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SIMULTANEOUS EXTENSIONS OF DIAZ-METCALF AND BUZANO INEQUALITIES

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Dedicated to Professor Kichi-Suke Saito in commemoration of his retirement

Abstract. We give a simultaneous extension of Diaz-Metcalf and Buzano inequalities: Let $z_1, \dots, z_m$ be nonzero vectors in a Hilbert space $\mathscr{H}$. Suppose that $x_1, \dots, x_n \in \mathscr{H}$ satisfy that for each $j = 1, \dots, m$ there exists a constant r_j such that $0 \le r_j \le \frac{\mathrm{Re}\langle x_i, z_j\rangle}{\|x_i\|}$ for $i = 1, \dots, n$. If $y_1, y_2 \in \mathscr{H}$ satisfy $\langle y_k, z_j\rangle = 0$ for $k = 1, 2$ and $j = 1, \dots, m$, then

$$\left|\left\langle \sum x_i, y_1\right\rangle \left\langle \sum x_i, y_2\right\rangle\right| + \left(\sum \frac{r_j^2}{c_j}\right)\left(\sum \|x_i\|\right)^2 \mathcal{B}(y_1, y_2) \le \mathcal{B}(y_1, y_2)\left\|\sum x_i\right\|^2,$$

where $\mathcal{B}(y_1, y_2) := \frac{1}{2}(\|y_1\|\,\|y_2\| + |\langle y_1, y_2\rangle|)$ and $c_j = \sum_h |\langle z_h, z_j\rangle|$ for $j = 1, \dots, m$.

As an application, we discuss a refinement of an extended Heinz-Kato-Furuta inequality. Moreover, we show some variant inequalities of it by Furuta inequality and chaotic order.

1. Introduction

About 50 years ago, Wilf [20] proposed a reverse arithmetic-geometric mean inequality for complex numbers: *For complex numbers $t_1, \dots, t_n$, suppose that*

$$|\arg t_i| \le \phi \le \frac{\pi}{2} \quad \textit{for } i = 1, \dots, n. \tag{1.1}$$

Then

$$|t_1 \cdot t_2 \cdots t_n|^{\frac{1}{n}} \le (\sec\phi)\frac{1}{n}|t_1 + t_2 + \cdots + t_n|. \tag{1.2}$$

As a matter of fact, the assumption (1.1) implies

$$\cos\phi \cdot (|t_1| + |t_2| + \cdots + |t_n|) \le |t_1 + t_2 + \cdots + t_n| \tag{1.3}$$

by which the conclusion (1.2) is obtained via the arithmetic-geometric mean inequality.

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Afterward, Diaz and Metcalf [2] advanced it to the case of vectors in a Hilbert space $\mathscr{H}$ with an inner product $\langle x, z\rangle$ as follows:

Diaz-Metcalf inequality. *Let z be a unit vector in $\mathscr{H}$. Suppose that $x_1, \ldots, x_n \in \mathscr{H}$ satisfy that there exists a constant r such that*

$$0 \le r \le \frac{\operatorname{Re}\langle x_i, z\rangle}{\|x_i\|} \quad \text{for } i = 1, \ldots, n.$$

Then

$$r \sum_i \|x_i\| \le \left\| \sum_i x_i \right\|.$$

In [9, Theorem 9], it was generalized by connecting the Selberg inequality, cf. [12]:

Theorem A. *Let $z_1, \ldots, z_m$ be vectors in $\mathscr{H}$. Suppose that $x_1, \ldots, x_n \in \mathscr{H}$ satisfy that for each $j = 1, \ldots, m$ there exists a constant r_j such that*

$$0 \le r_j \le \frac{\operatorname{Re}\langle x_i, z_j\rangle}{\|x_i\|} \quad \text{for } i = 1, \ldots, n.$$

If $y \in \mathscr{H}$ satisfies $\langle y, z_j\rangle = 0$ for $j = 1, \ldots, m$, then

$$|\langle x_1 + \cdots + x_n, y\rangle|^2 + \left(\sum_j \frac{r_j^2}{c_j}\right)\left(\sum_i \|x_i\|\right)^2 \|y\|^2 \le \left\|\sum_i x_i\right\|^2 \|y\|^2,$$

where $c_j = \sum_h |\langle z_h, z_j\rangle|$ for $j = 1, \ldots, m$.

