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ONE DIMENSIONAL PERTURBATION OF INVARIANT SUBSPACES IN THE HARDY SPACE OVER THE BIDISK I

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ABSTRACT. For an invariant subspace M_1 of the Hardy space H^2 over the bidisk $\mathbb{D}^2$, write $N_1 = H^2 \ominus M_1$. Let $\Omega(M_1) = M_1 \ominus (zM_1 + wM_1)$ and $\widetilde{\Omega}(N_1) = \{f \in N_1 : zf, wf \in M_1\}$. Then $\Omega(M_1) \neq \{0\}$, and $\Omega(M_1), \widetilde{\Omega}(N_1)$ are key spaces to study the structure of M_1. It is known that there is a nonzero $f_0 \in M_1$ such that $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$ is an invariant subspace. It is described the structures of $\Omega(M_2), \widetilde{\Omega}(N_2)$ using the words of $\Omega(M_1), \widetilde{\Omega}(N_1)$ and f_0. To do so, it occur many cases. We shall give examples for each cases.

1. Introduction

Let $H^2 = H^2(\mathbb{D}^2)$ be the Hardy space over the bidisk $\mathbb{D}^2$ with two variables z and w. Let T_z and T_w be the multiplication operators on H^2 by z and w, respectively. A nonzero closed subspace M of H^2 is said to be invariant if $T_zM \subset M$ and $T_wM \subset M$. The structure of invariant subspaces of H^2 is fairly complicated and in this moment it seems to be out of reach (see [1, 6, 7]).

Let M be an invariant subspace. Then by the Wold decomposition theorem, we have

$$M = \bigoplus_{n=0}^{\infty} w^n (M \ominus wM),$$

so the space $M \ominus wM$ contains a lot of informations of an invariant subspace M. In [7], R. Yang defined the operator F_z^M on $M \ominus wM$ by

$$F_z^M f = P_{M \ominus wM} T_z f, \quad f \in M \ominus wM,$$

where P_A is the orthogonal projection from H^2 onto $A \subset H^2$, and Yang called F_z^M the fringe operator on $M \ominus wM$. It is considered that the informations of M are encoded in the operator theoretic properties of F_z^M.

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We write $R_z^M = T_z|_M$ and $R_w^M = T_w|_M$. Then R_z^M, R_z^M are the operators on M. We set

$$\Omega(M) = M \ominus (zM + wM) = M \ominus \overline{zM + wM}. \tag{1.1}$$

Then $\Omega(M) \neq \{0\}$ (see for example [7, p. 532]). Let $N = H^2 \ominus M$. We also set

$$\widetilde{\Omega}(N) = \{f \in N : zf, wf \in M\}. \tag{1.2}$$

It is known that $\widetilde{\Omega}(N)$ may be empty. In [7], Yang showed that $\ker (F_z^M)^* = \Omega(M)$ and $\ker F_z^M = w\widetilde{\Omega}(N)$, where $(F_z^M)^*$ is the adjoint operator of F_z^M. When F_z^M on $M \ominus wM$ is a Fredholm operator,

$$\operatorname{ind} F_z^M = \dim \ker F_z^M - \dim \ker (F_z^M)^*$$

is called the Fredholm index of F_z^M, see [2] for the study of operator theory. So in this case, we have

$$\operatorname{ind} F_z^M = \dim\ \ker\ \Omega(M) - \dim\ \ker\ \widetilde{\Omega}(N).$$

There are a lot of examples of M satisfying that F_z^M on $M \ominus wM$ is Fredholm (see [4, 7, 8, 9]).

The smallest number of elements in M which generate M as an invariant subspace is called the rank of M. By (1.1), it is easy to see that the rank of M is greater than or equals to $\dim \ker \Omega(M)$. Motivated by these facts, we are interested in the structures of $\Omega(M)$ and $\widetilde{\Omega}(N)$.

Let M_1 be a nonzero invariant subspace of H^2. Then there is $f_0 \in M_1$ with $\|f_0\| = 1$ such that $M_2 := M_1 \ominus \mathbb{C} \cdot f_0$ is an invariant subspace (for example take f_0 in $\Omega(M_1)$). Our problem is what kind of changes of the structure of M_2 come from the ones of M_1. This problem is basic in the study of the structure of invariant subspaces. Let $N_j = H^2 \ominus M_j$ for $j = 1, 2$. We shall describe $\Omega(M_2), \widetilde{\Omega}(N_2)$ using the words of f_0, $\Omega(M_1)$ and $\widetilde{\Omega}(N_1)$. To do so, we need other notations;

$$\eta_0 := P_{M_1 \ominus wM_1} f_0, \quad \varphi_0 := P_{\widetilde{\Omega}(N_1)} T_z^* f_0, \quad \psi_0 := P_{\widetilde{\Omega}(N_1)} T_w^* f_0.$$

In Section 2, we shall give some facts which are used later. In Section 3, we shall describe $\Omega(M_2), \widetilde{\Omega}(N_2)$ under the condition "$f_0 \in \Omega(M_1)$". We need to divide the situation into several cases which depend on φ_0 and ψ_0. To describe $\Omega(M_2)$, we shall study under the additional assumption that $(F_z^{M_1})^*$ has closed range.

Suppose that $f_0 \notin \Omega(M_1)$. Since $\Omega(M_1) = (H^2 \ominus zM_1) \cap (H^2 \ominus wM_1)$, either $f_0 \notin M_1 \ominus zM_1$ or $f_0 \notin M_1 \ominus wM_1$. In Section 4, we shall describe $\Omega(M_2), \widetilde{\Omega}(N_2)$ under the condition "$f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$". Here we need to divide the situation into several cases which depend on f_0, φ_0 and ψ_0.

In Section 5, we shall describe $\Omega(M_2), \widetilde{\Omega}(N_2)$ under the condition "$f_0 \notin M_1 \ominus zM_1$ and $f_0 \notin M_1 \ominus wM_1$". Here we need to divide the situation into several cases which depend on η_0, φ_0 and ψ_0.

To prove our assertions, we use only elementary techniques. But we shall give examples which satisfy each condition given in Sections 3, 4 and 5. These examples will be some help for further investigation of invariant subspaces, and show us that the structure of invariant subspaces is not so simple.

In Section 6, we shall give some comments and problems on the related topics.

2. Preliminary

Let M be a nonzero invariant subspace of H^2. We have $(R_z^M)^* = P_M T_z^*|_M$ and $(R_w^M)^* = P_M T_w^*|_M$. Since $\ker(R_w^M)^* = M \ominus wM$, by (1.1) we have

$$\Omega(M) = \ker(R_z^M)^* \cap \ker(R_w^M)^*.$$

We also have

$$\Omega(M) = \{f \in M \ominus wM : \ker(R_z^M)^* f = 0\}.$$

Let $N = H^2 \ominus M$. Then we have $T_z^* N \subset N$ and $T_w^* N \subset N$. So

$$\Omega(M) = \{f \in M \ominus wM : T_z^* f \in N\}. \tag{2.1}$$

Since $w\widetilde{\Omega}(N) \subset M$, we have $w\widetilde{\Omega}(N) \subset M \ominus wM$. By (1.2), we have

$$\widetilde{\Omega}(N) = N \ominus (T_z^* N + T_w^* N) = N \ominus \overline{T_z^* N + T_w^* N}. \tag{2.2}$$

Let F_z^M on $M \ominus wM$ be the Fringe operator of M. We have that $(F_z^M)^* = (R_z^M)^* = P_M T_z^*$ on $M \ominus wM$. By [7, Proposition 4.4], we have the following.

Lemma 2.1. $\ker(F_z^M)^* = \Omega(M)$ *and* $\ker F_z^M = w\widetilde{\Omega}(N)$.

We shall use the following lemma in the proof of Theorem 3.1.

Lemma 2.2. *Suppose that $(F_z^M)^*$ has closed range. Then for every $f \in (M \ominus wM) \ominus w\widetilde{\Omega}(N)$, there is a unique function h in $(M \ominus wM) \ominus \Omega(M)$ such that $(R_z^M)^* h = f$.*

Proof. We have $(F_z^M)^* = (R_z^M)^*$ on $M \ominus wM$. By the assumption, $(F_z^M)^*$ is a one-to-one map from $(M \ominus wM) \ominus \ker(F_z^M)^*$ onto $(M \ominus wM) \ominus \ker F_z^M$. Hence by Lemma 2.1, we get the assertion. $\square$

For many examples of M, $(F_z^M)^*$ has closed range. We do not know an example of M for which $(F_z^M)^*$ does not have closed range.

Let M_1 be a nonzero invariant subspace of H^2 and $f_0 \in M_1$ with $\|f_0\| = 1$ such that $M_2 := M_1 \oplus \mathbb{C} \cdot f_0$ is an invariant subspace. We write $N_j = H^2 \ominus N_j$ for $j = 1, 2$. Since $f_0 \in N_2$, we have

$$T_z^* f_0, T_w^* f_0 \in N_2 = N_1 \oplus \mathbb{C} \cdot f_0.$$

3. The case $f_0 \in \Omega(M_1)$

In this section, we assume that $f_0 \in \Omega(M_1)$ and we shall study the structure of $\Omega(M_2)$ and $\widetilde{\Omega}(N_2)$. Recall that

$$\varphi_0 = P_{\widetilde{\Omega}(N_1)} T_z^* f_0 \quad \text{and} \quad \psi_0 = P_{\widetilde{\Omega}(N_1)} T_w^* f_0.$$

Lemma 3.1. *Suppose that $f_0 \in \Omega(M_1)$. Then we have the following.*

(i) $f_0 \in M_1 \ominus wM_1$ *and* $(R_z^{M_1})^* w f_0 \in M_1 \ominus wM_1$.

(ii) $\varphi_0 = 0$ *if and only if* $(R_z^{M_1})^* w f_0 \perp w\widetilde{\Omega}(N_1)$.

(iii) $\psi_0 = 0$ *if and only if* $f_0 \perp w\widetilde{\Omega}(N_1)$.

Proof. (i) Since $\Omega(M_1) \subset M_1 \ominus wM_1$, we have $f_0 \in M_1 \ominus wM_1$. Since $f_0 \in \Omega(M_1)$, we have $T_z^* f_0 \in N_1$. Hence $P_{M_1} w T_z^* f_0 \in M_1 \ominus wM_1$. Since $(R_z^{M_1})^* w f_0 = P_{M_1} w T_z^* f_0$, we have $(R_z^{M_1})^* w f_0 \in M_1 \ominus wM_1$.

(ii) We have that $\varphi_0 = 0$ if and only if $wT_z^* f_0 \perp w\widetilde{\Omega}(N_1)$. Since $w\widetilde{\Omega}(N_1) \subset M_1$, $wT_z^* f_0 \perp w\widetilde{\Omega}(N_1)$ if and only if $P_{M_1} w T_z^* f_0 \perp w\widetilde{\Omega}(N_1)$. Hence we get (ii).

(iii) We have that $\psi_0 = 0$ if and only if $T_w^* f_0 \perp \widetilde{\Omega}(N_1)$. Hence we get (iii). $\square$

Lemma 3.2. *If $f_0 \in \Omega(M_1)$, then*

$$M_2 \ominus wM_2 = \big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) \oplus \mathbb{C} \cdot w f_0.$$

Proof. Since $M_1 = M_2 \oplus \mathbb{C} \cdot f_0$, we have $(M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0 \subset M_2$. Since $(M_1 \ominus wM_1) \perp wM_2$, $(M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0 \subset M_2 \ominus wM_2$. Since $f_0 \in \Omega(M_1)$, we have $wf_0 \in M_2$, so $wf_0 \perp wM_2$. Hence

$$\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) \oplus \mathbb{C} \cdot w f_0 \subset M_2 \ominus wM_2.$$

To show the reverse inclusion, let $f \in M_2 \ominus wM_2$. Write $f = f_1 + cwf_0$, where $f_1 \in (M_2 \ominus wM_2) \ominus \mathbb{C} \cdot wf_0$ and $c \in \mathbb{C}$. Then $T_w^* f_1 \in N_2$. We have $\langle T_w^* f_1, f_0 \rangle = \langle f_1, wf_0 \rangle = 0$. Since $N_2 = N_1 \oplus \mathbb{C} \cdot f_0$, we have $T_w^* f_1 \in N_1$. Hence $f_1 \in M_1 \ominus wM_1$. Trivially we have $f_1 \perp f_0$. Therefore $f_1 \in (M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0$. Thus

$$M_2 \ominus wM_2 \subset \big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) \oplus \mathbb{C} \cdot w f_0,$$

so we get the assertion. $\square$

By the proof, the assertion of Lemma 3.2 holds if $f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$.

Lemma 3.3. *Suppose that $f_0 \in \Omega(M_1)$. Then we have the following.*

(i) $\Omega(M_2) = (\Omega(M_1) \ominus \mathbb{C} \cdot f_0) \oplus$
$$\big\{h \in \big((M_1 \ominus wM_1) \ominus \Omega(M_1)\big) \oplus \mathbb{C} \cdot wf_0 : (R_z^{M_1})^* h \in \mathbb{C} \cdot f_0\big\}.$$

(ii) $\widetilde{\Omega}(N_2) = \big(\widetilde{\Omega}(N_1) \ominus (\mathbb{C} \cdot \varphi_0 + \mathbb{C} \cdot \psi_0)\big) \oplus \mathbb{C} \cdot f_0$.

