Nihonkai Math. J.
Vol.27(2016), 167–180

VISCOSITY APPROXIMATION METHOD FOR QUASINONEXPANSIVE MAPPINGS WITH CONTRACTION-LIKE MAPPINGS

KOJI AOYAMA

ABSTRACT. We study the viscosity approximation method for a sequence of quasinonexpansive mappings with contraction-like mappings. We establish a strong convergence theorem and then we apply our result to approximate a solution of a split feasibility problem and a fixed point of a Lipschitz continuous pseudo-contraction.

1. Introduction

In this paper, an algorithm for finding a common fixed point of quasinonexpansive mappings in a Hilbert space is analyzed. In particular, we study convergence of a sequence $\{x_n\}$ in C defined by any point $x_1 \in C$ and

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n) S_n x_n \tag{1.1}$$

for $n \in \mathbb{N}$, where C is a nonempty closed convex subset of a Hilbert space, f_n is a contraction-like self-mapping of C, S_n is a quasinonexpansive self-mapping of C, and α_n is a real number in $[0, 1]$ for every $n \in \mathbb{N}$. The iterative algorithm (1.1) is called the viscosity approximation method [23] and has extensively studied; see [32, 25, 26, 27, 17, 20, 29, 22] and references therein.

It is known that the viscosity approximation method is closely related to the hybrid steepest descent method [33, 13] for solving a variational inequality problem over the fixed point set of nonexpansive or quasinonexpansive mappings; see [3, 8].

Recently, Hojo and Takahashi [16] considered the generalized split feasibility problem as follows: Find $z \in C$ such that $z = Uz$, $0 \in Bz$, and $Lz = TLz$, where C is a nonempty closed convex subset of a Hilbert space H_1, U is a widely more generalized hybrid self-mapping of C in the sense of Kawasaki and Takahashi [19], $B \subset H_1 \times H_1$ is a maximal monotone operator, T is a nonexpansive self-mapping of a Hilbert space H_2, and L is a bounded linear operator of H_1 into H_2. Then, under

2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 47H10, 41A65.
Key words and phrases. Viscosity approximation method, quasinonexpansive mapping, fixed point, pseudo-contraction.

some appropriate conditions, Hojo and Takahashi [16] established that the following iterative sequence $\{x_n\}$ converges strongly to a solution of the problem: $x_1 \in C$ and

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\Big(\alpha_n u_n + (1-\alpha_n) U J_{\lambda_n}\big(x_n - \lambda_n L^*(I-T)Lx_n\big)\Big) \tag{1.2}$$

for $n \in \mathbb{N}$, where α_n and β_n are real numbers in $[0,1]$, $\{u_n\}$ is a convergent sequence in C, λ_n is a positive real number, I is the identity mapping, and $J_{\lambda_n} = (I+\lambda_n B)^{-1}$. This iteration is also related the viscosity approximation method. Indeed, we can reduce (1.2) to (1.1) and the convergence of $\{x_n\}$ can be guaranteed by Theorem 3.1, which is a generalization of our previous result [8, Theorem 3.1]; see Sections 3 and 4 for more details.

This paper is organized as follows: In Section 2, we recall some definitions and list some lemmas in order to prove our main results. In Section 3, we present our main result (Theorem 3.1) and its corollaries. In Section 4, we consider the generalized split feasibility problem and apply our main result to approximate a solution of the problem. In section 5, we prove strong convergence theorem for a Lipschitz continuous pseudo-contraction by using our main result.

2. Preliminaries

Throughout the present paper, H denotes a real Hilbert space, $\langle \cdot, \cdot \rangle$ the inner product of H, $\|\cdot\|$ the norm of H, C a nonempty closed convex subset of H, I the identity mapping on H, and $\mathbb{N}$ the set of positive integers. Strong convergence of a sequence $\{x_n\}$ in H to $x \in H$ is denoted by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

Let $T\colon C \to H$ be a mapping. The set of fixed points of T is denoted by $\mathrm{F}(T)$. A mapping T is said to be *quasinonexpansive* if $\mathrm{F}(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in \mathrm{F}(T)$; T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is *demiclosed at* 0 if $Tp = 0$ whenever $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup p$ and $Tx_n \to 0$. We know that if $T\colon C \to H$ is quasinonexpansive, then $\mathrm{F}(T)$ is closed and convex; see [14, Theorem 1].

It is known that, for each $x \in H$, there exists a unique point $x_0 \in C$ such that

$$\|x - x_0\| = \min\{\|x - y\| : y \in C\}.$$

Such a point x_0 is denoted by $P_C(x)$ and P_C is called the *metric projection* of H onto C. It is also known that the metric projection P_C is nonexpansive and

$$\langle y - P_C(x), x - P_C(x) \rangle \leq 0 \tag{2.1}$$

for all $x \in H$ and $y \in C$; see [28].

Let $f\colon C \to C$ be a mapping, F a nonempty subset of C, and θ a real number in $[0,1)$. A mapping f is said to be a *θ-contraction with respect to F* [3] if

$\|f(x) - f(z)\| \le \theta \|x - z\|$ for all $x \in C$ and $z \in F$; f is said to be a *θ-contraction* if f is a θ-contraction with respect to C.

Let $A\colon C \to H$ be a mapping and ρ a positive real number. A mapping A is said to be *ρ-inverse strongly monotone* if $\langle x - y, Ax - Ay\rangle \ge \rho \|Ax - Ay\|^2$ for all $x, y \in C$.