On the other hand, we recall the Buzano inequality. For convenience, we denote by

$$\mathcal{B}(y_1, y_2) := \frac{1}{2}(\|y_1\|\,\|y_2\| + |\langle y_1, y_2\rangle|)$$

for $y_1, y_2 \in \mathscr{H}$. The inequality

$$|\langle x, y_1\rangle \langle x, y_2\rangle| \le \frac{1}{2}(\|y_1\|\,\|y_2\| + |\langle y_1, y_2\rangle|)\,\|x\|^2$$

holds for all $x, y_1, y_2 \in \mathscr{H}$, which includes the Schwarz inequality as in the case $y_1 = y_2$.

In our paper [8], we proposed a simultaneous extension of Selberg and Buzano inequalities:

Theorem B. *If $y_1, y_2 \in \mathscr{H}$ satisfy $\langle y_k, z_j\rangle = 0$ for $k = 1,2$ and given nonzero vectors $\{z_j; j = 1, 2, \ldots, m\} \subset \mathscr{H}$, then*

$$|\langle x, y_1\rangle \langle x, y_2\rangle| + \mathcal{B}(y_1, y_2) \sum_j \frac{|\langle x, z_j\rangle|^2}{\sum_h |\langle z_h, z_j\rangle|} \le \mathcal{B}(y_1, y_2) \|x\|^2 \tag{1.4}$$

holds for all $x \in \mathscr{H}$.

In this note, we propose a simultaneous extension of Theorems A and B related to Diaz-Metcalf and Buzano inequalities. As an application, we discuss a refinement of an extended Heinz-Kato-Furuta inequality. Moreover, we show some variant inequalities of it by Furuta inequality and chaotic order.

2. Simultaneous extension of Diaz-Metcalf and Buzano inequalities

We propose a simultaneous extension of Diaz-Metcalf and Buzano inequalities.

Theorem 2.1. *Let $z_1, \ldots, z_m$ be nonzero vectors in $\mathscr{H}$. Suppose that $x_1, \ldots, x_n \in \mathscr{H}$ satisfy that for each $j = 1, \ldots, m$ there exists a constant r_j such that*

$$0 \le r_j \le \frac{\operatorname{Re}\langle x_i, z_j\rangle}{\|x_i\|} \quad \text{for } i = 1, \ldots, n.$$

If $y_1, y_2 \in \mathscr{H}$ satisfy $\langle y_k, z_j\rangle = 0$ for $k = 1,2$ and $j = 1, \ldots, m$, then

$$\begin{aligned} &\left|\left\langle \sum_i x_i, y_1\right\rangle \left\langle \sum_i x_i, y_2\right\rangle\right| + \left(\sum_j \frac{r_j^2}{c_j}\right)\left(\sum_i \|x_i\|\right)^2 \mathcal{B}(y_1, y_2) \\ &\le \mathcal{B}(y_1, y_2) \left\|\sum_i x_i\right\|^2, \end{aligned} \tag{2.1}$$

where $c_j = \sum_h |\langle z_h, z_j\rangle|$ for $j = 1, \ldots, m$.

Proof. We have

$$\begin{aligned} &\mathcal{B}(y_1, y_2)\left\{\left\|\sum_i x_i\right\|^2 - \sum_j \frac{r_j^2}{c_j}\left(\sum_i \|x_i\|\right)^2\right\} \\ &\ge \mathcal{B}(y_1, y_2)\left(\left\|\sum_i x_i\right\|^2 - \sum_j \frac{(\operatorname{Re}\langle \sum_i x_i, z_j\rangle)^2}{c_j}\right) \end{aligned}$$

$$\geq \mathcal{B}(y_1, y_2)\left(\left\|\sum_i x_i\right\|^2 - \sum_j \frac{|\langle \sum_i x_i, z_j\rangle|^2}{c_j}\right)$$
$$\geq \left|\left\langle \sum_i x_i, y_1\right\rangle \left\langle \sum_i x_i, y_2\right\rangle\right| \qquad \text{(by Theorem B)}$$

as desired. □

Next, we propose a generalization of (2.1) as follows:

