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CONTROLLABILITY OF NONLINEAR IMPULSIVE SECOND ORDER INTEGRODIFFERENTIAL EVOLUTION SYSTEMS IN BANACH SPACES

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Abstract. This paper deals with the controllability of impulsive second order integrodifferential systems in Banach spaces. Sufficient conditions for the controllability are derived with the help of the fixed point theorem due to Sadovskii and the theory of strongly continuous cosine family of operators. An example is provided to show the effectiveness of the proposed results. Further, we study the controllability of second order integrodifferential evolution systems with impulses by using the Schaefer fixed-point theorem.

1. Introduction

In various real-world applications, there is a necessity given to steer processes in time. More and more it becomes acknowledged in science and engineering, that these processes exhibit discontinuities. Our paper on theory of control and on theory of dynamical systems gives a contribution to this natural or technical fact. One of the fundamental concepts in mathematical control theory is controllability which plays an important role in deterministic control theory and engineering because they have close connections to pole assignment, structural decomposition, quadratic optimal control, observer design and many other physical phenomena [2]. This concept leads to some very important conclusions regarding the behavior of linear and nonlinear dynamical systems. Most of the practical systems are nonlinear in nature and hence the study of nonlinear systems is important.

On the other hand, many systems are characterized by abrupt changes at certain moments due to instantaneous perturbations, which lead to impulsive effects. Such behavior is seen in a range of problems from: mechanics; chemotherapy; optimal control; ecology; industrial robotics; biotechnology; spread of disease; harvesting; medical models. The reader is referred to [8, 12, 22, 29] and references therein

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for some models and applications to the above areas. Impulsive dynamical systems exhibit the continuous evolutions of the states typically described by ordinary differential equations coupled with instantaneous state jumps or impulses. The presence of impulses implies that the trajectories of the system do not necessarily preserve the basic properties of the non-impulsive dynamical systems. To this end the theory of impulsive differential systems has emerged as an important area of investigation in applied sciences [14, 30]. It is well-known that the notation of “aftereffect” introduced in physics is very important. To model processes with aftereffect it is not sufficient to employ ordinary or partial differential equations. An approach to resolve this problem is to use integrodifferential equations. Integrodifferential equations arise in many engineering and scientific disciplines, often as approximations to partial differential equations, which represent much of the continuum phenomena. Many forms of these equations are possible. Some of the applications are unsteady aerodynamics and aeroelastic phenomena, viscoelastic panel in supersonic gas flow, fluid dynamics, electrodynamics of complex media, many models of population growth, polymer rheology, neural network modeling, sandwich system identification, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, theory of lossless transmission lines, theory of population dynamics, compartmental systems, nuclear reactors and mathematical modeling of hereditary phenomena. The theory of impulsive integrodifferential equations in the field of modern applied mathematics has made considerable headway in recent years, because the structure of its emergence has deep physical background and realistic mathematical models.

Impulsive control systems have been studied by several authors [9, 24, 27, 28]. In [24] the problem of controlling a physical object through impacts is studied, called impulsive manipulation, which arises in a number of robotic applications. In [27] the authors investigated the optimal harvesting policy for an ecosystem with impulsive harvest. For some recent references on different control strategies, including impulsive control, we refer the reader to [1, 6, 10, 16, 18] and the references therein.

Controllability problems for different types of nonlinear systems have been considered in many publications and monographs. The extensive list of these publications can be found, for example, in the papers [3, 4, 5, 15, 20]. The study of dynamical systems with impulsive effects has been an object of intensive investigations [7, 13, 17, 21]. Li et al.[13], using the Schaefer fixed point theorem, studied the controllability of impulsive functional differential systems in Banach spaces. In [21], sufficient conditions were formulated for the exact controllability of second-order nonlinear impulsive control systems. This paper is devoted to extending controllability results to impulsive second-order evolution systems. To be precise, in [5], the authors used Schaefer’s fixed point theorem to establish controllability results for second-order integrodifferential evolution systems in Banach spaces. Some papers on deterministic controllability problems contain a strict compactness assumption on the semigroup and cosine function, in this case the application of controllability results are restricted to finite dimensional space. Here, we obtain controllability results for impulsive second order integrodifferential systems with a noncompact condition on the cosine family of operators. Also, we establish the controllability conditions for integrodifferential evolution systems with impulsive conditions. However, the corresponding theory of impulsive integrodifferential equations in abstract spaces is still in its developing stage and many aspects of the theory remain to be addressed. To our best knowledge, there is no work reported on the controllability of nonlinear impulsive second order integrodifferential evolution systems in Banach spaces. To close the gap in this paper, we study this interesting problem.

2. Second Order Impulsive Delay Integrodifferential Systems

Before stating and proving the main result, we first introduce notations, definitions and preliminary facts which are used throughout this section. A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ defined on a Banach space X endowed with a norm $\|\cdot\|$. We denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated with $(C(t))_{t\in\mathbb{R}}$ which is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}.$$

Moreover, M and N are positive constants such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$, for every $t \in J$.

The notation $[D(A)]$ is the space $D(A) = \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$ endowed with the norm $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$.

Define $E = \{x \in X : C(t)x$ is once continuously differentiable in $t\}$ endowed with the norm $\|x\|_E = \|x\| + \sup\limits_{0\leq t\leq 1} \|AS(t)x\|$, $x \in E$. Then E is a Banach space. The operator-valued function

$$\mathcal{G}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this, it follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$, $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is a locally integrable function, then $y(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function which is a consequence of the fact that

$$\int_0^t \mathcal{G}(t-s)\begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \Big[\int_0^t S(t-s)x(s)ds, \int_0^t C(t-s)x(s)ds\Big]^T$$

defines an $(E \times X)$-valued continuous function.

The existence of solutions for the second order abstract Cauchy problem

$$x''(t) = Ax(t) + g(t), \qquad 0 \le t \le b, \tag{2.1}$$

$$x(0) = v, \quad x'(0) = w, \tag{2.2}$$

where $g : [0, b] \to X$ is an integrable function, has been discussed in [25]. Similarly the existence of solutions of semilinear second order abstract Cauchy problems has been treated in [26]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)v + S(t)w + \int_0^t S(t-s)g(s)ds, \quad 0 \le t \le b,$$

is called a mild solution of (2.1) – (2.2) and that when $v \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)v + C(t)w + \int_0^t C(t-s)g(s)ds \quad 0 \le t \le b.$$

The following properties are well known [25]:

(i) if $x \in X$ then $S(t)x \in E$ for every $t \in \mathbb{R}$.

(ii) if $x \in E$ then $S(t)x \in D(A), (d/dt)C(t)x = AS(t)x$ and $(d^2/dt^2)S(t)x = AS(t)x$ for every $t \in \mathbb{R}$.

(iii) if $x \in D(A)$ then $C(t)x \in D(A)$, and $(d^2/dt^2)C(t)x = AC(t)x = C(t)Ax$ for every $t \in \mathbb{R}$.

(iv) if $x \in D(A)$ then $S(t)x \in D(A)$, and $(d^2/dt^2)S(t)x = AS(t)x = S(t)Ax$ for every $t \in \mathbb{R}$.

To consider the impulsive conditions, it is convenient to introduce some additional concepts and notations.

Denote $J_0 = [0, t_1], J_k = (t_k, t_{k+1}], k = 1, 2, \ldots, m$. Let $I \subset \mathbb{R}$ be an interval. We define the following classes of functions :

$PC(I, X) = \{x : I \to X : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k),\ k = 1, 2, \ldots, m\}$.

For $x \in PC(I, X)$, take $\|x\|_{PC} = \sup_{t \in I} \|x(t)\|$, then $PC(I, X)$ is a Banach space.

Let $(Z, \|\cdot\|_Z)$, $(W, \|\cdot\|_W)$ be Banach spaces. The notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W endowed with the uniform operator norm denoted by $\|\cdot\|_{\mathcal{L}(Z,W)}$, and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. Moreover, $B_r(x : Z)$ denotes the closed ball with center at x and radius

$r > 0$ in Z and we write simply B_r when no confusion arises.

The following lemma is crucial in the proof of our main result.

Lemma 2.1 ([19]: Sadovskii's Fixed Point Theorem). *Let F be a condensing operator on a Banach space X. If $F(S) \subset S$ for a convex, closed and bounded set S of X, then F has a fixed point in S.*

This section is concerned with the study of controllability of delay integrodifferential system with impulsive conditions described in the form

$$\begin{aligned} x''(t) &= Ax(t) + Bu(t) + f\Big(t, x_t, \int_0^t a(t,s,x_s)ds\Big), \\ & \qquad t \in J = [0,b],\ t \neq t_k,\ k = 1,2,\ldots,m, && (2.3) \\ x_0 &= \phi \text{ on } [-r,0], \quad x'(0) = \eta \in X, && (2.4) \\ \triangle x(t_k) &= I_k(x(t_k^-)),\ k = 1,2,\ldots,m, && (2.5) \\ \triangle x'(t_k) &= J_k(x(t_k^-)),\ k = 1,2,\ldots,m, && (2.6) \end{aligned}$$

where A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ defined on a Banach space X. The control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space and $B : U \to X$ as a bounded linear operator; $a : J \times J \times D \to X,\ f : J \times D \times X \to X,\ I_k : X \to X,\ J_k : X \to X\ (k = 1,2,\ldots,m),\ \triangle\xi(t)$ represents the jump of a function $\xi(\cdot)$ at t, which is defined by $\triangle\xi(t) = \xi(t^+) - \xi(t^-)$. $D = \{\varphi : [-r,0] \to X,\ \varphi(t)$ is continuous everywhere except a finite number of points $\tilde{t}$ at which $\varphi(\tilde{t}^-),\ \varphi(\tilde{t}^+)$ exist and $\varphi(\tilde{t}^-) = \varphi(\tilde{t})\}$. $0 < t_1 < t_2 < \ldots < t_m < b,\ \phi : [-r,0] \to X$. For any continuous function x defined on $[-r,b] \setminus \{t_1,\ldots,t_m\}$ and any $t \in J$, we denote by x_t the element of D defined by $x_t(\theta) = x(t+\theta), -r \leq \theta \leq 0$. Here $x_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t.