Proof. (i) By (2.1),

$$\Omega(M_2) = \{h \in M_2 \ominus wM_2 : T_z^* h \in N_2\}.$$

Since $N_2 = N_1 \oplus \mathbb{C} \cdot f_0$, we have

$$\Omega(M_2) = \{h \in M_2 \ominus wM_2 : (R_z^{M_1})^* h \in \mathbb{C} \cdot f_0\}.$$

By Lemma 3.2,

$$\begin{aligned} M_2 \ominus wM_2 = (\Omega(M_1) \ominus \mathbb{C} \cdot f_0) \oplus \\ \big(\big((M_1 \ominus wM_1) \ominus \Omega(M_1)\big) \oplus \mathbb{C} \cdot wf_0\big). \end{aligned}$$

Since $f_0 \in \Omega(M_1)$, we have $(M_1 \ominus wM_1) \ominus \Omega(M_1) \subset M_2$ and $\Omega(M_1) \ominus \mathbb{C} \cdot f_0 \subset \Omega(M_2)$. Hence we have

$$\begin{aligned} \Omega(M_2) = (\Omega(M_1) \ominus \mathbb{C} \cdot f_0) \oplus \\ \{h \in \big((M_1 \ominus wM_1) \ominus \Omega(M_1)\big) \oplus \mathbb{C} \cdot wf_0 : (R_z^{M_1})^* h \in \mathbb{C} \cdot f_0\}. \end{aligned}$$

(ii) Since $f_0 \in \Omega(M_1)$, we have $\mathbb{C} \cdot f_0 \subset \widetilde{\Omega}(N_2)$. By (2.2), we have

$$\widetilde{\Omega}(N_2) = \{h \in \widetilde{\Omega}(N_1) : zh \perp f_0, wh \perp f_0\} \oplus \mathbb{C} \cdot f_0.$$

Hence we get (ii). □

The following follows directly from Lemma 3.3 (ii).

Corollary 3.1. *Suppose that $f_0 \in \Omega(M_1)$. Then we have the following.*

(i) *If $\varphi_0 = \psi_0 = 0$, then $\widetilde{\Omega}(N_2) = \widetilde{\Omega}(N_1) \oplus \mathbb{C} \cdot f_0$.*

(ii) *If $\varphi_0 \neq 0$ and $\psi_0 = 0$, then $\widetilde{\Omega}(N_2) = (\widetilde{\Omega}(N_1) \ominus \mathbb{C} \cdot \varphi_0) \oplus \mathbb{C} \cdot f_0$.*

(iii) *Suppose that $\varphi_0 \neq 0$ and $\psi_0 \neq 0$. If $\mathbb{C} \cdot \varphi_0 = \mathbb{C} \cdot \psi_0$, then*

$$\widetilde{\Omega}(N_2) = (\widetilde{\Omega}(N_1) \ominus \mathbb{C} \cdot \varphi_0) \oplus \mathbb{C} \cdot f_0.$$

(iv) *Suppose that $\varphi_0 \neq 0$ and $\psi_0 \neq 0$. If $\mathbb{C} \cdot \varphi_0 \neq \mathbb{C} \cdot \psi_0$, then*

$$\widetilde{\Omega}(N_2) = \big(\widetilde{\Omega}(N_1) \ominus (\mathbb{C} \cdot \varphi_0 + \mathbb{C} \cdot \psi_0)\big) \oplus \mathbb{C} \cdot f_0.$$

Theorem 3.1. *Suppose that $f_0 \in \Omega(M_1)$. Moreover suppose that $(F_z^{M_1})^*$ has closed range. Then we have the following.*

(i) *If $\varphi_0 = \psi_0 = 0$, then there are nonzero functions h_1 and h_2 (may be zero) in $(M_1 \ominus wM_1) \ominus \Omega(M_1)$ such that*

$$\Omega(M_2) = (\Omega(M_1) \ominus \mathbb{C} \cdot f_0) \oplus \big(\mathbb{C} \cdot h_1 + \mathbb{C} \cdot (h_2 \oplus wf_0)\big).$$

(ii) *If $\varphi_0 \neq 0$ and $\psi_0 = 0$, then there is a nonzero function h_3 in $(M_1 \ominus wM_1) \ominus \Omega(M_1)$ such that*

$$\Omega(M_2) = (\Omega(M_1) \ominus \mathbb{C} \cdot f_0) \oplus \mathbb{C} \cdot h_3.$$

(iii) *Suppose that* $\varphi_0 \neq 0$ *and* $\psi_0 \neq 0$. *If* $\mathbb{C} \cdot \varphi_0 = \mathbb{C} \cdot \psi_0$, *then there is a function* g_1 *in* $(M_1 \ominus wM_1) \ominus \Omega(M_1)$ *such that*

$$\Omega(M_2) = (\Omega(M_1) \ominus \mathbb{C} \cdot f_0) \oplus \mathbb{C} \cdot (g_1 \oplus wf_0).$$

(iv) *Suppose that* $\varphi_0 \neq 0$ *and* $\psi_0 \neq 0$. *If* $\mathbb{C} \cdot \varphi_0 \neq \mathbb{C} \cdot \psi_0$, *then*

$$\Omega(M_2) = \Omega(M_1) \ominus \mathbb{C} \cdot f_0.$$

Proof. (i) Since $\psi_0 = 0$, by Lemma 3.1 (iii) we have $f_0 \perp w\widetilde{\Omega}(N_1)$. Since $f_0 \in \Omega(M_1) \subset M_1 \ominus wM_1$, by Lemma 2.2 there is a unique nonzero function h_1 in $(M_1 \ominus wM_1) \ominus \Omega(M_1)$ satisfying $(R_z^{M_1})^* h_1 = f_0$. We note that

$$\big\{h \in (M_1 \ominus wM_1) \ominus \Omega(M_1) : (R_z^{M_1})^* h \in \mathbb{C} \cdot f_0\big\} = \mathbb{C} \cdot h_1. \tag{3.1}$$

Since $f_0 \in \Omega(M_1)$, by Lemma 3.1 (i) we have $(R_z^{M_1})^* wf_0 \in M_1 \ominus wM_1$. Since $\varphi_0 = 0$, by Lemma 3.1 (ii) we have $(R_z^{M_1})^* wf_0 \perp w\widetilde{\Omega}(N_1)$. Then by Lemma 2.2 again, there is a unique function h_2 in $(M_1 \ominus wM_1) \ominus \Omega(M_1)$ satisfying $(R_z^{M_1})^* h_2 = -(R_z^{M_1})^* wf_0$. Hence $(R_z^{M_1})^*(h_2 \oplus wf_0) = 0 \in \mathbb{C} \cdot f_0$.

Suppose that $(R_z^{M_1})^*(h \oplus wf_0) \in \mathbb{C} \cdot f_0$ for some $h \in (M_1 \ominus wM_1) \ominus \Omega(M_1)$. Then $(R_z^{M_1})^*(h - h_2) \in \mathbb{C} \cdot f_0$. Since $h - h_2 \in (M_1 \ominus wM_1) \ominus \Omega(M_1)$, by (3.1) we have $h - h_2 \in \mathbb{C} \cdot h_1$, and

$$h \oplus wf_0 \in h_2 + \mathbb{C} \cdot h_1 + wf_0 \subset \mathbb{C} \cdot h_1 + \mathbb{C} \cdot (h_2 \oplus wf_0).$$

By Lemma 3.3 (i), we get (i).

(ii) Since $f_0 \in \Omega(M_1)$, by Lemma 3.1 (i) we have $f_0 \in M_1 \ominus wM_1$ and $(R_z^{M_1})^* wf_0 \in M_1 \ominus wM_1$. Since $\psi_0 = 0$, by Lemma 3.1 (iii) $f_0 \perp w\widetilde{\Omega}(N_1)$. Then by Lemma 2.2, there is a unique nonzero function h_3 in $(M_1 \ominus wM_1) \ominus \Omega(M_1)$ such that $(R_z^{M_1})^* h_3 = f_0$. Since $\varphi_0 \neq 0$, by Lemma 3.1 (ii) we have $(R_z^{M_1})^* wf_0 \not\perp w\widetilde{\Omega}(N_1)$. Then by Lemma 2.2 again, $(R_z^{M_1})^* h \neq (R_z^{M_1})^* wf_0$ for any $h \in (M_1 \ominus wM_1) \ominus \Omega(M_1)$.

Suppose that there is $g \in (M_1 \ominus wM_1) \ominus \Omega(M_1)$ satisfying that $(R_z^{M_1})^*(g \oplus wf_0) = cf_0$ for some $c \in \mathbb{C}$. Then

$$\begin{aligned} (R_z^{M_1})^* wf_0 &= (R_z^{M_1})^*(g \oplus wf_0) - (R_z^{M_1})^* g = cf_0 - (R_z^{M_1})^* g \\ &= (R_z^{M_1})^*(ch_3 - g). \end{aligned}$$

Since $ch_3 - g \in (M_1 \ominus wM_1) \ominus \Omega(M_1)$, this contradicts the last paragraph. Hence by Lemma 3.3 (i), we get (ii).

(iii) Suppose that $\varphi_0 \neq 0$ and $\psi_0 \neq 0$. By the assumption, $\varphi_0 = c_1 \psi_0$ for some $c_1 \in \mathbb{C}$ with $c_1 \neq 0$. Then $P_{\widetilde{\Omega}(N_1)}(c_1 T_w^* f_0 - T_z^* f_0) = 0$, so

$$P_{w\widetilde{\Omega}(N_1)}(c_1 w T_w^* f_0 - w T_z^* f_0) = 0.$$

We have

$$P_{w\widetilde{\Omega}(N_1)} w T_z^* f_0 = P_{w\widetilde{\Omega}(N_1)} P_{M_1} T_z^* w f_0 = P_{w\widetilde{\Omega}(N_1)} (R_z^{M_1})^* w f_0$$

and $P_{w\widetilde{\Omega}(N_1)} w T_w^* f_0 = P_{w\widetilde{\Omega}(N_1)} f_0$. Then

$$P_{w\widetilde{\Omega}(N_1)}\big(c_1 f_0 - (R_z^{M_1})^* w f_0\big) = 0.$$

Hence $c_1 f_0 - (R_z^{M_1})^* w f_0 \perp w\widetilde{\Omega}(N_1)$. Since $f_0 \in \Omega(M_1)$, by Lemma 3.1 (i) we have

$$c_1 f_0 - (R_z^{M_1})^* w f_0 \in M_1 \ominus wM_1.$$

By Lemma 2.2, there is a unique function g_1 in $(M_1 \ominus wM_1) \ominus \Omega(M_1)$ such that

$$(R_z^{M_1})^* g_1 = c_1 f_0 - (R_z^{M_1})^* w f_0.$$

Hence

$$(R_z^{M_1})^*(g_1 \oplus w f_0) = c_1 f_0.$$

Since $\psi_0 \neq 0$, by Lemma 3.1 (iii) we have $f_0 \not\perp w\widetilde{\Omega}(N_1)$. Since $f_0 \in M_1 \ominus wM_1$, by Lemma 2.2 $(R_z^{M_1})^* h \notin \mathbb{C} \cdot f_0$ for any nonzero function $h \in (M_1 \ominus wM_1) \ominus \Omega(M_1)$. Hence by Lemma 3.3 (i), we get (iii).

(iv) By the assumption,

$$\mathbb{C} \cdot P_{\widetilde{\Omega}(N_1)} T_z^* f_0 \neq \mathbb{C} \cdot P_{\widetilde{\Omega}(N_1)} T_w^* f_0.$$

As the proof of (iii), we have

$$\mathbb{C} \cdot P_{w\widetilde{\Omega}(N_1)} (R_z^{M_1})^* w f_0 \neq \mathbb{C} \cdot P_{w\widetilde{\Omega}(N_1)} f_0. \tag{3.2}$$

As the last paragraph of (iii), $(R_z^{M_1})^* h \notin \mathbb{C} \cdot f_0$ for any nonzero function $h \in (M_1 \ominus wM_1) \ominus \Omega(M_1)$.

