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A SET-VALUED GENERALIZATION OF RICCERI'S THEOREM RELATED TO FAN-TAKAHASHI MINIMAX INEQUALITY

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Dedicated to Professor Kichi-Suke Saito on the occasion of his retirement

Abstract. In the paper, we propose Ricceri type theorem on Fan-Takahashi minimax inequality for set-valued maps by using the scalarization method proposed by Kuwano, Tanaka and Yamada based on a certain type of the set-relations.

1. Introduction

In Convex Analysis, there are several kinds of inequality theorems related to minimality or maximality. Fan-Takahashi minimax inequality theorem (see [1] in 1972 and [10] in 1976) is one of important results with many applications to other mathematical areas. It gives a value of upper-bound of minimax value of a two-variable function. Then, in [8], Ricceri proposes a reasonable substitute of assumptions for Fan-Takahashi minimax inequality, that is, he shows the same conclusion on the inequality under some different assumptions which contains a certain mutually exclusive condition to the assumption of the original Fan-Takahashi theorem.

On the other hand, Kuwano, Tanaka and Yamada in [5] propose four kinds of Fan-Takahashi minimax inequality theorem for set-valued maps. They use certain scalarization methods for set-valued maps, proposed in [4], based on set-relations in [2].

In 2015, we propose a certain Ricceri type theorem for set-valued maps with a set-relation $\not\leq^{(5)}_{\mathrm{int}\,C}$ in [6], in which we propose a certain equivalence between a simpler set and a nonsensitive set in the Ricceri's theorem. By using the same method used in [6], we could obtain another three types of set-valued generalized theorems.

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The aim of this paper is to research Ricceri type theorems for Fan-Takahashi minimax inequality into set-valued maps by a similar method above. We prove a result with respect to $\not\leq^{(3)}_{\mathrm{int}\,C}$ and some counter examples for the results with respect to $\leq^{(j)}_C$ $(j = 3, 5)$.

The organization of the paper is as follows. In Section 2, we recall some definitions and propositions on set-relations, and some tools of a scalarization method for sets. In Section 3, we introduce Ricceri type theorem (Theorem 3.2) on Fan-Takahashi minimax inequality. In Section 4, we propose generalized theorems of the two Ricceri type theorems for set-valued maps. Finally, we consider other cases with each set-relation and show counter examples on them.

2. Preliminaries

Throughout the paper, let E be a real topological vector space, V a linear subspace of E, D a non-empty subset of V, Y an ordered topological vector space, C an ordering cone in Y with $\mathrm{int}\,C \neq \emptyset$, θ_E (resp., θ_Y) the zero vector of E (resp., Y) and $\mathcal{V}(x)$ the open neighborhood system of a point x. Let F be a set-valued map from a real topological vector space into $2^Y \setminus \{\emptyset\}$.

Moreover, if S, T, U are three non-empty subsets of E, we put

$$I_{S,T,U} := \{x \in S \mid T \subseteq \cup_{\lambda>0} \lambda(x - U)\}.$$

Furthermore, we denote the algebraic sum and difference of any subsets A and B in Y by $A + B := \{a + b \mid a \in A, b \in B\}$ and $A - B := \{a - b \mid a \in A, b \in B\}$, respectively. Also, given $A \subset Y$, we write $tA := \{ta \mid a \in A\}$ for $t \in \mathbb{R}$ and $A + x := A + \{x\}$ for $x \in Y$.

At first, we introduce some set-relations by Kuroiwa, Tanaka and Ha.

Definition 2.1 (set-relation, [2])**.** For any nonempty sets A, $B \subset Y$, we write

$B \subset (A + C)$ by $A \leq^{(3)}_C B$;
$A \subset (B - C)$ by $A \leq^{(5)}_C B$.

Proposition 2.2 ([4])**.** *For any nonempty sets $A, B \subset Y$, the following statements hold for $j = 3$ and 5:*

(i) *$A \leq^{(j)}_C B$ implies $(A + y) \leq^{(j)}_C (B + y)$ for $y \in Y$, and $A \leq^{(j)}_C B$ implies $\alpha A \leq^{(j)}_C \alpha B$ for $\alpha > 0$;*
(ii) *$\leq^{(j)}_C$ is transitive;*
(iii) *$\leq^{(j)}_C$ is reflexive;*
(iv) *$A \leq^{(j)}_C B$ and $y_1 \leq_C y_2$ for $y_1, y_2 \in Y$ imply $A + y_1 \leq^{(j)}_C B + y_2$.*