Definition 2.1. A solution $x(\cdot) \in PC([-r,b],X)$ is said to be a mild solution of the abstract Cauchy problem $(2.3)-(2.6)$, if $x_0 = \phi$ on $[-r,0]$, the impulsive conditions $\triangle x(t_k) = I_k(x(t_k^-)),\ \triangle x'(t_k) = J_k(x(t_k^-)),\ k = 1,2,\ldots,m$, are satisfied and the following integral equation is verified :

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)\left[Bu(s) + f\Big(s, x_s, \int_0^s a(s,\tau,x_\tau)d\tau\Big)\right] ds \\ &\quad + \sum_{0<t_k<t} C(t-t_k)I_k(x(t_k^-)) + \sum_{0<t_k<t} S(t-t_k)J_k(x(t_k^-)), \quad t \in J. \end{aligned}$$

Definition 2.2. The system $(2.3)-(2.6)$ is said to be controllable on the interval J, if for every $x_0 = \phi \in PC([-r,0],X), x'(0) = \eta$ and $z_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $x(\cdot)$ of $(2.3)-(2.6)$ satisfies $x(b) = z_1$.

In order to establish the controllability result, we introduce the following technical hypothesis:

(H1) The function $f : J \times D \times X \to X$ satisfies the following conditions :

(i) $a(t, s, \cdot) : D \to X$ is continuous for each $t, s \in J$ and the function $a(\cdot, \cdot, x) : J \times J \to X$ is strongly measurable for each $x \in D$.

(ii) $f(t, \cdot, \cdot) : D \times X \to X$ is continuous for each $t \in J$ and the function $f(\cdot, x, y) : J \to X$ is strongly measurable for each $(x, y) \in D \times X$.

(iii) For every positive constant r, there exists $\alpha_r \in L^1(J)$ such that

$$\sup_{\|x\|,\|y\|\leq r} \|f(t, x, y)\| \leq \alpha_r(t), \text{ for a.e. } t \in J.$$

(iv) There exists an integrable function $n : J \to [0, \infty)$ such that

$$\Big\| \int_0^t a(t, s, \phi)ds \Big\| \leq n(t)\psi_2(\|\phi\|_{PC}), \quad \liminf_{\xi\to\infty} \frac{\psi_2(\xi)}{\xi} = \tilde{\Lambda} < \infty,$$

for almost all $t \in J, \phi \in PC([-r, 0], X)$, where $\psi_2 : [0, \infty) \to (0, \infty)$ is a continuous non-decreasing function.

(v) There exists an integrable function $m : J \to [0, \infty)$ such that

$$\|f(t, \phi, x)\| \leq m(t)\psi_1(\|\phi\|_{PC}) + \|x\|, \quad \liminf_{\xi\to\infty} \frac{\psi_1(\xi)}{\xi} = \Lambda < \infty,$$

for almost all $t \in J, \phi \in PC([-r, 0], X)$, where $\psi_1 : [0, \infty) \to (0, \infty)$ is a continuous non-decreasing function.

(vi) For each $t \in J$, the function $f(t, \cdot, \cdot) : D \times X \to X$ is completely continuous.

(H2) B is a continuous operator from U to X and the linear operator $W : L^2(J, U) \to X$, defined by

$$Wu = \int_0^b S(b - s)Bu(s)ds,$$

has a bounded invertible operator W^{-1} which takes values in $L^2(J, U)/kerW$ and there exists a positive constant M_1 such that $\|BW^{-1}\| \leq M_1$.

(H3) The impulsive functions satisfy the following conditions:

(i) The maps $I_k, J_k : X \to X, k = 1, 2, \dots, m$ are completely continuous and there exist continuous non-decreasing functions $\mu_k, \sigma_k : [0, \infty) \to (0, \infty), k = 1, 2, \dots, m$, such that

$$\|I_k(x)\| \leq \mu_k(\|x\|), \quad \|J_k(x)\| \leq \sigma_k(\|x\|), \quad x \in X.$$

(ii) There are positive constants L_1, L_2 such that

$$\begin{aligned} \|I_k(x_1) - I_k(x_2)\| &\leq L_1\|x_1 - x_2\|, \\ \|J_k(x_1) - J_k(x_2)\| &\leq L_2\|x_1 - x_2\|, \quad x_1, x_2 \in X, \quad k = 1, 2, \dots, m. \end{aligned}$$

Theorem 2.1. *Suppose that* $(H1)-(H3)$ *are satisfied. Then the system* $(2.3)-(2.6)$ *is controllable on* J *provided that*

$$(1+bNM_1)\left[N\Lambda\int_0^b m(s)ds + N\tilde{\Lambda}\int_0^b n(s)ds + \sum_{k=1}^m (ML_1+NL_2)\right] < 1.$$

Proof. Using the assumption $(H2)$, for an arbitrary function $x(\cdot)$, we define the control

$$\begin{aligned} u(t) &= W^{-1}\Bigg[z_1 - C(b)\phi(0) - S(b)\eta - \int_0^b S(b-s)f\Big(s, x_s, \int_0^s a(s,\tau,x_\tau)d\tau\Big)ds \\ &\quad - \sum_{k=1}^m C(b-t_k)I_k(x(t_k^-)) - \sum_{k=1}^m S(b-t_k)J_k(x(t_k^-))\Bigg](t). \end{aligned}$$

We shall now show that when using this control the operator Φ defined by

$$\begin{aligned} \Phi x(t) &= C(t)\phi(0) + S(t)\eta + \int_0^t S(t-\xi)BW^{-1}\Bigg[z_1 - C(b)\phi(0) - S(b)\eta \\ &\quad - \int_0^b S(b-s)f\Big(s, x_s, \int_0^s a(s,\tau,x_\tau)d\tau\Big)ds - \sum_{k=1}^m C(b-t_k)I_k(x(t_k^-)) \\ &\quad - \sum_{k=1}^m S(b-t_k)J_k(x(t_k^-))\Bigg](\xi)d\xi + \int_0^t S(t-s)f\Big(s, x_s, \int_0^s a(s,\tau,x_\tau)d\tau\Big)ds \\ &\quad + \sum_{0<t_k<t} C(t-t_k)I_k(x(t_k^-)) + \sum_{0<t_k<t} S(t-t_k)J_k(x(t_k^-)), \quad t \in J, \end{aligned}$$

has a fixed point. This fixed point is then a mild solution of the system $(2.3)-(2.6)$. Clearly $(\Phi x)(b) = z_1$ which means that the control u steers the system from the initial function ϕ to z_1 in time b, provided we can obtain a fixed point of the operator Φ which implies that the system is controllable.

For $\phi \in PC([-r,0],X)$, we define $\hat{\phi} \in PC([-r,b],X)$ by

$$\hat{\phi}\,(t) = \begin{cases} C(t)\phi(0), & 0 \le t \le b, \\ \phi(t), & -r \le t \le 0. \end{cases}$$

If $x(t) = y(t) + \hat{\phi}(t), t \in [-r,b]$, it is easy to see that y satisfied $y_0 = 0$ and

$$\begin{aligned} y(t) &= S(t)\eta + \int_0^t S(t-\xi)BW^{-1}\Bigg[z_1 - C(b)\phi(0) - S(b)\eta \\ &\quad - \int_0^b S(b-s)f\Big(s, y_s + \hat{\phi}_s, \int_0^s a(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau\Big)ds \end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=1}^{m} C(b-t_k) I_k(y(t_k^-) + \hat{\phi}(t_k^-)) - \sum_{k=1}^{m} S(b-t_k) J_k(y(t_k^-) + \hat{\phi}(t_k^-)) \Big] (\xi) d\xi \\
&+ \int_0^t S(t-s) f\Big(s, y_s + \hat{\phi}_s, \int_0^s a(s,\tau, y_\tau + \hat{\phi}_\tau) d\tau\Big) ds \\
&+ \sum_{0<t_k<t} C(t-t_k) I_k(y(t_k^-) + \hat{\phi}(t_k^-)) + \sum_{0<t_k<t} S(t-t_k) J_k(y(t_k^-) + \hat{\phi}(t_k^-))
\end{aligned}
$$

if and only if x satisfies

$$
\begin{aligned}
x(t) \;=\; & C(t)\phi(0) + S(t)\eta + \int_0^t S(t-\xi) B W^{-1} \Big[z_1 - C(b)\phi(0) - S(b)\eta \\
& - \int_0^b S(b-s) f\Big(s, x_s, \int_0^s a(s,\tau,x_\tau) d\tau\Big) ds - \sum_{k=1}^{m} C(b-t_k) I_k(x(t_k^-)) \\
& - \sum_{k=1}^{m} S(b-t_k) J_k(x(t_k^-)) \Big] (\xi) d\xi + \int_0^t S(t-s) f\Big(s, x_s, \int_0^s a(s,\tau,x_\tau) d\tau\Big) ds \\
& + \sum_{0<t_k<t} C(t-t_k) I_k(x(t_k^-)) + \sum_{0<t_k<t} S(t-t_k) J_k(x(t_k^-))
\end{aligned}
$$

and $x(t) = \phi(t),\ t \in [-r, 0]$.