Corollary 2.2. *Let $T = U|T|$ be the polar decomposition of an operator T on $\mathscr{H}$. Let $z_1, \ldots, z_m$ be nonzero vectors in $\mathscr{H}$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1 \geq \alpha$. Suppose that $x_1, \ldots, x_n \in \mathscr{H}$ satisfy that for each $j = 1, \ldots, m$ there exists a constant r_j such that*

$$0 \leq r_j \leq \frac{\mathrm{Re}\,\langle Tx_i, z_j\rangle}{\||T|^\alpha x_i\|} \quad \left(\textit{resp. } 0 \leq r_j \leq \frac{\mathrm{Re}\,\langle |T|^{2\alpha} x_i, z_j\rangle}{\||T|^\alpha x_i\|}\right) \quad \textit{for } i = 1, \ldots, n.$$

If $y_1, y_2 \in \mathscr{H}$ satisfy $\left\langle |T^*|^{\beta+1-\alpha} y_k, z_j\right\rangle = 0$ (resp. $\left\langle T|T|^{\alpha+\beta-1} z_j, y_k\right\rangle = 0$) for $k = 1, 2$ and $j = 1, \ldots, m$, then

$$\left|\left\langle \sum_i T|T|^{\alpha+\beta-1} x_i, y_1\right\rangle \left\langle \sum_i T|T|^{\alpha+\beta-1} x_i, y_2\right\rangle\right| + \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right)\left(\sum_j \frac{r_j^2}{c_j}\right)\left(\sum_i \||T|^\alpha x_i\|\right)^2 \leq \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right)\left\|\sum_i |T|^\alpha x_i\right\|^2 \tag{2.2}$$

where $c_j = \sum_h |\left\langle |T^*|^{2(1-\alpha)} z_h, z_j\right\rangle|$ (resp. $c_j = \sum_h |\langle |T|^{2\alpha} z_h, z_j\rangle|$) for $j = 1, \ldots, m$.

Proof. We apply Theorem 2.1 by replacing x_i, z_j, y_k to $|T|^\alpha x_i$, $|T|^{1-\alpha} U^* z_j$, $U^*|T^*|^\beta y_k$ (resp. $U|T|^\alpha x_i, U|T|^\alpha z_j, |T^*|^\beta y_k$). □

3. Extensions of Heinz-Kato-Furuta inequality

In [15], Furuta extended the Heinz-Kato inequality:

The Heiz-Kato-Furuta inequality. Let A and B be positive operators on $\mathscr{H}$. If T satisfies $T^*T \leq A^2$ and $TT^* \leq B^2$, then

$$|\left\langle T|T|^{\alpha+\beta-1} x, y\right\rangle| \leq \|A^\alpha x\| \left\|B^\beta y\right\|$$

holds for all $x, y \in \mathscr{H}$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geq 1$. In addition, if A and B are invertible, then $\alpha + \beta \geq 1$ is unnecessary.

Afterwards, several authors have generalized it, e.g. [9], [10], [11].

In this section, we apply Corollary 2.2 to extend the Heinz-Kato-Furuta inequality. To do this, we use the following lemma in our paper [8] ⊠

Lemma C. *If $TT^* \leq B^2$ for some $B \geq 0$, then for $\beta \in [0,1]$*

$$\mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \leq \left\|B^\beta y_1\right\| \left\|B^\beta y_2\right\|$$

holds for all $y_1, y_2 \in \mathscr{H}$.

Now the following inequality follows from Corollary 2.2 and Lemma C:

Corollary 3.1. *Let $T = U|T|$ be the polar decomposition of an operator T on $\mathscr{H}$. Let $z_1, \ldots, z_m$ be nonzero vectors in $\mathscr{H}$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geq 1$. Suppose that $x_1, \ldots, x_n \in \mathscr{H}$ satisfy that for each $j = 1, \ldots, m$ there exists a constant r_j such that*

$$0 \leq r_j \leq \frac{\operatorname{Re}\langle Tx_i, z_j\rangle}{\||T|^\alpha x_i\|} \quad \left(\textit{resp. } 0 \leq r_j \leq \frac{\operatorname{Re}\langle |T|^{2\alpha} x_i, z_j\rangle}{\||T|^\alpha x_i\|}\right) \quad \textit{for } i = 1, \ldots, n.$$