We recall some definitions of C-notions which are referred in [7]. A subset A in Y is said to be C-convex (resp., C-closed) if $A + C$ is convex (resp., closed); C-proper if $A + C \neq Y$. Moreover, A is said to be C-bounded if for each $U \in \mathcal{V}(\theta_Y)$ there exists $t \geq 0$ such that $A \subset tU + C$. Furthermore, we say that a set-valued map F is each C-notion mentioned above on a topological vector space E, if the set $F(x)$ for each $x \in E$ has the property of the corresponding C-notion. A set A (resp., a map F) is said to be finitely notion if $A \cap V$ (resp., $F(\cdot) \cap V$) has the notion for any finite subspace V on Y.

Definition 2.3 (type (j) C-convexity)**.** For each $j = 3, 5$, a set-valued map F is called a *type (j) C-convex function* if for each $x, y \in E$ and $\lambda \in (0, 1)$,

$$F(\lambda x + (1 - \lambda)y) \leq_C^{(j)} \lambda F(x) + (1 - \lambda)F(y).$$

Definition 2.4 (type (j) C-concavity)**.** For each $j = 3, 5$, a set-valued map F is called a *type (j) C-concave function* if for each $x, y \in E$ and $\lambda \in (0, 1)$,

$$\lambda F(x) + (1 - \lambda)F(y) \leq_C^{(j)} F(\lambda x + (1 - \lambda)y).$$

Above two definitions include the C-convexity and C-concavity for vector-valued functions with respect to $\leq_C$ (where $z_1 \leq_C z_2$ denotes $z_2 - z_1 \in C$), respectively.

Definition 2.5 (C-continuity, [7])**.**

(i) F is called a *C-lower continuous function* if for each $\bar{x} \in E$ and open set W with $F(\bar{x}) \cap W \neq \emptyset$, there exists $U \in \mathcal{V}(\bar{x})$ such that $F(y) \cap (W + C) \neq \emptyset$ for all $y \in U$.

(ii) F is called a *C-upper continuous function* if for each $\bar{x} \in E$ and open set W with $F(\bar{x}) \subset W$, there exists $U \in \mathcal{V}(\bar{x})$ such that $F(y) \subset W + C$ for all $y \in U$.

Next, we introduce the definition of two types of nonlinear scalarizing functions for sets.

Definition 2.6 (unified scalarization for sets, [4])**.** Let A and V' be nonempty subsets in Y and direction $k \in \operatorname{int} C$. For each $j = 3, 5$, we define scalarizing functions $I_{k,V'}^{(j)}$ and $S_{k,V'}^{(j)} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$I_{k,V'}^{(j)}(A) := \inf \left\{ t \in \mathbb{R} \;\middle|\; A \leq_C^{(j)} (tk + V') \right\} \quad \text{and}$$
$$S_{k,V'}^{(j)}(A) := \sup \left\{ t \in \mathbb{R} \;\middle|\; (tk + V') \leq_C^{(j)} A \right\},$$

respectively. They are called unified scalarizing functions for sets.

Proposition 2.7 ([4])**.** *Let A, B and V' be nonempty subsets in Y and $k \in \operatorname{int} C$. Then, for each $j = 3, 5$,*

$$A \leq_C^{(j)} B \quad \textit{implies} \quad I_{k,V'}^{(j)}(A) \leq I_{k,V'}^{(j)}(B) \quad \textit{and} \quad S_{k,V'}^{(j)}(A) \leq S_{k,V'}^{(j)}(B).$$

Proposition 2.8 ([5])**.** *Let A and V' be nonempty subsets in Y and $k \in \operatorname{int} C$. Then, the following statements hold:*

(i) *If A is C-bounded and V' is C-proper then $S^{(3)}_{k,V'}(A) \in \mathbb{R}$;*
(ii) *If A is C-proper and V' is C-bounded then $I^{(3)}_{k,V'}(A) \in \mathbb{R}$;*
(iii) *If A is $(-C)$-bounded and V' is $(-C)$-proper then $I^{(5)}_{k,V'}(A) \in \mathbb{R}$.*

For each $x \in E$ and $j = 3, 5$, we consider the following composite functions:

$$(I^{(j)}_{k,V'} \circ F)(x) := I^{(j)}_{k,V'}(F(x));$$
$$(S^{(j)}_{k,V'} \circ F)(x) := S^{(j)}_{k,V'}(F(x)).$$

Then, we can get the following properties between a set-valued map F and the composite function $S^{(j)}_{k,V'} \circ F$.