Let B be a set-valued mapping of H into H. The effective domain of B is denoted by $\mathrm{D}(B)$, that is, $\mathrm{D}(B) = \{x \in H : Bx \ne \emptyset\}$; the set of zeros of B is denoted by $B^{-1}0$, that is, $B^{-1}0 = \{z \in \mathrm{D}(B) : Bz \ni 0\}$. We identify B with its graph $\{(x, y) \in H \times H : x \in \mathrm{D}(B),\ y \in Bx\}$. A set-valued mapping $B \subset H \times H$ is said to be a *monotone operator* if $\langle x - u, y - v\rangle \ge 0$ for all (x, y), $(u, v) \in B$. A monotone operator $B \subset H \times H$ is said to be *maximal* if $B = B'$ whenever $B' \subset H \times H$ is a monotone operator and $B \subset B'$. Let $B \subset H \times H$ be a maximal monotone operator and $\lambda > 0$. It is known that $(I + \lambda B)^{-1}$ is a single-valued mapping of H into $\mathrm{D}(B)$; see [28].

A function $\tau\colon \mathbb{N} \to \mathbb{N}$ is said to be *eventually increasing* [4] if $\lim_{n\to\infty} \tau(n) = \infty$ and $\tau(n) \le \tau(n + 1)$ for all $n \in \mathbb{N}$. The following is clear from the definition.

Lemma 2.1. *Let $\tau\colon \mathbb{N} \to \mathbb{N}$ be an eventually increasing function and $\{\xi_n\}$ a sequence of real numbers such that $\xi_n \to \xi$. Then $\xi_{\tau(n)} \to \xi$.*

The following is directly obtained from [21, Lemma 3.1].

Lemma 2.2 ([4, Lemma 3.4])**.** *Let $\{\xi_n\}$ be a sequence of nonnegative real numbers which is not convergent. Then there exist $N \in \mathbb{N}$ and an eventually increasing function $\tau\colon \mathbb{N} \to \mathbb{N}$ such that $\xi_{\tau(n)} \le \xi_{\tau(n)+1}$ for all $n \in \mathbb{N}$ and $\xi_n \le \xi_{\tau(n)+1}$ for all $n \ge N$.*

Under the assumptions of Lemma 2.2, we cannot choose a strictly increasing function τ; see [4, Example 3.3].

Let $\{T_n\}$ be a sequence of mappings of C into H such that $F = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$ is nonempty. Then $\{T_n\}$ is said to be *strongly quasinonexpansive type* [8] if each T_n is quasinonexpansive and $T_n x_n - x_n \to 0$ whenever $\{x_n\}$ is a bounded sequence in C and $\|x_n - p\| - \|T_n x_n - p\| \to 0$ for some point $p \in F$; $z \in C$ is said to be an *asymptotic fixed point* of $\{T_n\}$ if there exist a sequence $\{x_n\}$ in C and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $T_n x_n - x_n \to 0$ and $x_{n_i} \rightharpoonup z$; see [1]. The set of asymptotic fixed points of $\{T_n\}$ is denoted by $\hat{\mathrm{F}}(\{T_n\})$. It is clear that $F \subset \hat{\mathrm{F}}(\{T_n\})$.

Remark 2.3. It is known that $\{T_n\}$ is strongly quasinonexpansive type if and only if it is a strongly relatively nonexpansive sequence in the sense of [4, 10]; see [8, Remark 2.5]. It is also known that $F = \hat{\mathrm{F}}(\{T_n\})$ if and only if $\{T_n\}$ satisfies the condition (Z); see [1, Proposition 6]. Recall that $\{T_n\}$ is said to satisfy the *condition (Z)* if every weak cluster point of $\{x_n\}$ belongs to F whenever $\{x_n\}$ is a bounded sequence in C such that $T_n x_n - x_n \to 0$; see [2, 4, 9]. We know some examples of strongly quasinonexpansive type sequences with the condition (Z); see [10], [8, Example 4.5], and [6, 2].

We know the following lemmas:

Lemma 2.4 ([8, Lemma 2.6])**.** *Let $\{T_n\}$ be a sequence of mappings of C into H such that $F = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$ is nonempty, $\tau \colon \mathbb{N} \to \mathbb{N}$ an eventually increasing function, and $\{z_n\}$ a bounded sequence in C such that $\|z_n - p\| - \|T_{\tau(n)} z_n - p\| \to 0$ for some $p \in F$. If $\{T_n\}$ is strongly quasinonexpansive type, then $T_{\tau(n)} z_n - z_n \to 0$.*

Lemma 2.5 ([4, Lemma 3.6])**.** *Let $\{T_n\}$ be a sequence of mappings of C into H such that $F = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$ is nonempty, $\tau \colon \mathbb{N} \to \mathbb{N}$ an eventually increasing function, and $\{z_n\}$ a bounded sequence in C such that $T_{\tau(n)} z_n - z_n \to 0$. Suppose that $\hat{\mathrm{F}}(\{T_n\}) = F$. Then every weak cluster point of $\{z_n\}$ belongs to F.*

The following lemma is similar to [4, Lemma 3.7]. For the sake of completeness, we give the proof.

Lemma 2.6. *Let $\{T_n\}$ be a sequence of mappings of C into H, F a nonempty closed convex subset of H, $\{z_n\}$ a bounded sequence in C such that $T_n z_n - z_n \to 0$, and $\{u_n\}$ a sequence in H such that $u_n \to u \in H$. Suppose that every weak cluster point of $\{z_n\}$ belongs to F. Then*

$$\limsup_{n\to\infty} \langle T_n z_n - w, u_n - w \rangle \le 0,$$

where $w = P_F(u)$.