Assume that

$$(R_z^{M_1})^*(g \oplus w f_0) \in \mathbb{C} \cdot f_0$$

for some $g \in (M_1 \ominus wM_1) \ominus \Omega(M_1)$. Since $(R_z^{M_1})^* g \in M_1 \ominus wM_1$, $(R_z^{M_1})^* w f_0 \in M_1 \ominus wM_1$, so we may write

$$(R_z^{M_1})^* w f_0 = p \oplus c_1 f_0 \in \big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) \oplus \mathbb{C} \cdot f_0. \tag{3.3}$$

Then $(R_z^{M_1})^* g = -p \oplus c_2 f_0$ for some $c_2 \in \mathbb{C}$. We have

$$(R_z^{M_1})^*(M_1 \ominus wM_1) \perp \ker F_z^{M_1}.$$

By Lemma 2.1,

$$(R_z^{M_1})^*(M_1 \ominus wM_1) \perp w\widetilde{\Omega}(N_1).$$

Hence $-p \oplus c_2 f_0 \perp w\widetilde{\Omega}(N_1)$, so $P_{w\widetilde{\Omega}(N_1)} p = c_2 P_{w\widetilde{\Omega}(N_1)} f_0$. By (3.3),

$$\begin{aligned} P_{w\widetilde{\Omega}(N_1)} (R_z^{M_1})^* w f_0 &= P_{w\widetilde{\Omega}(N_1)} p + c_1 P_{w\widetilde{\Omega}(N_1)} f_0 \\ &= (c_1 + c_2) P_{w\widetilde{\Omega}(N_1)} f_0. \end{aligned}$$

Since $\varphi_0 \neq 0$ and $\psi_0 \neq 0$, by Lemma 3.1 (ii) and (iii) we have $P_{w\widetilde{\Omega}(N_1)} (R_z^{M_1})^* w f_0 \neq 0$ and $P_{w\widetilde{\Omega}(N_1)} f_0 \neq 0$. Hence the above equations contradict (3.2). Therefore there are no $g \in (M_1 \ominus wM_1) \ominus \Omega(M_1)$ satisfying that $(R_z^{M_1})^*(g \oplus wf_0) \in \mathbb{C} \cdot f_0$. By Lemma 3.3 (i), we get (iv). $\square$

When $\varphi_0 = 0$ and $\psi_0 \neq 0$ in Corollary 3.1 and Theorem 3.1, we can describe $\widetilde{\Omega}(N_2)$ and $\Omega(M_2)$ exchanging variables z and w in Corollary 3.1 (ii) and Theorem 3.1 (ii), respectively. We do not know whether in Theorem 3.1 (iii) we can take g_1 as $g_1 \neq 0$, and this is equivalent to $(R_z^{M_1})^* wf_0 \notin \mathbb{C} \cdot f_0$.

We shall show the examples which satisfy each conditions in Corollary 3.1 and Theorem 3.1.

Example 3.1. (i-1) Let $M_1 = z^2H^2 + wH^2$, $f_0 = w$ and $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$. Then M_1 and M_2 are invariant subspaces. We have

$$M_1 \ominus wM_1 = z^2H^2(z) \oplus \mathbb{C} \cdot zw \oplus \mathbb{C} \cdot w,$$

where $H^2(z)$ is the z-variable Hardy space, $\Omega(M_1) = \mathbb{C} \cdot z^2 \oplus \mathbb{C} \cdot w$, $f_0 \in \Omega(M_1)$ and $\widetilde{\Omega}(N_1) = \mathbb{C} \cdot z$. Hence $T_z^* f_0 = 0 \perp \widetilde{\Omega}(N_1)$ and $T_w^* f_0 = 1 \perp \widetilde{\Omega}(N_1)$, so $\varphi_0 = \psi_0 = 0$. In the proof of Theorem 3.1 (i), h_2 belongs to $(M_1 \ominus wM_1) \ominus \Omega(M_1)$ and $(R_z^{M_1})^* h_2 = (R_z^{M_1})^* wf_0$. In this case, we have $(R_z^{M_1})^* wf_0 = 0$, so $h_2 \in \Omega(M_1)$ and $h_2 = 0$. Note that

$$(R_z^{M_1})^*(M_1 \ominus wM_1) = z^2H^2(z) \oplus \mathbb{C} \cdot w.$$

(i-2) Let

$$M_1 = z^2 b_\alpha(z)H^2 + b_\alpha(z)wH^2 + w^2H^2,$$

where $b_\alpha = (z-\alpha)/(1-\overline{\alpha}z)$ and $\alpha \in \mathbb{D}$ with $0 < |\alpha| < 1$. Then

$$\Omega(M_1) = \mathbb{C} \cdot z^2 b_\alpha(z) \oplus \mathbb{C} \cdot b_\alpha(z)w.$$

Take $f_0 = b_\alpha(z)w \in \Omega(M_1)$. We have $\widetilde{\Omega}(N_1) = \mathbb{C} \cdot zb_\alpha(z)$. Then $T_z^* f_0 \perp \widetilde{\Omega}(N_1)$ and $T_w^* f_0 \perp \widetilde{\Omega}(N_1)$, so $\varphi_0 = \psi_0 = 0$. We have

$$M_1 \ominus wM_1 = z^2 b_\alpha(z)H^2(z) \oplus \mathbb{C} \cdot zb_\alpha(z)w \oplus \mathbb{C} \cdot \frac{w^2}{1-\overline{\alpha}z}$$

and

$$(M_1 \ominus wM_1) \ominus \Omega(M_1) = z^3 b_\alpha(z)H^2(z) \oplus \mathbb{C} \cdot zb_\alpha(z)w \oplus \mathbb{C} \cdot \frac{w^2}{1-\overline{\alpha}z}.$$

Take $h_2 = w^2/(1-\overline{\alpha}z)$. Then $h_2 \in (M_1 \ominus wM_1) \ominus \Omega(M_1)$ and $h_2 \neq 0$. We have

$$(R_z^{M_1})^* wf_0 = (R_z^{M_1})^* b_\alpha(z)w^2 = \frac{\alpha}{1-\overline{\alpha}z} w^2 = (R_z^{M_1})^* h_2.$$

Note that

$$(R_z^{M_1})^*(M_1 \ominus wM_1) = z^2 b_\alpha(z)H^2(z) \oplus \mathbb{C} \cdot \frac{w^2}{1-\overline{\alpha}z}.$$

(ii) Let $M_1 = zH^2 + wH^2$, $f_0 = z$ and $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$. Then M_1 and M_2 are invariant subspaces. We have $M_1 \ominus wM_1 = zH^2(z) \oplus \mathbb{C} \cdot w$, $\Omega(M_1) = \mathbb{C} \cdot z \oplus \mathbb{C} \cdot w$,

$f_0 \in \Omega(M_1)$ and $\widetilde{\Omega}(N_1) = \mathbb{C} \cdot 1$. Hence $T_z^* f_0 = 1 \not\perp \widetilde{\Omega}(N_1)$ and $T_w^* f_0 = 0 \perp \widetilde{\Omega}(N_1)$, so $\varphi_0 \neq 0$ and $\psi_0 = 0$. Note that $(R_z^{M_1})^*(M_1 \ominus wM_1) = zH^2(z)$.

(iii) Let $M_1 = zH^2 + wH^2$, $f_0 = z + w$ and $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$. Then M_1, M_2 are invariant subspaces. We have $\Omega(M_1) = \mathbb{C} \cdot z \oplus \mathbb{C} \cdot w$, $f_0 \in \Omega(M_1)$ and $\widetilde{\Omega}(N_1) = \mathbb{C} \cdot 1$. Hence $T_z^* f_0 = T_w^* f_0 = 1 \not\perp \widetilde{\Omega}(N_1)$, so $\varphi_0 \neq 0, \psi_0 \neq 0$ and $\mathbb{C} \cdot \varphi_0 = \mathbb{C} \cdot \psi_0$. We have that $c_1 = 1$ in the proof of Theorem 3.1 (iv). Hence

$$(R_z^{M_1})^* w f_0 - c_1 f_0 = w - (z + w) = -z.$$

We also have

$$(M_1 \ominus wM_1) \ominus \Omega(M_1) = z^2 H^2(z).$$

Let $g_1 = -z^2 \in (M_1 \ominus wM_1) \ominus \Omega(M_1)$. Then $g_1 \neq 0$ and

$$(R_z^{M_1})^* g_1 = (R_z^{M_1})^* w f_0 - c_1 f_0.$$

(iv) Let $M_1 = z^2 H^2 + zwH^2 + w^2 H^2$, $f_0 = zw$ and $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$. Then M_1, M_2 are invariant subspaces. We have

$$M_1 \ominus wM_1 = z^2 H^2(z) \oplus \mathbb{C} \cdot zw \oplus \mathbb{C} \cdot w^2,$$

$$\Omega(M_1) = \mathbb{C} \cdot z^2 \oplus \mathbb{C} \cdot zw \oplus \mathbb{C} \cdot w^2,$$

$f_0 \in \Omega(M_1)$ and $\widetilde{\Omega}(N_1) = \mathbb{C} \cdot z + \mathbb{C} \cdot w$. Hence $T_z^* f_0 = w \not\perp \widetilde{\Omega}(N_1)$, $T_w^* f_0 = z \not\perp \widetilde{\Omega}(N_1)$ and $P_{\widetilde{\Omega}(N_1)} T_z^* f_0 = w \neq z = P_{\widetilde{\Omega}(N_1)} T_w^* f_0$. Therefore $\varphi_0 \neq 0, \psi_0 \neq 0$ and $\mathbb{C} \cdot \varphi_0 \neq \mathbb{C} \cdot \psi_0$. Note that $(R_z^{M_1})^*(M_1 \ominus wM_1) = z^2 H^2(z)$. □

4. The case that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$

If $f_0 \notin \Omega(M_1)$, then either $f_0 \notin M_1 \ominus zM_1$ or $f_0 \notin M_1 \ominus wM_1$. In this section, we assume that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$. Since $M_1 \ominus M_2 = \mathbb{C} \cdot f_0$, there is $\alpha_0 \in \mathbb{C}$ with $\alpha_0 \neq 0$ satisfying that

$$(R_z^{M_1})^* f_0 = \alpha_0 f_0. \tag{4.1}$$

We shall study the structure of $\Omega(M_2)$ and $\widetilde{\Omega}(N_2)$. Let

$$\sigma_0 = P_{M_1 \ominus zM_1} f_0.$$

Since $\Omega(M_1) \subset M_1 \ominus zM_1$, we have $P_{\Omega(M_1)} \sigma_0 = P_{\Omega(M_1)} f_0$.

Lemma 4.1. *Suppose that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$. Then $\sigma_0 \neq 0$ and $f_0 = \sigma_0/(1 - \alpha_0 z)$.*

Proof. Since $(R_z^{M_1})^* f_0 = \alpha_0 f_0$, we have $f_0 = \alpha_0 z f_0 + \sigma_0$. Then we get the assertion. □

Lemma 4.2. *Suppose that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$. Then*

$$M_2 \ominus wM_2 = \big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) \oplus \mathbb{C} \cdot wf_0.$$

Proof. Since $M_1 \ominus M_2 = \mathbb{C} \cdot f_0$ and $wf_0 \perp f_0$, we have $wf_0 \in M_2$. In the same way as the proof of Lemma 3.2, we have the assertion. $\square$

Lemma 4.3. *Suppose that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$. Then we have the following.*

(i) $\Omega(M_2) = \big\{f \in (M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0 : (R_z^{M_1})^* f \in \mathbb{C} \cdot f_0\big\}$.

(ii) $\widetilde{\Omega}(N_2) = (N_1 \oplus \mathbb{C} \cdot f_0) \ominus \big(\overline{T_z^* N_1 + T_w^* N_1} + \mathbb{C} \cdot T_z^* f_0\big)$.

Proof. (i) By (2.1),

$$\Omega(M_2) = \big\{f \in M_2 \ominus wM_2 : T_z^* f \in N_2\big\}.$$

Since $N_2 = N_1 \oplus \mathbb{C} \cdot f_0$, we have

$$\Omega(M_2) = \big\{f \in M_2 \ominus wM_2 : (R_z^{M_1})^* f \in \mathbb{C} \cdot f_0\big\}.$$

By Lemma 4.2,

$$\Omega(M_2) = \big\{f \in \big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) \oplus \mathbb{C} \cdot wf_0 : (R_z^{M_1})^* f \in \mathbb{C} \cdot f_0\big\}.$$

Suppose that $(R_z^{M_1})^*(g \oplus wf_0) \in \mathbb{C} \cdot f_0$ for some $g \in (M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0$. We have $(R_z^{M_1})^* g = 0$ and $f_0 \in M_1 \ominus wM_1$. Hence $(R_z^{M_1})^* wf_0 \in M_1 \ominus wM_1$.

By (4.1), we may write $T_z^* f_0 = \alpha_0 f_0 \oplus f_1$ for some $f_1 \in N_1$. Then

$$(R_z^{M_1})^* wf_0 = P_{M_1} w T_z^* f_0 = \alpha_0 wf_0 + P_{M_1} wf_1.$$

Since $f_1 \in N_1$, $P_{M_1} wf_1 \in M_1 \ominus wM_1$. Hence $\alpha_0 wf_0 \in M_1 \ominus wM_1$, so $\alpha_0 = 0$. This contradicts $\alpha_0 \neq 0$. Therefore $(R_z^{M_1})^*(g \oplus wf_0) \notin \mathbb{C} \cdot f_0$ for any $g \in (M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0$. Hence we get (i).