Proposition 2.9 ([3])**.** *If F is type (3) C-concave, then for each fixed $(k, V) \in (\operatorname{int} C) \times (2^Y \setminus \{\emptyset\})$, $S^{(3)}_{k,V'} \circ F$ is a concave function on E.*

Proposition 2.10 ([9])**.** *For each fixed $(k, V') \in (\operatorname{int} C) \times (2^Y \setminus \{\emptyset\})$, the following statements hold:*

(i) (a) *If F is $(-C)$-lower continuous on E then $S^{(3)}_{k,V'} \circ F$ is upper semicontinuous in E,*
(b) *if F is C-upper continuous on E then $S^{(3)}_{k,V'} \circ F$ is lower semicontinuous in E;*
(ii) (a) *If F is $(-C)$-lower continuous on E then $I^{(3)}_{k,V'} \circ F$ is upper semicontinuous in E,*
(b) *if F is C-upper continuous on E then $I^{(3)}_{k,V'} \circ F$ is lower semicontinuous in E;*
(iii) (a) *If F is C-lower continuous on E then $I^{(5)}_{k,V'} \circ F$ is lower semicontinuous in E,*
(b) *if F is $(-C)$-upper continuous on E then $I^{(5)}_{k,V'} \circ F$ is upper semicontinuous in E.*

3. Ricceri's theorems on Fan-Takahashi minimax inequality

At first, we recall the following two theorems.

Theorem 3.1 (Fan-Takahashi minimax inequality, [10])**.** *Let E be a real Hausdorff topological vector space, X a non-empty compact convex subset of E and f a real function on $X \times X$ satisfying the following conditions:*

(1) *for every $x \in X$, the function $f(x, \cdot)$ is (quasi) concave in X;*
(2) *for every $y \in X$, the function $f(\cdot, y)$ is lower semicontinuous in X;*
(3) *for every $x \in X$, $f(x, x) \leq 0$.*

Then, there exists $\hat{x} \in X$ such that $f(\hat{x}, y) \leq 0$ for all $y \in X$.

Theorem 3.2 (Ricceri type theorem for Fan-Takahashi minimax inequality, [8]). *Let E be a real topological vector space, X a non-empty compact convex subset of E, $\theta_E \in X$ and f a real-valued function on $X \times E$ satisfying the following conditions:*

(1) *for every $x \in X$, the function $f(x, \cdot)$ is concave in E and $f(x, \theta_E) = 0$;*
(2) *for every $y \in E$, the function $f(\cdot, y)$ is lower semicontinuous in X;*
(3) *for every $x \in X$ such that $X \setminus \cup_{\lambda>0}\lambda(x - X) \neq \emptyset$, one has $f(x, x) > 0$.*

Then, there exists $\hat{x} \in X$ such that $f(\hat{x}, y) \leq 0$ for all $y \in X$.

Ricceri proposes Theorem 3.2 above which is a reasonable substitute of Theorem 3.1. Clearly, both third conditions in the theorems can not occur at the same time. However, the two theorems have the same result, and so they are mutually exclusive. Also, a set-valued version of Fan-Takahashi minimax inequality theorem is proposed by the scalarization method in [5].

A set-valued Ricceri type theorem can be proved by a similar method (see [6]). In the next section, we obtain a set-valued version of Theorem 3.2 as a corollary of a set-valued version of the following theorem, that is, the origin of Theorem 3.2.

We note that about an operator A from X into M_V, where M_V is the set of all real-valued functions on V, we often consider A as a two-variable real-valued function on $X \times V$. Then, we write $(A(x))(y)$ by $A(x, y)$ for each $x \in X$ and $y \in V$.

Theorem 3.3 ([8]). *Let E be a real topological vector space, X a non-empty finitely closed and convex subset of E, K a finitely compact subset of X with $\theta_E \in K$, $\tilde{\tau}$ a topology on K with respect to which K is compact and f a real-valued function on $X \times V$. We assume that f satisfies the following conditions:*

(1) *for every $x \in X$, the function $f(x, \cdot)$ is concave in V;*
(2) *the function $f(\cdot, y)$ is finitely lower semicontinuous in X for every $y \in (X - X) \cap V$, is $\tilde{\tau}$-lower semicontinuous in K for every $y \in D$, is finitely continuous in X and $\tilde{\tau}$-continuous in K for $y = \theta_E$.*

Then, for any convex real-valued function ψ on V with $\psi(\theta_E) = 0$ and

$$f(x, x) > f(x, \theta_E) + \psi(x) \qquad \text{for all} \quad x \in (X \cap V) \setminus I_{K,D,X},$$

there exists $\hat{x} \in K$ such that

$$f(\hat{x}, y) \leq f(\hat{x}, \theta_E) + \psi(y) \qquad \text{for all} \quad y \in D.$$

4. Ricceri type theorem for set-valued maps

In this section, we study Ricceri type theorems for set-valued maps. Throughout this section, we assume that $\operatorname{int} C \neq \emptyset$. In [6], we propose Ricceri type theorem for set-valued maps and its corollary as follows.