Proof. Since $T_n z_n - z_n \to 0$, $u_n \to u$, and $\{z_n\}$ is bounded, there exists a weakly convergent subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty} \langle T_n z_n - w, u_n - w \rangle = \lim_{i\to\infty} \langle z_{n_i} - w, u - w \rangle .$$

Let z be the weak limit of $\{z_{n_i}\}$. By assumption, we see that $z \in F$. Thus (2.1) shows that

$$\lim_{i\to\infty} \langle z_{n_i} - w, u - w \rangle = \langle z - w, u - w \rangle \le 0.$$

This completes the proof. ☐

The following lemma is a well-known result; see [31, 5].

Lemma 2.7. *Let $\{\xi_n\}$ be a sequence of nonnegative real numbers, $\{\delta_n\}$ a sequence of real numbers, and $\{\beta_n\}$ a sequence in $[0, 1]$. Suppose that $\xi_{n+1} \le (1-\beta_n)\xi_n + \beta_n \delta_n$ for every $n \in \mathbb{N}$, $\limsup_{n\to\infty} \delta_n \le 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then $\xi_n \to 0$.*

3. Main results

Firstly, we prove the following theorem by using the technique developed in [8]; see also [4].

Theorem 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{S_n\}$ a sequence of self-mappings of C, F the set of common fixed points of $\{S_n\}$, and $\{\alpha_n\}$ a sequence in $(0,1]$. Suppose that $\{S_n\}$ is strongly quasinonexpansive type, F is nonempty, $\hat{\mathrm{F}}(\{S_n\}) = F$, $\alpha_n \to 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{f_n\}$ be a sequence of self-mappings of C and $\theta \in [0,1)$. Suppose that each f_n is a θ-contraction with respect to F and there exists $u \in C$ such that $f_n \circ P_F(u) \to u$. Then the sequence $\{x_n\}$ defined by $x_1 \in C$ and (1.1) for $n \in \mathbb{N}$ converges strongly to $P_F(u)$.*

Proof. We first show that $\{x_n\}$, $\{S_n x_n\}$, and $\{f_n(x_n)\}$ are bounded and obtain some inequalities. Set

$$w = P_F(u),\ \beta_n = \alpha_n\big(1 + (1-2\theta)(1-\alpha_n)\big)$$

and

$$\gamma_n = \alpha_n^2 \left\| f_n(x_n) - w \right\|^2 + 2\alpha_n(1-\alpha_n) \left\langle S_n x_n - w, f_n(w) - w \right\rangle$$

for $n \in \mathbb{N}$. Since f_n is a θ-contraction with respect to F, S_n is quasinonexpansive, and $w \in F \subset \mathrm{F}(S_n)$, we have

$$\begin{aligned}
\|x_{n+1} - w\| &\le \alpha_n \|f_n(x_n) - w\| + (1-\alpha_n)\|S_n x_n - w\| \\
&\le \alpha_n(\|f_n(x_n) - f_n(w)\| + \|f_n(w) - w\|) + (1-\alpha_n)\|x_n - w\| \\
&\le \big(1 - \alpha_n(1-\theta)\big)\|x_n - w\| + \alpha_n(1-\theta)\frac{\|f_n(w) - w\|}{1-\theta}.
\end{aligned} \tag{3.1}$$

From the assumption that $\{f_n(w)\}$ is convergent, $M = \sup_n \|f_n(w) - w\| / (1-\theta) < \infty$, and thus, by induction on n, it holds that

$$\|S_n x_n - w\| \le \|x_n - w\| \le \max\{\|x_1 - w\|, M\},$$

for every $n \in \mathbb{N}$. Therefore we conclude that $\{x_n\}$ and $\{S_n x_n\}$ are bounded. Thus $\{f_n(x_n)\}$ is also bounded. Since

$$\begin{aligned}
\langle S_n x_n - w, f_n(x_n) - w\rangle &\le \|S_n x_n - w\| \|f_n(x_n) - f_n(w)\| + \langle S_n x_n - w, f_n(w) - w\rangle \\
&\le \theta \|x_n - w\|^2 + \langle S_n x_n - w, f_n(w) - w\rangle,
\end{aligned}$$

we have

$$\begin{aligned}
\|x_{n+1} - w\|^2 &= \alpha_n^2 \|f_n(x_n) - w\|^2 + (1-\alpha_n)^2 \|S_n x_n - w\|^2 \\
&\qquad + 2\alpha_n(1-\alpha_n)\langle S_n x_n - w, f_n(x_n) - w\rangle \\
&\le \alpha_n^2 \|f_n(x_n) - w\|^2 + \big((1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)\theta\big)\|x_n - w\|^2 \\
&\qquad + 2\alpha_n(1-\alpha_n)\langle S_n x_n - w, f_n(w) - w\rangle.
\end{aligned}$$

Hence

$$\|x_{n+1} - w\|^2 \leq (1 - \beta_n) \|x_n - w\|^2 + \gamma_n \tag{3.2}$$

holds for every $n \in \mathbb{N}$. Moreover, we can check that $0 < \beta_n \leq 1$ for every $n \in \mathbb{N}$,

$$2\alpha_n(1 - \alpha_n)/\beta_n \to 1/(1 - \theta), \text{ and } \alpha_n^2 \|f_n(x_n) - w\|^2 /\beta_n \to 0 \tag{3.3}$$

in the same way as in [7, Lemma 3.3].