(ii) We have $N_2 = N_1 \oplus \mathbb{C} \cdot f_0$. Hence

$$\overline{T_z^* N_2 + T_w^* N_2} = \overline{T_z^* N_1 + T_w^* N_1} + \mathbb{C} \cdot T_z^* f_0 + \mathbb{C} \cdot T_w^* f_0.$$

By (4.1), we have $f_0 \perp \ker F_z^{M_1}$, so by Lemma 2.1 $f_0 \perp w\widetilde{\Omega}(N_1)$. Hence $T_w^* f_0 \perp \widetilde{\Omega}(N_1)$. Since $f_0 \in M_1 \ominus wM_1$, we have $T_w^* f_0 \in N_1$. By (2.2), $T_w^* f_0 \in \overline{T_z^* N_1 + T_w^* N_1}$. Therefore

$$\overline{T_z^* N_2 + T_w^* N_2} = \overline{T_z^* N_1 + T_w^* N_1} + \mathbb{C} \cdot T_z^* f_0.$$

By (2.2) again, we get (ii). $\square$

Theorem 4.1. *Suppose that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$. Then we have the following.*

(i) *If $f_0 \perp \Omega(M_1)$, then $\Omega(M_2) = \Omega(M_1)$.*

(ii) *If* $f_0 \not\perp \Omega(M_1)$*, then there is a nonzero function* h_0 *in* $(M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0$ *such that*

$$\Omega(M_2) = \big(\Omega(M_1) \ominus \mathbb{C} \cdot P_{\Omega(M_1)} f_0\big) \oplus \mathbb{C} \cdot h_0.$$

Proof. (i) Suppose that there is $h \in (M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0$ satisfying $(R_z^{M_1})^* h = f_0$. By (4.1), we have that $(R_z^{M_1})^*(h - f_0/\alpha_0) = 0$, so $h - f_0/\alpha_0 \in \Omega(M_1)$. Since $f_0 \perp \Omega(M_1)$, we have

$$0 = \langle h - f_0/\alpha_0, f_0 \rangle = -\|f_0\|^2/\alpha_0.$$

This contradicts $f_0 \neq 0$. Hence by Lemma 4.3 (i), we have

$$\begin{aligned} \Omega(M_2) &= \big\{f \in (M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0 : (R_z^{M_1})^* f = 0\big\} \\ &= \big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) \cap \Omega(M_1) = \Omega(M_1). \end{aligned}$$

(ii) By the assumption, we have $P_{\Omega(M_1)} f_0 \neq 0$. Let

$$h_0 = \frac{1}{\alpha_0}\Big(f_0 - \frac{\|f_0\|^2}{\|P_{\Omega(M_1)} f_0\|^2} P_{\Omega(M_1)} f_0\Big) \in M_1 \ominus wM_1.$$

Since $f_0 \notin \Omega(M_1)$, we have that $h_0 \neq 0$ and

$$\langle h_0, f_0 \rangle = \frac{1}{\alpha_0}(\|f_0\|^2 - \|f_0\|^2) = 0.$$

Hence $h_0 \in (M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0$, $h_0 \in M_2 \ominus wM_2$ and

$$(R_z^{M_1})^* h_0 = \frac{1}{\alpha_0}(R_z^{M_1})^* f_0 = f_0.$$

Moreover we have $h_0 \in \Omega(M_2)$. Therefore by Lemma 4.3 (i), we have

$$\begin{aligned} \Omega(M_2) &= \big\{f \in (M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0 : (R_z^{M_1})^* f = 0\big\} \oplus \mathbb{C} \cdot h_0 \\ &= \big(\big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_0\big) \cap \Omega(M_1)\big) \oplus \mathbb{C} \cdot h_0 \\ &= \big(\Omega(M_1) \ominus \mathbb{C} \cdot P_{\Omega(M_1)} f_0\big) \oplus \mathbb{C} \cdot h_0. \end{aligned}$$

□

Recall that $\varphi_0 = P_{\widetilde{\Omega}(N_1)} T_z^* f_0$.

Theorem 4.2. *Suppose that* $f_0 \notin M_1 \ominus zM_1$ *and* $f_0 \in M_1 \ominus wM_1$*. Then we have the following.*

(i) *If* $\varphi_0 = 0$*, then* $\widetilde{\Omega}(N_2) = \widetilde{\Omega}(N_1)$*.*

(ii) *If* $\varphi_0 \neq 0$*, then*

$$\widetilde{\Omega}(N_2) = (\widetilde{\Omega}(N_1) \oplus \mathbb{C} \cdot f_0) \ominus \mathbb{C} \cdot (\varphi_0 \oplus \alpha_0 f_0).$$

Proof. (i) Since $\varphi_0 = 0$, we have $T_z^* f_0 \perp \widetilde{\Omega}(N_1)$. Then by (2.2), $P_{N_1} T_z^* f_0 \in \overline{T_z^* N_1 + T_w^* N_1}$. Since $T_z^* f_0 \in N_2 = N_1 \oplus \mathbb{C} \cdot f_0$, we have

$$T_z^* f_0 \in \overline{T_z^* N_1 + T_w^* N_1} \oplus \mathbb{C} \cdot f_0,$$

so

$$\overline{T_z^* N_1 + T_w^* N_1} + \mathbb{C} \cdot T_z^* f_0 = \overline{T_z^* N_1 + T_w^* N_1} \oplus \mathbb{C} \cdot f_0.$$

Then by Lemma 4.3 (ii),

$$\begin{aligned}\widetilde{\Omega}(N_2) &= (N_1 \oplus \mathbb{C} \cdot f_0) \ominus \left(\overline{T_z^* N_1 + T_w^* N_1} + \mathbb{C} \cdot T_z^* f_0\right) \\ &= (N_1 \oplus \mathbb{C} \cdot f_0) \ominus \left(\overline{T_z^* N_1 + T_w^* N_1} \oplus \mathbb{C} \cdot f_0\right) \\ &= N_1 \ominus \overline{T_z^* N_1 + T_w^* N_1} = \widetilde{\Omega}(N_1) \qquad \text{by (2.2).}\end{aligned}$$

(ii) Suppose that $\varphi_0 \neq 0$. We have

$$\begin{aligned}&\overline{T_z^* N_1 + T_w^* N_1} + \mathbb{C} \cdot T_z^* f_0 \\ &= \overline{T_z^* N_1 + T_w^* N_1} \oplus \mathbb{C} \cdot (P_{\widetilde{\Omega}(N_1)} T_z^* f_0 \oplus \alpha_0 f_0) \\ &= \overline{T_z^* N_1 + T_w^* N_1} \oplus \mathbb{C} \cdot (\varphi_0 \oplus \alpha_0 f_0).\end{aligned}$$

Hence by Lemma 4.3 (ii),

$$\begin{aligned}\widetilde{\Omega}(N_2) &= (N_1 \oplus \mathbb{C} \cdot f_0) \ominus \left(\overline{T_z^* N_1 + T_w^* N_1} + \mathbb{C} \cdot T_z^* f_0\right) \\ &= (\widetilde{\Omega}(N_1) \oplus \mathbb{C} \cdot f_0) \ominus \mathbb{C} \cdot (\varphi_0 \oplus \alpha_0 f_0) \qquad \text{by (2.2) again.}\end{aligned}$$

□

We shall show four examples which satisfy each conditions in the proof of Theorems 4.1 and 4.2.

Example 4.1. (i) Let

$$M_1 = \frac{z-a}{1-\bar{a}z} H^2 + wH^2, \quad 0 < |a| < 1,$$

$f_0 = \frac{w}{1-\bar{a}z}$ and $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$. Then M_1 and M_2 are invariant subspaces. We have

$$M_1 \ominus zM_1 = \mathbb{C} \cdot \frac{z-a}{1-\bar{a}z} \oplus wH^2(w)$$

and

$$M_1 \ominus wM_1 = \frac{z-a}{1-\bar{a}z} H^2(z) \oplus \mathbb{C} \cdot \frac{w}{1-\bar{a}z}.$$

Then $f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$. We have $\Omega(M_1) = \mathbb{C} \cdot \frac{z-a}{1-\bar{a}z}$ (see [5]). Then $f_0 \perp \Omega(M_1)$. We have $\widetilde{\Omega}(N_1) = \{0\}$, so $\varphi_0 = P_{\widetilde{\Omega}(N_1)} T_z^* f_0 = 0$.

(ii) Let

$$M_1 = z\frac{z-a}{1-\bar{a}z} H^2 + zwH^2 + w^2 H^2, \quad 0 < |a| < 1,$$

$f_0 = \frac{zw}{1-\overline{a}z}$ and $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$. Then M_1 and M_2 are invariant subspaces. We have

$$M_1 \ominus zM_1 = \mathbb{C} \cdot z\frac{z-a}{1-\overline{a}z} \oplus \mathbb{C} \cdot zw \oplus w^2 H^2(w)$$

and

$$M_1 \ominus wM_1 = z\frac{z-a}{1-\overline{a}z}H^2(z) \oplus \mathbb{C} \cdot \frac{zw}{1-\overline{a}z} \oplus \mathbb{C} \cdot w^2.$$

Then $f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$. We have

$$\Omega(M_1) = \mathbb{C} \cdot z\frac{z-a}{1-\overline{a}z} \oplus \mathbb{C} \cdot w^2.$$

Hence $f_0 \perp \Omega(M_1)$. We have $\widetilde{\Omega}(N_1) = \mathbb{C} \cdot w$ and $T_z^* f_0 = w/(1-\overline{a}z)$, so $\varphi_0 = P_{\widetilde{\Omega}(N_1)} T_z^* f_0 \neq 0$.

(iii) Let

$$M_1 = \frac{z-\alpha}{1-\overline{\alpha}z}H^2 + \frac{w-\beta}{1-\overline{\beta}w}H^2, \quad 0 < |\alpha| < 1, \ 0 < |\beta| < 1,$$

$f_0 = \frac{1}{1-\overline{\alpha}z}\frac{w-\beta}{1-\overline{\beta}w}$ and $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$. Then M_1 and M_2 are invariant subspaces. We have

$$M_1 \ominus zM_1 = \mathbb{C} \cdot \frac{z-\alpha}{1-\overline{\alpha}z}\frac{1}{1-\overline{\beta}w} \oplus \frac{w-\beta}{1-\overline{\beta}w}H^2(w)$$

and

$$M_1 \ominus wM_1 = \frac{z-\alpha}{1-\overline{\alpha}z}H^2(z) \oplus \mathbb{C} \cdot \frac{1}{1-\overline{\alpha}z}\frac{w-\beta}{1-\overline{\beta}w}.$$

Then $f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$. We have

$$\Omega(M_1) = \mathbb{C} \cdot \Big(\frac{-\overline{\beta}}{1-|\beta|^2}\frac{w-\beta}{1-\overline{\beta}w} + \frac{-\overline{\alpha}}{1-\overline{\beta}w}\frac{z-\alpha}{1-\overline{\alpha}z} \Big)$$

(see [5]). Then $f_0 \not\perp \Omega(M_1)$. We also have $\widetilde{\Omega}(N_1) = \{0\}$, so $\varphi_0 = 0$.

(iv) Let

$$M_1 = z\frac{z-\alpha}{1-\overline{\alpha}z}H^2 + w\frac{w-\beta}{1-\overline{\beta}w}H^2, \quad 0 < |\alpha| < 1, \ 0 < |\beta| < 1,$$

$f_0 = \frac{z}{1-\overline{\alpha}z}\frac{z-\alpha}{1-\overline{\alpha}z}$ and $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$. Then M_1 and M_2 are invariant subspaces. We have

$$\begin{aligned} M_1 \ominus zM_1 \;\; = \;\; & \mathbb{C} \cdot z\frac{z-\alpha}{1-\overline{\alpha}z}\frac{1}{1-\overline{\beta}w} \oplus \mathbb{C} \cdot z\frac{z-\alpha}{1-\overline{\alpha}z}\frac{w-\beta}{1-\overline{\beta}w} \\ & \oplus w\frac{w-\beta}{1-\overline{\beta}w}H^2(w) \end{aligned}$$

and

$$\begin{aligned} M_1 \ominus wM_1 &= \mathbb{C} \cdot \frac{1}{1-\overline{\alpha}z} w \frac{w-\beta}{1-\overline{\beta}w} \oplus \mathbb{C} \cdot \frac{z-\alpha}{1-\overline{\alpha}z} w \frac{w-\beta}{1-\overline{\beta}w} \\ &\qquad \oplus z \frac{z-\alpha}{1-\overline{\alpha}z} H^2(z). \end{aligned}$$

Hence $f_0 \notin M_1 \ominus zM_1$ and $f_0 \in M_1 \ominus wM_1$. We have

$$\Omega(M_1) = \mathbb{C} \cdot z \frac{z-\alpha}{1-\overline{\alpha}z} + \mathbb{C} \cdot w \frac{w-\beta}{1-\overline{\beta}w}.$$

This shows that $f_0 \not\perp \Omega(M_1)$. We have

$$\widetilde{\Omega}(N_1) = \mathbb{C} \cdot \frac{z-\alpha}{1-\overline{\alpha}z} \frac{w-\beta}{1-\overline{\beta}w}.$$

Then

$$\Big\langle T_z^* f_0, \frac{z-\alpha}{1-\overline{\alpha}z} \frac{w-\beta}{1-\overline{\beta}w} \Big\rangle = \Big\langle \frac{1}{1-\overline{\alpha}z} \frac{z-\alpha}{1-\overline{\alpha}z}, \frac{z-\alpha}{1-\overline{\alpha}z} \frac{w-\beta}{1-\overline{\beta}w} \Big\rangle = -\overline{\beta}.$$

Hence $T_z^* f_0 \not\perp \widetilde{\Omega}(N_1)$ and $\varphi_0 \neq 0$. $\square$

When $f_0 \in M_1 \ominus zM_1$ and $f_0 \notin M_1 \ominus wM_1$, exchanging the variables z and w in Lemma 4.3, Theorems 4.1, 4.2 and 4.1 we have the corresponding results.