If $T^*T \leq A^2$ and $TT^* \leq B^2$ for some $A, B \geq 0$, and $y_1, y_2 \in \mathscr{H}$ satisfy

$$\left\langle |T^*|^{\beta+1-\alpha} y_k, z_j\right\rangle = 0 \quad (\textit{resp. } \left\langle T|T|^{\alpha+\beta-1} z_j, y_k\right\rangle = 0)$$

for $k = 1, 2$ and $j = 1, \ldots, m$, then

$$\begin{aligned} &\left|\left\langle \sum_i T|T|^{\alpha+\beta-1} x_i, y_1\right\rangle \left\langle \sum_i T|T|^{\alpha+\beta-1} x_i, y_2\right\rangle\right| \\ &\quad + \mathcal{B}\left(|T^*|^\beta y_1, |T^*|^\beta y_2\right) \left(\sum_j \frac{r_j^2}{c_j}\right) \left(\sum_i \||T|^\alpha x_i\|\right)^2 \\ &\quad \leq \left\|B^\beta y_1\right\| \left\|B^\beta y_2\right\| \left\|\sum_i A^\alpha x_i\right\|^2 \end{aligned} \tag{3.1}$$

where $c_j = \sum_h |\left\langle |T^*|^{2(1-\alpha)} z_h, z_j\right\rangle|$ (resp. $c_j = \sum_h |\langle |T|^{2\alpha} z_h, z_j\rangle|$) for $j = 1, \ldots, m$.

Next we cite the Furuta inequality [13] for convenience:

The Furuta inequality.
If $A \geq B \geq 0$, then for each $r \geq 0$,

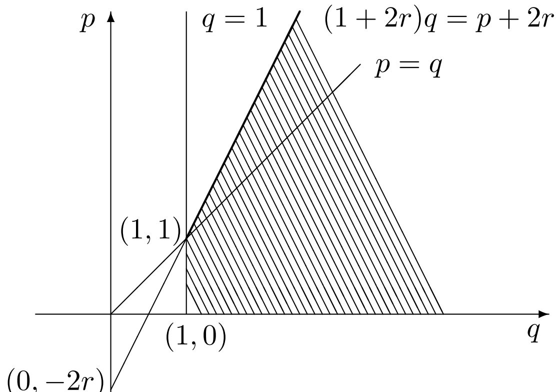
(i) $$(B^r A^p B^r)^{\frac{1}{q}} \geq (B^r B^p B^r)^{\frac{1}{q}}$$

and

(ii) $$(A^r A^p A^r)^{\frac{1}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with

$$(1+2r)q \geq p + 2r.$$



Figure

We refer [17] and [3] for mean theoretic proofs of it, and [14] for a one-page proof. The best possibility of the domain drawn in the Figure is proved by Tanahashi [18]. The Heinz -Kato-Furuta inequality has been extended by the use of the Furuta inequality in [16].

Now, we have the following extension of Corollary 2.2 by the Furuta inequality:

Theorem 3.2. *Let A be a positive operator on $\mathscr{H}$ and $T = U|T|$ be the polar decomposition of an operator T on $\mathscr{H}$ such that $T^*T \leq A^2$. Let $z_1, \ldots, z_m$ be nonzero vectors in $\mathscr{H}$ and $\alpha, \beta \geq 0$ with $(1+r)\alpha + (1+s)\beta \geq 1 \geq (1+r)\alpha$ for each $r, s \geq 0$. Suppose that $x_1, \ldots, x_n \in \mathscr{H}$ satisfy that for each $j = 1, \ldots, m$ there exists a constants r_j such that*

$$0 \leq r_j \leq \frac{\operatorname{Re}\langle Tx_i, z_j\rangle}{\||T|^{(1+r)\alpha}x_i\|} \qquad \left(\textit{resp. } 0 \leq r_j \leq \frac{\operatorname{Re}\left\langle |T|^{2(1+r)\alpha}x_i, z_j\right\rangle}{\||T|^{(1+r)\alpha}x_i\|}\right) \qquad \textit{for } i = 1, \ldots, n.$$

