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METRICS ON S^3 SUCH THAT BRIESKORN CURVE IS A GEODESIC

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1. Introduction.

Let $(a_1, a_2, \dots, a_{n+1})$ be an $(n+1)$-tuple of positive integers with $a_i \geq 2 (i = 1, 2, \dots, n+1)$. $B^{2n-1}(a_1, a_2, \dots, a_{n+1}) \subset \mathbb{C}^{n+1}$ is said to be a $(2n-1)$-dimensional Brieskorn manifold if it satisfies the following two equations.

$$|z_1|^2 + |z_2|^2 + \dots + |z_{n+1}|^2 = 1, \tag{1}$$

$$(z_1)^{a_1} + (z_2)^{a_2} + \dots + (z_{n+1})^{a_{n+1}} = 0. \tag{2}$$

In particular, $B^1(a_1, a_2)$ is called a Brieskorn curve on a unit sphere S^3.

Brieskorn manifolds have interesting properties in the topological and differential view points. For example they have S^1-actions with the singular orbits as the G-manifolds, some of these are exotic spheres and they have many normal contact metric structures(cf.[2],[4] and [5]).

Let x^1, x^2, x^3, x^4 be a local coordinate system of $\mathbb{R}^4$. We put $x^1 = \cos\theta^1$, $x^2 = \sin\theta^1 \cos\theta^2$, $x^3 = \sin\theta^1 \sin\theta^2 \cos\theta^3$, $x^4 = \sin\theta^1 \sin\theta^2 \sin\theta^3$, where $\theta^1, \theta^2 \in (0, \pi)$ and $\theta^3 \in (-\pi, \pi)$. Then the usual metric on S^3 is defined by

$$ds^2 = (d\theta^1)^2 + \sin^2\theta^1 (d\theta^2)^2 + \sin^2\theta^1 \sin^2\theta^2 (d\theta^3)^2. \tag{3}$$

In general, Brieskorn curve $B^1(p, q)$ is not a geodesic on S^3 with the usual metric (3).

The purpose of the present paper is to describe an adapted metric g such that Brieskorn curve $B^1(p, q)$ is a geodesic on sphere S^3.

Theorem. *A metric on S^3 such that Brieskorn curve $B^1(p, q)$ is a geodesic is given by*

$$ds^2 = p^2(d\theta^1)^2 + p^2 \sin^2\theta^2 (d\theta^2)^2 + q^2 \sin^2\theta^1 \sin^2\theta^2 (d\theta^3)^2,$$

where p and q are integers with $p \geq q \geq 2$.

Moreover Brieskorn curve $B^1(p, q)$ is

$$(\exp(\sqrt{-1}s/p), \exp(\sqrt{-1}s/q))B_0 \quad \textit{for all } s,$$

where B_0 is a point of $B^1(p, q)$.

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2. Preliminaries.

Denoting by x^1, x^2, x^3, x^4 a real coordinate of $\mathbb{C}^2$ such that $z^1 = x^1 + \sqrt{-1}x^2$ and $z^2 = x^3 + \sqrt{-1}x^4$, the identification of $\mathbb{C}^2$ with $\mathbb{R}^4$ will always be done by means of the correspondence $(z^1, z^2) \to (x^1, x^2, x^3, x^4)$.

Let S^3 be a 3-dimensional unit sphere defined by

$$|z_1|^2 + |z_2|^2 = 1, \tag{4}$$

with the usual metric induced by the Euclidean metric on $\mathbb{R}^4$.

Furthermore put $x^1 = \cos\theta^1$, $x_2 = \sin\theta^1\cos\theta^2$, $x^3 = \sin\theta^1\sin\theta^2\cos\theta^3$, $x^4 = \sin\theta^1\sin\theta^2\sin\theta^3$, where $\theta^1, \theta^2 \in (0, \pi)$ and $\theta^3 \in (-\pi, \pi)$. As against the usual metric (3) on S^3, we shall define new Riemannian metric by

$$ds^2 = (f_1)^2(d\theta^1)^2 + (f_2)^2\sin^2\theta^1(d\theta^2)^2 + (f_3)^2\sin^2\theta^1\sin^2\theta^2(d\theta^3)^2, \tag{5}$$

where f_1 is a function on S^3 with respect to θ^1, f_2 and f_3 are constants. $B^1(p,q)$ is called a Brieskorn curve if it satisfies the equation (4) and an equation defined by

$$(z_1)^p + (z_2)^q = 0, \tag{6}$$

where p and q are integers with $p \geq q \geq 2$. By conjugate complex in (6), $x = \sin^2\theta^1\sin^2\theta^2$ satisfies

$$(1-x)^p = x^q. \tag{7}$$

Then we can easily prove the following lemma.