5. The case that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \notin M_1 \ominus wM_1$

In this section, we assume that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \notin M_1 \ominus wM_1$ and we shall study the structure of $\Omega(M_2)$ and $\widetilde{\Omega}(N_2)$. Let

$$\eta_0 = P_{M_1 \ominus wM_1} f_0 \quad \text{and} \quad \sigma_0 = P_{M_1 \ominus zM_1} f_0.$$

Since $\Omega(M_1) = (M_1 \ominus zM_1) \cap (M_1 \ominus wM_1)$, we have

$$P_{\Omega(M_1)} \eta_0 = P_{\Omega(M_1)} \sigma_0 = P_{\Omega(M_1)} f_0.$$

By (4.1), $(R_z^{M_1})^* f_0 = \alpha_0 f_0$ for some $\alpha_0 \in \mathbb{D}$ with $\alpha_0 \neq 0$. Similarly we have that $(R_w^{M_1})^* f_0 = \beta_0 f_0$ for some $\beta_0 \in \mathbb{D}$ with $\beta_0 \neq 0$.

Lemma 5.1. *Suppose that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \notin M_1 \ominus wM_1$. Then we have the following.*

(i) $\eta_0 \neq 0$ *and* $f_0 = \eta_0/(1-\beta_0 w)$.
(ii) *Either* $\eta_0 \notin \Omega(M_1)$ *or* $\sigma_0 \notin \Omega(M_1)$.

Proof. Since $(R_w^{M_1})^* f_0 = \beta_0 f_0$, we have $f_0 = \beta_0 w f_0 + \eta_0$. (i) follows from this fact.

To show (ii), suppose that $\eta_0 \in \Omega(M_1)$ and $\sigma_0 \in \Omega(M_1)$. Since $\Omega(M_1) = (M_1 \ominus zM_1) \cap (M_1 \ominus wM_1)$, we have

$$\begin{aligned} \eta_0 &= P_{\Omega(M_1)}\eta_0 = P_{\Omega(M_1)}P_{M_1 \ominus wM_1} f_0 \\ &= P_{\Omega(M_1)}P_{M_1 \ominus zM_1} f_0 = \sigma_0. \end{aligned}$$

By (i), $f_0 = \eta_0/(1-\beta_0 w)$, and $f_0 = \sigma_0/(1-\alpha_0 z)$. Hence $\eta_0/(1-\beta_0 w) = \sigma_0/(1-\alpha_0 z)$, so $(\alpha_0 z - \beta_0 w)\eta_0 = 0$. Since $\alpha_0\beta_0 \neq 0$, we have $\eta_0 = 0$. This contradicts $\eta_0 \neq 0$. $\square$

By Lemma 5.1 (ii), we may assume that $\eta_0 \notin \Omega(M_1)$. Similarly as Lemma 5.1 (i), we have $\sigma_0 \neq 0$ and $f_0 = \sigma_0/(1-\alpha_0 z)$. When $\sigma_0 \notin \Omega(M_1)$, exchanging variables z and w we can get the similar result.

Lemma 5.2. *Suppose that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \notin M_1 \ominus wM_1$. If $\eta_0 \notin \Omega(M_1)$, then*

$$M_2 \ominus wM_2 = \big((M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0\big) \oplus \mathbb{C}\cdot\Big(f_0 - \frac{1}{1-|\beta_0|^2}\eta_0\Big).$$

Proof. By Lemma 5.1 (i), we have $\eta_0 \neq 0$ and $f_0 = \bigoplus_{n=0}^{\infty} \beta_0^n \eta_0 w^n$. Since $\|f_0\| = 1$ by the starting assumption, we have $\|\eta_0\|^2/(1-|\beta_0|^2) = 1$. Let

$$g_0 = f_0 - \frac{1}{1-|\beta_0|^2}\eta_0 \in M_1.$$

Then $g_0 \neq 0$. We have

$$\langle g_0, f_0\rangle = 1 - \frac{1}{1-|\beta_0|^2}\langle \eta_0, f_0\rangle = 1 - \frac{1}{1-|\beta_0|^2}\langle \eta_0, \eta_0\rangle = 0$$

and $(R_w^{M_1})^* g_0 = \beta_0 f_0$. Hence $g_0 \in M_2 \ominus wM_2$. Since $(M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0 \perp f_0$, we have

$$(M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0 \subset M_2 \ominus wM_2.$$

Therefore

$$\big((M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0\big) \oplus \mathbb{C}\cdot g_0 \subset M_2 \ominus wM_2.$$

To show the reverse inclusion, let $g \in M_2 \ominus wM_2$. Then $(R_w^{M_1})^* g = cf_0$ for some $c \in \mathbb{C}$. If $c = 0$, then $g \in M_1 \ominus wM_1$. Since $g \perp f_0$, we have that

$$\langle g, \eta_0\rangle = \langle g, f_0 - \beta_0 w f_0\rangle = -\overline{\beta_0}\langle (R_w^{M_1})^* g, f_0\rangle = 0.$$

Hence $g \in (M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0$.

Suppose that $c \neq 0$. Since $(R_w^{M_1})^* g_0 = \beta_0 f_0$, we have $(R_w^{M_1})^*(g/c - g_0/\beta_0) = 0$, so $g/c - g_0/\beta_0 \in M_1 \ominus wM_1$. Since $g \perp f_0$ and $g_0 \perp f_0$, we have that

$$\begin{aligned} \langle g/c - g_0/\beta_0, \eta_0\rangle &= \langle g/c - g_0/\beta_0, f_0 - \beta_0 w f_0\rangle \\ &= -\overline{\beta_0}\langle (R_w^{M_1})^*(g/c - g_0/\beta_0), f_0\rangle = 0. \end{aligned}$$

Hence $g/c - g_0/\beta_0 \in (M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0$, so

$$g \in \big((M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0\big) \oplus \mathbb{C}\cdot g_0.$$

Thus we get the assertion. $\square$

Theorem 5.1. *Suppose that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \notin M_1 \ominus wM_1$. Moreover suppose that $\eta_0 \notin \Omega(M_1)$. Then we have the following.*

(i) *There is h_0 in $M_2 \ominus wM_2$ satisfying that*

$$\Omega(M_2) = \big(\Omega(M_1) \ominus \mathbb{C}\cdot P_{\Omega(M_1)} f_0\big) \oplus \mathbb{C}\cdot h_0.$$

.

(ii) *If $\eta_0 \perp \Omega(M_1)$, then $h_0 = 0$ and $\Omega(M_2) = \Omega(M_1)$.*

(iii) *If $\eta_0 \not\perp \Omega(M_1)$, then $h_0 \neq 0$.*

Proof. We put

$$g_0 = f_0 - \frac{1}{1-|\beta_0|^2}\eta_0 \in M_2 \ominus wM_2.$$

Then by Lemma 5.2,

$$M_2 \ominus wM_2 = \big((M_1 \ominus wM_1) \oplus \mathbb{C}\cdot\eta_0\big) \oplus g_0$$

and $g_0 \neq 0$.

(i) By (2.1), we have

$$\Omega(M_2) = \big\{f \in M_2 \ominus wM_2 : T_z^* f \in N_2\big\}.$$

Since $N_2 = N_1 \oplus \mathbb{C}\cdot f_0$, we have that

$$\Omega(M_2) = \big\{f \in M_2 \ominus wM_2 : (R_z^{M_1})^* f \in \mathbb{C}\cdot f_0\big\}.$$

Hence there is h_0 in $M_2 \ominus wM_2$ such that

$$\Omega(M_2) = \big\{f \in M_2 \ominus wM_2 : (R_z^{M_1})^* f = 0\big\} \oplus \mathbb{C}\cdot h_0. \tag{5.1}$$

We have that $f_0 \in (R_z^{M_1})^*(M_2 \ominus wM_2)$ if and only if $h_0 \neq 0$, and in this case we may assume that $(R_z^{M_1})^* h_0 = f_0$. We have

$$(R_z^{M_1})^* g_0 = \alpha_0 f_0 - \frac{1}{1-|\beta_0|^2}(R_z^{M_1})^*\eta_0.$$

Note that $(R_z^{M_1})^*(M_1 \ominus wM_1) \subset M_1 \ominus wM_1$. Since $f_0 \notin M_1 \ominus wM_1$, $(R_z^{M_1})^* g_0 \notin M_1 \ominus wM_1$, and by Lemma 5.2 we have

$$\begin{aligned}&\big\{f \in M_2 \ominus wM_2 : (R_z^{M_1})^* f = 0\big\}\\ &= \big\{f \in (M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0 : (R_z^{M_1})^* f = 0\big\}.\end{aligned}$$

We also have

$$\begin{aligned}\big\{f \in (M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0 : (R_z^{M_1})^* f = 0\big\} \\ &= \Omega(M_1) \ominus \mathbb{C}\cdot P_{\Omega(M_1)}\eta_0 \\ &= \Omega(M_1) \ominus \mathbb{C}\cdot P_{\Omega(M_1)} f_0.\end{aligned}$$

Hence by (5.1), we get (i).

(ii) By Lemma 5.1 (i), we have $\eta_0 \neq 0$. Suppose that $\eta_0 \perp \Omega(M_1)$, i.e., $f_0 \perp \Omega(M_1)$. Then $P_{\Omega(M_1)} f_0 = P_{\Omega(M_1)}\eta_0 = 0$ and

$$\Omega(M_1) \ominus \mathbb{C}\cdot P_{\Omega(M_1)} f_0 = \Omega(M_1).$$

Since $\eta_0 \in M_1 \ominus wM_1$ and $\eta_0 \perp \Omega(M_1)$, we have $(R_z^{M_1})^*\eta_0 \neq 0$.

Suppose that $(R_z^{M_1})^* h = c(R_z^{M_1})^*\eta_0$ for some nonzero $h \in (M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0$ and $c \in \mathbb{C}$ with $c \neq 0$. Then $(R_z^{M_1})^*(h - c\eta_0) = 0$. Since $h - c\eta_0 \in M_1 \ominus wM_1$, we have $h - c\eta_0 \in \Omega(M_1)$. Since $\eta_0 \perp \Omega(M_1)$,

$$0 = \langle h - c\eta_0, \eta_0\rangle = -c\|\eta_0\|^2 \neq 0.$$

This contradiction shows that there are no such h and c.

To show $h_0 = 0$, suppose that $h_0 \neq 0$. As mentioned in the proof of (i), we may consider that $(R_z^{M_1})^* h_0 = f_0$. Since $g_0 \in M_2 \ominus wM_2$, by Lemma 5.2 we may write $h_0 = F \oplus dg_0$ for some $F \in (M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0$ and $d \in \mathbb{C}$. Since

$$(R_z^{M_1})^* g_0 = \alpha_0 f_0 - \frac{1}{1-|\beta_0|^2}(R_z^{M_1})^*\eta_0,$$

we have that

$$\begin{aligned} f_0 &= (R_z^{M_1})^* h_0 = (R_z^{M_1})^* F + d(R_z^{M_1})^* g_0 \\ &= (R_z^{M_1})^* F + \alpha_0 d f_0 - \frac{d}{1-|\beta_0|^2}(R_z^{M_1})^*\eta_0.\end{aligned}$$

Since $\eta_0, F \in M_1 \ominus wM_1$, we have

$$(1-\alpha_0 d) f_0 = (R_z^{M_1})^* F - \frac{d}{1-|\beta_0|^2}(R_z^{M_1})^*\eta_0 \in M_1 \ominus wM_1.$$

Since $f_0 \notin M_1 \ominus wM_1$, we have $\alpha_0 d = 1$ and $d \neq 0$. Hence

$$(R_z^{M_1})^* F = \frac{d}{1-|\beta_0|^2}(R_z^{M_1})^*\eta_0 \neq 0.$$

This contradicts the fact given in the last paragraph. Hence $h_0 = 0$. Therefore by (i), we get (ii).

(iii) Suppose that $\eta_0 \not\perp \Omega(M_1)$. Then $P_{\Omega(M_1)} f_0 = P_{\Omega(M_1)}\eta_0 \neq 0$. We have

$$\begin{aligned}&\Big\langle \eta_0 - \frac{\|\eta_0\|^2}{\|P_{\Omega(M_1)} f_0\|^2} P_{\Omega(M_1)} f_0, \eta_0 \Big\rangle \\ &\qquad = \|\eta_0\|^2 - \frac{\|\eta_0\|^2}{\|P_{\Omega(M_1)} f_0\|^2} \|P_{\Omega(M_1)} f_0\|^2 = 0.\end{aligned}$$

Putting

$$h = \frac{1}{1-|\beta_0|^2}\Big(\eta_0 - \frac{\|\eta_0\|^2}{\|P_{\Omega(M_1)} f_0\|^2} P_{\Omega(M_1)} f_0\Big),$$

we have $h \in (M_1 \ominus wM_1) \ominus \mathbb{C}\cdot\eta_0$. Since $\eta_0 \notin \Omega(M_1)$, we have $h \neq 0$. By Lemma 5.2, $h + g_0 \in M_2 \ominus wM_2$. We have

$$\begin{aligned}(R_z^{M_1})^*(h+g_0) &= \frac{1}{1-|\beta_0|^2}(R_z^{M_1})^*\eta_0 + (R_z^{M_1})^* g_0 \\ &= \frac{1}{1-|\beta_0|^2}(R_z^{M_1})^*\eta_0 + \alpha_0 f_0 - \frac{1}{1-|\beta_0|^2}(R_z^{M_1})^*\eta_0 \\ &= \alpha_0 f_0.\end{aligned}$$

Hence $h + g_0 \in \Omega(M_2)$. Since $\alpha_0 f_0 \neq 0$, we also have $h + g_0 \notin \Omega(M_1)$. Therefore by Theorem 5.1 (i), we get $h_0 \neq 0$. $\Box$

In the last part, we shall study the structure of $\widetilde{\Omega}(N_2)$. Recall that $\varphi_0 = P_{\widetilde{\Omega}(N_1)} T_z^* f_0$ and $\psi_0 = P_{\widetilde{\Omega}(N_1)} T_w^* f_0$.