Define $PC_0 = \{y \in PC([-r,b],X) : y_0 = 0\}$ and $\Psi : PC_0 \to PC_0$ by $(\Psi y)(t) = 0,\ -r \le t \le 0$ and

$$
\begin{aligned}
(\Psi y)(t) = & S(t)\eta + \int_0^t S(t-\xi) B W^{-1} \Big[z_1 - C(b)\phi(0) - S(b)\eta - \int_0^b S(b-s) \\
& \times f\Big(s, y_s + \hat{\phi}_s, \int_0^s a(s,\tau, y_\tau + \hat{\phi}_\tau) d\tau\Big) ds - \sum_{k=1}^{m} C(b-t_k) I_k(y(t_k^-) + \hat{\phi}(t_k^-)) \\
& - \sum_{k=1}^{m} S(b-t_k) J_k(y(t_k^-) + \hat{\phi}(t_k^-)) \Big] (\xi) d\xi + \int_0^t S(t-s) \\
& \times f\Big(s, y_s + \hat{\phi}_s, \int_0^s a(s,\tau, y_\tau + \hat{\phi}_\tau) d\tau\Big) ds + \sum_{0<t_k<t} C(t-t_k) I_k(y(t_k^-) + \hat{\phi}(t_k^-)) \\
& + \sum_{0<t_k<t} S(t-t_k) J_k(y(t_k^-) + \hat{\phi}(t_k^-)),\ t \in J.
\end{aligned}
$$

Obviously, the operator Φ has a fixed point if and only if Ψ has a fixed point. So we have to prove that Ψ has a fixed point.

Now we claim that there exists $r > 0$ such that $\Psi(B_r(0, PC_0)) \subseteq B_r(0, PC_0)$. If we assume that this assertion is false, then for each $r > 0$, we can choose $x^r \in B_r(0, PC_0)$ such that $\|(\Psi y^r)(t^r)\| > r$.

$$
\begin{aligned}
r &< \|(\Psi y^r)(t^r)\| \\
&\le N\|\eta\| + NM_1\int_0^{t^r}\Bigg[\|z_1\| + M\|\phi(0)\| + N\|\eta\| + N\int_0^b \Big[m(s)\psi_1(\|y_s^r + \hat{\phi}_s\|) \\
&\quad + n(s)\psi_2(\|y_s^r + \hat{\phi}_s\|)\Big]ds + M\sum_{k=1}^m \Big[\|I_k(y^r(t_k^-) + \hat{\phi}(t_k^-)) - I_k(\hat{\phi}(t_k^-))\| \\
&\quad + \|I_k(\hat{\phi}(t_k^-))\|\Big] + N\sum_{k=1}^m \Big[\|J_k(y^r(t_k^-) + \hat{\phi}(t_k^-)) - J_k(\hat{\phi}(t_k^-))\| \\
&\quad + \|J_k(\hat{\phi}(t_k^-))\|\Big]\Bigg]d\xi + N\int_0^{t^r}\Big[m(s)\psi_1(\|y_s^r + \hat{\phi}_s\|) + n(s)\psi_2(\|y_s^r + \hat{\phi}_s\|)\Big]ds \\
&\quad + M\sum_{k=1}^m \Big[\|I_k(y^r(t_k^-) + \hat{\phi}(t_k^-)) - I_k(\hat{\phi}(t_k^-))\| + \|I_k(\hat{\phi}(t_k^-))\|\Big] \\
&\quad + N\sum_{k=1}^m \Big[\|J_k(y^r(t_k^-) + \hat{\phi}(t_k^-)) - J_k(\hat{\phi}(t_k^-))\| + \|J_k(\hat{\phi}(t_k^-))\|\Big] \\
&\le N\|\eta\| + bNM_1\Bigg[\|z_1\| + M\|\phi(0)\| + N\|\eta\| + N\int_0^b \Big[m(s)\psi_1(r + \|\hat{\phi}_s\|) \\
&\quad + n(s)\psi_2(r + \|\hat{\phi}_s\|)\Big]ds + M\sum_{k=1}^m \big[L_1 r + \|I_k(\hat{\phi}(t_k^-))\|\big] + N\sum_{k=1}^m \big[L_2 r \\
&\quad + \|J_k(\hat{\phi}(t_k^-))\|\big]\Bigg] + N\int_0^b \Big[m(s)\psi_1(r + \|\hat{\phi}_s\|) + n(s)\psi_2(r + \|\hat{\phi}_s\|)\Big]ds \\
&\quad + M\sum_{k=1}^m \big[L_1 r + \|I_k(\hat{\phi}(t_k^-))\|\big] + N\sum_{k=1}^m \big[L_2 r + \|J_k(\hat{\phi}(t_k^-))\|\big]
\end{aligned}
$$

and hence

$$
(1 + bNM_1)\left[N\Lambda\int_0^b m(s)ds + N\tilde{\Lambda}\int_0^b n(s)ds + \sum_{k=1}^m (ML_1 + NL_2)\right] \ge 1,
$$

which contradicts our assumption.

Let $r > 0$ be such that $\Psi(B_r(0, PC_0)) \subseteq B_r(0, PC_0)$. In order to prove that Ψ is a condensing map on $B_r(0, PC_0)$ into $B_r(0, PC_0)$. Consider the decomposition $\Psi = \Psi_1 + \Psi_2$ where

$$
\Psi_1 y(t) = S(t)\eta + \sum_{0<t_k<t} C(t - t_k)I_k(y(t_k^-) + \hat{\phi}(t_k^-)) + \sum_{0<t_k<t} S(t - t_k)J_k(y(t_k^-) + \hat{\phi}(t_k^-)),
$$

$$
\Psi_2 y(t) = \int_0^t S(t - s)\Big[f\Big(s, y_s + \hat{\phi}_s, \int_0^s a(s, \tau, y_\tau + \hat{\phi}_\tau)d\tau\Big) + Bu(s)\Big]ds.
$$

Now

$$
\begin{aligned}
\|Bu(s)\| &\leq M_1\Bigg[\|z_1\| + M\|\phi(0)\| + N\|\eta\| + N\int_0^b \alpha_r(s)ds \\
&\quad + M\sum_{k=1}^{m}\mu_k\big(\|y(t_k^-)\| + \|\hat{\phi}(t_k^-)\|\big) + N\sum_{k=1}^{m}\sigma_k\big(\|y(t_k^-)\| + \|\hat{\phi}(t_k^-)\|\big)\Bigg] \\
&\leq M_1\Bigg[\|z_1\| + M\|\phi(0)\| + N\|\eta\| + N\int_0^b \alpha_r(s)ds \\
&\quad + \sum_{k=1}^{m}(M\mu_k + N\sigma_k)(r + \|\hat{\phi}(t_k^-)\|)\Bigg] = \tilde{A}_0.
\end{aligned}
$$

From [15, Lemma 3.1], we infer that $\varPsi_2$ is completely continuous. Moreover, from the estimate

$$
\|\varPsi_1 v - \varPsi_1 w\| \leq \sum_{k=1}^{m}(ML_1 + NL_2)\|v - w\|,
$$

it follows that $\varPsi_1$ is a contraction on $B_r(0, PC_0)$ which implies that $\varPsi$ is a condensing operator on $B_r(0, PC_0)$.