Lemma 1. *Let x be real number with $0 \leq x \leq 1$, and p, q positive integers with $p \geq q \geq 2$. Then an equation*

$$F(x) = (1-x)^p - x^q \tag{8}$$

has the unique solution $x_0 \in (0,1)$ such that $F(x_0) = 0$.

We put $\alpha = \sin\theta^1\sin\theta^2 (> 0)$. Hence from Lemma 1, the constant α have the following properties.

(i) $(1-\alpha^2)^p = \alpha^{2q}$,

(ii) $0 < \alpha^2 \leq \frac{1}{2}$, and $\alpha^2 = \frac{1}{2}$ if and only if $p = q$,

(iii) α only depends on p and q.

Therefore we hold that $\sin^2\theta^1(s)\sin^2\theta^2(s)$ is a constant along Brieskorn curve $s \to (\theta^1(s), \theta^2(s), \theta^3(s))$.

Now from (5), we get that a geodesic $s \to (\theta^1(s), \theta^2(s), \theta^3(s))$ satisfies the following ordinary differntial equations.

$$(9) \qquad \frac{d^2\theta^1}{ds^2} + \frac{1}{f_1}\frac{df_1}{d\theta^1}\left(\frac{d^2\theta^2}{ds^2}\right)^2 - \left(\frac{f_2}{f_1}\right)^2 \sin\theta^1\cos\theta^1\left(\frac{d\theta^2}{ds}\right)^2 - \left(\frac{f_3}{f_1}\right)^2 \sin\theta^1\cos\theta^1\sin^2\theta^2\left(\frac{d\theta^3}{ds}\right)^2 = 0,$$

$$(10) \qquad \frac{d^2\theta^2}{ds^2} + 2\cot\theta^1\frac{d\theta^1}{ds}\frac{d\theta^2}{ds} - \left(\frac{f_3}{f_2}\right)^2 \sin\theta^2\cos\theta^2\left(\frac{d\theta^3}{ds}\right)^2 = 0,$$

$$(11) \qquad \frac{d^2\theta^3}{ds^2} + 2\cot\theta^1\frac{d\theta^1}{ds}\frac{d\theta^3}{ds} + 2\cot\theta^2\frac{d\theta^2}{ds}\frac{d\theta^3}{ds} = 0,$$

where a parameter s is arc length of a geodesic.

3. Proof of theorem.

Assume that Brieskorn curve $s \to (\theta^1(s), \theta^2(s), \theta^3(s))$ has the unit speed geodesic. We shall determine a function f_1, and constants f_2 and f_3 in (5)

(A) Since (11) implies $\dfrac{d}{ds}\left(\sin^2\theta^1\sin^2\theta^2\dfrac{d\theta^3}{ds}\right) = 0$, we find

$$(12) \qquad \sin^2\theta^1\sin^2\theta^2\frac{d\theta^3}{ds} = \text{constant}(\neq 0).$$

Then we have easily the following lemma.

Lemma 2. *The following conditions are equivalent.*

(a) $\sin^2\theta^1\sin^2\theta^2 = \text{constant}\ (\neq 0)$,

(b) $\dfrac{d\theta^3}{ds} = \text{constant}(\neq 0)$,

(c) $\cot\theta^1\dfrac{d\theta^1}{ds} + \cot\theta^2\dfrac{d\theta^2}{ds} = 0.$

As we have $\sin^2\theta^1(s)\sin^2\theta^2(s) = \text{constant}$ along Brieskorn curve, we get

$$(13) \qquad \frac{d\theta^3}{ds} = \beta\,(= \text{constant} \neq 0).$$

Therefore we obtain

$$(14) \qquad \theta^3(s) = \beta s + \gamma,$$

where γ is a constant.

