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PURE COMPLETELY POSITIVE MAPS AS A DUAL OBJECT OF C^*–ALGEBRAS

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Dedicated to prof M.Takesaki's 60th birthday

ABSTRACT

Let A and B be unital C^*-algebras,

$$CP(A, M_n) = \{\phi | \phi \text{ is a pure completely positive map from } A \text{ to } M_n \text{ with } Tr\phi(I) = 1\}$$

and α be a natural action induced by $SU(n)$ on $CP(A, M_n)$.

It is proved that

Theorem If $\psi: CP(B, M_n) \cup \{0\} \to CP(A, M_n) \cup \{0\}, (n \geq 3)$, a bijection with $\psi(0) = 0$, is α-invariant, preserves trasition probabilities and ψ and ψ^{-1} are uniformly continuous, then ψ gives rise to a *-isomorphism between A and B.

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§1. Introduction

The starting point of this paper is Shultz's paper[6] in which he proved:

Theorem Let A and B be C^*-algebras. Suppose that $P(A)$ and $P(B)$ are pure state spaces of A and B, $\psi: P(B) \cup \{0\} \to P(A) \cup \{0\}$ a bijection with $\psi(0) = 0$. Then ψ is induced by a $*$-isomorphism of A onto B iff ψ and ψ^{-1} are uniformly continuous and ψ preserves orientation and transition probabilities.

Let A and B be unital C^*-algebras, n be fixed integer, $n \geq 3$. Set

$$CP(A, M_n) = \{\phi | \phi \text{ is a pure completely positive map from } A \text{ to } M_n \text{ with } Tr\phi(I) = 1\}.$$

First of all, in section 3 we consider the non-commutative version of the above theorem, that is, we cosider $CP(A, M_n)$ in the place of $P(A)$. Suppose that α is a natural action induced by $SU(n)$ on $CP(A, M_n)$. Namely we define

$$\alpha_g \phi(x) = g\phi(x)g^{-1}.\forall x \in A, g \in SU(n), \phi \in CP(A, M_n).$$

Theorem If $\psi: CP(B, M_n) \cup \{0\} \to CP(A, M_n) \cup \{0\}$, $(n \geq 3)$, a bijection with $\psi(0) = 0$, is α-invariant, preserves transition probabilities and ψ and ψ^{-1} are uniformly continuous, then ψ gives rise to a $*$-isomorphism between A and B.

The motivation for the above theorem is as follows. Alfsen and Shultz [6] defined the notion orientation of the state spaces of a C^*-algebra, and proved that the state space with the orientation can determine the structure of the C^*-algebra. Author [1] considered the matrix algebra of a C^*-algebra instead of the state space and defined the notion of α-invariance with which the matrix algebra can determine the structure of the C^*-algebra. In the theorem we used the α-invariance in the place of the orientation and set $CP(A, M_n)$ as the non-commutative version of the pure state space. The theorem obtained can be regarded as a non-commutative Shultz theorem.

Recently the theory of pure completely bounded and completely positive maps is developing rapidly. This is another motivation of our paper.

In section 4, we provide with a counterexample to show that the condition $n \geq 3$ in Theorem 3.1 is essential.

§2. Preliminary

In [1] ,we considered the matrix algebra of a C^*-algebra instead of the state space and defines the notion of α-invariance with which the matrix algebra can determine the structure of the C^*-algebra. For later use and completeness , we give a sketch of the proof.

Let A, B be unital C^*-algebras

$$\mathcal{K}_A = \{\varphi \mid \varphi \text{ are completely positive maps from } M_n(C) \to A \text{ with } \|a(\varphi)\| \leq 1\},$$

$$a(\varphi) = \begin{pmatrix} \varphi(e_{11}) & \cdots & \varphi(e_{1n}) \\ \vdots & \ddots & \vdots \\ \varphi(e_{n1}) & \cdots & \varphi(e_{nn}) \end{pmatrix},$$

where $\{e_{ij}\}$ is the matrix unit of $M_n(C)$.

Suppose that $SU(n)$ is the set of all $n \times n$ unimodular unitary matrices and α is the automorphism group on $M_n(C)$ defined by

$$\alpha_g(x) = gxg^{-1}, \qquad x \in M_n(C), g \in SU(n).$$

Using $(\alpha_g\varphi)(x) = \varphi(\alpha_g^{-1}(x))$, $\varphi \in \mathcal{K}_A, x \in M_n(C)$, α induces an action on $\mathcal{K}_A$.