Theorem 4.1 ([6]). *Let E be a real topological vector space, Y an ordered topological vector space with ordering cone C, X a non-empty finitely closed and convex subset of E, K a finitely compact subset of X with $\theta_E \in K$, $\tilde{\tau}$ a topology on K with respect to which K is compact and F a set-valued map from $X \times V$ to $2^Y \setminus \{\emptyset\}$. We assume that F satisfies the following conditions:*

(1) *F is $(-C)$-proper;*
(2) *for every $x \in X$, the map $F(x, \cdot)$ is type (5) C-concave in V;*
(3) *the map $F(\cdot, y)$ is finitely C-lower continuous in X for every $y \in (X - X) \cap V$, $\tilde{\tau}$-C-lower continuous in K for every $y \in D$, the map $F(\cdot, \theta_Y)$ is vector-valued map, finitely continuous in X and $\tilde{\tau}$-continuous in K.*

Then, for any C-convex vector-valued map ψ from V to Y with $\psi(\theta_E) = \theta_Y$ and

$$F(x, \theta_E) + \psi(x) \leq^{(5)}_{\mathrm{int}\, C} F(x, x) \qquad \text{for all} \quad x \in (X \cap V) \setminus I_{K,D,X},$$

there exists $\hat{x} \in K$ such that

$$F(\hat{x}, \theta_E) + \psi(y) \not\leq^{(5)}_{\mathrm{int}\, C} F(\hat{x}, y) \qquad \text{for all} \quad y \in D.$$

Corollary 4.2 ([6]). *Let E be a real topological vector space, Y an ordered topological vector space with ordering cone C, X a non-empty compact convex subset of E, $\theta_E \in X$ and F a set-valued map from $X \times E$ to $2^Y \setminus \{\emptyset\}$ satisfying the following conditions:*

(1) *F is $(-C)$-proper;*
(2) *for every $x \in X$, $F(x, \cdot)$ is type (5) C-concave in E and $F(x, \theta_E) = \{\theta_Y\}$;*
(3) *for every $y \in E$, $F(\cdot, y)$ is C-lower continuous in X;*
(4) *for every $x \in X$ such that $X \setminus \cup_{\lambda > 0} \lambda(x - X) \neq \emptyset$, one has $\{\theta_Y\} \leq^{(5)}_{\mathrm{int}\, C} F(x, x)$.*

Then, there exists $\hat{x} \in X$ such that $\{\theta_Y\} \not\leq^{(5)}_{\mathrm{int}\, C} F(\hat{x}, y)$ for all $y \in X$.

The conclusion of Corollary 4.2 is of the form $\{\theta_Y\} \not\leq^{(5)}_{\mathrm{int}\, C} F(\hat{x}, y)$ with respect to the set-relation type (5). We consider another three kinds of Ricceri type theorem for set-valued maps with the following conclusions:

(i) $\{\theta_Y\} \not\leq^{(3)}_{\mathrm{int}\, C} F(\hat{x}, y)$;
(ii) $F(\hat{x}, y) \leq^{(5)}_{C} \{\theta_Y\}$;
(iii) $F(\hat{x}, y) \leq^{(3)}_{C} \{\theta_Y\}$.

4.1. Observation for the case (i)

We show the following theorem and its corollary with respect to the case (i).