(B) From (10),(13), Lemma 2 and

$$\sin^2\theta^2\frac{d}{ds}\left(\frac{1}{\sin^2\theta^2}\frac{d\theta^2}{ds}\right) = -2\cot\theta^2\left(\frac{d\theta^2}{ds}\right)^2 + \frac{d^2\theta^2}{ds^2},$$

we have that (10) implies

$$\frac{d}{ds}\left(\frac{1}{\sin^2\theta^2}\frac{d\theta^2}{ds}\right)=\left(\frac{f_3}{f_2}\beta\right)^2\cot\theta^2. \tag{15}$$

Put $g(s)=\cot\theta^2(s)$. Then

$$\frac{d}{ds}g(s)=-\frac{1}{\sin^2\theta^2}\frac{d\theta^2}{ds}. \tag{16}$$

By using (15) and (16), we get

$$\frac{d^2}{ds^2}g(s)=-\frac{d}{ds}\left(\frac{1}{\sin^2\theta^2}\frac{d\theta^2}{ds}\right)=-\left(\frac{f_3}{f_2}\beta\right)^2 g(s). \tag{17}$$

Then the differntial equation above has the solution

$$g(s)=A\sin(Bs+\varphi), \tag{18}$$

where $A>0, B=(f_3/f_2)\beta$ and φ are constants. Using $g(s)=\cot\theta^2(s)$, (16) and (18), we obtain

$$\frac{d\theta^2}{ds}=-\frac{AB\cos(Bs+\varphi)}{1+A^2\sin^2(Bs+\varphi)}. \tag{19}$$

The solution of differential equation (19) is

$$\theta^2(s)=\operatorname{arccot}(A\sin(Bs+\varphi)). \tag{20}$$

(C) From a constant $\alpha=\sin\theta^1(s)\sin\theta^2(s)$, $0<\theta^1<\pi$ and (20), we get

$$\sin\theta^1(s)=\alpha\sqrt{1+A^2\sin^2(Bs+\varphi)}. \tag{21}$$

Therefore we obtain

$$\theta^1(s)=\arcsin\left(\alpha\sqrt{1+A^2\sin^2(Bs+\varphi)}\right). \tag{22}$$

By using (21) or (22), we have

$$\frac{d\theta^1}{ds}=\frac{\alpha A^2B\sin(Bs+\varphi)\cos(Bs+\varphi)}{\sqrt{1+A^2\sin^2(Bs+\varphi)}\sqrt{(1-\alpha^2)-\alpha^2A^2\sin^2(Bs+\varphi)}} \tag{23}$$

By virtue of (21), we choose a constant A such that

$$\alpha^2(1+A^2)=1. \tag{24}$$

By using (6),(14),(20) and (22), we choose constant $\gamma = (p\varphi - \pi)/q$. Then we obtain

$$(25) \qquad \frac{f_3}{f_2} = \frac{q}{p}, \quad B = \frac{q}{p}\beta \quad \text{and} \quad \varphi = \frac{1}{p}(q\gamma + \pi).$$

Consequently, we get

$$(26) \qquad \begin{aligned} x^1 &= \epsilon\sqrt{1-\alpha^2}\cos(Bs+\varphi), & x^2 &= \epsilon\sqrt{1-\alpha^2}\sin(Bs+\varphi), \\ x^3 &= \epsilon\alpha\cos(\beta s+\varphi), & x^4 &= \epsilon\alpha\sin(\beta s+\varphi), \end{aligned}$$

where $\epsilon = \pm 1$. By assumption of Brieskorn curve with the unit speed, and using (5),(13), (19),(23),(24) and (25), we obtain

$$(27) \quad (f_1)^2 = \frac{1}{A^2B^2\sin^2(Bs+\varphi)}\left(1 + A^2\sin^2(Bs+\varphi) - \left(\frac{q}{p}\right)^2(f_2)^2\beta^2\right).$$

Here we choose constants β and f_2 such that

$$(28) \qquad \beta = \frac{1}{q} \quad \text{and} \quad f_2 = p.$$

Then from (25), (27) and (28), we have

$$(29) \qquad B = \frac{1}{p}, \quad f_3 = q \quad \text{and} \quad f_1 = p.$$

Now from (25), (26), (28) and (29), we have that (6) implies

$$(30) \quad \left(\sqrt{1-\alpha^2}\exp\sqrt{-1}\left(\frac{s}{p} + \frac{q\gamma+\pi}{p}\right)\right)^p + \left(\alpha\exp\sqrt{-1}\left(\frac{s}{q}+\gamma\right)\right)^q = 0.$$

Hence from (29), we find that for all s

$$(31) \quad \left(\sqrt{1-\alpha^2}\exp\sqrt{-1}\left(\frac{s}{p} + \frac{q\gamma+\pi}{p}\right),\ a\exp\sqrt{-1}\left(\frac{s}{q}+\gamma\right)\right) \in S^3$$

is in Brieskorn curve $B^1(p,q)$. Therefore theorem is completely proved.

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