Theorem 1 [1] Let A, B be C^*-algebras. For $n \geq 3$, if Φ is an $\alpha-$invariant affine isomorphism from $\mathcal{K}_A$ to $\mathcal{K}_B$, $\Phi(0) = 0$, ($\alpha-$invariance means that $\alpha\Phi = \Phi\alpha$), then A and B are $^*-$isomorphic.

Proof.

1)By Choi-Effros theory,

$$\mathcal{K}_A = \{a \in (M_n \otimes A)^+ : \|a\| \leq 1\}$$

2)We can extend Φ as an α-invariant positively preserving isometry from $M_n \otimes A$ onto $M_n \otimes B$.

3)If $x \in A$, then $uxu^* = x$ for every $u \in SU(n)$. From α-invariance

$$\Phi(x) = u\Phi(x)u^*, x \in A, u \in SU(n)$$

such that $\Phi(A) = B$.

4)From Kadison Isometry theory [8] p335, let z be the central projection of B such that $x \in M_n \otimes A \to \Phi(x)z$ is multiplicative and $x \to \Phi(x)z^{\perp}$ is anti-multiplicative. By use of α- invariance, we can prove $z^{\perp} = 0$. □

§3. MAIN THEOREM

In this section we consider pure completely positive maps in the place of pure states and obtain a theorem as follows.

First of all we give some notations. Let $COP(A,B)$ be all completely positive maps from C^*-algebra A to C^*-algebra B.

Definition 3.1 A completely positive map ϕ from C^*-algebra A to C^*-algebra B is said to be pure if for every $\psi \in COP(A,B)$, $\psi \leq \phi$ implies $\psi = \lambda\phi$ for some $\lambda \geq 0$.

Let A and B be unital C^*-algebras, n a fixed integer ($n \geq 3$).

$$CP(A, M_n) = \{\phi | \phi \text{ is a pure completely positive map from } A \text{ to } M_n \text{ with } Tr\phi(I) = 1\}.$$

If $SU(n)$ is the unimodular unitary group, we can define an action on $CP(A, M_n)$ by $SU(n)$ as follows.

$$\alpha_g\phi(x) = g\phi(x)g^{-1}. \forall x \in A, g \in SU(n), \phi \in CP(A, M_n).$$

Definition 3.2 If x and y are unit vectors in a Hilbert space, the transition probability between the vector states ω_x and ω_y ($\omega_x(\cdot) = (\cdot x|x)$) on $B(H)$ is defined to be $(\omega_x|\omega_y) = |(x|y)|^2$.

If $\pi: A \to B(H)$ is an irreducible representation of C^*-algebra A, then the transition probability between the pure states $\omega_x \cdot \pi$ and $\omega_y \cdot \pi$ is again defined to be $|(x|y)|^2$.

If σ and τ are arbitrary pure states on A, let u_σ and u_τ be their support projections in A^{**}, we then define $(\sigma|\tau) = < u_\sigma, \tau > = < u_\tau, \sigma >$. Let $L(A, M_n)$ be the vector space of linear functions from A to M_n and $L(A, M_n)^{\oplus}$ the cone of all completely positive maps from A to M_n. From the theory of completely positive map, there is an order isomorphism between $L(A, M_n)$ and $(M_n(A))^*$ (with respect to $L(A, M_n)^{\oplus}$). The restriction of this order isomorphism to $CP(A, M_n)$ can be viewed as a map from $CP(A, M_n)$ to $P(A \otimes M_n)$ (the pure state space of $A \otimes M_n$), which is denoted by γ. The topology on $P(A \otimes M_n)$ is hereditied to $CP(A, M_n)$.

Definition 3.3 If $\phi, \psi \in CP(A, M_n)$, the transition probability between ϕ and ψ is defined to be $(\gamma(\phi), \gamma(\psi))$.

Theorem 3.1 Suppose that Ψ is a bijection

$$\Psi: CP(B, M_n) \cup \{0\} \to CP(A, M_n) \cup \{0\}$$

$$\Psi(0) = 0. \ (n \geq 3)$$

If Ψ is α-invariant, preserves transition probabilities, and Ψ and Ψ^{-1} are uniformly continuous, then Ψ gives rise to a *-isomorphism between A and B.

Proof. By the above remark, there exists a map γ between $CP(A, M_n)$ and $P(M_n(A))$ (the pure state space of $M_n(A)$). Ψ can be viewed as a bijection from $P(M_n(B)) \cup \{0\}$ to $P(M_n(A)) \cup \{0\}$.