If $y_1, y_2 \in \mathscr{H}$ satisfy $\left\langle |T^*|^{(1+s)\beta+1-(1+r)\alpha}y_k, z_j\right\rangle = 0$ (resp. $\left\langle T|T|^{(1+r)\alpha+(1+s)\beta-1}z_j, y_k\right\rangle = 0$) for $k = 1, 2$ and $j = 1, \ldots, m$, then

$$\begin{aligned}
&\left|\left\langle \sum_i T|T|^{(1+r)\alpha+(1+s)\beta-1}x_i, y_1\right\rangle \left\langle \sum_i T|T|^{(1+r)\alpha+(1+s)\beta-1}x_i, y_2\right\rangle\right| \\
&\quad + \mathcal{B}\left(|T^*|^{(1+s)\beta}y_1, |T^*|^{(1+s)\beta}y_2\right)\left(\sum_j \frac{r_j^2}{c_j}\right)\left(\sum_i \||T|^{(1+r)\alpha}x_i\|\right)^2 \qquad (3.2)\\
&\quad \leq \mathcal{B}\left(|T^*|^{(1+s)\beta}y_1, |T^*|^{(1+s)\beta}y_2\right)\left\langle \left(|T|^r A^{2p}|T|^r\right)^{\frac{(1+r)\alpha}{p+r}} \sum_i x_i, \sum_i x_i\right\rangle,
\end{aligned}$$

where $p \geq 1$ and $c_j = \sum_h \left|\left\langle |T^*|^{2(1-(1+r)\alpha)}z_h, z_j\right\rangle\right|$ (resp. $c_j = \sum_h \left|\left\langle |T|^{2(1+r)\alpha}z_h, z_j\right\rangle\right|$) for $j = 1, \ldots, m$.

Proof. By replacing α and β to $\alpha_1 = (1+r)\alpha$ and $\beta_1 = (1+s)\beta$, respectively in Corollary 2.2, we have

$$\begin{aligned}\Bigg|\left\langle \sum_i T|T|^{\alpha_1+\beta_1-1}x_i, y_1\right\rangle \left\langle \sum_i T|T|^{\alpha_1+\beta_1-1}x_i, y_2\right\rangle\Bigg| \\ + \mathcal{B}\left(|T^*|^{\beta_1}y_1, |T^*|^{\beta_1}y_2\right)\left(\sum_j \frac{r_j^2}{c_j}\right)\left(\sum_i \||T|^{\alpha_1}x_i\|\right)^2 \\ \leq \mathcal{B}\left(|T^*|^{\beta_1}y_1, |T^*|^{\beta_1}y_2\right)\left\langle |T|^{2\alpha_1}\sum_i x_i, \sum_i x_i\right\rangle\end{aligned}$$

where $c_j = \sum_h |\left\langle |T^*|^{2(1-\alpha_1)}z_h, z_j\right\rangle|$ (resp. $c_j = \sum_h |\langle |T|^{2\alpha_1}z_h, z_j\rangle|$) for $j = 1, \dots, m$. Next we replace A, B, r and q to A^2, $|T|^2$, $\frac{r}{2}$ and $\frac{p+r}{(1+r)\alpha}$, respectively in the Furuta inequality. Then we have

$$|T|^{2\alpha_1} = |T|^{2(1+r)\alpha} \leq (|T|^r A^{2p}|T|^r)^{\frac{(1+r)\alpha}{p+r}}.$$

Combining them, we obtain the inequality (3.2). □

We remark that the condition $(1+r)\alpha + (1+s)\beta \geq 1$ in above is unnecessary if T is either positive or invertible.

From the operator monotonicity of the logarithmic function, we introduced the chaotic order among positive invertible operators by $A \gg B$ if $\log A \geq \log B$ in [4], and obtained a characterization of the chaotic order in terms of Furuta's type operator inequality [5], [6] and [7]. We show a variant of Corollary 2.2 by chaotic order. For this, we use the following characterization of the chaotic order which is an extension of Ando's theorem [4], [5], [6], [7] and [19] for a polished proof.

Theorem D. *For positive invertible operators A and B, $A \gg B$ if and only if*

$$(B^r A^p B^r)^{\frac{1}{q}} \geq (B^r B^p B^r)^{\frac{1}{q}}$$

holds for $q \geq 1$, $p, r \geq 0$ with $2rq \geq p + 2r$.