We next show that $\{\|x_n - w\|\}$ is convergent. Assume that $\{\|x_n - w\|\}$ is not convergent. Then Lemma 2.2 implies that there exist $N \in \mathbb{N}$ and an eventually increasing function $\tau\colon \mathbb{N} \to \mathbb{N}$ such that

$$\left\|x_{\tau(n)} - w\right\| \leq \left\|x_{\tau(n)+1} - w\right\| \tag{3.4}$$

for every $n \in \mathbb{N}$ and

$$\|x_n - w\| \leq \left\|x_{\tau(n)+1} - w\right\| \tag{3.5}$$

for every $n \geq N$. Since $S_{\tau(n)}$ is quasinonexpansive and $w \in F \subset \mathrm{F}(S_{\tau(n)})$, $\alpha_n \to 0$, and $\{f_{\tau(n)}(x_{\tau(n)})\}$ is bounded, it follows from (3.4), (3.1), and Lemma 2.1 that

$$\begin{aligned}
0 &\leq \left\|x_{\tau(n)} - w\right\| - \left\|S_{\tau(n)}x_{\tau(n)} - w\right\| \\
&\leq \left\|x_{\tau(n)+1} - w\right\| - \left\|S_{\tau(n)}x_{\tau(n)} - w\right\| \\
&\leq \alpha_{\tau(n)} \left\|f_{\tau(n)}(x_{\tau(n)}) - w\right\| \to 0
\end{aligned}$$

as $n \to \infty$. Thus Lemma 2.4 implies that $S_{\tau(n)}x_{\tau(n)} - x_{\tau(n)} \to 0$. Furthermore, Lemma 2.5 implies that every weak cluster point of $\{x_{\tau(n)}\}$ belongs to F. Noticing that $f_{\tau(n)}(w) = f_{\tau(n)} \circ P_F(u) \to u$, we conclude from Lemma 2.6 that

$$\limsup_{n\to\infty} \left\langle S_{\tau(n)}x_{\tau(n)} - w, f_{\tau(n)}(w) - w\right\rangle \leq 0. \tag{3.6}$$

Since

$$\begin{aligned}
\left\|x_{\tau(n)+1} - w\right\|^2 &\leq (1 - \beta_{\tau(n)}) \left\|x_{\tau(n)} - w\right\|^2 + \gamma_{\tau(n)} \\
&\leq (1 - \beta_{\tau(n)}) \left\|x_{\tau(n)+1} - w\right\|^2 + \gamma_{\tau(n)}
\end{aligned}$$

by (3.2) and (3.4), we deduce that

$$\left\|x_{\tau(n)+1} - w\right\|^2 \leq \frac{\gamma_{\tau(n)}}{\beta_{\tau(n)}} \tag{3.7}$$

for every $n \in \mathbb{N}$. Using (3.5), (3.7), (3.3), and (3.6), we obtain

$$\limsup_{n\to\infty} \|x_n - w\|^2 \leq \limsup_{n\to\infty} \left\|x_{\tau(n)+1} - w\right\|^2 \leq \limsup_{n\to\infty} \frac{\gamma_{\tau(n)}}{\beta_{\tau(n)}} \leq 0,$$

which is a contradiction. Therefore, $\{\|x_n - w\|\}$ is convergent.

Lastly, we show that $x_n \to P_F(u)$. Since S_n is quasinonexpansive, $w \in \mathrm{F}(T_n)$, $\alpha_n > 0$, $\{\|x_n - w\|\}$ is convergent, $\alpha_n \to 0$, and $\{f_n(x_n)\}$ is bounded, it follows from (3.1) that

$$\begin{aligned} 0 &\le \|x_n - w\| - \|S_n x_n - w\| \\ &\le \|x_n - w\| - (1-\alpha_n)\|S_n x_n - w\| \\ &\le \|x_n - w\| - \|x_{n+1} - w\| + \alpha_n \|f_n(x_n) - w\| \to 0 \end{aligned}$$

as $n \to \infty$. Thus $S_n x_n - x_n \to 0$ because $\{S_n\}$ is strongly quasinonexpansive type and $\{x_n\}$ is bounded. Since $\hat{\mathrm{F}}(\{S_n\}) = F$, Lemma 2.6 implies that

$$\limsup_{n\to\infty} \langle S_n x_n - w, f_n(w) - w\rangle \le 0. \tag{3.8}$$

Using (3.3) and (3.8), we have $\limsup_{n\to\infty} \gamma_n/\beta_n \le 0$. Note that

$$\|x_{n+1} - w\|^2 \le (1-\beta_n)\|x_n - w\|^2 + \beta_n \frac{\gamma_n}{\beta_n}$$

by (3.2) and $\sum_{n=1}^{\infty} \beta_n = \infty$ by [7, Lemma 3.3]. Taking into account Lemma 2.7, we conclude that $x_n - w \to 0$. This completes the proof. □

The following results are direct consequences of Theorem 3.1:

Corollary 3.2. *Let H, C, $\{S_n\}$, F, and $\{\alpha_n\}$ be the same as in Theorem 3.1. Let $\{u_n\}$ be a strongly convergent sequence in C and u the limit of $\{u_n\}$. Then the sequence $\{x_n\}$ defined by $x_1 \in C$ and $x_{n+1} = \alpha_n u_n + (1-\alpha_n) S_n x_n$ for $n \in \mathbb{N}$ converges strongly to $P_F(u)$.*

