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A NOTE ON THE EXTENSIONS OF THE INVERSION MAP TO THE ABSORBING ELEMENTS OF A SEMIGROUP

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Abstract. Given a non-trivial automorphism (resp. anti-automorphism) of a semigroup, we study its homomorphic (resp. anti-homomorphic) extensions to a larger semigroup by considering the images of the absorbing elements. Then we exhibit some examples to show the application of the results obtained.

1. Introduction

We are interested in homomorphic extensions of the inversion map to the "*absorbing elements*" of a semigroup. There is an easy result on this subject which can be described as follows.

Let $\mathbf{K}$ be the set of real numbers or the set of complex numbers. Then $\mathbf{K}$ is a semigroup with the ordinary multiplication. In this case, the inversion map ϕ defined by $\phi(t) = 1/t$ $(t \neq 0)$ has a unique homomorphic extension to $\mathbf{K}$. In fact, the map $\tilde{\phi}$ defined by

$$\tilde{\phi}(t) = \begin{cases} 1/t & (t \neq 0) \\ 0 & (t = 0) \end{cases}$$

is an automorphism of $\mathbf{K}$ which extends ϕ. Let ψ be any homomorphic extension of ϕ to $\mathbf{K}$. Then $\psi(0)$ must be an idempotent of $\mathbf{K}$, and hence $\psi(0) = 0$ or $\psi(0) = 1$. If $\psi(0) = 1$, then $2 = \psi(1/2)\psi(0) = \psi(0) = 1$, a contradiction. Therefore $\psi(0) = 0$ holds, so that $\psi = \tilde{\phi}$ as required.

In this note, we wish to extend such a result to a more general setting in order to characterize the homomorphic (resp. anti-homomorphic) extensions of a non-trivial automorphism (resp. anti-automorphism) of a semigroup to a larger semigroup in terms of the images of the absorbing elements (see Theorem 1 and Propositions 1

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and 2). Moreover, some illustrative examples (see Examples 1, 2 and 3) are given as application.

2. Results

Let S be a semigroup with binary operation $*$ and T a sub-semigroup of S. An automorphism (or anti-automorphism) φ of T is said to be trivial if $\varphi(x) = e$ for all $x \in T$ and for some idempotent $e \in T$. We denote by $Z_l(T,S)$ the set of the elements $s \in S$ such that $s * t = s$ for all $t \in T$. Similarly we denote by $Z_r(T,S)$ the set of the elements $s \in S$ such that $t * s = s$ for all $t \in T$. We also define

$$Z(T,S) = Z_l(T,S) \cup Z_r(T,S).$$

Then we have the following

Theorem 1. *Let ϕ be a non-trivial automorphism (resp. anti-automorphism) of T and ψ be any homomorphic (resp. anti-homomorphic) extension of ϕ to S. If T is cancellative and unital, then $\psi(z) \notin T$ for all $z \in Z(T,S)$.*

Proof. Suppose that T is cancellative and unital. If $Z(T,S)$ is empty, then the statement is clearly verified. So let $Z(T,S)$ be non-empty and assume that there exists $z_0 \in Z_l(T,S)$ such that $\psi(z_0) \in T$ (the proof for $z_0 \in Z_r(T,S)$ is similar and is accordingly omitted). We also assume that ψ is an homomorphic extension of ϕ (the proof for ψ anti-homomorphic is similar and accordingly omitted). Then we take $x \in T$ arbitrarily. Since

$$\psi(z_0) * \phi(x) = \psi(z_0) * \psi(x) = \psi(z_0 * x) = \psi(z_0),$$

it follows that

$$\psi(z_0) * \{\phi(x) * y\} = \{\psi(z_0) * \phi(x)\} * y = \psi(z_0) * y$$

holds for all $y \in T$. Since T is cancellative, we deduce that $\phi(x) * y = y$ for all $y \in T$. Accordingly $\phi(x)$ is a left identity element for T. Then $\phi(x)$ must be the identity element of T because T is unital. Consequently ϕ must be trivial. This contradicts the fact that ϕ is non-trivial, and the theorem is proved. $\square$

We may want to relax the conditions on T and take T to be only right-cancellative (or left-cancellative), then we have to be more specific in the assumption of the theorem and we can prove the following.