Theorem 5.2. *Suppose that $f_0 \notin M_1 \ominus zM_1$ and $f_0 \notin M_1 \ominus wM_1$. Then we have the following.*

(i) *If $\varphi_0 = \psi_0 = 0$, then $\widetilde{\Omega}(N_2) = \widetilde{\Omega}(N_1)$.*

(ii) *If either $\varphi_0 \neq 0$ or $\psi_0 \neq 0$, then $\varphi_0 \neq 0$ and $\psi_0 \neq 0$.*

(iii) *If $\varphi_0 \neq 0$ and $\psi_0 \neq 0$, then*

$$\frac{\overline{\alpha_0}}{\|\varphi_0\|^2}\varphi_0 = \frac{\overline{\beta_0}}{\|\psi_0\|^2}\psi_0$$

and

$$\widetilde{\Omega}(N_2) = \big(\widetilde{\Omega}(N_1) \ominus \mathbb{C}\cdot\varphi_0\big) \oplus \mathbb{C}\cdot\Big(f_0 - \frac{\overline{\alpha_0}}{\|\varphi_0\|^2}\varphi_0\Big).$$

Proof. Let $\xi \in \widetilde{\Omega}(N_1) \ominus (\mathbb{C}\cdot\varphi_0 + \mathbb{C}\cdot\psi_0)$. Since $z\xi \in M_1$, by the definition of φ_0 we have

$$\langle z\xi, f_0\rangle = \langle \xi, T_z^* f_0\rangle = \langle \xi, P_{\widetilde{\Omega}(N_1)} T_z^* f_0\rangle = \langle \xi, \varphi_0\rangle = 0.$$

Similarly we have $\langle w\xi, f_0\rangle = 0$. Note that $M_2 = M_1 \ominus \mathbb{C}\cdot f_0$. Then $z\xi, w\xi \in M_2$. Hence $\xi \in \widetilde{\Omega}(N_2)$. Thus

$$\widetilde{\Omega}(N_1) \ominus (\mathbb{C}\cdot\varphi_0 + \mathbb{C}\cdot\psi_0) \subset \widetilde{\Omega}(N_2).$$

Since $N_2 = N_1 \oplus \mathbb{C} \cdot f_0$, we have

$$\widetilde{\Omega}(N_2) = \big(\widetilde{\Omega}(N_1) \ominus (\mathbb{C} \cdot \varphi_0 + \mathbb{C} \cdot \psi_0)\big) \oplus \Lambda, \tag{5.2}$$

where

$$\Lambda = \big\{h \in \mathbb{C} \cdot \varphi_0 + \mathbb{C} \cdot \psi_0 + \mathbb{C} \cdot f_0 : zh \perp f_0, wh \perp f_0\big\}. \tag{5.3}$$

Suppose that $c_1\varphi_0 + c_2\psi_0 \in \widetilde{\Omega}(N_2)$ for some $c_1, c_2 \in \mathbb{C}$. Then $z(c_1\varphi_0 + c_2\psi_0) \in M_2$, and $z(c_1\varphi_0 + c_2\psi_0) \perp f_0$. Hence $c_1\varphi_0 + c_2\psi_0 \perp T_z^* f_0$, so $c_1\varphi_0 + c_2\psi_0 \perp P_{\widetilde{\Omega}(N_1)} T_z^* f_0 = \varphi_0$. Similarly we have $c_1\varphi_0 + c_2\psi_0 \perp \psi_0$. Hence $c_1\varphi_0 + c_2\psi_0 = 0$.

(i) Suppose that $\varphi_0 = \psi_0 = 0$. Since $\|f_0\| = 1$ and $(R_z^{M_1})^* f_0 = \alpha_0 f_0$, we have

$$\langle z f_0, f_0\rangle = \langle f_0, (R_z^{M_1})^* f_0\rangle = \langle f_0, \alpha_0 f_0\rangle = \overline{\alpha_0} \neq 0.$$

Hence by (5.3), we have $\Lambda = \{0\}$, so by (5.2) we get (i).

(ii) We assume that $\psi_0 \neq 0$. Recall that $(R_w^{M_1})^* f_0 = \beta_0 f_0$. Since $\psi_0 = P_{\widetilde{\Omega}(N_1)} T_w^* f_0$, we have

$$\begin{aligned} \Big\langle T_w^* f_0, f_0 - \frac{\overline{\beta_0}}{\|\psi_0\|^2}\psi_0\Big\rangle &= \beta_0 - \frac{\beta_0}{\|\psi_0\|^2}\langle T_w^* f_0, \psi_0\rangle \\ &= \beta_0 - \frac{\beta_0}{\|\psi_0\|^2}\|\psi_0\|^2 = 0. \end{aligned}$$

Hence

$$T_w^* f_0 \perp f_0 - \frac{\overline{\beta_0}}{\|\psi_0\|^2}\psi_0. \tag{5.4}$$

Since $z\psi_0 \in M_1 \ominus zM_1$ and $w\psi_0 \in M_1 \ominus wM_1$, we also have that

$$\begin{aligned} \beta_0\langle f_0, z\psi_0\rangle &= \langle (R_w^{M_1})^* f_0, z\psi_0\rangle = \langle T_w^* f_0, z\psi_0\rangle \\ &= \langle f_0, zw\psi_0\rangle = \langle T_z^* f_0, w\psi_0\rangle = \alpha_0\langle f_0, w\psi_0\rangle \\ &= \alpha_0\langle P_{\widetilde{\Omega}(N_1)} T_w^* f_0, \psi_0\rangle = \alpha_0\|\psi_0\|^2. \end{aligned}$$

Hence

$$\langle \varphi_0, \psi_0\rangle = \langle T_z^* f_0, \psi_0\rangle = \langle f_0, z\psi_0\rangle = \frac{\alpha_0}{\beta_0}\|\psi_0\|^2 \neq 0. \tag{5.5}$$

This shows that $\varphi_0 \neq 0$. Similarly if $\varphi_0 \neq 0$, then $\psi_0 \neq 0$.

(iii) Suppose that $\varphi_0 \neq 0$ and $\psi_0 \neq 0$. We have that

$$\begin{aligned} &\Big\langle T_z^* f_0, f_0 - \frac{\overline{\beta_0}}{\|\psi_0\|^2}\psi_0\Big\rangle \\ &= \langle T_z^* f_0, f_0\rangle - \frac{\beta_0}{\|\psi_0\|^2}\langle T_z^* f_0, \psi_0\rangle \\ &= \langle (R_z^{M_1})^* f_0, f_0\rangle - \frac{\beta_0}{\|\psi_0\|^2}\frac{\alpha_0}{\beta_0}\|\psi_0\|^2 \qquad \text{by (5.5)} \\ &= \langle \alpha_0 f_0, f_0\rangle - \alpha_0 = \alpha_0 - \alpha_0 = 0. \end{aligned}$$

Then

$$T_z^* f_0 \perp f_0 - \frac{\overline{\beta_0}}{\|\psi_0\|^2}\psi_0.$$

Therefore by (5.4) and (5.5),

$$f_0 - \frac{\overline{\beta_0}}{\|\psi_0\|^2}\psi_0 \in \Lambda.$$

Similarly, we have

$$f_0 - \frac{\overline{\alpha_0}}{\|\varphi_0\|^2}\varphi_0 \in \Lambda.$$

Hence by (5.3),

$$\frac{\overline{\alpha_0}}{\|\varphi_0\|^2}\varphi_0 - \frac{\overline{\beta_0}}{\|\psi_0\|^2}\psi_0 \in \Lambda,$$

so by (5.2) we have

$$\frac{\overline{\alpha_0}}{\|\varphi_0\|^2}\varphi_0 - \frac{\overline{\beta_0}}{\|\psi_0\|^2}\psi_0 \in \widetilde{\Omega}(N_2).$$

By the second paragraph of the proof,

$$\frac{\overline{\alpha_0}}{\|\varphi_0\|^2}\varphi_0 = \frac{\overline{\beta_0}}{\|\psi_0\|^2}\psi_0.$$

Then

$$\Lambda = \mathbb{C} \cdot \left(f_0 - \frac{\overline{\alpha_0}}{\|\varphi_0\|^2}\varphi_0\right)$$

and by (5.2), we get

$$\widetilde{\Omega}(N_2) = \left(\widetilde{\Omega}(N_1) \ominus \mathbb{C} \cdot \varphi_0\right) \oplus \mathbb{C} \cdot \left(f_0 - \frac{\overline{\alpha_0}}{\|\varphi_0\|^2}\varphi_0\right).$$

□

We shall show examples which satisfy each conditions in Theorems 5.1 and 5.2.

Example 5.1. (i) Let

$$M_1 = zH^2 + \frac{w-\beta}{1-\overline{\beta}w}H^2, \quad 0 < |\beta| < 1,$$

$f_0 = \frac{z}{1-\overline{\alpha}z}\frac{1}{1-\overline{\beta}w}$ for some $0 < |\alpha| < 1$ and $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$. Then

$$M_2 = z\frac{z-\alpha}{1-\overline{\alpha}z}H^2 + \frac{w-\beta}{1-\overline{\beta}w}H^2,$$

so M_1 and M_2 are invariant subspaces. We have

$$M_1 \ominus zM_1 = \mathbb{C} \cdot \frac{z}{1-\overline{\beta}w} \oplus \frac{w-\beta}{1-\overline{\beta}w}H^2(w)$$

and

$$M_1 \ominus wM_1 = zH^2(z) \oplus \mathbb{C} \cdot \frac{w-\beta}{1-\overline{\beta}w}.$$

Then $f_0 \notin M_1 \ominus zM_1$, $f_0 \notin M_1 \ominus wM_1$ and

$$\eta_0 = P_{M_1 \ominus wM_1} f_0 = \frac{z}{1 - \overline{\alpha} z}.$$

We have

$$\Omega(M_1) = \mathbb{C} \cdot \frac{w - \beta}{1 - \overline{\beta} w} \quad \text{and} \quad \eta_0 \perp \Omega(M_1).$$

Since $N_1 = \mathbb{C} \cdot 1/(1 - \overline{\beta} w)$, we have $\widetilde{\Omega}(N_1) = \{0\}$. Hence $\varphi_0 = \psi_0 = 0$.

(ii) Let $\alpha \in \mathbb{D}$ with $\alpha \neq 0$ and

$$M_1 = z^2 \frac{z - \alpha}{1 - \overline{\alpha} z} H^2 + z^2 w H^2 + z w^2 H^2 + w^2 \frac{w - \alpha}{1 - \overline{\alpha} w} H^2.$$

Then M_1 is an invariant subspace and

$$\begin{aligned} M_1 \ominus zM_1 &= \mathbb{C} \cdot z^2 \frac{z - \alpha}{1 - \overline{\alpha} z} \oplus \mathbb{C} \cdot z^2 w \oplus \mathbb{C} \cdot z \frac{w^2}{1 - \overline{\alpha} w} \\ &\qquad \oplus \mathbb{C} \cdot w^2 \frac{w - \alpha}{1 - \overline{\alpha} w} H^2(w) \end{aligned}$$

and

$$\begin{aligned} M_1 \ominus wM_1 &= z^2 \frac{z - \alpha}{1 - \overline{\alpha} z} H^2(z) \oplus \mathbb{C} \cdot \frac{z^2}{1 - \overline{\alpha} z} w \oplus \mathbb{C} \cdot z w^2 \\ &\qquad \oplus \mathbb{C} \cdot w^2 \frac{w - \alpha}{1 - \overline{\alpha} w}. \end{aligned}$$

Hence

$$\Omega(M_1) = \mathbb{C} \cdot z^2 \frac{z - \alpha}{1 - \overline{\alpha} z} \oplus \mathbb{C} \cdot w^2 \frac{w - \alpha}{1 - \overline{\alpha} w}.$$

Let

$$f_0 = \frac{z^2}{1 - \overline{\alpha} z} w \oplus z \frac{w^2}{1 - \overline{\alpha} w}.$$

Then $f_0 \in M_1$, $f_0 \notin M_1 \ominus zM_1$ and $f_0 \notin M_1 \ominus wM_1$. Let $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$. We have

$$M_2 = z^2 \frac{z - \alpha}{1 - \overline{\alpha} z} H^2 + z^2 w^2 H^2 + w^2 \frac{w - \alpha}{1 - \overline{\alpha} w} H^2,$$

so M_2 is an invariant subspace. Moreover we have $f_0 \perp \Omega(M_1)$, so we get $\eta_0 \perp \Omega(M_1)$.

