^{-1}(\Delta')$ with Δ' since Δ' is not affected by μ.

The divisor $\mu'(C)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|\mu'(C)|} : \tilde{V} \to \mathbb{P}^1$. Then E_m becomes a section of Φ and $\mu^{-1}(D) - E_m$ is contained in fibers of Φ. Since $\rho(\tilde{V}) = \rho(V) + m = 2 + \#(\mu^{-1}(D) - (E_m + \mu'(C)))$ and each irreducible component of $\mu^{-1}(D) - (E_m + \mu'(C)) = \Delta' + E_1 + \cdots + E_{m-1}$ has self-intersection number ≤ -2, we know that every singular fiber of Φ consists only of a (-1)-curve and some (one or two) connected components of $\operatorname{Supp}(\Delta' + E_1 + \cdots + E_{m-1})$.

If Φ has no singular fibers, then $\operatorname{Supp} \Delta' = \emptyset$ and so S is smooth. Assume that Φ has a singular fiber F. If $\operatorname{Supp} F$ does not contain $\operatorname{Supp}(E_1 + \cdots + E_{m-1})$, then the component F' of F meeting E_m is a (-1)-curve. Since E_m is a section of Φ, $\operatorname{Supp} F$ contains a (-1)-curve other than F'. This is a contradiction. Hence we see that $m \geq 2$ and $\operatorname{Supp} F$ contains $E_1, \ldots, E_{m-1}$. In particular, F is the unique singular fiber of Φ. By the remark as in the previous paragraph, $\operatorname{Supp} F$ has a (-1)-curve, say F_0, and $\operatorname{Supp} F - F_0 = \operatorname{Supp}(\Delta' + E_1 + \cdots + E_{m-1})$. So $\operatorname{Supp} \Delta'$ is conntected if $\operatorname{Sing} S \neq \emptyset$ and hence $\#\operatorname{Sing} S \leq 1$.

Case 2: C **is not smooth.** We use the same notations as in the proof of Theorem 1.1. $\Delta = \pi^{-1}(C)$ can be expressed as $\Delta = H + C' + T_1 + T_2$, where H is the unique

(-1)-curve in $\mathrm{Supp}(\Delta - C')$ and we assume that T_1 is linear. By Theorem 1.1 (1), $C'^2 \geq -1$. We consider the following subcases separately.

Subcase 1: $C'^2 = 0$. Then the divisor C' defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|C'|} : V \to \mathbb{P}^1$, H becomes a section of Φ and $D - H$ is contained in fibers of Φ. Since $\rho(V) = 2 + \#(D - (C' + H))$ and every irreducible component of $D - (C' + H)$ has self-intersection number ≤ -2, we see that every singular fiber of Φ consists only of a (-1)-curve and some connected components of $\mathrm{Supp}(D - (C' + H))$. Let F_1 and F_2 be the fiber of Φ containing T_1 and T_2, respectively. Then F_1 and F_2 are singular fibers of Φ and $F_1 \neq F_2$. By using the same argument as in Case 1, we know that F_1 and F_2 exhaust the singular fibers of Φ and that $\mathrm{Supp}\, F_i$ $(i = 1, 2)$ contains at most one connected component of $\mathrm{Supp}\, \Delta'$. Therefore, $\#\, \mathrm{Sing}\, S \leq 2$.

Subcase 2: $C'^2 > 0$. Set $P = C' \cap H$ and $m = C'^2$. Let $\mu : \tilde{V} \to V$ be the composite of blowing-ups over P such that $\mu'(C')^2 = 0$ and $\mu^{-1}(P) = E_1 + \cdots + E_m$ is a linear chain of $\mathbb{P}^1$'s with $E_i^2 = -2$, $E_i E_{i+1} = 1$ $(i = 1, \ldots, m-1)$ and $E_m^2 = -1$. We know that $\rho(\tilde{V}) = \rho(V) + m = 2 + \#(\mu^{-1}(D) - (\mu'(C') + E_m))$ and that every irreducible component of $\mathrm{Supp}(\mu^{-1}(D) - (\mu'(C') + E_m))$ has self-intersection number ≤ -2. Since $\mu'(C')^2 = 0$, $\mu'(C)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|\mu'(C)|} : \tilde{V} \to \mathbb{P}^1$. Then E_m becomes a section of Φ and $\mathrm{Supp}(\mu^{-1}(D) - (\mu'(C') + E_m))$ is contained in fibers of Φ. By using the same argument as in Case 1 (regarding $E_1 + \cdots + E_{m-1} + \mu'(H + T_2 + T_3)$ as $E_1 + \cdots + E_{m-1}$ in Case 1), we see that $\#\, \mathrm{Sing}\, S \leq 1$.