Proposition 1. (i) *Let ϕ be a non-trivial automorphism of T and ψ be any homomorphic extension of ϕ to S. If T is left (resp. right)-cancellative and has a unique left (resp. right) identity element, then $\psi(z) \notin T$ for all $z \in Z_l(T,S)$ (resp. $Z_r(T,S)$).*

(ii) *Let ϕ be a non-trivial anti-automorphism of T and ψ be any anti-homomorphic extension of ϕ to S. If T is left (resp. right)-cancellative and has a unique left (resp. right) identity element, then $\psi(z) \notin T$ for all $z \in Z_r(T,S)$ (resp. $Z_l(T,S)$).*

Proof. The proof is a straightforward adaptation of the argument used in the proof of Theorem 1 and it is accordingly omitted. □

In particular, under the assumptions of Proposition 1 we have $S \neq T$ whenever either $Z_l(T,S)$ or $Z_r(T,S)$ are non-empty (note that we may have $Z_l(T,S) \subseteq T$ or $Z_r(T,S) \subseteq T$). A consequence of Proposition 1 is presented in Proposition 2 here below. In the sequel we will denote by $Z_l(S)$ (resp. $Z_r(S)$) the set of the left (resp. right) absorbing elements of S.

Proposition 2. (i) *Let ϕ be a non-trivial automorphism of T and ψ be any homomorphic extension of ϕ to S. Assume that $S = T \cup Z_r(S)$ (resp. $= T \cup Z_l(S)$) and that $Z_l(T,S) = Z_l(S)$ (resp. $Z_r(T,S) = Z_r(S)$). If T is left (resp. right)-cancellative and has a unique left (resp. right) identity element, then there exists $z_0 \in Z_r(S)$ (resp. $Z_l(S)$) such that $\psi(z) = z_0$ for all $z \in Z_l(S)$ (resp. $Z_r(S)$).*

(ii) *Let ϕ be a non-trivial anti-automorphism of T and ψ be any anti-homomorphic extension of ϕ to S. Assume that $S = T \cup Z_r(S)$ (resp. $= T \cup Z_l(S)$) and that $Z_r(T,S) = Z_r(S)$ (resp. $Z_l(T,S) = Z_l(S)$). If T is left (resp. right)-cancellative and has a unique left (resp. right) identity element, then there exists $z_0 \in Z_r(S)$ (resp. $Z_l(S)$) such that*

(ii-1) *$\psi(z) = z_0$ for all $z \in Z_r(S)$ (resp. $Z_l(S)$);*
(ii-2) *$z_0 * \psi(s) = z_0$ (resp. $\psi(s) * z_0 = z_0$) for all $s \in S$.*

Proof. (i) Assume that $S = T \cup Z_r(S)$ and $Z_l(T,S) = Z_l(S)$ and that T is left-cancellative and has a unique left identity element. We show that $\psi(z_1) = \psi(z_2)$ for all $z_1, z_2 \in Z_l(S)$. Indeed, take $z_1, z_2 \in Z_l(S)$ arbitrarily. By Proposition 1-(i) and our assumptions we must have $\psi(z_1) \in Z_r(S)$. Then $\psi(z_1) = \psi(z_2) * \psi(z_1) = \psi(z_2 * z_1) = \psi(z_2)$ as required. Accordingly, there exists $z_0 \in Z_r(S)$ such that $\psi(z) = z_0$ for all $z \in Z_l(S)$. The proof of the other case is similar and accordingly omitted.

(ii) Assume that $S = T \cup Z_r(S)$ and $Z_r(T,S) = Z_r(S)$ and that T is left-cancellative and has a unique left identity element. To prove (ii-1) we show that $\psi(z_1) = \psi(z_2)$ for all $z_1, z_2 \in Z_r(S)$. Indeed, take $z_1, z_2 \in Z_r(S)$ arbitrarily. By Proposition 1-(ii) and our assumptions we must have $\psi(z_1) \in Z_r(S)$. Then $\psi(z_1) = \psi(z_2) * \psi(z_1) = \psi(z_1 * z_2) = \psi(z_2)$ as required. Accordingly, there exists $z_0 \in Z_r(S)$ such that $\psi(z) = z_0$ for all $z \in Z_r(S)$. The proof of the other case is similar and accordingly omitted.