Theorem 4.3. *Let E be a real topological vector space, Y an ordered topological vector space with ordering cone C, X a non-empty finitely closed and convex subset* *of E, K a finitely compact subset of X with $\theta_E \in K$, $\tilde{\tau}$ a topology on K with respect to which K is compact and F a set-valued map from $X \times V$ to $2^Y \setminus \{\emptyset\}$. We assume that F satisfies the following conditions:*

(1) *F is compact-valued;*
(2) *for every $x \in X$, the map $F(x,\cdot)$ is type (3) C-concave in V;*
(3) *the map $F(\cdot,y)$ is finitely C-upper continuous in X for every $y \in (X-X)\cap V$ and $\tilde{\tau}$-C-upper continuous in K for every $y \in D$, the map $F(\cdot,\theta_Y)$ is vector-valued map, finitely continuous in X and $\tilde{\tau}$ continuous in K.*

Then, for any C-convex vector-valued map ψ from V to Y with $\psi(\theta_E) = \theta_Y$ and

$$\psi(x) \leq^{(3)}_{\operatorname{int} C} F(x,x) - F(x,\theta_E) \qquad \text{for all} \quad x \in (X \cap V) \setminus I_{K,D,X},$$

there exists $\hat{x} \in K$ such that

$$\psi(y) \not\leq^{(3)}_{\operatorname{int} C} F(\hat{x},y) - F(\hat{x},\theta_E) \qquad \text{for all} \quad y \in D.$$

Proof. Let $V' := \{\theta_Y\}$, $k \in \operatorname{int} C$ be fixed. We consider the set-valued map B from $X \times V$ to $2^Y \setminus \{\emptyset\}$ defined by

$$B(x,y) := F(x,y) - F(x,\theta_E) - \psi(y).$$

We consider the composite function $S^{(3)}_{k,V'} \circ B$, and we denote it by A. By Proposition 2.8, A has real-valued images without $\pm\infty$. Then, there exists $\hat{x} \in K$ such that $A(\hat{x},y) \leq 0$ for all $y \in D$ if A holds the following conditions:

(a) $A(x,\cdot)$ is concave for any $x \in X$;
(b) $A(x,\theta_E) = 0$;
(c) $A(\cdot,y)$ is finitely lower semicontinuous in X for any $y \in (X-X)\cap V$;
(d) $A(\cdot,y)$ is $\tilde{\tau}$-lower semicontinuous in K for any $y \in D$;
(e) $A(x,x) > 0$ for all $x \in (X \cap V) \setminus I_{K,D,X}$.

We show each proof of the five statements above.

(a) By assumption (2), the C-concavity of vector-valued function $-\psi$ and the C-convexity of V', it follows from (iv) of Proposition 2.2 that $B(x,\cdot)$ is type (3) C-concave in V. From Proposition 2.9, it follows that $S^{(3)}_{k,V'} \circ B(x,\cdot)$ is concave.

(b) Since $F(x,\theta_E)$ is singleton and $\psi(\theta_E) = \theta_Y$, we get $B(x,\theta_E) = \{\theta_Y\} = V'$. Clearly, $S^{(3)}_{k,V'}(V') = 0$ is always true.

(c) Let $y \in (X-X)\cap V$ be fixed. For each finite dimensional subspace S in E, we take $x \in X \cap S$ and an open subset W of Y with $(F(x,y) - F(x,\theta_E)) \subset W$. Since $F(x,y)$ is compact set, $F(x,y) - F(x,\theta_E)$ is compact. Hence, there exist W_1, any open neighborhood of $F(x,y)$, and W_2, any open neighborhood of $-F(x,\theta_E)$, such that $W_1 + W_2 \subset W$. By the C-upper continuity of $F(\cdot,y)$ and $F(\cdot,\theta_E)$, there exist open neighborhoods $U^{(1)}_x$ and $U^{(2)}_x$ of x such that $F(z_1,y) \subset (W_1 + C)$ and

$(-F(z_2,\theta_E)) \subset (W_2+C)$ for any $z_1 \in U_x^{(1)}$ and $z_2 \in U_x^{(2)}$. We put $U_x := U_x^{(1)} \cap U_x^{(2)}$, then U_x is an open neighborhood of x and

$$(F(z,y) - F(z,\theta_E)) \subset (W_1 + W_2 + C) \qquad \text{for all } z \in U_x.$$

We know $(W_1+W_2) \subset W$, and then $F(\cdot,y)-F(\cdot,\theta_E)$ is finitely C-upper continuous. Thus, $B(\cdot,y)$ is (finitely) C-upper continuous. Hence, $S^{(3)}_{k,V'} \circ B(\cdot,y)$ is (finitely) lower semicontinuous, which is proved by (i)-(b) of Proposition 2.10,

(d) It can be proved in a similar way to the proof of (c).