Subcase 3: $C'^2 = -1$. Set $P := H \cap T_2$, where we assume that T_1 is linear. (Of course, T_2 may be linear.) Since C' and H are (-1)-curves and $H + T_1$ is a linear chain of $\mathbb{P}^1$'s, we infer from [17, Corollary 2.2.11.1 (p.82)] that there exists a birational morphism $\mu : \tilde{V} \to V$ that is a composite of blowing-ups over P such that the following conditions are satisfied:

(i) $\mu^{-1}(P) \cup \mathrm{Supp}\, \mu'(H + T_1)$ is a linear chain.

(ii) There exists an effective divisor F such that $\mathrm{Supp}\, F \subset \mu^{-1}(P) \cup \mathrm{Supp}\, \mu'(C' + H + T_1)$ and F defines a $\mathbb{P}^1$-fibration $\Phi = \Phi_{|F|} : \tilde{V} \to \mathbb{P}^1$.

Let E_m be the component of $\mathrm{Supp}\, \mu^*(\Delta) \setminus \mathrm{Supp}\, F$ meeting F. Then E_m becomes a section of Φ. Since $\rho(\tilde{V}) = 2 + \#(\mu^{-1}(D) - (\mu'(C') + E_m))$ and every irreducible component of $\mu^{-1}(D) - (\mu'(C') + E_m)$ has self-intersection number ≤ -2, we know that Φ has at most one singular fiber other than F by using the same argument as in Case 1 (see Subcase 2). Since $\mathrm{Supp}\, F \cap \mathrm{Supp}\, \mu^{-1}(\Delta') = \emptyset$ and every singular fiber of Φ contains at most one connected component of $\mathrm{Supp}\, \mu'(\Delta')$, we know that $\#\, \mathrm{Sing}\, S \leq 1$. □

It is well-known that the fundamental group of smooth points of a log del Pezzo surface is finite. See Gurjar–Zhang [8], [9] (see also [3] for another short proof of the result). As a consequence of Theorem 1.1, we obtain the following result.

Proposition 4.3. *Let S be a $\mathbb{Q}$-homology projective plane. Assume that there exists an irreducible curve C in $S^0 = S - \operatorname{Sing} S$ such that $\overline{\kappa}(S^0 - C) = -\infty$. Then S^0 is smply connected.*

Proof. We use the same notations in Theorem 1.1, here we note that C is then a rational cuspidal curve with $\# \operatorname{Sing} C \leq 1$.

Assume that $S^0 - C$ is affine ruled. Then, as seen in the proof of Proposition 4.2, we have a sequence of blowing-ups over a point of Δ, say $\mu : \tilde{V} \to V$, and an effective divisor F with $\operatorname{Supp} F \subset \operatorname{Supp} \mu^{-1}(\Delta)$ that defines a $\mathbb{P}^1$-fibration $\Phi = \Phi_{|F|} : \tilde{V} \to \mathbb{P}^1$. Then $\mu^{-1}(\Delta')$ is contained in fibers of Φ. Moreover, every fiber G of Φ contains an irreducible component not contained in $\operatorname{Supp} \mu^{-1}(\Delta')$ such that its coefficient in G equals one. Hence we know that $\pi_1(\tilde{V} - \mu^{-1}(\Delta')) = (1)$. Therefore, $\pi_1(S^0) = \pi_1(V - \operatorname{Supp} \Delta') = \pi_1(\tilde{V} - \mu^{-1}(\Delta')) = (1)$.

Assume next that $S^0 - C$ is not affine ruled. Then the weighted dual graph of $D = \Delta + \Delta'$ is one of (a)–(g) in Theorem 1.1. We consdier the case (g) only. The other cases can be treated similarly. Let $\Delta = C' + H + T_{11} + T_{12} + T_2$ and $\Delta' = D_0 + D_1 + D_2 + D_3 + D_4 + D_5$ be the irreducible decompositions of Δ and Δ' and we may assume that the weighted dual graph of $\Delta + \Delta'$ is given in Figure 3, where we omit the weight corresponding to a (-2)-curve.

Figure 3

As seen from [16, pp. 40–41], we know that there exists a (-1)-curve E such that $ED = 2$, $E\Delta = ET_{12} = 1$ and $E\Delta' = ED_2 = 1$. Then the divisor $F = E+T_2+T_{12}+2T_{11}+3H$ defines a $\mathbb{P}^1$-fibration $\Phi = \Phi_{|F|} : V \to \mathbb{P}^1$. Then D_2 is a section of Φ and $\operatorname{Supp}(\Delta' - D_2)$ is contained in a fiber of Φ. Then $\pi_1(V - \operatorname{Supp} \Delta') = (1)$. Hence $\pi_1(S^0) = \pi_1(V - \operatorname{Supp} \Delta') = (1)$. $\square$

Acknowledgements. The author would like to express his gratitude to the referee for giving useful comments and suggestions to improve the paper.

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Received June 11, 2017
Revised October 9, 2017