To show (ii-2), let $s \in S$. Since $z_0 \in Z_r(S)$ and $\psi(z_0) = z_0$, it follows that $z_0 * \psi(s) = \psi(z_0) * \psi(s) = \psi(s * z_0) = \psi(z_0) = z_0$, as required. The proof of the other case is similar and accordingly omitted. $\square$

3. Examples

The first example is obtained with the help of Theorem 1. In this example, we give a complete description of every homomorphic extension of the inversion map of the ordinary multiplicative group of positive numbers to the semigroup obtained by adding three points to it.

The second example is obtained with the help of Proposition 1. In this example we give a complete description of every homomorphic extension of an automorphism of the injection semigroup on a given set to the semigroup obtained by adding the constant maps to it.

The third example is obtained with the help of Proposition 2. In this example we provide a complete description of every anti-homomorphic extension of an anti-automorphism of the surjection semigroup on a given set to the semigroup obtained by adding the constant maps to it.

Finally, we have included a remark, based of the third example, where we show that there exist no anti-homomorphic extension of an anti-isomorphism of the bijection group on a given set to the semigroup obtained by adding the constant maps to it. In particular, there exist no anti-homomorphic extension of the inversion map of the bijection group to the semigroup obtained by adding the constant maps to it.

Example 1. Let $T = (0, +\infty)$ be the ordinary multiplicative group of positive numbers and put $S = (0, +\infty) \cup \{0, +\infty, \omega\}$, where $+\infty$ and ω are symbols. We equip S with a semigroup structure by means of the binary operation $*$ which coincides with the usual product of real numbers on $[0, +\infty) \times [0, +\infty)$ and such that

$$s * +\infty = +\infty * s = +\infty \ (0 < \forall s < +\infty),$$

$$+\infty * +\infty = +\infty, \ 0 * +\infty = +\infty * 0 = \omega,$$

and

$$s * \omega = \omega * s = \omega \ (\forall s \in S).$$

Then T is a sub-semigroup of S. We denote by ϕ the inversion map from T to itself which takes t to $\phi(t) = 1/t$. We wish to characterize homomorphic extension of ϕ to S.

As a first step we observe that $Z(T, S) = \{0, +\infty, \omega\}$. Then, by Theorem 1, we deduce that every extension ψ of ϕ maps $\{0, +\infty, \omega\}$ to itself. In addition, it must be

$$\psi(0)*\psi(+\infty)=\psi(+\infty)*\psi(0)=\psi(\omega)\,,$$
$$\psi(0)*\psi(\omega)=\psi(\omega)*\psi(0)=\psi(\omega)\,,$$
$$\psi(\omega)*\psi(+\infty)=\psi(+\infty)*\psi(\omega)=\psi(\omega)\,.$$

Accordingly, we find that every extension ψ coincides with one of the maps $\psi_1, \dots, \psi_9$ defined by the following conditions: $\psi_i|_T=\phi$ for all $i\in\{1,\dots,9\}$, and

$$\begin{array}{lll}
\psi_1(0)=0\,, & \psi_1(+\infty)=0\,, & \psi_1(\omega)=0\,,\\
\psi_2(0)=0\,, & \psi_2(+\infty)=+\infty\,, & \psi_2(\omega)=\omega\,,\\
\psi_3(0)=0\,, & \psi_3(+\infty)=\omega\,, & \psi_3(\omega)=\omega\,,\\
\psi_4(0)=+\infty\,, & \psi_4(+\infty)=0\,, & \psi_4(\omega)=\omega\,,\\
\psi_5(0)=+\infty\,, & \psi_5(+\infty)=+\infty\,, & \psi_5(\omega)=+\infty\,,\\
\psi_6(0)=+\infty\,, & \psi_6(+\infty)=\omega\,, & \psi_6(\omega)=\omega\,,\\
\psi_7(0)=\omega\,, & \psi_7(+\infty)=0\,, & \psi_7(\omega)=\omega\,,\\
\psi_8(0)=\omega\,, & \psi_8(+\infty)=+\infty\,, & \psi_8(\omega)=\omega\,,\\
\psi_9(0)=\omega\,, & \psi_9(+\infty)=\omega\,, & \psi_9(\omega)=\omega\,.
\end{array}$$