Hence by the Sadovskii's fixed point theorem, $\varPsi$ has a fixed point $\overline{y}$ in PC_0. Let

$$
x(t) = \overline{y}(t) + \hat{\phi}(t), \quad t \in [-r, b].
$$

Then $x(\cdot)$ is a fixed point of the operator $\varPhi$ which is a mild solution of the problem $(2.3) - (2.6)$. This completes the proof. □

Corollary 2.1. *If all conditions of Theorem 2.1 hold except that (H3) replaced by the following one,*
(C1) : there exist positive constants c_k, d_k, f_k, g_k *and constants* $\theta_k, \delta_k \in (0,1), k = 1, 2, \ldots, m$ *such that for each* $\phi \in X$*,*

$$
\|I_k(\phi)\| \leq c_k + d_k(\|\phi\|)^{\theta_k}, \quad k = 1, 2, \ldots, m,
$$

and

$$
\|J_k(\phi)\| \leq f_k + g_k(\|\phi\|)^{\delta_k}, \quad k = 1, 2, \ldots, m,
$$

then the system $(2.3) - (2.6)$ *is controllable on* J *provided that*

$$
(1 + bNM_1)\left[N\Lambda\int_0^b m(s)ds + N\tilde{\Lambda}\int_0^b n(s)ds\right] < 1.
$$

Example 2.1. Consider the following impulsive partial delay integrodifferential equation of the form

$$\frac{\partial^2}{\partial t^2}z(t,\xi) = \frac{\partial^2}{\partial \xi^2}z(t,\xi) + \hat{\mu}(t,\xi) + \frac{\sin z(t-r,\xi)}{(1+t)(1+t^2)} + \frac{z(t-r,\xi)}{(1+t)(1+t^2)}\int_0^t e^{-z(s-r,\xi)}ds, \tag{2.7}$$

for $t \in J = [0,b], 0 \le \xi \le 1$, subject to the initial conditions

$$\begin{aligned} z(t,0) &= z(t,1) = 0, \quad t \ge 0, \\ z(t,\xi) &= \phi(t,\xi), \text{ for } -r \le t \le 0, \\ \frac{\partial}{\partial t}z(0,\xi) &= z_0(\xi), \\ \triangle z(t_k)(\xi) &= \int_0^{t_k} e_k(t_k-s)z(s,\xi)ds, \quad k=1,2,\dots,m, \\ \triangle z'(t_k)(\xi) &= \int_0^{t_k} \tilde{e}_k(t_k-s)z(s,\xi)ds, \quad k=1,2,\dots,m, \end{aligned}$$

where $e_k, \tilde{e}_k \in C(\mathbb{R},\mathbb{R})$.

We have to show that there exists a control $\hat{\mu}$ which steers (2.7) from any specified initial state to the final state in a Banach space X.

Let $X = L^2[0,1]$ and let A be an operator defined by $Az = z''$ with domain

$$D(A) = \{z \in X : z, z' \text{ are absolutely continuous }, z'' \in X, z(0) = z(1) = 0\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$ on X. Moreover, A has a discrete spectrum with eigenvalues of the form $-n^2$, $n \in \mathbb{N}$, and the corresponding normalized eigenfunctions given by $w_n(\zeta) := \sqrt{2}\sin n\zeta$. Also the following properties hold :

(a) The set of functions $\{w_n : n \in \mathbb{N}\}$ forms an orthonormal basis of X.
(b) If $z \in D(A)$, then $Az = \sum_{n=1}^{\infty} -n^2\langle z, w_n\rangle w_n$.
(c) For $z \in X, C(t)z = \sum_{n=1}^{\infty}\cos(nt)\langle z, w_n\rangle w_n$. The associated sine family is given by $S(t)z = \sum_{n=1}^{\infty}\frac{\sin(nt)}{n}\langle z, w_n\rangle w_n, z \in X$.

Let

$$\begin{aligned} \int_0^t a(t,s,z_s)(\xi)ds &= \frac{z(t-r,\xi)}{(1+t)(1+t^2)}\int_0^t e^{-z(s-r,\xi)}ds, \\ f\Big(t, z_t, \int_0^t a(t,s,z_s)ds\Big)(\xi) &= \frac{1}{(1+t)(1+t^2)}\Big[\sin z(t-r,\xi) \\ &\quad + z(t-r,\xi)\int_0^t e^{-z(s-r,\xi)}ds\Big]. \end{aligned}$$

Also define the operators I_k and J_k

$$I_k(\psi)(\xi) = \int_0^1 e_k(s)\psi(s,\xi)ds, \quad J_k(\psi)(\xi) = \int_0^1 \tilde{e}_k(s)\psi(s,\xi)ds.$$

Further, we have

$$\left| \frac{1}{(1+t)(1+t^2)} \left[\sin z(t-r,\xi) + z(t-r,\xi) \int_0^t e^{-z(s-r,\xi)} ds \right] \right| \leq \frac{1}{(1+t^2)} |z|.$$

Assume that the bounded linear operator $B : U \subset J \to X$ is defined by

$$(Bu)(t)(\xi) = \hat{\mu}(t,\xi), \quad 0 \leq \xi \leq 1.$$

Further, the linear operator W is given by

$$(Wu)(\xi) = \sum_{n=1}^{\infty} \int_0^1 \frac{1}{n} \sin ns \langle \hat{\mu}(s,\xi), w_n \rangle w_n ds, \quad 0 \leq \xi \leq 1.$$

Assume that this operator has a bounded inverse operator W^{-1} in $L^2(J,U)/kerW$. With the choice of $A,\ B,\ W,\ f,\ I_k$ and J_k, (2.3) – (2.6) is the abstract formulation of (2.7). Hence the second order impulsive system (2.7) is controllable on J.

3. Second Order Impulsive Integrodifferential Evolution Systems

The main objective of this section is to study the controllability of systems governed by a second order integrodifferential evolution equation with impulsive conditions of the form

$$\begin{aligned} x''(t) &= A(t)x(t) + Bu(t) + f(t,x(t),x'(t)) + \int_0^t h(t,s,x(s),x'(s))ds, \\ &\quad t \in J = [0,T],\ t \neq t_k,\ k = 1,2,\dots,m, && (3.1) \\ x(0) &= x_0, \quad x'(0) = y_0, && (3.2) \\ \triangle x(t_k) &= I_k(x(t_k), x'(t_k^-)),\ k = 1,2,\dots,m, && (3.3) \\ \triangle x'(t_k) &= J_k(x(t_k), x'(t_k^-)),\ k = 1,2,\dots,m, && (3.4) \end{aligned}$$

where $x_0, y_0 \in X, A(t) : X \to X$ is a closed densely defined operator. The control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space and $B : U \to X$ as a bounded linear operator ; $f(\cdot),\ h(\cdot),\ I_k(\cdot)$ and $J_k(\cdot)$ are appropriate functions and the jump $\triangle\xi(t)$ of the function $\xi(\cdot)$ at t defined by $\triangle\xi(t) = \xi(t^+) - \xi(t^-)$.

Let X denote a real reflexive Banach space and, for each $t \in [0,T]$, let $A(t) : X \to X$ be a closed densely defined operator. The fundamental solution for the second-order evolution equation,

$$x''(t) = A(t)x(t), \tag{3.5}$$

has been developed by Kozak [11]. Let us assume that the domain of $A(t)$ does not depend on $t \in [0,T]$ and denote it by $D(A)$ (for each $t \in [0,T], D(A(t)) = D(A)$).

Definition 3.1. A family S of bounded linear operators $S(t,s) : X \to X,\ t,s \in [0,T]$, is called a fundamental solution of a second order equation (3.5) if :

(Z1) For each $x \in X$, the mapping $[0,T] \times [0,T] \ni (t,s) \to S(t,s)x \in X$ is of class C^1 and
(i) for each $t \in [0,T], S(t,t) = 0$,
(ii) for all $t,s \in [0,T]$, and for each $x \in X$,

$$\frac{\partial}{\partial t}S(t,s)\Big|_{t=s} x = x, \qquad \frac{\partial}{\partial s}S(t,s)\Big|_{t=s} x = -x.$$

(Z2) For all $t,s \in [0,T]$, if $x \in D(A)$, then $S(t,s)x \in D(A)$, the mapping $[0,T] \times [0,T] \ni (t,s) \to S(t,s)x \in X$ is of class C^2 and
(i) $\frac{\partial^2}{\partial t^2}S(t,s)x = A(t)S(t,s)x$,
(ii) $\frac{\partial^2}{\partial s^2}S(t,s)x = S(t,s)A(s)x$,
(iii) $\frac{\partial}{\partial s}\frac{\partial}{\partial t}S(t,s)\Big|_{t=s} x = 0$.

(Z3) For all $t,s \in [0,T]$, if $x \in D(A)$, then $\frac{\partial}{\partial s}S(t,s)x \in D(A)$, there exists $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}S(t,s)x$, $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}S(t,s)x$ and
(i) $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}S(t,s)x = A(t)\frac{\partial}{\partial s}S(t,s)x$,
(ii) $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}S(t,s)x = \frac{\partial}{\partial t}S(t,s)A(s)x$
and the mapping $[0,T] \times [0,T] \ni (t,s) \to A(t)\frac{\partial}{\partial s}S(t,s)x$ is continuous.

We now consider some notations and definitions concerning impulsive differential equations. In what follows we put $t_0 = 0, t_{m+1} = T$ and we denote by $\mathcal{PC}$ the space formed by the functions $x : J \to X$ such that $x(\cdot)$ is continuous at $t \neq t_k, x(t_k^-) = x(t_k)$ and $x(t_k^+)$ exists for all $k = 1,2,\ldots,m$. It is clear that $\mathcal{PC}$ endowed with the norm $\|x\|_{\mathcal{PC}} = \sup\limits_{t\in J}\|x(t)\|$ is a Banach space. Similarly, $\mathcal{PC}^1$ will be the space of the functions $x(\cdot) \in \mathcal{PC}$ such that $x(\cdot)$ is continuously differentiable on $J \setminus \{t_k : k = 1,\ldots,m\}$ and the lateral derivatives $x'_R(t) = \lim\limits_{s\to 0^+}\frac{x(t+s)-x(t^+)}{s}, x'_L(t) = \lim\limits_{s\to 0^-}\frac{x(t+s)-x(t^-)}{s}$ are continuous functions on $[t_k, t_{k+1})$ and $(t_k, t_{k+1}]$ respectively. Next, for $x \in \mathcal{PC}^1$ we represent by $x'(t)$ the left derivative at $t \in (0,T]$ and by $x'(0)$ the right derivative at zero. It is easy to see that $\mathcal{PC}^1$ provided with the norm $\|x\|_{\mathcal{PC}^1} = \|x\|_{\mathcal{PC}} + \|x'\|_{\mathcal{PC}}$ is a Banach space.