We have $\widetilde{\Omega}(N_1) = \mathbb{C} \cdot zw$, and

$$\langle T_z^* f_0, zw \rangle = \langle f_0, z^2 w \rangle = \left\langle \frac{z^2}{1 - \overline{\alpha} z} w, z^2 w \right\rangle = 1.$$

Hence $\varphi_0 = P_{\widetilde{\Omega}(N_1)} T_z^* f_0 = zw \neq 0$.

(iii) Let $\alpha, \beta \in \mathbb{D}$ satisfy $\alpha \neq 0, \beta \neq 0$ and $\alpha \neq \beta$. Let $M_1 = \overline{(z - w) H^2}$ and $M_2 = \{f \in M_1 : f(\alpha, \beta) = 0\}$. Then M_1, M_2 are invariant subspaces,

$$M_2 = \overline{(z - w) \Big(\frac{z - \alpha}{1 - \overline{\alpha} z} H^2 + \frac{w - \beta}{1 - \overline{\beta} w} H^2 \Big)}$$

and

$$M_1 \ominus M_2 = \mathbb{C} \cdot P_{M_1} \frac{1}{1-\overline{\alpha} z} \frac{1}{1-\overline{\beta} w}.$$

Put

$$f_0 = P_{M_1} \frac{1}{1-\overline{\alpha} z} \frac{1}{1-\overline{\beta} w}.$$

We have $f_0 \not\perp zM_1$ and $f_0 \not\perp wM_1$. Hence $f_0 \notin M_1 \ominus zM_1$ and $f_0 \notin M_1 \ominus wM_1$. We have $\Omega(M_1) = \mathbb{C} \cdot (z-w)$ and

$$\begin{aligned} \langle z-w, \eta_0 \rangle &= \langle z-w, P_{M_1 \ominus wM_1} f_0 \rangle = \langle z-w, f_0 \rangle \\ &= \left\langle z-w, \frac{1}{1-\overline{\alpha} z} \frac{1}{1-\overline{\beta} w} \right\rangle = \alpha - \beta \neq 0. \end{aligned}$$

Hence $\eta_0 \notin \Omega(M_1)$ and $\eta_0 \not\perp \Omega(M_1)$.

Since $\widetilde{\Omega}(N_1) = \{0\}$, we have that $\varphi_0 = \psi_0 = 0$.

(iv) Let α, β be nonzero numbers in $\mathbb{D}$. Let $M_1 = zH^2 + wH^2$, $f_0 = P_{M_1} \frac{1}{1-\overline{\alpha} z} \frac{1}{1-\overline{\beta} w}$ and $M_2 = M_1 \ominus \mathbb{C} \cdot f_0$. Then $M_2 = \{f \in M_1 : f(\alpha, \beta) = 0\}$ and M_1, M_2 are invariant subspaces . Since $f_0 \not\perp zM_1$ and $f_0 \not\perp wM_1$, we have $f_0 \notin M_1 \ominus zM_1$ and $f_0 \notin M_1 \ominus wM_1$. We have $\Omega(M_1) = \mathbb{C} \cdot z + \mathbb{C} \cdot w$, and

$$\begin{aligned} \langle z, \eta_0 \rangle &= \langle z, P_{M_1 \ominus wM_1} f_0 \rangle = \langle z, f_0 \rangle \\ &= \left\langle z, \frac{1}{1-\overline{\alpha} z} \frac{1}{1-\overline{\beta} w} \right\rangle = \alpha \neq 0. \end{aligned}$$

Hence $\eta_0 \notin \Omega(M_1)$ and $\eta_0 \not\perp \Omega(M_1)$.

Since $\widetilde{\Omega}(N_1) = \mathbb{C} \cdot 1$, we have

$$\langle 1, \varphi_0 \rangle = \left\langle 1, P_{\widetilde{\Omega}(N_1)} T_z^* f_0 \right\rangle = \langle z, f_0 \rangle \neq 0,$$

so $\varphi_0 \neq 0$. □

6. Related topics and problems

[1] Fredholm fringe operators.

Proposition 6.1. *Let M_1 be an invariant subspace of H^2 and $f_0 \in M_1$ such that $M_2 := M_1 \ominus \mathbb{C} \cdot f_0$ is an invariant subspace. Then $F_z^{M_1}$ on $M_1 \ominus wM_1$ is a Fredholm operator if and only if so is $F_z^{M_2}$ on $M_2 \ominus wM_2$. In this case, we have* $\operatorname{ind} F_z^{M_1} = \operatorname{ind} F_z^{M_2}$.

Proof. There is a unique function f_1 (except constant multiplication) in $M_2 \ominus wM_2$ such that $(R_z^{M_1})^* f_1 \in \mathbb{C} \cdot f_0$ and

$$(M_2 \ominus wM_2) \ominus \mathbb{C} \cdot f_1 \subset M_1 \ominus wM_1.$$

Then we have

$$M_2 \ominus wM_2 = \big((M_1 \ominus wM_1) \ominus \mathbb{C} \cdot P_{M_1 \ominus wM_1} f_0\big) \oplus \mathbb{C} \cdot f_1.$$

There is also a unique function f_2 (except constant multiplication) in $M_1 \ominus wM_1$ such that $P_{M_1 \ominus wM_1} f_0 \perp zh$ for every $h \in (M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_2$, and there is a unique function f_3 (except constant multiplication) in $(M_1 \ominus wM_1) \ominus \mathbb{C} \cdot f_2$ such that $f_1 \perp zh$ for every

$$h \in (M_1 \ominus wM_1) \ominus (\mathbb{C} \cdot f_2 + \mathbb{C} \cdot f_3).$$

Let

$$L = (M_1 \ominus wM_1) \ominus \big(\mathbb{C} \cdot P_{M_1 + wM_1} f_0 + \mathbb{C} \cdot f_2 + \mathbb{C} \cdot f_3\big).$$

Since $\dim (M_1 \ominus wM_1) = \infty$, we have $L \neq \{0\}$. For every $g \in L$, we have $g \in M_2 \ominus wM_2$ and

$$F_z^{M_2} g = P_{M_2 \ominus wM_2} zg = P_{M_1 \ominus wM_1} zg = F_z^{M_1} g.$$

Then $F_z^{M_1}|_L = F_z^{M_2}|_L$. Let A_1 be the operator on $M_1 \ominus wM_1$ defined by

$$A_1 = \begin{cases} F_z^{M_1} & \text{on } L \\ 0 & \text{on } (M_1 \ominus wM_1) \ominus L \end{cases}$$

and A_2 be the operator on $M_2 \ominus wM_2$ defined by

$$A_2 = \begin{cases} F_z^{M_2} & \text{on } L \\ 0 & \text{on } (M_2 \ominus wM_2) \ominus L. \end{cases}$$

Since $F_z^{M_1}$ on $M_1 \ominus wM_1$ and A_1 differ by a finite rank operator, their Fredholmness and index are identical (see [2]). Similarly Fredholmness and index of $F_z^{M_2}$ on $M_2 \ominus wM_2$ and A_2 are identical. As a result, we get the assertion. □

Corollary 6.1. *Let L_1, L_2 be invariant subspaces of H^2 such that $L_1 \subset L_2$ and dim $(L_2 \ominus L_1) < \infty$. Then $F_z^{L_1}$ on $L_1 \ominus wL_1$ is a Fredholm operator if and only if so is $F_z^{L_2}$ on $L_2 \ominus wL_2$. In this case, we have* $\operatorname{ind} F_z^{L_1} = \operatorname{ind} F_z^{L_2}$.

Question 1. Let M be an invariant subspace of H^2 satisfying $\dim \Omega(M) < \infty$. Is F_z^M on $M \ominus wM$ a Fredholm operator?

When F_z^M on $M \ominus wM$ is a Fredholm operator, the Fredholm index of F_z^M is defined by

$$\operatorname{ind} F_z^M = \dim \ker F_z^M - \dim \ker F_z^{M*}.$$

For a nonzero function f in H^2, we denote by $[f]$ the smallest invariant subspace of H^2 containing f, that is, $[f] = \overline{f \cdot \mathbb{C}[z, w]}$, where $\mathbb{C}[z, w]$ is the polynomial ring. Similarly for a subset E of H^2, we denote by $[E]$ the smallest invariant subspace of H^2 containing E.

Question 2. Is $F_z^{[f]}$ on $[f] \ominus w[f]$ a Fredholm operator for any nonzero $f \in H^2$?

In [7], Yang showed that F_z^M on $M \ominus wM$ has closed range if and only if $zM + wM$ is closed.

Question 3. Is $z[f] + w[f]$ closed for any $f \in H^2$?

When f is an inner function, it is easy to see that $F_z^{[f]}$ on $[f] \ominus w[f]$ is Fredholm and ind $F_z^{[f]} = -1$.

[2] One dimensional perturbation.

Let M be an invariant subspace of H^2 satisfying $M \subsetneqq H^2$ and $N = H^2 \ominus M$. As mentioned in the introduction, there is a nonzero function f_0 in M such that $M \ominus \mathbb{C} \cdot f_0$ is an invariant subspace. First, we shall show that there are a lot of such f_0. Write $D_z = \partial/\partial z$ and $D_w = \partial/\partial w$.

Example 6.1. Take $(\alpha, \beta) \in \mathbb{D}^2$. For each $f \in M$, let $\Gamma(f)$ be the family of pairs of nonnegative integers (n, m) such that $(D_z^n D_w^m f)(\alpha, \beta) \neq 0$. Let $\Gamma_M = \bigcup_{f \in M} \Gamma(f)$. Then $\Gamma_M \neq \emptyset$, and if $(n, m) \in \Gamma_M$, then $(n+1, m) \in \Gamma_M$ and $(n, m+1) \in \Gamma_M$. Moreover if $(n, m) \notin \Gamma_M$, then $(n-1, m) \notin \Gamma_M$ and $(n, m-1) \notin \Gamma_M$. Take $(n_1, m_1) \in \Gamma_M$ satisfying that

$$n_1 + m_1 = \min\{n + m : (n, m) \in \Gamma_M\}.$$

Set

$$M_{(\alpha,\beta)} = \big\{f \in M : (D_z^n D_w^m f)(\alpha, \beta) = 0\big\}.$$

Then $M_{(\alpha,\beta)}$ is an invariant subspace and $M_{(\alpha,\beta)} \subsetneqq M$. It is easy to see that $M = M_{(\alpha,\beta)} \oplus \mathbb{C} \cdot f_{(\alpha,\beta)}$ for some $f_{(\alpha,\beta)} \in M$ with $f_{(\alpha,\beta)} \neq 0$. $\square$

As a counterpart, one may ask whether there is a nonzero function g in N such that $M \oplus \mathbb{C} \cdot g$ is an invariant subspace. If $\widetilde{\Omega}(N) \neq \{0\}$ and $g \in \widetilde{\Omega}(N)$, then trivially $M \oplus \mathbb{C} \cdot g$ is an invariant subspace. For $f \in H^2$, we denote by $Z(f)$ the zero set of f in $\mathbb{D}^2$. For a closed subset $E \subset \mathbb{D}^2$, let

$$M_E = \{f \in H^2 : f = 0 \text{ on } E\}$$

Proposition 6.2. *Let E be a connected closed subset of $\mathbb{D}^2$ containing more than one point. If $M_E \neq \{0\}$, then $M_E \oplus \mathbb{C} \cdot g$ is not an invariant subspace for any nonzero function g in $H^2 \ominus M_E$.*

Proof. Suppose that $M_E \oplus \mathbb{C} \cdot g$ is an invariant subspace for some nonzero $g \in H^2 \ominus M_E$. Since $g \notin M_E$, we have $E \setminus Z(g) \neq \emptyset$. By the assumption on E, there are $\alpha, \beta \in E$ such that $\alpha \neq \beta$, $g(\alpha) \neq 0$ and $g(\beta) \neq 0$. Take a polynomial p such that $p(\alpha) \neq p(\beta)$. We have $pg \in M_E \oplus \mathbb{C} \cdot g$, so $pg - cg \in M_E$ for some $c \in \mathbb{C}$. Hence $p(\alpha)g(\alpha) - cg(\alpha) = 0$ and $p(\beta)g(\beta) - cg(\beta) = 0$, so $p(\alpha) = c = p(\beta)$. This is a contradiction. $\square$

Let $E = \{(\alpha, \alpha) : \alpha \in \mathbb{D}\}$. It is known that $M_E = [z - w]$. So $[z - w] \oplus \mathbb{C} \cdot g$ is not an invariant subspace for any nonzero function g in $H^2 \ominus [z - w]$.

Proposition 6.3. *Let $\varphi(z), \psi(w)$ be nonconstant one variable inner functions and $M = \varphi(z)H^2 + \psi(w)H^2$. Then there is a nonzero function g in N such that $M \oplus \mathbb{C} \cdot g$ is an invariant subspace if and only if both $\varphi(z), \psi(w)$ have Blaschke factors.*

Proof. Suppose that $\varphi(\alpha) = \psi(\beta) = 0$ for some $(\alpha, \beta) \in \mathbb{D}^2$. Let $b_\alpha(z) = (z - \alpha)/(1 - \overline{\alpha}z)$ and $b_\beta(w) = (w - \beta)/(1 - \overline{\beta}w)$. Then $\varphi_1(z) := \varphi(z)/b_\alpha(z)$ and $\psi_1(w) := \psi(w)/b_\beta(w)$ are one variable inner functions. We have

$$g := \varphi_1(z)\frac{1}{1 - \overline{\alpha}z}\psi_1(w)\frac{1}{1 - \overline{\beta}w} \in N,$$

$$zg = \varphi(z)\psi_1(w)\frac{1}{1 - \overline{\beta}w} + \alpha g \in M \oplus \mathbb{C} \cdot g$$

and

$$wg = \varphi_1(z)\frac{1}{1 - \overline{\alpha}z}\psi(w) + \beta g \in M \oplus \mathbb{C} \cdot g.$$

Then $M \oplus \mathbb{C} \cdot g$ is an invariant subspace.