Proof. For each $n \in \mathbb{N}$, let $f_n\colon C \to C$ be a mapping defined by $f_n(x) = u_n$ for all $x \in C$. Then it is clear that each f_n is a 0-contraction and $f_n \circ P_F(u) = u_n \to u$. Therefore, Theorem 3.1 implies the conclusion. □

Corollary 3.3 ([8, Theorem 3.1])**.** *Let H, C, $\{S_n\}$, F, and $\{\alpha_n\}$ be the same as in Theorem 3.1. Let $\{f_n\}$ be a sequence of self-mappings of C and $\theta \in [0,1)$. Suppose that each f_n is a θ-contraction with respect to F and $\{f_n(z) : n \in \mathbb{N}\}$ is a singleton for every $z \in F$. Then the sequence $\{x_n\}$ defined by $x_1 \in C$ and (1.1) for $n \in \mathbb{N}$ converges strongly to w, where w is a unique fixed point of $P_F \circ f_1$.*

Proof. Since $P_F \circ f_1$ is a contraction on F, $P_F \circ f_1$ has a unique fixed point $w \in F$. Noting that $\{f_n(w) : n \in \mathbb{N}\}$ is a singleton, we see that $f_n(w) = f_1(w)$ for every $n \in \mathbb{N}$. Thus, $f_n \circ P_F(f_1(w)) = f_n(w) = f_1(w)$ for every $n \in \mathbb{N}$. Therefore, Theorem 3.1 implies the conclusion. □

4. Application to a split feasibility problem

In this section, we consider the generalized split feasibility problem studied in Hojo and Takahashi [16] as follows:

Problem 4.1. *Let H_1 and H_2 be two Hilbert spaces, C a nonempty closed convex subset of H_1, $U\colon C \to H_1$ a quasinonexpansive mapping, $B \subset H_1 \times H_1$ a maximal monotone operator, $T\colon H_2 \to H_2$ a nonexpansive mapping, and $L\colon H_1 \to H_2$ a bounded linear operator such that $L \neq 0$. Then find $z \in \mathrm{F}(U) \cap B^{-1}0 \cap L^{-1}\,\mathrm{F}(T)$.*

Hojo and Takahashi [16] dealt with a special case of this problem and they established some strong convergence results for the problem.

In this section, we show a strong convergence theorem for Problem 4.1. Before proving it, we show the following theorem by using Corollary 3.2.

Theorem 4.2. *Let H be a Hilbert space, C a nonempty closed convex subset of H, ρ a positive real number, $A\colon H \to H$ a ρ-inverse strongly monotone mapping, $B \subset H \times H$ a maximal monotone operator, $U\colon C \to H$ a quasinonexpansive mapping, $\{u_n\}$ a sequence in H, $\{\alpha_n\}$ a sequence in $(0,1]$, $\{\beta_n\}$ a sequence in $[a,b]$, and $\{\lambda_n\}$ a sequence in $[c,d]$, where $0 < a \leq b < 1$ and $0 < c \leq d < 2\rho$. Suppose that $F = \mathrm{F}(U) \cap (A+B)^{-1}0$ is nonempty, $\mathrm{D}(B) \subset C$, $I - U$ is demiclosed at 0, $u_n \to u$, $\alpha_n \to 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{x_n\}$ be a sequence in H defined by $x_1 \in H$ and*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\bigl(\alpha_n u_n + (1-\alpha_n) U J_{\lambda_n}(x_n - \lambda_n A x_n)\bigr)$$

for $n \in \mathbb{N}$, where $J_{\lambda_n} = (I + \lambda_n B)^{-1}$. Then $\{x_n\}$ converges strongly to $P_F(u)$.

Proof. Set $T_n = U J_{\lambda_n}(I - \lambda_n A)$. Since J_{λ_n} and $I - \lambda_n A$ are strongly nonexpansive in the sense of [12], [12, Proposition 1.1] shows that $J_{\lambda_n}(I - \lambda_n A)$ is strongly nonexpansive. Thus it follows from [6, Lemma 5.8], [24, Lemma 2.3], and [9, Lemma 3.2] that $\mathrm{F}(T_n) = F$ and T_n is quasinonexpansive for every $n \in \mathbb{N}$. By the definition of $\{x_n\}$, we see that

$$\begin{aligned} x_{n+1} &= \gamma_n u_n + (1-\gamma_n)\left[\frac{\beta_n}{1-\gamma_n} x_n + \left(1 - \frac{\beta_n}{1-\gamma_n}\right) T_n x_n\right] \\ &= \gamma_n u_n + (1-\gamma_n) S_n x_n \end{aligned}$$

for every $n \in \mathbb{N}$, where $\gamma_n = \alpha_n(1-\beta_n)$ and

$$S_n = \frac{\beta_n}{1-\gamma_n} I + \left(1 - \frac{\beta_n}{1-\gamma_n}\right) T_n.$$

Then it is not hard to verify that

$$0 < \inf_n \frac{\beta_n}{1-\gamma_n}, \sup_n \frac{\beta_n}{1-\gamma_n} < 1,$$

$\gamma_n \to 0$, and $\sum_{n=1}^{\infty} \gamma_n = \infty$. Thus it follows from $\bigcap_{n=1}^{\infty} \mathrm{F}(T_n) = F$ and [10, Theorem 3.8] that $\{S_n\}$ is strongly quasinonexpansive type and $\hat{\mathrm{F}}(\{S_n\}) = F$. Therefore, Corollary 3.2 implies the conclusion. $\square$