For $x \in \mathcal{PC}$, we denote by $\tilde{x}_k, k = 0,1,\ldots,m$, the unique continuous function

$\tilde{x}_k \in C([t_k, t_{k+1}]; X)$ such that

$$\tilde{x}_k(t) = \begin{cases} x(t), & \text{for } t \in (t_k, t_{k+1}], \\ x(t_k^+), & \text{for } t = t_k. \end{cases}$$

The proof is based on the following fixed point theorem.

Lemma 3.1 ([23]: Schaefer's Theorem)**.** *Let E be a normed linear space. Let $F : E \to E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set, and let*

$$\zeta(F) = \{x \in E : x = \lambda F x \text{ for some } 0 < \lambda < 1\}.$$

Then, either $\zeta(F)$ is unbounded or F has a fixed point.

Definition 3.2. A function $x \in \mathcal{PC}^1$ is said to be a mild solution of problem $(3.1) - (3.4)$ if $x(t) \in D(A(t))$, for each $t \in [0,T]$ and if it satisfies the following integral equation,

$$\begin{aligned} x(t) \;=\; & -\frac{\partial}{\partial s} S(t,s)\Big|_{s=0} x_0 + S(t,0) y_0 + \int_0^t S(t,s) B u(s) ds \\ & + \int_0^t S(t,s) f(s, x(s), x'(s)) ds + \int_0^t \int_0^s S(t,s) h(s, \tau, x(\tau), x'(\tau)) d\tau ds \\ & - \sum_{0<t_k<t} \frac{\partial}{\partial s} S(t, t_k) I_k(x(t_k), x'(t_k^-)) + \sum_{0<t_k<t} S(t, t_k) J_k(x(t_k), x'(t_k^-)), \; t \in J. \end{aligned}$$

Definition 3.3. The system $(3.1) - (3.4)$ is said to be controllable on the interval J, if for every $x_0, y_0 \in D(A)$ and $z_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the mild solution $x(\cdot)$ of $(3.1) - (3.4)$ satisfies $x(T) = z_1$.

To investigate the controllability of problem $(3.1) - (3.4)$, we use the following assumptions:

- **(A1)** $x(t) \in D(A(t))$, for each $t \in [0,T]$.
- **(A2)** There exists a fundamental solution $S(t,s)$ of (3.5).
- **(A3)** $S(t,s)$ is compact for each $t, s \in [0,T]$ and there exist positive constants M, M^* and N, N^*, such that
  $M = \sup\{||S(t,s)|| : t, s \in J\}$, $M^* = \sup\left\{||\frac{\partial}{\partial s} S(t,s)|| : t, s \in J\right\}$ and
  $N = \sup\left\{||\frac{\partial}{\partial t} S(t,s)|| : t, s \in J\right\}$, $N^* = \sup\left\{||\frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,s)|| : t, s \in J\right\}$, respectively.
- **(A4)** B is a continuous operator from U to X and the linear operator $W : L^2(J, U) \to X$, defined by

$$Wu = \int_0^T S(T,s) B u(s) ds,$$

has a bounded invertible operator W^{-1} which takes values in $L^2(J,U)/kerW$ and there exists a positive constant M_1 such that $\|BW^{-1}\| \leq M_1$.

(A5) $f(t,\cdot,\cdot): X\times X \to X$ is continuous for each $t\in J$ and the function $f(\cdot,x,y)$: $J\to X$ is strongly measurable for each $(x,y)\in X\times X$.

(A6) $h(t,s,\cdot,\cdot): X\times X\to X$ is continuous for each $t,s\in J$ and the function $g(\cdot,\cdot,x,y): J\times J\to X$ is strongly measurable for each $(x,y)\in X\times X$.

(A7) For every positive constant k, there exists $\alpha_k\in L^1(J)$ such that

$$\sup_{\|x\|,\|y\|\leq k} \|f(t,x,y)\| \leq \alpha_k(t), \text{ for a.e. } \ t\in J.$$

(A8) For every positive constant k, there exists $\beta_k\in L^1(J)$ such that

$$\sup_{\|x\|,\|y\|\leq k} \Big\| \int_0^t h(t,s,x,y)ds\Big\| \leq \beta_k(t), \text{ for a.e. } \ t\in J.$$

(A9) There exists an integrable function $p: J\to[0,\infty)$ such that

$$\|f(t,x,y)\| \leq p(t)\Omega(\|x\|+\|y\|), \ t\in J, \quad x,y\in X,$$

where $\Omega:[0,\infty)\to(0,\infty)$ is a continuous non-decreasing function.

(A10) There exists an integrable function $q: J\to[0,\infty)$ such that

$$\Big\|\int_0^t h(t,s,x,y)ds\Big\| \leq q(t)\Omega_0(\|x\|+\|y\|), \ t\in J, \quad x,y\in X,$$

where $\Omega_0:[0,\infty)\to(0,\infty)$ is a continuous non-decreasing function.

(A11) The impulsive functions satisfy the following conditions:

a) The functions $I_k, J_k: X\times X\to X, k=1,2,\ldots,m$ are continuous.

b)There exist positive constants $a_k^l, b_k^l, l=1,2, k=1,2,\ldots,m$ such that

$$\|I_k(x,x')\| \leq a_k^1(\|x\|+\|x'\|)+a_k^2, \quad \|J_k(x,x')\|\leq b_k^1(\|x\|+\|x'\|)+b_k^2, \text{ for } x,x'\in X.$$

(A12) $\mu=\sum_{k=1}^m\Big[(M^*+N^*)a_k^1+(M+N)b_k^1\Big]<1$ and

$$(M+N)\int_0^T\hat{\phi}(s)ds < \int_c^\infty \frac{ds}{\Omega(s)+\Omega_0(s)}, \text{ where } \hat{\phi}(t)=\max\Big[\frac{1}{1-\mu}p(t),\frac{1}{1-\mu}q(t)\Big],$$

$$\begin{aligned} M_2 \ =& \ \Big[\|z_1\|+M^*\|x_0\|+M\|y_0\|+M\int_0^T p(s)\Omega(\|x(s)\|+\|x'(s)\|)ds \\ &+M\int_0^T q(s)\Omega_0(\|x(s)\|+\|x'(s)\|)ds+\sum_{k=1}^m(M^*a_k^2+Mb_k^2) \\ &+\sum_{k=1}^m(M^*a_k^1+Mb_k^1)\big(\|x(t_k)\|+\|x'(t_k^-)\|\big)\Big] \end{aligned}$$

$$\text{and } c \ = \ \frac{1}{1-\mu}\Big[(M^*+N^*)\big[\|x_0\|+\sum_{k=1}^m a_k^2\big]+(M+N)\big[\|y_0\|+M_1M_2T+\sum_{k=1}^m b_k^2\big]\Big].$$

Theorem 3.1. *If the assumptions* $(A1)-(A12)$ *are satisfied, then the system* $(3.1)-(3.4)$ *is controllable on* J.

Proof. Consider the space $Z = \mathcal{PC}^1(J,X)$ with norm $\|x\|^* = \max\{\|x\|, \|x'\|\}$. Using the assumption $(A4)$, for an arbitrary function $x(\cdot)$, define the control

$$\begin{aligned} u(t) \ &= \ W^{-1}\Bigg[z_1 + \frac{\partial}{\partial s}S(T,s)\Big|_{s=0}x_0 - S(T,0)y_0 - \int_0^T S(T,s)f(s,x(s),x'(s))ds \\ &\quad - \int_0^T\int_0^s S(T,s)h(s,\tau,x(\tau),x'(\tau))d\tau ds + \sum_{k=1}^m \frac{\partial}{\partial s}S(T,t_k)I_k(x(t_k),x'(t_k^-)) \\ &\quad - \sum_{k=1}^m S(T,t_k)J_k(x(t_k),x'(t_k^-))\Bigg](t). \end{aligned}$$

Using this control we shall now show that the operator $\Phi : Z \to Z$ defined by

$$\begin{aligned} (\Phi x)(t) \ &= \ -\frac{\partial}{\partial s}S(t,s)\Big|_{s=0}x_0 + S(t,0)y_0 + \int_0^t S(t,\eta)BW^{-1}\Bigg[z_1 \\ &\quad + \frac{\partial}{\partial s}S(T,s)\Big|_{s=0}x_0 - S(T,0)y_0 - \int_0^T S(T,s)f(s,x(s),x'(s))ds \\ &\quad - \int_0^T\int_0^s S(T,s)h(s,\tau,x(\tau),x'(\tau))d\tau ds \\ &\quad + \sum_{k=1}^m \frac{\partial}{\partial s}S(T,t_k)I_k(x(t_k),x'(t_k^-)) - \sum_{k=1}^m S(T,t_k)J_k(x(t_k),x'(t_k^-))\Bigg](\eta)d\eta \\ &\quad + \int_0^t S(t,s)f(s,x(s),x'(s))ds + \int_0^t\int_0^s S(t,s)h(s,\tau,x(\tau),x'(\tau))d\tau ds \\ &\quad - \sum_{0<t_k<t} \frac{\partial}{\partial s}S(t,t_k)I_k(x(t_k),x'(t_k^-)) + \sum_{0<t_k<t} S(t,t_k)J_k(x(t_k),x'(t_k^-)), \end{aligned}$$

$t \in J$, has a fixed point. This fixed point is then a mild solution of the system $(3.1)-(3.4)$.