We now show the chaotic version of Corollary 2.2 by applying Theorem D:

Theorem 3.3. *Let A be a positive operator on $\mathscr{H}$ and $T = U|T|$ be the polar decomposition of an operator T on $\mathscr{H}$ such that $T^*T \ll A^2$. Let $z_1, \dots, z_m$ be nonzero vectors in $\mathscr{H}$ and $\alpha, \beta \in [0,1]$ with $r\alpha + s\beta \geq 1 \geq r\alpha$ for each $r, s \geq 0$. Suppose that $x_1, \dots, x_n \in \mathscr{H}$ satisfy that for each $j = 1, \dots, m$ there exists a* *constant* r_j *such that*

$$0 \le r_j \le \frac{\mathrm{Re}\,\langle Tx_i, z_j\rangle}{\||T|^{r\alpha}x_i\|} \quad \left(\textit{resp. } 0 \le r_j \le \frac{\mathrm{Re}\,\langle |T|^{2r\alpha}x_i, z_j\rangle}{\||T|^{r\alpha}x_i\|}\right) \quad \textit{for } i = 1, \ldots, n.$$

If $y_1, y_2 \in \mathscr{H}$ *satisfy* $\left\langle |T^*|^{s\beta+1-r\alpha}y_k, z_j\right\rangle = 0$ (*resp.* $\left\langle T|T|^{r\alpha+s\beta-1}z_j, y_k\right\rangle = 0$) *for* $k = 1, 2$ *and* $j = 1, \ldots, m$, *then*

$$\begin{aligned} &\left|\left\langle \sum_i T|T|^{r\alpha+s\beta-1}x_i, y_1\right\rangle \left\langle \sum_i T|T|^{r\alpha+s\beta-1}x_i, y_2\right\rangle\right| \\ &\quad + \mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\left(\sum_j \frac{r_j^2}{c_j}\right)\left(\sum_i \||T|^{r\alpha}x_i\|\right)^2 \qquad (3.3) \\ &\le \mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\left\langle (|T|^r A^{2p}|T|^r)^{\frac{r\alpha}{p+r}} \sum_i x_i, \sum_i x_i\right\rangle, \end{aligned}$$

where $p \ge 0$ *and* $c_j = \sum_h |\left\langle |T^*|^{2(1-r\alpha)}z_h, z_j\right\rangle|$ (*resp.* $c_j = \sum_h |\langle |T|^{2r\alpha}z_h, z_j\rangle|$) *for* $j = 1, \ldots, m$.

Proof. By replacing α and β to $r\alpha$ and $s\beta$, respectively in Corollary 2.2, we have

$$\begin{aligned} &\left|\left\langle \sum_i T|T|^{r\alpha+s\beta-1}x_i, y_1\right\rangle \left\langle \sum_i T|T|^{r\alpha+s\beta-1}x_i, y_2\right\rangle\right| \\ &\quad + \mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\left(\sum_j \frac{r_j^2}{c_j}\right)\left(\sum_i \||T|^{r\alpha}x_i\|\right)^2 \\ &\le \mathcal{B}\left(|T^*|^{s\beta}y_1, |T^*|^{s\beta}y_2\right)\left\langle |T|^{2r\alpha} \sum_i x_i, \sum_i x_i\right\rangle \end{aligned}$$

where $c_j = \sum_h |\left\langle |T^*|^{2(1-r\alpha)}z_h, z_j\right\rangle|$ (resp. $c_j = \sum_h |\langle |T|^{2r\alpha}z_h, z_j\rangle|$) for $j = 1, \ldots, m$.

Moreover we replace A, B, r and q to A^2, $|T|^2$, $\frac{r}{2}$ and $\frac{p+r}{r\alpha}$, respectively in Theorem D. Then we have

$$|T|^{2r\alpha} \le (|T|^r A^{2p}|T|^r)^{\frac{r\alpha}{p+r}}.$$

Combining inequalities above, we obtain the desired inequality (3.3). □

Next we interpolate between Theorems 3.2 and 3.3 by the use of Furuta's type operator inequality which interpolates the Furuta inequality and Theorem D.