Using Theorem 4.2 and other known results, we obtain the following:

Theorem 4.3. *Let H_1, H_2, C, B, U, T, and L be the same as in Problem 4.1. Let L^* be the adjoint operator of L and F the set of solutions of Problem 4.1. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be the same as in Theorem 4.2. Let $\{u_n\}$ be a sequence in H_1 and $\{\lambda_n\}$ a sequence in $[c, d]$, where $0 < c \leq d < 1/\|L\|^2$. Suppose that F is nonempty, $\mathrm{D}(B) \subset C$, $I - U$ is demiclosed at 0, and $u_n \to u$. Let $\{x_n\}$ be a sequence in H_1 defined by $x_1 \in H_1$ and*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\Big(\alpha_n u_n + (1 - \alpha_n) U J_{\lambda_n}\big(x_n - \lambda_n L^*(I - T)Lx_n\big)\Big)$$

for $n \in \mathbb{N}$, where $J_{\lambda_n} = (I + \lambda_n B)^{-1}$. Then $\{x_n\}$ converges strongly to $P_F(u)$.

Proof. Set $A = L^*(I - T)L$. From [30, Lemma 3.3] and [30, Lemma 3.4], we know that A is $1/(2\|L\|^2)$-inverse strongly monotone and $B^{-1}0 \cap L^{-1}\mathrm{F}(T) = (A + B)^{-1}0$. Thus $F = \mathrm{F}(U) \cap (A + B)^{-1}0$. Therefore, Theorem 4.2 implies the conclusion. $\square$

Remark 4.4. The main result in [16], [16, Theorem 3.1], is a direct consequence of Theorem 4.3. Indeed, suppose that a mapping U is the same as in [16, Theorem 3.1], that is, α, β, γ, δ, ϵ, ζ, η are real numbers such that

$$\alpha + \beta + \gamma + \delta \geq 0, \ \alpha + \beta > 0, \ \text{and} \ \zeta + \eta \geq 0,$$

and U is an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$-widely more generalized hybrid mapping in the sense of Kawasaki and Takahashi [19]. In this case, it is known that U is quasinonexpansive; see [19, Lemma 5.3] and [15, Lemma 4.1]. It is also known that $I - U$ is demiclosed at 0; see [15, Lemma 4.2].

5. Strong convergence theorem for a pseudo-contraction

In this section, applying Theorem 3.1, we prove a strong convergence theorem for a pseudo-contraction.

Throughout this section, let C be a nonempty closed convex subset of a Hilbert space H and $T\colon C \to C$ an η-Lipschitz continuous pseudo-contraction [11], that is, we assume that η is a positive real number and

$$\|Tx - Ty\| \leq \eta \|x - y\| \ \text{and} \ \|Tx - Ty\|^2 \leq \|x - y\|^2 + \|x - Tx - (y - Ty)\|^2$$

for all $x, y \in C$. In this case, it is clear that $I - T$ is monotone, that is,

$$\langle x - y, (I - T)x - (I - T)y \rangle \geq 0$$

for all $x, y \in C$.

We know the following fact. For the sake of completeness, we give the proof.

Lemma 5.1. *$I - T$ is demiclosed at* 0.

Proof. Let $\{z_n\}$ be a sequence in C such that $z_n - Tz_n \to 0$ and $z_n \rightharpoonup z$. Note that $z \in C$. Set $y_\lambda = \lambda Tz + (1-\lambda)z$ for $\lambda \in (0,1)$. Then it follows from the monotonicity of $I - T$ that

$$\begin{aligned}\lambda \langle Tz - z, (I-T)y_\lambda \rangle &= \langle y_\lambda - z, (I-T)y_\lambda \rangle \\ &= \lim_{n\to\infty} \langle y_\lambda - z_n, (I-T)y_\lambda \rangle \\ &\geq \lim_{n\to\infty} \langle y_\lambda - z_n, (I-T)z_n \rangle = 0,\end{aligned}$$

and hence $\langle Tz - z, (I-T)y_\lambda \rangle \geq 0$ for all $\lambda \in (0,1)$. Since T is continuous and $y_\lambda \to z$ as $\lambda \downarrow 0$,

$$-\|z - Tz\|^2 = \langle Tz - z, z - Tz \rangle = \lim_{\lambda\downarrow 0} \langle Tz - z, (I-T)y_\lambda \rangle \geq 0,$$

and therefore, $(I-T)z = 0$. $\square$

The following lemma was established in Ishikawa [18]:

Lemma 5.2. *Let $U\colon C \to C$ be a mapping defined by*

$$U = \lambda T\big(\mu T + (1-\mu)I\big) + (1-\lambda)I,$$

where $0 \leq \lambda \leq \mu \leq 1$. Then

$$\lambda\mu(1 - 2\mu - \mu^2\eta^2)\|x - Tx\|^2 \leq \|x - z\|^2 - \|Ux - z\|^2$$

for all $x \in C$ and $z \in \mathrm{F}(T)$.