Clearly, $(\Phi x)(T) = z_1$, which means that the control u steers the system from the initial state x_0 to z_1 in time T , provided we can obtain a fixed point of the operator Φ which implies that the system is controllable.

In order to study the controllability problem for system $(3.1)-(3.4)$, we have to apply the Schaefer fixed-point theorem to the following operator equation,

$$x(t) = \lambda\Phi x(t), \qquad \lambda \in (0,1).$$

Let $\sigma(t) = \sup\limits_{s\in[0,t]} \|x(s)\|$ and $\tau(t) = \sup\limits_{s\in[0,t]} \|x'(s)\|$. Then from

$$
\begin{aligned}
x(t) = & -\lambda \frac{\partial}{\partial s} S(t,s)\Big|_{s=0} x_0 + \lambda S(t,0) y_0 + \lambda \int_0^t S(t,\eta) B W^{-1} \Bigg[z_1 \\
& + \frac{\partial}{\partial s} S(T,s)\Big|_{s=0} x_0 - S(T,0) y_0 - \int_0^T S(T,s) f(s,x(s),x'(s)) ds \\
& - \int_0^T \int_0^s S(T,s) h(s,\tau,x(\tau),x'(\tau)) d\tau ds \\
& + \sum_{k=1}^m \frac{\partial}{\partial s} S(T,t_k) I_k(x(t_k), x'(t_k^-)) - \sum_{k=1}^m S(T,t_k) J_k(x(t_k), x'(t_k^-)) \Bigg] (\eta) d\eta \\
& + \lambda \int_0^t S(t,s) f(s,x(s),x'(s)) ds + \lambda \int_0^t \int_0^s S(t,s) h(s,\tau,x(\tau),x'(\tau)) d\tau ds \\
& - \lambda \sum_{0<t_k<t} \frac{\partial}{\partial s} S(t,t_k) I_k(x(t_k), x'(t_k^-)) + \lambda \sum_{0<t_k<t} S(t,t_k) J_k(x(t_k), x'(t_k^-)), t \in J.
\end{aligned}
$$

we have,

$$
\begin{aligned}
\|x(t)\| \le \; & M^* \|x_0\| + M \|y_0\| + M \int_0^t \|BW^{-1}\| \Bigg[\|z_1\| + M^* \|x_0\| + M \|y_0\| \\
& + M \int_0^T p(s) \Omega(\|x(s)\| + \|x'(s)\|) ds + M \int_0^T q(s) \Omega_0(\|x(s)\| + \|x'(s)\|) ds \\
& + \sum_{k=1}^m (M^* a_k^2 + M b_k^2) + \sum_{k=1}^m (M^* a_k^1 + M b_k^1) \big(\|x(t_k)\| + \|x'(t_k^-)\| \big) \Bigg] d\eta \\
& + M \int_0^t p(s) \Omega(\|x(s)\| + \|x'(s)\|) ds + M \int_0^t q(s) \Omega_0(\|x(s)\| + \|x'(s)\|) ds \\
& + \sum_{0<t_k<t} (M^* a_k^2 + M b_k^2) + \sum_{0<t_k<t} (M^* a_k^1 + M b_k^1) \big(\|x(t_k)\| + \|x'(t_k^-)\| \big),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\sigma(t) \le \; & M^* \|x_0\| + M \|y_0\| + M M_1 M_2 T + M \int_0^t p(s) \Omega(\|x(s)\| + \|x'(s)\|) ds \\
& + M \int_0^t q(s) \Omega_0(\|x(s)\| + \|x'(s)\|) ds + \sum_{k=1}^m (M^* a_k^2 + M b_k^2) \\
& + \sum_{k=1}^m (M^* a_k^1 + M b_k^1) \big(\|x(t_k)\| + \|x'(t_k^-)\| \big).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
x'(t) = & -\lambda \frac{\partial}{\partial t}\frac{\partial}{\partial s}S(t,s)\Big|_{s=0}x_0 + \lambda \frac{\partial}{\partial t}S(t,0)y_0 + \lambda \int_0^t \frac{\partial}{\partial t}S(t,\eta)BW^{-1}\Bigg[z_1 \\
& + \frac{\partial}{\partial s}S(T,s)\Big|_{s=0}x_0 - S(T,0)y_0 - \int_0^T S(T,s)f(s,x(s),x'(s))ds \\
& - \int_0^T \int_0^s S(T,s)h(s,\tau,x(\tau),x'(\tau))d\tau ds \\
& + \sum_{k=1}^m \frac{\partial}{\partial s}S(T,t_k)I_k(x(t_k),x'(t_k^-)) - \sum_{k=1}^m S(T,t_k)J_k(x(t_k),x'(t_k^-))\Bigg](\eta)d\eta \\
& + \lambda \int_0^t \frac{\partial}{\partial t}S(t,s)f(s,x(s),x'(s))ds + \lambda \int_0^t \int_0^s \frac{\partial}{\partial t}S(t,s)h(s,\tau,x(\tau),x'(\tau))d\tau ds \\
& - \lambda \sum_{0<t_k<t} \frac{\partial}{\partial t}\frac{\partial}{\partial s}S(t,t_k)I_k(x(t_k),x'(t_k^-)) \\
& + \lambda \sum_{0<t_k<t} \frac{\partial}{\partial t}S(t,t_k)J_k(x(t_k),x'(t_k^-)), t \in J.
\end{aligned}
$$

Thus we have,

$$
\begin{aligned}
\|x'(t)\| \le & N^*\|x_0\| + N\|y_0\| + N\int_0^t \|BW^{-1}\|\Bigg[\|z_1\| + M^*\|x_0\| + M\|y_0\| \\
& + M\int_0^T p(s)\Omega(\|x(s)\| + \|x'(s)\|)ds + M\int_0^T q(s)\Omega_0(\|x(s)\| + \|x'(s)\|)ds \\
& + \sum_{k=1}^m (M^*a_k^2 + Mb_k^2) + \sum_{k=1}^m (M^*a_k^1 + Mb_k^1)\big(\|x(t_k)\| + \|x'(t_k^-)\|\big)\Bigg]d\eta \\
& + N\int_0^t p(s)\Omega(\|x(s)\| + \|x'(s)\|)ds + N\int_0^t q(s)\Omega_0(\|x(s)\| + \|x'(s)\|)ds \\
& + \sum_{0<t_k<t} (N^*a_k^2 + Nb_k^2) + \sum_{0<t_k<t} (N^*a_k^1 + Nb_k^1)\big(\|x(t_k)\| + \|x'(t_k^-)\|\big),
\end{aligned}
$$

and hence

$$
\begin{aligned}
\tau(t) \le & N^*\|x_0\| + N\|y_0\| + NM_1M_2T + N\int_0^t p(s)\Omega(\|x(s)\| + \|x'(s)\|)ds \\
& + N\int_0^t q(s)\Omega_0(\|x(s)\| + \|x'(s)\|)ds + \sum_{k=1}^m (N^*a_k^2 + Nb_k^2) \\
& + \sum_{k=1}^m (N^*a_k^1 + Nb_k^1)\big(\|x(t_k)\| + \|x'(t_k^-)\|\big).
\end{aligned}
$$

From the assumption on μ and the previous estimates, it follows that

$$\begin{aligned}\sigma(t)+\tau(t) &\le c+\frac{1}{1-\mu}\Big[(M+N)\Big[\int_0^t p(s)\Omega(\sigma(s)+\tau(s))ds \\ &\qquad + \int_0^t q(s)\Omega_0(\sigma(s)+\tau(s))ds\Big]\Big]\end{aligned}$$

Let $\rho(t)=\sigma(t)+\tau(t),\ t\in J$. Then $\rho(0)=c$ and

$$\begin{aligned}\rho'(t) &\le \frac{1}{1-\mu}\Big[(M+N)\big[p(t)\Omega(\sigma(t)+\tau(t))+q(t)\Omega_0(\sigma(t)+\tau(t))\big]\Big] \\ &= \hat{\phi}(t)(M+N)\big[\Omega(\rho(t))+\Omega_0(\rho(t))\big],\ t\in J.\end{aligned}$$

This implies

$$\int_{\rho(0)}^{\rho(t)}\frac{ds}{\Omega(s)+\Omega_0(s)}\le (M+N)\int_0^T\hat{\phi}(s)ds<\int_c^{\infty}\frac{ds}{\Omega(s)+\Omega_0(s)},\ t\in J.$$

This inequality implies that there is a constant K such that

$$\sigma(t)+\tau(t)=\rho(t)\le K,\quad t\in J.$$

and hence

$$\|x\|^*=\max\{\|x\|,\|x'\|\}\le K,$$

where K depends only on T and on the functions p,q,Ω and Ω_0.