The atomic part of $M_n(A)^{**}$ is a direct sum of type I factors $c_i M_n(A)^{**} \equiv B(H_i)$ for each i(c_i is the central support of some pure state of $M_n \otimes A$ in $M_n \otimes A^{**}$). The pure states in $c_i^{-1}(1)$ are a maximal set of mutually equivalent pure states, and all such maximal sets occur in this way. It follows that Ψ carries the pure normal states of $c_i M_n(A)^{**}$ onto those of some type I factor $d_i M_n(B)^{**}$, a direct summand of $M_n(B)^{**}$.

By Wigner's theorem [6,p.499], there exists a unique affine isomorphism between normal state spaces of $c_i M_n(A)^{**}$ and $d_i M_n(B)^{**}$, and is induced by a *-isomorphism or *-anti-isomorphism $\Phi_i: c_i M_n(A)^{**} \to d_i M_n(B)^{**}$, which induces $\Psi_i: d_i^{-1}(1) \to c_i^{-1}(1)$. Since Ψ_i is α-invariant, Φ_i is α-invariant. It follows that Φ_i is a *-isomorphism. Now the direct sum $\Phi = \oplus \Phi_i$ will map the atomic part of $M_n(A)^{**}$ *-isomorphically onto that of $M_n(B)^{**}$, and induces Ψ. If A is a C^*-algebra, we denote by A_u the set of elements

$a \in z_A A^{**}$ (z_A is the central projection in A^{**} such that $z_A A^{**}$ is the atomic part of A^{**}) such that a, $a^* a$, aa^* are uniformly continuous on $P(A) \cup \{0\}$. We say that A is **weakly perfect** if $A_u = z_A A$.

Let $z_{M_n(A)}$ denote the central projection in $M_n(A)^{**}$ such that $z_{M_n(A)} M_n(A)^{**}$ is the atomic part of $M_n(A)^{**}$.

By [6,Theorem 1,p.507], every C^*-algebra is weakly perfect so that $M_n(A)$ is weakly perfect. Since Ψ and Ψ^{-1} are uniformly continuous, Ψ induces an isometry from $z_{M_n(A)} M_n(A)$ to $z_{M_n(B)} M_n(B)$. Since $P(M_n(A))$ annihilates $(I - z_{M_n(A)}) M_n(A)$, Ψ induces an isometry from $M_n(A)$ to $M_n(B)$, which is α-invariant. By Theorem 1 we can conclude that $A \cong B$. □

If we consider M_{n^∞} in the place of M_n (where M_{n^∞} is a UHF-algebra of type n^∞), we can get a generalization of above theorem. We give some notations. Let A be a unital C^*-algebra.

$$CP(A, M_{n^\infty}) = \{\phi | \phi \text{ is a completely positive map from } A \text{ to } M_{n^\infty} \text{ with } Tr\phi(I) = 1\}$$

(Tr is the trace in M_{n^∞})

$$M_{n^\infty} = \overline{\cup_{k=1}^{\infty} \varphi_k(M_{n^k})},$$

where $\{\varphi_k\}$ are embedings from M_{n^k} to M_{n^∞}.

$$SSU(\infty) = \cup_{k=1}^{\infty} \varphi_k(SU(n^k)).$$

By use of $SSU(\infty)$, we can define a natural action on $CP(A, M_{n^\infty})$ as follows,

$$\alpha_g \phi(x) = g\phi(x)g^{-1}. \forall x \in A, g \in SSU(\infty), \phi \in CP(A, M_{n^\infty}),$$

which denoted by α.

When we consider M_{n^∞} in the place of M_n, we can obtain an order isomorphism between $L(A, M_{n^\infty})$ and $(M_{n^\infty} \otimes A)^*$ (with respect to $L(A, M_{n^\infty})^{\oplus}$)[2]. The restriction of this order isomorphism to $CP(A, M_{n^\infty})$ can be viewed as a map from $CP(A, M_{n^\infty})$

to $P(A \otimes M_{n^\infty})$ (the pure state space of of $A \otimes M_{n^\infty}$). In the same way we can define the transition probabilities between the elements in $CP(A, M_{n^\infty})$. Then we can get the following theorem.

Theorem 3.2 Suppose that Ψ is a bijection

$$\Psi: CP(B, M_{n^\infty}) \cup \{0\} \to CP(A, M_{n^\infty}) \cup \{0\}$$

$$\Psi(0) = 0.$$

If Ψ is α-invariant, preserves transition probabilities and Ψ and Ψ^{-1} are uniformly continuous, then Ψ induces a *-isomorphism between A and B.