Using Lemmas 5.1 and 5.2, we obtain the following:

Lemma 5.3. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of real numbers such that*

$$0 < \inf_n \lambda_n,\ \lambda_n \leq \mu_n \text{ for all } n \in \mathbb{N}, \text{ and } \sup_n \mu_n < \frac{-1 + \sqrt{1+\eta^2}}{\eta^2}.$$

Let $U_n\colon C \to C$ be a mapping defined by

$$U_n = \lambda_n T\big(\mu_n T + (1-\mu_n)I\big) + (1-\lambda_n)I \tag{5.1}$$

for $n \in \mathbb{N}$. Suppose that $\mathrm{F}(T) \neq \emptyset$. Then the following hold:

(1) $\mathrm{F}(U_n) = \mathrm{F}(T)$ *for every* $n \in \mathbb{N}$*;*
(2) $\{U_n\}$ *is strongly quasinonexpansive type;*
(3) $\hat{\mathrm{F}}(\{U_n\}) = \mathrm{F}(T)$.

Proof. Set $\rho = (\inf_n \lambda_n)^2\bigl(1 - 2\sup_n \mu_n - (\sup_n \mu_n)^2\eta^2\bigr)$. By assumption, it is clear that

$$\lambda_n\mu_n(1 - 2\mu_n - \mu_n^2\eta^2) \geq \rho > 0$$

for all $n \in \mathbb{N}$.

We first prove (1). Let $w \in \mathrm{F}(U_n)$ and $z \in \mathrm{F}(T)$. Then it follows from Lemma 5.2 that

$$0 \leq \rho \|w - Tw\|^2 \leq \|w - z\|^2 - \|U_n w - z\|^2 = 0.$$

Hence we have $w = Tw$, and thus $\mathrm{F}(U_n) \subset \mathrm{F}(T)$. On the other hand, it is obvious that $\mathrm{F}(U_n) \supset \mathrm{F}(T)$. Therefore, (1) holds.

We next show (2). Let $x \in C$ and $w \in \mathrm{F}(U_n)$. Then Lemma 5.2 and (1) imply that $\|U_n x - w\| \leq \|x - w\|$. Therefore each U_n is quasinonexpansive. Let $\{y_n\}$ be a bounded sequence in C such that $\|y_n - z\| - \|U_n y_n - z\| \to 0$ for $z \in \bigcap_{n=1}^{\infty} \mathrm{F}(U_n) = \mathrm{F}(T)$. Since $\{y_n\}$ and $\{U_n y_n\}$ are bounded, it follows from Lemma 5.2 that

$$0 \leq \rho \|y_n - Ty_n\|^2 \leq (\|y_n - z\| - \|U_n y_n - z\|)(\|y_n - z\| + \|U_n y_n - z\|) \to 0,$$

and thus $y_n - Ty_n \to 0$. Since $\lambda_n, \mu_n \in [0, 1]$ and T is η-Lipschitz continuous,

$$\begin{aligned}
\|y_n - U_n y_n\| &= \lambda_n \bigl\|T\bigl(\mu_n T y_n + (1 - \mu_n) y_n\bigr) - y_n\bigr\| \\
&\leq \bigl\|T\bigl(\mu_n T y_n + (1 - \mu_n) y_n\bigr) - T y_n\bigr\| + \|T y_n - y_n\| \\
&\leq (\eta + 1) \|T y_n - y_n\| \to 0.
\end{aligned}$$

Therefore, $\{U_n\}$ is strongly quasinonexpansive type.

Lastly, we show (3). It is obvious that $\hat{\mathrm{F}}(\{U_n\}) \supset \mathrm{F}(T)$. Let $z \in \hat{\mathrm{F}}(\{U_n\})$ and $p \in \mathrm{F}(T)$. Then there exists a sequence $\{z_n\}$ in C and a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_n - U_n z_n \to 0$ and $z_{n_i} \rightharpoonup z$. Since $\{z_{n_i}\}$ is bounded and

$$0 \leq \|z_{n_i} - p\| - \|U_{n_i} z_{n_i} - p\| \leq \|z_{n_i} - U_{n_i} z_{n_i}\| \to 0,$$

it follows from Lemma 5.2 that

$$0 \leq \rho \|z_{n_i} - T z_{n_i}\|^2 \leq (\|z_{n_i} - p\| - \|U_{n_i} z_{n_i} - p\|)(\|z_{n_i} - p\| + \|U_{n_i} z_{n_i} - p\|) \to 0,$$

and thus $z_{n_i} - T z_{n_i} \to 0$. By Lemma 5.1, we conclude that $z \in \mathrm{F}(T)$. This means that $\hat{\mathrm{F}}(\{U_n\}) \subset \mathrm{F}(T)$. $\square$

Using Theorem 3.1 and Lemma 5.3, we obtain the following:

Theorem 5.4. *Let C be a nonempty closed convex subset of a Hilbert space H, $T\colon C \to C$ an η-Lipschitz continuous pseudo-contraction, and $U_n\colon C \to C$ a mapping defined by* (5.1) *for $n \in \mathbb{N}$, where $\{\lambda_n\}$ and $\{\mu_n\}$ are the same as in Lemma 5.3. Suppose that $\mathrm{F}(T) \neq \emptyset$. Let $\{\alpha_n\}$ and θ be the same as in Theorem 3.1 and f_n a*

θ-contraction with respect to $\mathrm{F}(T)$ *for* $n \in \mathbb{N}$. *Suppose that there exists* $u \in C$ *such that* $f_n \circ P_{\mathrm{F}(T)}(u) \to u$. *Let* $\{x_n\}$ *be a sequence in* C *defined by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n) U_n x_n$$

for $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to* $P_{\mathrm{F}(T)}(u)$.

Remark 5.5. In Theorem 5.4, $P_{\mathrm{F}(T)}$ is well-defined because $\mathrm{F}(T) = \mathrm{F}(U_n)$ and U_n is quasinonexpansive by Lemma 5.3.

