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REPRESENTATION OF HARMONIC FUNCTIONS WITH POINT SINGULARITY

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Abstract. A classical result on the representation of harmonic functions with singularity in $\mathbb{R}^n$ is proved using the distributions instead of the series expansion of a harmonic function.

1. Introduction

M. Brelot (pp. 189-202 in [1]) has described the behaviour of harmonic function in the neighborhood of a point singularity in $\mathbb{R}^n$, $n \geq 2$. For that study, he makes an extensive use of the Laurent-type series expansion of harmonic function around the singular point.

We point out in this note, how these results could be obtained using the theory of distributions instead of the series expansion. This method is elegant and allows a unified version of the results unlike in the series expansion where the case $\mathbb{R}^2$ has to be treated a little differently from the case $\mathbb{R}^n$, $n \geq 3$.

If Δ is the Laplacian operator, the fundamental solution F_n in $\mathbb{R}^n$, $n \geq 1$, is

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given by $\Delta F_n = \delta$, where δ is the Dirac measure at 0 and

$$F_n(x) = \begin{cases} \frac{1}{2}|x| & \text{if } n = 1 \\ \frac{1}{2\pi}\log|x| & \text{if } n = 2 \\ \frac{-1}{(n-2)\sigma_n}\frac{1}{|x|^{n-2}} & \text{if } n \geq 3, \sigma_n \text{ is the surface area of the unit sphere in } \mathbb{R}^n \end{cases}$$

Interpreting the linear (resp. the convex) functions in $\mathbb{R}$ as harmonic (resp. subharmonic) functions, we will prove the following theorem for all $\mathbb{R}^n$, $n \geq 1$.

Theorem: Let u be harmonic in the punctured unit sphere $B_n^\star$, $n \geq 1$, such that $u(x) = o\left(\frac{1}{|x|^{n-1}}\right)$ when $x \to 0$. Then $u(x)$ is of the form $u(x) = v(x) + \alpha F_n(x)$ where $v(x)$ is harmonic in the unit sphere B_n and α is a constant.

2. The Proof of Theorem

Let B_n (resp. $B_n^\star$) denote the unit sphere $|x| < 1$ (resp. the punctured unit sphere $0 < |x| < 1$) in $\mathbb{R}^n$.

Lemma: Let $u(x)$ be a harmonic function in $B_n^\star$, $n \geq 1$, such that $u(x) = o\left(\frac{1}{|x|^{n-1}}\right)$ when $x \to 0$, then $grad u = o\left(\frac{1}{|x|^n}\right)$ when $x \to 0$

Proof: When $n = 1$, u is of the form

$$u(x) = \begin{cases} ax & \text{if } x > 0 \\ bx & \text{if } x < 0 \end{cases}$$

where a and b are real. If we take $grad u = \frac{du}{dx}$, then $grad u = o\left(\frac{1}{|x|}\right)$.

When $n \geq 2$, given ϵ, by hypothesis, there exists an $r > 0$ such that $|u(x)| \leq \frac{\epsilon}{|x|^{n-1}}$ if $|x| \leq r$. Let x_0 be a point such that $|x_0| = \rho < \frac{2}{3}r$. Then if s is the sphere $|x - x_0| = \frac{\rho}{2}$, $\max_{x \in s}|u(x)| \leq \epsilon\frac{2^{n-1}}{\rho^{n-1}}$. Consequently, (see p. 198, M. Brelot [1]),

$$|grad\, u \text{ at } x_0| \leq \frac{A}{\rho}\frac{\epsilon 2^{n-1}}{\rho^{n-1}}.$$

ϵ being arbitrary, this proves that

$$|gradu| = o\left(\frac{1}{|x|^n}\right) \quad \text{when} x \to 0.$$

Proof of theorem; When $n = 1$, this theorem can be proved quite simply as follows: By hypothesis, u is of the form

$$u(x) = \begin{cases} ax & \text{if } 0 < x < 1 \\ bx & \text{if } -1 < x < 0 \end{cases}$$

i.e. $u(x) = \frac{a+b}{2}x + \frac{a-b}{2}|x|$. Hence $u(x) = v(x) + (a-b)F_1(x)$, where $v(x)$ is harmonic in $|x| < 1$.

When $n \geq 2$, since $|x|^{n-1}u(x)$ is bounded in a neighbourhood of 0, $u(x)$ is locally integrable in B_n and hence defines a distribution in B_n.

We will calculate Δu in the sense of distribution. Let $\varphi \in C_0^\infty(B_n)$ be a test function.

$$\text{Then} \quad < \Delta u, \varphi > \; = \; < u, \Delta\varphi >$$
$$= \lim_{r\to 0} \int_{|x| \geq r} u.\Delta\varphi dv$$

$$< \Delta u, \varphi >= \lim_{r\to 0}[\int_{|x| \geq r} \varphi\Delta u dv + \int_{|x| = r} [u\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial u}{\partial n}]d\sigma \qquad (\star)$$

(i) since $\Delta u = 0$ when $|x| \geq r > 0$, then

$$\int_{|x| \geq r} \varphi\Delta u dv = 0$$

(ii) On $|x| = r$, $d\sigma = r^{n-1}dw$, where dw is the small surface area on the unit sphere consequently, since $u(x) = o\left(\frac{1}{|x|^{n-1}}\right)$ and $\frac{\partial\varphi}{\partial n}$ is bounded on $|x| = r$, then we have

$$\lim_{r\to 0} \int_{|x| = r} u\frac{\partial\varphi}{\partial n}d\sigma = 0$$

(iii) When $|x| = r$ is small, $|\varphi(x) - \varphi(0)| \leq A|x|$ where A is real positive. From the lemma above we have $\left|\frac{\partial u}{\partial n}\right| = o\left(\frac{1}{|x|^n}\right)$. If we set $\alpha = -\int_{|x| = r} \frac{\partial u}{\partial n} d\sigma$, then α is constant multiple of the flux of u which is independent of $r > 0$, since u is harmonic in $B_n^\star$.

$$\text{Consequently} \qquad |\int_{|x| = r} \varphi(x) \frac{\partial u}{\partial n} d\sigma + \alpha\varphi(0)|$$
$$= |\int_{|x| = r} (\varphi(x) - \varphi(0)) \frac{\partial u}{\partial n} d\sigma|$$
$$\leq \int_{|x| = r} |\varphi(x) - \varphi(0)||\frac{\partial u}{\partial n}| d\sigma \rightarrow 0 \text{ as } r \rightarrow 0$$

Thus, going back to the equation $(\star)$ we conclude that:

$$< \Delta u, \varphi > = < \alpha\delta, \varphi > .$$

Since φ is an arbitrary test function, then $\Delta u = \alpha\delta$ and consequently $u = v + \alpha F_n$ where $\Delta v = 0$ i.e. v is a harmonic function in B_n.

Corollary. Let u be a harmonic function in $B_n^\star$, $n \geq 2$, such that $u(x) = o(F_n(x))$ when $x \rightarrow 0$. Then u extends as a harmonic function in B_n.

References

[1] M. Brelot, Eléments de la théorie classique du potentiel. C.D.U. Paris, 3 édition 1965.

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