In the second step we prove that the operator $\Phi: Z\to Z$ is a completely continuous operator. Let $B_k=\{x\in Z:\|x\|^*\le k\}$ for some $k\ge 1$. We first show that Φ maps B_k into an equicontinuous family. Let $x\in B_k$ and $t_1,t_2\in J$. Then, if $0<t_1<t_2\le T$, we have

$$\begin{aligned}&\|(\Phi x)(t_1)-(\Phi x)(t_2)\| \\ &\le \Big\|\frac{\partial}{\partial s}[S(t_1,s)-S(t_2,s)]\Big|_{s=0}x_0\Big\|+\|[S(t_1,0)-S(t_2,0)]y_0\| \\ &+\int_0^{t_1}\|S(t_1,\eta)-S(t_2,\eta)\|\|BW^{-1}\|\Big[\|z_1\|+\Big\|\frac{\partial}{\partial s}S(T,s)\Big|_{s=0}x_0\Big\| \\ &+\|S(T,0)y_0\|+\int_0^T\|S(T,s)\|\alpha_k(s)ds+\int_0^T\|S(T,s)\|\beta_k(s)ds \\ &+\sum_{k=1}^m\Big\|\frac{\partial}{\partial s}S(T,t_k)\Big\|\big[a_k^1(\sigma(t)+\tau(t))+a_k^2\big]+\sum_{k=1}^m\|S(T,t_k)\|\big[b_k^1(\sigma(t)+\tau(t))+b_k^2\big]\Big]d\eta \\ &+\int_{t_1}^{t_2}\|S(t_2,\eta)\|\|BW^{-1}\|\Big[\|z_1\|+\Big\|\frac{\partial}{\partial s}S(T,s)\Big|_{s=0}x_0\Big\|+\|S(T,0)y_0\|\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \|S(T,s)\|\alpha_k(s)ds+\int_0^T \|S(T,s)\|\beta_k(s)ds+\sum_{k=1}^m \Big\|\frac{\partial}{\partial s}S(T,t_k)\Big\| \\
&\times\big[a_k^1(\sigma(t)+\tau(t))+a_k^2\big]+\sum_{k=1}^m \|S(T,t_k)\|\big[b_k^1(\sigma(t)+\tau(t))+b_k^2\big]\Bigg]d\eta \\
&+\int_0^{t_1}\|S(t_1,s)-S(t_2,s)\|\alpha_k(s)ds+\int_{t_1}^{t_2}\|S(t_2,s)\|\alpha_k(s)ds \\
&+\int_0^{t_1}\|S(t_1,s)-S(t_2,s)\|\beta_k(s)ds+\int_{t_1}^{t_2}\|S(t_2,s)\|\beta_k(s)ds \\
&+\sum_{0<t_k<t_1}\Big\|\frac{\partial}{\partial s}[S(t_1,t_k)-S(t_2,t_k)]\Big\|\big[a_k^1(\sigma(t)+\tau(t))+a_k^2\big]+\sum_{t_1\le t_k<t_2}\Big\|\frac{\partial}{\partial s}S(t_2,t_k)\Big\| \\
&\times\big[a_k^1(\sigma(t)+\tau(t))+a_k^2\big]+\sum_{0<t_k<t_1}\|S(t_1,t_k)-S(t_2,t_k)\|\big[b_k^1(\sigma(t)+\tau(t))+b_k^2\big] \\
&+\sum_{t_1\le t_k<t_2}\|S(t_2,t_k)\|\big[b_k^1(\sigma(t)+\tau(t))+b_k^2\big] \\
&\to 0 \quad \text{as} \quad t_1\to t_2,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&\|(\Phi x)'(t_1)-(\Phi x)'(t_2)\| \\
&\le \Big\|\frac{\partial}{\partial s}\Big[\frac{\partial}{\partial t_1}S(t_1,s)-\frac{\partial}{\partial t_2}S(t_2,s)\Big]\Big|_{s=0}x_0\Big\|+\Big\|\Big[\frac{\partial}{\partial t_1}S(t_1,0)-\frac{\partial}{\partial t_2}S(t_2,0)\Big]y_0\Big\| \\
&+\int_0^{t_1}\Big\|\Big[\frac{\partial}{\partial t_1}S(t_1,\eta)-\frac{\partial}{\partial t_2}S(t_2,\eta)\Big]\Big\|\|BW^{-1}\|\Bigg[\|z_1\|+\Big\|\frac{\partial}{\partial s}S(T,s)\Big|_{s=0}x_0\Big\| \\
&+\|S(T,0)y_0\|+\int_0^T\|S(T,s)\|\alpha_k(s)ds+\int_0^T\|S(T,s)\|\beta_k(s)ds \\
&+\sum_{k=1}^m\Big\|\frac{\partial}{\partial s}S(T,t_k)\Big\|\big[a_k^1(\sigma(t)+\tau(t))+a_k^2\big]+\sum_{k=1}^m\|S(T,t_k)\|\big[b_k^1(\sigma(t)+\tau(t))+b_k^2\big]\Bigg]d\eta \\
&+\int_{t_1}^{t_2}\Big\|\frac{\partial}{\partial t_2}S(t_2,\eta)\Big\|\|BW^{-1}\|\Bigg[\|z_1\|+\Big\|\frac{\partial}{\partial s}S(T,s)\Big|_{s=0}x_0\Big\|+\|S(T,0)y_0\| \\
&+\int_0^T\|S(T,s)\|\alpha_k(s)ds+\int_0^T\|S(T,s)\|\beta_k(s)ds+\sum_{k=1}^m\Big\|\frac{\partial}{\partial s}S(T,t_k)\Big\| \\
&\times\big[a_k^1(\sigma(t)+\tau(t))+a_k^2\big]+\sum_{k=1}^m\|S(T,t_k)\|\big[b_k^1(\sigma(t)+\tau(t))+b_k^2\big]\Bigg]d\eta \\
&+\int_0^{t_1}\Big\|\Big[\frac{\partial}{\partial t_1}S(t_1,s)-\frac{\partial}{\partial t_2}S(t_2,s)\Big]\Big\|\alpha_k(s)ds+\int_{t_1}^{t_2}\Big\|\frac{\partial}{\partial t_2}S(t_2,s)\Big\|\alpha_k(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{t_1}\Big\|\Big[\frac{\partial}{\partial t_1}S(t_1,s)-\frac{\partial}{\partial t_2}S(t_2,s)\Big]\Big\|\beta_k(s)ds+\int_{t_1}^{t_2}\Big\|\frac{\partial}{\partial t_2}S(t_2,s)\Big\|\beta_k(s)ds\\
&+\sum_{0<t_k<t_1}\Big\|\frac{\partial}{\partial s}\Big[\frac{\partial}{\partial t_1}S(t_1,t_k)-\frac{\partial}{\partial t_2}S(t_2,t_k)\Big]\Big\|\big[a_k^1(\sigma(t)+\tau(t))+a_k^2\big]\\
&+\sum_{t_1\le t_k<t_2}\Big\|\frac{\partial}{\partial s}\Big[\frac{\partial}{\partial t_2}S(t_2,t_k)\Big]\Big\|\big[a_k^1(\sigma(t)+\tau(t))+a_k^2\big]\\
&+\sum_{0<t_k<t_1}\Big\|\frac{\partial}{\partial t_1}S(t_1,t_k)-\frac{\partial}{\partial t_2}S(t_2,t_k)\Big\|\big[b_k^1(\sigma(t)+\tau(t))+b_k^2\big]\\
&+\sum_{t_1\le t_k<t_2}\Big\|\frac{\partial}{\partial t_2}S(t_2,t_k)\Big\|\big[b_k^1(\sigma(t)+\tau(t))+b_k^2\big]\\
&\to 0 \quad \text{as} \quad t_1\to t_2.
\end{aligned}
$$

Thus, Φ maps B_k into an equicontinuous family of functions. It is easy to see that the family ΦB_k is uniformly bounded.