Example 2. Let X be an arbitrary set with ${}^\sharp X\geq 2$. Let T denote the set of the injective maps from X to itself. For all $x\in X$ let z_x denote the map from X to itself defined by $z_x(y)=x$ for all $y\in X$. Let $Z=\{z_x\,:\,x\in X\}$ be the set of such maps z_x. Then $T\cap Z=\varnothing$ because ${}^\sharp X\geq 2$. Put $S=T\cup Z$. We endow S with a semigroup structure by means of the binary operation $\circ$ which takes a pair of functions (h,g) to the functional composition $h\circ g$ (defined by $(h\circ g)(x)=h(g(x))$ for all $x\in X$). Then $(T,\circ)$ is a left-cancellative unital semigroup. Also we have

$$Z=Z_l(S)=Z_l(T,S)\,.$$

In fact, $Z\subseteq Z_l(S)\subseteq Z_l(T,S)$ will be evident. If $f\in Z_l(T,S)\cap T$, then $f\circ f=f$, so f is the identity map of X (because T is left-cancellative). Hence T is composed only by the identity map, which contradicts ${}^\sharp X\geq 2$. As consequence, $Z_l(T,S)\cap T$ is empty and $Z_l(T,S)\subseteq Z$. Then we obtain the desired equalities.

Let now ϕ be an automorphism from T to itself and put

$$C_\phi=\{f:X\to X\,|\,f\circ h=\phi(h)\circ f\ \ (\forall h\in T)\}.$$

We first assume that ϕ is non-trivial. By Proposition 1-(i) we can verify that if C_ϕ is empty, then there exist no homomorphic extension of ϕ to S, and that if C_ϕ is non-empty, then any homomorphic extension ψ of ϕ to S can be expressed as

$$\psi(h)=\begin{cases}\phi(h) & (h\in T)\\ z_{f(x)} & (h=z_x\in Z)\end{cases}\tag{1}$$

for a unique element $f \in C_\phi$. Indeed, let ψ be a homomorphic extension of ϕ to S. By Proposition 1-(i), we have $\psi(Z) \subseteq Z$, and then for an arbitrary element $x \in X$, we can find a unique element $f(x)$ in X such that $\psi(z_x) = z_{f(x)}$. Therefore we have

$$z_{f(h(x))} = \psi(z_{h(x)}) = \psi(h \circ z_x) = \psi(h) \circ \psi(z_x) = \phi(h) \circ z_{f(x)} = z_{\phi(h)(f(x))},$$

hence $f(h(x)) = \phi(h)(f(x))$ for all $x \in X$ and $h \in T$. In other words, $f \in C_\phi$. Moreover, we can easily see that the map from S to itself defined by (1) is a homomorphic extension of ϕ to S when $f \in C_\phi$. Therefore the desired result follows immediately from the above observation.

We next assume that ϕ is trivial. Then we can verify that any homomorphic extension ψ of ϕ to S can be expressed as either

$$\psi(h) = id_X \ (h \in S), \tag{2}$$

where id_X is the identity map from X to itself, or

$$\psi(h) = \begin{cases} id_X & (h \in T) \\ z_a & (h \in Z) \end{cases} \tag{3}$$

for a unique element $a \in X$. Indeed, $\phi(h) = id_X$ holds for all $h \in T$ because id_X is a unique idempotent of T. We first consider the case of $\psi(Z) \nsubseteq Z$. We can choose an element $z_a \in Z$ with $\psi(z_a) \in T$. Then $\psi(z_a) \circ \psi(z_a) = \psi(z_a)$ and hence $\psi(z_a) = id_X$. Now take $z_x \in Z$ arbitrarily and choose an element $h \in T$ with $h(a) = x$. If there exists $y \in X$ with $\psi(z_x) = z_y$, then

$$id_X = \phi(h) = \psi(h) \circ \psi(z_a) = \psi(h \circ z_a) = \psi(z_{h(a)}) = \psi(z_x) = z_y,$$

a contradiction because $\sharp X \geq 2$. Therefore we conclude that $\psi(Z) \subseteq T$. Consequently $\psi(z) = id_X$ for all $z \in Z$ as observed above. In other words, ψ is expressed as (2). We next consider the case when $\psi(Z) \subseteq Z$. In this case, there exists a map $f : X \to X$ such that $\psi(z_x) = z_{f(x)}$ for all $x \in X$. Then