Proof. Ψ can be viewed as a bijection from $P(B \otimes M_{n^\infty}) \cup \{0\}$ to $P(A \otimes M_{n^\infty}) \cup \{0\}$ with $\Psi(0) = 0$. Following the proof in Theorem 3.1, we can get an α-invariant isometry between $A \otimes M_{n^\infty}$ and $B \otimes M_{n^\infty}$.

Fixing an integer $n^k \geq 3$, there is an isomorphism κ from M_{n^∞} to $M_{n^k} \otimes M_{n^\infty}$. we have a diagram as follows:

$$\begin{array}{ccc} M_{n^\infty} \otimes A & \xrightarrow{\Psi} & M_{n^\infty} \otimes B \\ \kappa \otimes I \downarrow & & \downarrow \kappa \otimes I \\ M_{n^k} \otimes M_{n^\infty} \otimes A & \xrightarrow{\Psi'} & M_{n^k} \otimes M_{n^\infty} \otimes B \end{array}$$

in which

$$\Psi' = (\kappa \otimes I) \circ \Psi \circ (\kappa^{-1} \otimes I)$$

Since $SU(n^k) \otimes SU(\infty) \subset SU(\infty)$, so Ψ' is an α'-invariant map, where α' is a natural action induced by $SU(n^k) \otimes SU(\infty)$, that is

$$\alpha_{u_1 \otimes u_2}: x \mapsto (u_1 \otimes u_2)(x)(u_1^* \otimes u_2^*),$$

$u_1 \in SU(n^k), u_2 \in SU(\infty), x \in M_{n^k} \otimes M_{n^\infty} \otimes A$.

Next thing we should prove is $\Psi'(M_{n^k} \otimes A) \subseteq M_{n^k} \otimes B$. We set $\alpha' = \alpha_1' \otimes \alpha_2'$ in which α_1' is an action induced by $SU(n^k)$, α_2' is the one induced by $SU(\infty)$. α_2'-invariance implies that

$$(I_{n^k} \otimes u)\Psi'(x)(I_{n^k} \otimes u^*) = \Psi'(x).$$

$u \in SU(\infty)$, $x \in M_{n^k} \otimes A$ (I_{n^k} is the identity in $M_{n^k}(\mathbb{C})$) such that $\Psi'(x) \in M_{n^k} \otimes B$.

It follows that $\Psi'(M_{n^k} \otimes A) \subseteq M_{n^k} \otimes B$ and Ψ' is α_1'-invariant. We can arrive that A is $*$-isomorphic to B. □

§4. Counter-example

In this section, we will present an example to show that the condition $n \geq 3$ is essential for theorem 3.1.

Theorem 4.1 There are C^*-algebras A and B, and map Ψ from $CP(B, M_2) \cup \{0\}$to $CP(A, M_2) \cup \{0\}$, with $\Psi(0) = 0$, such that Ψ preserves the transition probabilities, Ψ and Ψ^{-1} are uniformly continuous, Ψ is α-invariant, but Ψ does not give rise to a $*$-isomorphism between A and B.

Proof. Suppose that A and B are C^*-algebras such that A is anti-isomorphic to B,

$$\pi : A \to B.$$

In M_2, define

$$\sigma\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

Then σ is an anti-automorphism of M_2 of order 2 such that $\sigma(u) = u^*$, $u \in SU(2)$. Then $\Psi_* = \sigma \otimes \pi$ induces an α-invariant isomorphism from $M_2(A)_1^+$ to $M_2(B)_1^+$ with $\Psi_*(0) = 0$. If $\Psi_1 = (\Psi_*)^t$ and $\Psi = \Psi_1 |_{CP(B,M_2)\cup\{0\}}$, Ψ is an α-invariant isometry so that Ψ, Ψ^{-1} are uniformly continuous.

Following the proof of Theorem 3.1, for $n = 2$, $\Phi_i : c_i M_2(A)^{**} \to d_i M_2(B)^{**}$, induced by $\Psi_i : d_i^{-1}(1) \to c_i^{-1}(1)$, will be a $*$-isomorphism or a $*$-anti-isomorphism. Note That every $*$-isomorphism (or $*$-anti-isomorphism) induces an affine isomorphism of their state space, which then preserves transition probabilities for pure states. So Ψ preserves transition probabilities. But Ψ induces a $*$-anti-isomorphism between A and B. □

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