Next we show $\overline{\Phi B_k}$ is compact. Since we have shown ΦB_k is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that Φ maps B_k into a precompact set in X. Let $0 < t \le T$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$
\begin{aligned}
(\Phi_\epsilon x)(t) = &-\frac{\partial}{\partial s}S(t,s)\Big|_{s=0}x_0+S(t,0)y_0+\int_0^{t-\epsilon}S(t,\eta)BW^{-1}\Big[z_1\\
&+\frac{\partial}{\partial s}S(T,s)\Big|_{s=0}x_0-S(T,0)y_0-\int_0^T S(T,s)f(s,x(s),x'(s))ds\\
&-\int_0^T\int_0^s S(T,s)h(s,\tau,x(\tau),x'(\tau))d\tau ds+\sum_{k=1}^m\frac{\partial}{\partial s}S(T,t_k)I_k(x(t_k),x'(t_k^-))\\
&-\sum_{k=1}^m S(T,t_k)J_k(x(t_k),x'(t_k^-))\Big](\eta)d\eta+\int_0^{t-\epsilon}S(t,s)f(s,x(s),x'(s))ds\\
&+\int_0^{t-\epsilon}\int_0^s S(t,s)h(s,\tau,x(\tau),x'(\tau))d\tau ds-\sum_{0<t_k<t}\frac{\partial}{\partial s}S(t,t_k)I_k(x(t_k),x'(t_k^-))\\
&+\sum_{0<t_k<t}S(t,t_k)J_k(x(t_k),x'(t_k^-)),\ t\in J.
\end{aligned}
$$

Since $S(t,s)$ is a compact operator, the set $Y_\epsilon(t)=\{(\Phi_\epsilon x)(t) : x\in B_k\}$ is precompact in X, for every ϵ, $0<\epsilon<t$. Moreover, for every $x\in B_k$, we have

$$
\begin{aligned}
&\|(\Phi x)(t)-(\Phi_\epsilon x)(t)\| \\
&\qquad \le \int_{t-\epsilon}^{t} \|S(t,\eta)\|\|BW^{-1}\| \Bigg[\|z_1\| + \Big\| \frac{\partial}{\partial s} S(T,s)\Big|_{s=0} x_0 \Big\| + \|S(T,0)y_0\| \\
&\qquad + \int_0^T \|S(T,s)\|\alpha_k(s)ds + \int_0^T \|S(T,s)\|\beta_k(s)ds + \sum_{k=1}^{m} \Big\| \frac{\partial}{\partial s} S(T,t_k) \Big\| \\
&\qquad \times \left[a_k^1(\sigma(t)+\tau(t)) + a_k^2\right] + \sum_{k=1}^{m} \|S(T,t_k)\| \left[b_k^1(\sigma(t)+\tau(t)) + b_k^2\right] \Bigg] d\eta \\
&\qquad + \int_{t-\epsilon}^{t} \|S(t,s)\|\alpha_k(s)ds + \int_{t-\epsilon}^{t} \|S(t,s)\|\beta_k(s)ds \;\to 0 \quad \text{as} \quad \epsilon \to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(\Phi x)'(t)-(\Phi_\epsilon x)'(t)\| \\
&\qquad \le \int_{t-\epsilon}^{t} \Big\| \frac{\partial}{\partial t} S(t,\eta) \Big\| \|BW^{-1}\| \Bigg[\|z_1\| + \Big\| \frac{\partial}{\partial s} S(T,s)\Big|_{s=0} x_0 \Big\| + \|S(T,0)y_0\| \\
&\qquad + \int_0^T \|S(T,s)\|\alpha_k(s)ds + \int_0^T \|S(T,s)\|\beta_k(s)ds + \sum_{k=1}^{m} \Big\| \frac{\partial}{\partial s} S(T,t_k) \Big\| \\
&\qquad \times \left[a_k^1(\sigma(t)+\tau(t)) + a_k^2\right] + \sum_{k=1}^{m} \|S(T,t_k)\| \left[b_k^1(\sigma(t)+\tau(t)) + b_k^2\right] \Bigg] d\eta \\
&\qquad + \int_{t-\epsilon}^{t} \Big\| \frac{\partial}{\partial t} S(t,s) \Big\| \alpha_k(s)ds + \int_{t-\epsilon}^{t} \Big\| \frac{\partial}{\partial t} S(t,s) \Big\| \beta_k(s)ds \;\to 0 \quad \text{as} \quad \epsilon \to 0.
\end{aligned}
$$

Therefore, there are precompact sets arbitrarily close to the set $\{(\Phi x)(t) : x \in B_k\}$. Hence, the set $\{(\Phi_\epsilon x)(t) : x \in B_k\}$ is precompact in X.

It remains to show that $\Phi : Z \to Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \to x$ in Z. Then there is an integer r such that $\|x_n(t)\| \le r, \|x_n'(t)\| \le r$ for all n and $t \in J$, so $\|x(t)\| \le r, \|x'(t)\| \le r$ and $x, x' \in B_r$. By $A(5)$ and $A(11)$, we have
(i) I_k and $J_k,\ k = 1, 2, \ldots, m$ are continuous.
(ii) $f(t, x_n(t), x_n'(t)) \to f(t, x(t), x'(t))$ for each $t \in J$ and since

$$\|f(t, x_n(t), x_n'(t)) - f(t, x(t), x'(t))\| \le 2\alpha_r(t).$$

(iii) $h(t, s, x_n(s), x_n'(s)) \to h(t, s, x(s), x'(s))$ for each $t, s \in J$ and since

$$\Big\| \int_0^t [h(t, s, x_n(s), x_n'(s)) - h(t, s, x(s), x'(s))] ds \Big\| \le 2\beta_r(t).$$

we have by dominated convergence theorem,

$$
\begin{aligned}
&\|\Phi x_n - \Phi x\| \\
&\quad \le \int_0^t \bigg\| S(t,\eta)BW^{-1}\bigg[\int_0^T S(T,s)\Big[f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))\Big]ds \\
&\quad + \int_0^T\int_0^s S(T,s)\Big[h(s,\tau,x_n(\tau),x_n'(\tau)) - h(s,\tau,x(\tau),x'(\tau))\Big]d\tau ds \\
&\quad + \sum_{k=1}^m \frac{\partial}{\partial s}S(T,t_k)\Big[I_k(x_n(t_k),x_n'(t_k^-)) - I_k(x(t_k),x'(t_k^-))\Big] \\
&\quad + \sum_{k=1}^m S(T,t_k)\Big[J_k(x_n(t_k),x_n'(t_k^-)) - J_k(x(t_k),x'(t_k^-))\Big]\bigg](\eta)\bigg\| d\eta \\
&\quad + \int_0^t \|S(t,s)\big[f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))\big]\| ds \\
&\quad + \int_0^t\int_0^s \|S(t,s)\big[h(s,\tau,x_n(\tau),x_n'(\tau)) - h(s,\tau,x(\tau),x'(\tau))\big]\| d\tau ds \\
&\quad + \sum_{0<t_k<t} \Big\|\frac{\partial}{\partial s}S(t,t_k)\Big[I_k(x_n(t_k),x_n'(t_k^-)) - I_k(x(t_k),x'(t_k^-))\Big]\Big\| \\
&\quad + \sum_{0<t_k<t} \|S(t,t_k)\big[J_k(x_n(t_k),x_n'(t_k^-)) - J_k(x(t_k),x'(t_k^-))\big]\| \\
&\quad \to 0 \ \text{ as n} \to \infty,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&\|(\Phi x_n)' - (\Phi x)'\| \\
&\quad \le \int_0^t \bigg\| \frac{\partial}{\partial t}S(t,\eta)BW^{-1}\bigg[\int_0^T S(T,s)\Big[f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))\Big]ds \\
&\quad + \int_0^T\int_0^s S(T,s)\Big[h(s,\tau,x_n(\tau),x_n'(\tau)) - h(s,\tau,x(\tau),x'(\tau))\Big]d\tau ds \\
&\quad + \sum_{k=1}^m \frac{\partial}{\partial s}S(T,t_k)\Big[I_k(x_n(t_k),x_n'(t_k^-)) - I_k(x(t_k),x'(t_k^-))\Big] \\
&\quad + \sum_{k=1}^m S(T,t_k)\Big[J_k(x_n(t_k),x_n'(t_k^-)) - J_k(x(t_k),x'(t_k^-))\Big]\bigg](\eta)\bigg\| d\eta \\
&\quad + \int_0^t \Big\|\frac{\partial}{\partial t}S(t,s)\Big[f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))\Big]\Big\| ds \\
&\quad + \int_0^t\int_0^s \Big\|\frac{\partial}{\partial t}S(t,s)\Big[h(s,\tau,x_n(\tau),x_n'(\tau)) - h(s,\tau,x(\tau),x'(\tau))\Big]\Big\| d\tau ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{0<t_k<t} \left\| \frac{\partial}{\partial t}\frac{\partial}{\partial s} S(t,t_k)\Big[I_k(x_n(t_k),x_n'(t_k^-)) - I_k(x(t_k),x'(t_k^-))\Big]\right\| \\
&+ \sum_{0<t_k<t} \left\| \frac{\partial}{\partial t} S(t,t_k)\Big[J_k(x_n(t_k),x_n'(t_k^-)) - J_k(x(t_k),x'(t_k^-))\Big]\right\| \\
&\to 0 \quad \text{as n} \to \infty.
\end{aligned}
$$

Thus, Φ is continuous. This completes the proof that Φ is completely continuous.

Finally the set $\zeta(\Phi) = \{x \in Z : x = \lambda\Phi x, \lambda \in (0,1)\}$ is bounded, as we proved in the first step. Hence, by the Schaefer fixed-point theorem, the operator Φ has a fixed point in Z. This means that any fixed point of Φ is a mild solution of (3.1) − (3.4) on J satisfying $(\Phi x)(t) = x(t)$. Thus the system (3.1) − (3.4) is controllable on J. □

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