Suppose that $\varphi(z)$ is a singular inner function. Moreover assume that $M \oplus \mathbb{C} \cdot g$ is an invariant subspace for some nonzero $g \in N$. If $zg \in M$, then $zg \in \varphi(z)H^2$. Since $\varphi(0) \neq 0$, we have $g \in \varphi(z)H^2$ and this is a contradiction. Hence $P_{\mathbb{C}\cdot g}zg = cg$ for some $c \in \mathbb{D}$ with $c \neq 0$. This shows that $P_{\mathbb{C}\cdot g}\varphi(z)g = \varphi(c)g$. Since $\varphi(z)g \in M$, we have $P_{\mathbb{C}\cdot g}\varphi(z)g = 0$, so $\varphi(c) = 0$. This is a contradiction. Therefore there are no nonzero $g \in N$ such that $M \oplus \mathbb{C} \cdot g$ is an invariant subspace. $\square$

Let M be an invariant subspace of H^2 satisfying $M \subsetneqq H^2$. Suppose that there is a nonzero function g in N such that $M \oplus \mathbb{C} \cdot g$ is an invariant subspace. Then there are $\alpha, \beta \in \mathbb{D}$ such that $(z - \alpha)g \in M$ and $(w - \beta)g \in M$. Hence $(p - p(\alpha, \beta))g \in M$ for every polynomial p.

An invariant subspace L_1 of L_2 is said to be unitarily equivalent if there is a unitary module map U from L_1 onto L_2, that is, $T_zU = UT_z$ and $T_wU = UT_w$ on L_1. In this case, it is known that there is a unimodular function θ on $\partial\mathbb{D} \times \partial\mathbb{D}$ such that $L_2 = \theta L_1$ (see [1, 3]).

Proposition 6.4. *Let M be an invariant subspace of H^2 satisfying $M \subsetneqq H^2$. Suppose that there is a nonzero function g in $H^2 \ominus M$ such that $M \oplus \mathbb{C} \cdot g$ is an invariant subspace. If L is an invariant subspace of H^2 which is unitarily equivalent to M, then there is a nonzero function g_1 in $H^2 \ominus L$ such that $L \oplus \mathbb{C} \cdot g_1$ is an invariant subspace.*

Proof. Let θ be a unimodular function on $\partial\mathbb{D} \times \partial\mathbb{D}$ such that $L = \theta M \subset H^2$ By the fact above Proposition 6.4, there is $\alpha, \beta \in \mathbb{D}$ such that $(z - \alpha)g \in M$ and

$(z-\beta)g \in M$. Then $(z-\alpha)\theta g \in L \subset H^2$ and $(z-\beta)\theta g \in H^2$. Hence $\theta g \in H^2$. Since $g \perp M$, we have $\theta g \perp \theta M = L$, so $\theta g \in H^2 \ominus L$. Since $L \oplus \mathbb{C} \cdot \theta g = \theta(M \oplus \mathbb{C} \cdot g)$, $L \oplus \mathbb{C} \cdot \theta g$ is an invariant subspace. □

Proposition 6.4 shows that the property of M "there is a nonzero function g in $H^2 \ominus M$ such that $M \oplus \mathbb{C} \cdot g$ is an invariant subspace " is *invariant* for unitary module maps.

Question 4. Let $f \in H^2$ satisfy $\{0\} \neq [f] \subsetneqq H^2$. Is $[f] \oplus \mathbb{C} \cdot g$ not an invariant subspace for any $g \in H^2 \ominus [f]$ with $g \neq 0$?

Question 5. Let $f \in H^2$ satisfy $\{0\} \neq [f] \subsetneqq H^2$. Is $\widetilde{\Omega}(H^2 \ominus [f]) = \{0\}$?

Question 6. Characterize an invariant subspace M such that $M \oplus \mathbb{C} \cdot g$ is not an invariant subspace for any nonzero function g in N.

Question 7. Let f, h be functions in H^2 such that $[f] \subsetneqq [h]$. Is $\dim([h] \ominus [f]) = \infty$?

[3] Ranks of invariant subspaces.

Let M_1 be an invariant subspace of H^2 and $f_0 \in M_1$ with $\|f_0\| = 1$ such that $M_2 := M_1 \ominus \mathbb{C} \cdot f_0$ is an invariant subspace. We denote by $\operatorname{rank} M_1$ the rank of M_1, that is, $\operatorname{rank} M_1$ (may be ∞) is the smallest number of elements in M_1 which generate M_1 as an invariant subspace.

Proposition 6.5. $\operatorname{rank} M_1 - 1 \leq \operatorname{rank} M_2 \leq \operatorname{rank} M_1 + 1$.

Proof. It is easy to see that $\operatorname{rank} M_1 \leq \operatorname{rank} M_2 + 1$. So, when $\operatorname{rank} M_1 = \infty$ we get the assertion.

Suppose that $m := \operatorname{rank} M_1 < \infty$. Let $f_1, f_2, \cdots, f_m \in M_1$ such that $[f_1, f_2, \cdots, f_m] = M_1$. We may assume that $f_1 \not\perp f_0$. If $f_j \not\perp f_0$ for some $2 \leq j \leq m$, replacing f_j by

$$f_j - \frac{\langle f_j, f_0 \rangle}{\|f_0\|^2} f_0,$$

we may assume that $f_j \perp f_0$ for every $2 \leq j \leq m$, that is, $f_j \in M_2$ for every $2 \leq j \leq m$. Since $M_1 \ominus \mathbb{C} \cdot f_0$ is an invariant subspace, there are $\alpha, \beta \in \mathbb{D}^2$ such that $(z-\alpha)f_0 \in M_2$ and $(w-\beta)f_0 \in M_2$. Hence $(z-\alpha)f_1 \in M_2$ and $(w-\beta)f_1 \in M_2$. We shall show that

$$\big[(z-\alpha)f_1, (w-\beta)f_1, f_2, \cdots, f_m\big] = M_2. \tag{6.1}$$

Let $h \in M_2$. Then there are sequences of polynomials

$$\{p_{1,k}\}_{k \geq 1}, \{p_{2,k}\}_{k \geq 1}, \cdots, \{p_{m,k}\}_{k \geq 1}$$

such that

$$\lim_{k\to\infty}\sum_{\ell=1}^{m} p_{\ell,k} f_\ell = h.$$

We have

$$0 = \langle h, f_0\rangle = \lim_{k\to\infty}\sum_{\ell=1}^{m}\langle p_{\ell,k} f_\ell, f_0\rangle = \lim_{k\to\infty}\langle p_{1,k} f_1, f_0\rangle.$$

Let

$$p_{1,k}(z,w) = \sum_{i,j} c_{k,i,j}(z-\alpha)^i(w-\beta)^j$$

be the Taylor expansion of $p_{1,k}$ at (α,β). Then

$$0 = \lim_{k\to\infty}\langle p_{1,k} f_1, f_0\rangle = \lim_{k\to\infty} c_{k,0,0}\langle f_1, f_0\rangle.$$

Since $\langle f_1, f_0\rangle \neq 0$, $c_{k,0,0}\to 0$ as $k\to\infty$. Hence

$$h = \lim_{k\to\infty}\sum_{\ell=1}^{m} p_{\ell,k} f_\ell = \lim_{k\to\infty}\Big((p_{1,k} - c_{k,0,0}) f_1 + \sum_{\ell=2}^{m} p_{\ell,k} f_\ell\Big).$$

Since

$$(p_{1,k} - c_{k,0,0}) f_1 \in [(z-\alpha) f_1, (w-\beta) f_1],$$

we have

$$h \in \big[(z-\alpha) f_1, (w-\beta) f_1, f_2, \cdots, f_m\big].$$

Thus we get (6.1), so

$$\operatorname{rank} M_2 \le m+1 = \operatorname{rank} M_1 + 1.$$

□

Example 6.2. (i) Let $M_1 = H^2$ and $f_0 = 1$. Then $M_2 := M_1 \ominus \mathbb{C}\cdot 1 = zH^2 + wH^2$ is an invariant subspace. It is easy to check that $\operatorname{rank} M_1 = 1$ and $\operatorname{rank} M_2 = 2$.

(ii) Let $M_3 = z^2H^2 + wH^2$. Then $M_2 \ominus \mathbb{C}\cdot z = M_3$ is an invariant subspace. We have $\operatorname{rank} M_2 = 2 = \operatorname{rank} M_3$.

(iii) Let $M_1 = z^2H^2 + zwH^2 + w^2H^2$ and $f_0 = zw$. We have $\operatorname{rank} M_1 = 3$. Since $M_2 := M_1 \ominus \mathbb{C}\cdot f_0 = z^2H^2 + w^2H^2$, we have $\operatorname{rank} M_2 = 2$. □

Suppose that $\operatorname{rank} M_1 = 1$, that is, $M_1 = [f]$ for some nonzero $f \in H^2$. Then $\operatorname{rank} M_2 \ge 1$.

Question 8. Do there exist M_1 and $f_0 \in M_1$ such that $\operatorname{rank} M_1 = \operatorname{rank} M_2 = 1$?

Question 9. Do there exist M_1 and $f_0 \in M_1$ such that $\operatorname{rank} M_1 = 2$ and $\operatorname{rank} M_2 = 1$?

Question 10. Let $f \in H^2$ be a nonzero function and $f_0 \in [f]$ be a nonzero function such that $M_2 := [f] \ominus \mathbb{C} \cdot f_0$ is an invariant subspace. Does $\operatorname{rank} M_2 = 2$ hold?

These questions have some connection with Questions 4 and 7.

Let $N_j = H^2 \ominus M_j$ for $j = 1, 2$. Since $T_z^* N_j \subset N_j$ and $T_w^* N_j \subset N_j$, we may consider $\operatorname{rank} N_j$ for the operators T_z^*, T_w^*. In the similar way as Proposition 6.5, we can prove the following.

Proposition 6.6. *Suppose that $M_1 \neq H^2$. Then we have*

$$\operatorname{rank} N_1 - 1 \leq \operatorname{rank} N_2 \leq \operatorname{rank} N_1 + 1.$$

Example 6.3. (i) Let $M_1 = zH^2 + wH^2$ and $f_0 = z$. We have $N_1 = \mathbb{C} \cdot 1$ and $N_2 = \mathbb{C} \cdot 1 + \mathbb{C} \cdot z$. Hence $\operatorname{rank} N_1 = 1 = \operatorname{rank} N_2$.

(ii) Let $M_1 = z^2H^2 + zwH^2 + w^2H^2$ and $f_0 = zw$. We have $N_1 = \mathbb{C} \cdot z + \mathbb{C} \cdot w + \mathbb{C} \cdot 1$ and $N_2 = \mathbb{C} \cdot z + \mathbb{C} \cdot w + \mathbb{C} \cdot 1 + \mathbb{C} \cdot zw$. Hence $\operatorname{rank} N_1 = 2$ and $\operatorname{rank} N_2 = 1$.

(iii) Let $M_1 = z^2H^2 + zwH^2 + w^2H^2 + \mathbb{C} \cdot (z + w)$ and $f_0 = z + w$. We have $N_1 = \mathbb{C} \cdot (z - w) + \mathbb{C} \cdot 1$ and $N_2 = \mathbb{C} \cdot z + \mathbb{C} \cdot w + \mathbb{C} \cdot 1$. Hence $\operatorname{rank} N_1 = 1$ and $\operatorname{rank} N_2 = 2$. □

In the forthcoming paper, we shall study relationship of ranks of the cross commutators on M_1, M_2 and on N_1, N_2, respectively.

References

[1] X. Chen and K. Guo, *Analytic Hilbert Modules*, Chapman & Hall/CRC, Boca Raton, FL, 2003.

[2] J. Conway, *A Course in Operator Theory*, Grad. Stud. Math., Vol. 21, Amer. Math. Soc., RI, 2000.

[3] R. Douglas and K. Yan, *On the rigidity of Hardy submodules*, Integral Equations Operator Theory **13** (1990), 350–363.

[4] K. Guo and R. Yang, *The core function of submodules over the bidisk*, Indiana Univ. Math. J. **53** (2004), 205–222.

[5] K. H. Izuchi, *Rank-one commutators on invariant subspaces of the Hardy space on the bidisk III*, Hokkaido Math. J. **38** (2009), 663–681.

[6] W. Rudin, *Function Theory in Polydiscs*, Benjamin, New York, 1969.

[7] R. Yang, *Operator theory in the Hardy space over the bidisk (III)*, J. Funct. Anal. **186** (2001), 521–545.

[8] R. Yang, *Beurling's phenomenon in two variables*, Integral Equations Operator Theory **48** (2004), 411–423.

[9] R. Yang, *On two variable Jordan block II*, Integral Equations Operator Theory **56** (2006), 431–449.

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