(e) If $(X \cap V) \setminus I_{K,D,X} = \emptyset$, then (e) is true. Otherwise, for each $x \in (X \cap V) \setminus I_{K,D,X}$, by assumption, we have

$$\{\theta_Y\} = V' \leq^{(3)}_{\mathrm{int}\, C} B(x,x).$$

Thus, $B(x,x) \subset V' + \mathrm{int}\, C$. Since $V' - \mathrm{int}\, C$ is open, there exists $t > 0$ such that $B(x,x) - tk \subset V' + \mathrm{int}\, C$, which implies that $0 < t \leq (S^{(3)}_{k,V'} \circ B)(x,x) = A(x)(x)$.

Therefore, in the same way as the proof of Theorem 3.3, we obtain there exists $\hat{x} \in K$ such that $A(\hat{x},y) = (S^{(3)}_{k,V'} \circ B)(\hat{x},y) \leq 0$ for all $y \in D$. By the definition of $S^{(3)}_{k,V'}$ and Proposition 2.7, for each $y \in D$ and $s > 0$,

$$B(\hat{x},y) \quad \not\subseteq \quad \{\theta_Y\} + sk + C.$$

By $\cup_{s>0}(sk + C) = \mathrm{int}\, C$, we obtain $\{\theta_Y\} \not\leq^{(3)}_{\mathrm{int}\, C} B(\hat{x},y)$. Thus,

$$\psi(y) \not\leq^{(3)}_{\mathrm{int}\, C} F(\hat{x},y) - F(\hat{x},\theta_E).$$

∎

Corollary 4.4. *Let E be a real topological vector space, Y an ordered topological vector space with ordering cone C, X a non-empty compact convex subset of E, $\theta_E \in X$ and F a set-valued map from $X \times E$ to $2^Y \setminus \{\emptyset\}$ satisfying the following conditions:*

1. *F is compact-valued;*
2. *for every $x \in X$, $F(x,\cdot)$ is type (3) C-concave in E and $F(x,\theta_E) = \{\theta_Y\}$;*
3. *for every $y \in E$, $F(\cdot,y)$ is C-upper continuous in X:*
4. *for every $x \in X$ such that $X \setminus \cup_{\lambda>0}\lambda(x - X) \neq \emptyset$, one has $\{\theta_Y\} \leq^{(3)}_{\mathrm{int}\, C} F(x,x)$.*

Then, there exists $\hat{x} \in X$ such that $\{\theta_Y\} \not\leq^{(3)}_{\mathrm{int}\, C} F(\hat{x},y)$ for all $y \in X$.

The conclusion has the form $\{\theta_Y\} \not\leq^{(3)}_{\mathrm{int}\, C} F(\hat{x},y)$, that is, the reverse form of assumption (4) in Corollary 4.4.

4.2. Observation for the cases (ii) and (iii)

On the other hand, the cases (ii) and (iii) don't give us naturally assumptions that reach each consequence. Specifically, in order to use Propositions 2.8 and 2.10, suitable four assumptions for the case (ii) (resp., (iii)), which correspond to assumptions (1), (2), (3) and (4) in Corollary 4.4, are expected to be compact-valued, type (5) (resp., type (3)) $(-C)$-concavity, C-lower (resp., C-upper) continuity, and $F(x,x) \not\leq_C^{(5)} \{\theta_Y\}$ (resp., $F(x,x) \not\leq_C^{(3)} \{\theta_Y\}$). However, we can indicate the following example which satisfies those conditions but doesn't deduce the consequence for the cases (ii) nor (iii).

Example 4.5. Let $X := [-1,1] \subset R$, $C := R^2_+$, $F : X \times X \to R^2$ defined as $F(x,y) := \{(y, 1-2^y)\}$ for every $x, y \in X$. Then, F is a compact-valued map, $F(x,\cdot)$ is type (j) $(-C)$-concave and $F(x,0) = \{(0,0)\}$ for every $x \in X$ and $j = 3,5$, $F(\cdot,y)$ is C-lower and C-upper continuous for every $y \in X$. $\{x \in X \mid X \setminus \cup_{\lambda>0}\lambda(x-X) \neq \emptyset\} = \{-1,1\}$, also we have $F(-1,-1) = \{(-1,\frac{1}{2})\} \not\leq_C^{(j)} \{(0,0)\}$ and $F(1,1) = \{(1,-1)\} \not\leq_C^{(j)} \{(0,0)\}$ for each $j = 3,5$. However, for every $x \in X$, there is $y \in X$ such that $F(x,y) \not\subset -C$, that is, there is no $\hat{x} \in X$ such that $F(\hat{x},y) \leq_C^{(j)} \{(0,0)\}$ for all $y \in X$ and $j = 3,5$.

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