Theorem 3.4. *Let* A *be a positive operator on* $\mathscr{H}$ *and* $T = U|T|$ *be the polar decomposition of an operator* T *on* $\mathscr{H}$ *such that* $|T|^{2\delta} \le A^{2\delta}$ *for some* $\delta \in (0, 1]$. *Let* $z_1, \ldots, z_m$ *be nonzero vectors in* $\mathscr{H}$ *and* $\alpha, \beta \in [0, 1]$ *with* $(\delta + r)\alpha + (\delta + s)\beta \ge$

$1 \geq (\delta+r)\alpha$ *for each* $r, s \geq 0$. *Suppose that* $x_1, \ldots, x_n \in \mathscr{H}$ *satisfy that for each* $j = 1, \ldots, m$ *there exists a constant* r_j *such that*

$$0 \leq r_j \leq \frac{\operatorname{Re}\langle Tx_i, z_j\rangle}{\||T|^{(\delta+r)\alpha}x_i\|} \quad \left(\textit{resp. } 0 \leq r_j \leq \frac{\operatorname{Re}\left\langle |T|^{2(\delta+r)\alpha}x_i, z_j\right\rangle}{\||T|^{(\delta+r)\alpha}x_i\|}\right) \quad \textit{for } i = 1, \ldots, n.$$

If $y_1, y_2 \in \mathscr{H}$ *satisfy* $\left\langle |T^*|^{(\delta+s)\beta+1-(\delta+r)\alpha}y_k, z_j\right\rangle = 0$ (*resp.* $\left\langle T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}z_j, y_k\right\rangle = 0$) *for* $k = 1, 2$ *and* $j = 1, \ldots, m$, *then*

$$\begin{aligned} &\left|\left\langle \sum_i T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x_i, y_1\right\rangle \left\langle \sum_i T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x_i, y_2\right\rangle\right| \\ &\quad + \mathcal{B}\left(|T^*|^{(\delta+s)\beta}y_1, |T^*|^{(\delta+s)\beta}y_2\right)\left(\sum_j \frac{r_j^2}{c_j}\right)\left(\sum_i \||T|^{(\delta+r)\alpha}x_i\|\right)^2 \\ &\quad \leq \mathcal{B}\left(|T^*|^{(\delta+s)\beta}y_1, |T^*|^{(\delta+s)\beta}y_2\right)\left\langle \left(|T|^r A^{2p}|T|^r\right)^{\frac{(\delta+r)\alpha}{p+r}}\sum_i x_i, \sum_i x_i\right\rangle, \end{aligned} \tag{3.4}$$

where $p \geq \delta$ *and* $c_j = \sum_h |\left\langle |T^*|^{2(1-(\delta+r)\alpha)}z_h, z_j\right\rangle|$ (*resp.* $c_j = \sum_h |\left\langle |T|^{2(\delta+r)\alpha}z_h, z_j\right\rangle|$) *for* $j = 1, \ldots, m$.

Proof. By Corollary 2.2, we have

$$\begin{aligned} &\left|\left\langle \sum_i T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x_i, y_1\right\rangle \left\langle \sum_i T|T|^{(\delta+r)\alpha+(\delta+s)\beta-1}x_i, y_2\right\rangle\right| \\ &\quad + \mathcal{B}\left(|T^*|^{(\delta+s)\beta}y_1, |T^*|^{(\delta+s)\beta}y_2\right)\left(\sum_j \frac{r_j^2}{c_j}\right)\left(\sum_i \||T|^{(\delta+r)\alpha}x_i\|\right)^2 \\ &\quad \leq \mathcal{B}\left(|T^*|^{(\delta+s)\beta}y_1, |T^*|^{(\delta+s)\beta}y_2\right)\left\langle |T|^{2(\delta+r)\alpha}\sum_i x_i, \sum_i x_i\right\rangle \end{aligned}$$

where $c_j = \sum_h |\left\langle |T^*|^{2(1-(\delta+r)\alpha)}z_h, z_j\right\rangle|$ (resp. $c_j = \sum_h |\left\langle |T|^{2(\delta+r)\alpha}z_h, z_j\right\rangle|$) for $j = 1, \ldots, m$. Moreover the following inequality is known in [6]:

$$|T|^{2(\delta+r)\alpha} \leq \left(|T|^r A^{2p}|T|^r\right)^{\frac{(\delta+r)\alpha}{p+r}}.$$

Combining above inequalities, we obtain the desired inequality (3.4). □

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