$$\begin{aligned} z_{(f \circ h)(x)} &= \psi(z_{h(x)}) = \psi(h \circ z_x) = \psi(h) \circ \psi(z_x) \\ &= id_X \circ \psi(z_x) = \psi(z_x) = z_{f(x)} \end{aligned}$$

holds for all $x \in X$ and $h \in T$. This implies that $f \circ h = f$ for all $h \in T$, hence $f \in C_\phi$. Note that $C_\phi = Z$. In fact, it is evident that $Z \subseteq C_\phi$. To show the converse inclusion, let $f \in C_\phi$. For any $x, y \in X$, choose $h \in T$ with $h(x) = y$. Then $f(x) = f \circ h(x) = f(h(x)) = f(y)$. In other words, f is a constant function, so $f \in Z$ as required. These observations imply that ψ is expressed as (3). Of course, it is clear that ψ expressed by (3) is a homomorphic extension of ϕ to S.

Example 3. Let X be an arbitrary set with $\sharp X \geq 2$. Let T be the set of the surjective maps from X to itself. Let $Z = \{z_x : x \in X\}$ be as in Example 2 and let $S = T \cup Z$. We endow S with a semigroup structure by means of the binary operation $\circ$ as in Example 2. Then $(T, \circ)$ is a right-cancellative unital semigroup such that $S = T \cup Z_l(S)$ and $Z = Z_l(T, S) = Z_l(S)$ as observed in Example 2. Now let ϕ be an anti-automorphism from T to itself and put

$$\mathrm{Fix}(\phi(T)) = \{a \in X : \phi(h)(a) = a \ (\forall h \in T)\}.$$

We first assume that ϕ is non-trivial. By Proposition 2-(ii) we can verify that if $\mathrm{Fix}(\phi(T))$ is empty, then there exist no anti-homomorphic extension of ϕ to S, and that if $\mathrm{Fix}(\phi(T))$ is non-empty, then any anti-homomorphic extension ψ of ϕ to S can be expressed as

$$\psi(h) = \begin{cases} \phi(h) & (h \in T) \\ z_a & (h \in Z) \end{cases} \tag{4}$$

for a unique element $a \in \mathrm{Fix}(\phi(T))$. Indeed, let ψ be an anti-homomorphic extension of ϕ to S. By Proposition 2-(ii) and by the definition of Z, there exists $a \in X$ such that $\psi(z) = z_a$ for all $z \in Z$ and $\psi(h) \circ z_a = z_a$ for all $h \in S$. It follows that $\phi(h)(a) = a$ for all $h \in T$ and thus $\mathrm{Fix}(\phi(\mathrm{T}))$ is not empty. Moreover, we can easily see that the map from S to itself defined by (4) is an anti-homomorphic extension of ϕ to S when $a \in \mathrm{Fix}(\phi(T))$. Therefore the desired result follows immediately from the above observation.

We next assume that ϕ is trivial. Then $\phi(h) = id_X$ holds for all $h \in T$ because id_X is a unique idempotent of T. In addition, any anti-homomorphic extension ψ of ϕ to S can be expressed either as (2) or as (3) (the proof follows the footprints of the argument used in Example 2 and it is accordingly omitted). We also note that in this case $\mathrm{Fix}(\phi(\mathrm{T})) = \mathrm{X}$ and thus (3) can be seen as a special case of (4).

Remark. We observe that the statement proved in Example 3 remains valid if we replace the set T of all the surjective maps from X to itself by another set, which we still denote by T and which is closed under the composition operator $\circ$ and consists of some (but not necessarily all) surjective maps from X to itself. In particular, if we denote by T the set of all the bijective maps from X to itself and we take an anti-isomorphism ϕ from T to itself, then we can see that the corresponding set $\mathrm{Fix}(\phi(T))$ is empty (since $\phi(T) = T$ and X contains at least two elements a and b we can always find a bijection h from X to X such that $\phi(h)(a) = b \neq a$). Accordingly there exist no anti-homomorphic extension of ϕ to S.

In particular, there exist no anti-homomorphic extension to S of the inversion map from T to itself which takes a function h to its inverse function h^{-1}.

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