Remark 5.6. The assumptions on T can be relaxed in Theorem 5.4: It suffices to assume that T is Lipschitz continuous,

$$\|Tx - z\|^2 \leq \|x - z\|^2 + \|Tx - x\|^2$$

for all $x \in C$ and $z \in \mathrm{F}(T)$, and $I - T$ is demiclosed at 0.

Acknowledgement. The author would like to express his deep appreciation to the anonymous referee for helpful suggestions and comments on the manuscript.

References

[1] K. Aoyama, *Asymptotic fixed points of sequences of quasi-nonexpansive type mappings*, in Banach and function spaces III (ISBFS2009), Yokohama Publ., Yokohama, 2011, pp. 343–350.

[2] K. Aoyama and Y. Kimura, *Strong convergence theorems for strongly nonexpansive sequences*, Appl. Math. Comput. **217** (2011), 7537–7545.

[3] K. Aoyama and Y. Kimura, *Viscosity approximation methods with a sequence of contractions*, Cubo **16** (2014), 9–20.

[4] K. Aoyama, Y. Kimura, and F. Kohsaka, *Strong convergence theorems for strongly relatively nonexpansive sequences and applications*, J. Nonlinear Anal. Optim. **3** (2012), 67–77.

[5] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[6] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, *On a strongly nonexpansive sequence in Hilbert spaces*, J. Nonlinear Convex Anal. **8** (2007), 471–489.

[7] K. Aoyama and F. Kohsaka, *Strongly relatively nonexpansive sequences generated by firmly nonexpansive-like mappings*, Fixed Point Theory Appl. (2014), 2014:95, 13pp.

[8] K. Aoyama and F. Kohsaka, *Viscosity approximation process for a sequence of quasinonexpansive mappings*, Fixed Point Theory Appl. (2014), 2014:17, 11pp.

[9] K. Aoyama, F. Kohsaka, and W. Takahashi, *Strong convergence theorems by shrinking and hybrid projection methods for relatively nonexpansive mappings in Banach spaces*, in Nonlinear analysis and convex analysis, Yokohama Publ., Yokohama, 2009, pp. 7–26.

[10] K. Aoyama, F. Kohsaka, and W. Takahashi, *Strongly relatively nonexpansive sequences in Banach spaces and applications*, J. Fixed Point Theory Appl. **5** (2009), 201–224.

[11] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 197–228.

[12] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. **3** (1977), 459–470.

[13] A. Cegielski and R. Zalas, *Methods for variational inequality problem over the intersection of fixed point sets of quasi-nonexpansive operators*, Numer. Funct. Anal. Optim. **34** (2013), 255–283.

[14] W. G. Dotson Jr, *Fixed points of quasi-nonexpansive mappings*, J. Austral. Math. Soc. **13** (1972), 167–170.

[15] M. Hojo, T. Suzuki, and W. Takahashi, *Fixed point theorems and convergence theorems for generalized hybrid non-self mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 363–376.

[16] M. Hojo and W. Takahashi, *Generalized split feasibility problem governed by widely more generalized hybrid mappings in Hilbert spaces*, Nihonkai Math. J. **25** (2014), 127–146.

[17] S. Iemoto and W. Takahashi, *Viscosity approximation methods for equilibrium problem and nonexpansive mapping in Hilbert spaces.* Yokohama Math. J. **54** (2008), 113–129.

[18] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[19] T. Kawasaki and W. Takahashi, *Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 71–87.

[20] M. Kikkawa and W. Takahashi, *Strong convergence theorems by the viscosity approximation method for a countable family of nonexpansive mappings*, Taiwanese J. Math. **12** (2008), 583–598.

[21] P.-E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal. **16** (2008), 899–912.

[22] P.-E. Maingé, *The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces*, Comput. Math. Appl. **59** (2010), 74–79.

[23] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl. **241** (2000), 46–55.

[24] K. Nakajo and W. Takahashi, *Approximation of a zero of maximal monotone operators in Hilbert spaces*, in Nonlinear analysis and convex analysis, Yokohama Publ., Yokohama, 2002, pp. 303–341.

[25] T. Suzuki, *Moudafi's viscosity approximations with Meir-Keeler contractions*, J. Math. Anal. Appl. **325** (2007), 342–352.

[26] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl. **331** (2007), 506–515.

[27] W. Takahashi, *Viscosity approximation methods for resolvents of accretive operators in Banach spaces*, J. Fixed Point Theory Appl. **1** (2007), 135–147.

[28] W. Takahashi, *Introduction to nonlinear and convex analysis*, Yokohama Publ., Yokohama, 2009.

[29] W. Takahashi, *Viscosity approximation methods for countable families of nonexpansive mappings in Banach spaces*, Nonlinear Anal. **70** (2009), 719–734.

[30] W. Takahashi, H.-K. Xu, and J.-C. Yao, *Iterative methods for generalized split feasibility problems in Hilbert spaces*, Set-Valued Var. Anal. **23** (2015), 205–221.

[31] H.-K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. **66** (2002), 240–256.

[32] H.-K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298** (2004), 279–291.

[33] I. Yamada and N. Ogura, *Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings*, Numer. Funct. Anal. Optim. **25** (2004), 619–655.

(Koji Aoyama) Department of Economics, Chiba University, Yayoi-cho, Inage-ku, Chiba-shi, Chiba 263-8522, Japan.
E-mail address: aoyama@le.chiba-u.ac.jp

Received March 4, 2016
Revised May 23, 2016