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A THIRD ORDER DIFFERENTIAL EQUATION AND REPRESENTABLE POLES

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Abstract It is showed in this note that if a third order differential equation $w''' = \lambda(z)w'' + R(z,w)w' = \lambda(z)w'' + \frac{P(z,w)}{Q(z,w)}w'$, where $\lambda(z)$ is a meromorphic function and $P(z,w)$ and $Q(z,w)$ are polynomials in w with meromorphic coefficients, possesses an admissible solution $w(z)$, then $w(z)$ satisfies a linear differential equation, a second order equation of Painlevé type, or first order equation of the form $c(z)(w')^2 + B(z,w)w' + A(z,w) = 0$, where $B(z,w)$ and $A(z,w)$ are polynomials in w having small coefficients with respect to $w(z)$. The main tools of the proof are lemmas on representable poles.

1. Introduction

In this note, we will treat algebraic differential equations with admissible solutions in the complex plane. The Malmquist–Yosida–Steinmetz type theorems have been studied by means of the Nevanlinna theory. During the last two decades several mathematicians gave remarkable improvements. We can find them, for instance, in Laine [8, Chapters 9–13].

In this note, we use standard notations in the Nevanlinna theory (see e.g. [2], [8], [10]). Let $f(z)$ be a meromorphic function. As usual, $m(r,f)$, $N(r,f)$, and $T(r,f)$ denote the proximity function, the counting function, and the characteristic function of $f(z)$, respectively.

A function $\varphi(r)$, $0 \leqq r < \infty$, is said to be $S(r,f)$ if there is a set $E \subset \mathbb{R}^+$ of finite linear measure such that $\varphi(r) = o(T(r,f))$ as $r \to \infty$ with $r \notin E$.

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A meromorphic function $a(z)$ is *small* with respect to $f(z)$ if $T(r, a) = S(r, f)$. In the below, $\mathcal{M} = \{a(z)\}$ denotes a given finite collection of meromorphic functions. A transcendental meromorphic function $f(z)$ is *admissible* with respect to $\mathcal{M}$ if $T(r, a) = S(r, f)$ for any $a(z) \in \mathcal{M}$.

Let $c \in \mathbb{C} \cup \{\infty\}$. We call z_0 a c-point of $f(z)$ if $f(z_0) - c = 0$. Suppose that a transcendental meromorphic function $f(z)$ is admissible with respect to $\mathcal{M}$. A c-point z_0 of $f(z)$ is an *admissible c-point* with respect to $\mathcal{M}$ if $a(z_0) \neq 0, \infty$ for any $a(z) \in \mathcal{M}$.

Suppose $N(r, c; f) \neq S(r, f)$ for a $c \in \mathbb{C} \cup \{\infty\}$. Let P be a property. We denote by $n_{\mathrm{P}}(r, c; f)$ the number of c-points in $|z| \leqq r$ that admit the property P. The integrated counting function $N_{\mathrm{P}}(r, c; f)$ is defined in the usual fashion. If

$$N(r, c; f) - N_{\mathrm{P}}(r, c; f) = S(r, f),$$

then we say that *almost all* c-points admit the property P.

We define an admissible solution of the equation

$$\Omega(z, w, w', \ldots, w^{(n)}) = \sum_{J \in \mathcal{J}} \Phi_J = \sum_{J \in \mathcal{J}} c_J(z) w^{j_0} (w')^{j_1} \cdots (w^{(n)})^{j_n} = 0, \tag{1.1}$$

where $\mathcal{J}$ is a finite set of multi-indices $J = (j_0, j_1, \ldots, j_n)$, and $c_J(z)$ are meromorphic functions. Let $\mathcal{M}_{(1.1)}$ be the collection of the coefficients of $\Omega(z, w, w', \ldots, w^{(n)})$ in (1.1), say $\mathcal{M}_{(1.1)} := \{c_J(z) | J \in \mathcal{J}\}$. A meromorphic solution $w(z)$ of the equation (1.1) is an *admissible solution* if $w(z)$ is admissible with respect to $\mathcal{M}_{(1.1)}$.

The basic Test-Power test gives us an information of the dominant behavior of an admissible solution in a neighbourhood of its admissible pole. The ideas which are contained in Steinmetz [12, Lemma 1, pp. 47–48] oriented us towards the constructions of some auxiliary functions that play important roles. In Section 2, we will give some generalizations of the Steinmetz lemma by means of further investigations of higher order terms in the Laurent series of the admissible solution in a neighbourhood of its admissible pole. In Section 3, we will give an application of the lemmas, which we study in Section 2, to a third order differential equation.

2. Lemmas on the representable poles

In the first part of this section, we summarize our results on representable simple poles in Ishizaki [4], [5].

Let $f(z)$ be a transcendental meromorphic function and let $R(z)$ and $\alpha(z)$ be small functions with respect to $f(z)$. Let z_0 be a simple pole of $f(z)$. We say that z_0 is *representable in the first sense by $R(z)$ and $\alpha(z)$*, if

$$f(z) = \frac{R(z_0)}{z - z_0} + \alpha(z_0) + O(z - z_0), \qquad \text{as } z \to z_0$$

in a neighbourhood of z_0. For the sake of simplicity, we call such a simple pole an S1-*pole*. The Steinmetz lemma [12] could be rewritten as follows in terms of "S1-pole" defined here.

Lemma 2.1. *Let $w(z)$ be a transcendental meromorphic function. If almost all poles of $w(z)$ are S1-poles and if $w(z)$ satisfies $m(r,w) = S(r,w)$, then $w(z)$ satisfies a Riccati equation.*

For the definition of S2-pole, we introduce the following further material. Let λ_1, λ_0 be complex constants and let $\mathbf{L}$ be a set of linear transformations of a quantity R,

$$\mathbf{L} = \mathbf{L}_{(\lambda_1,\lambda_0)} = \left\{ L = \frac{l_1 R + l_2}{l_3 R + l_4} \;\middle|\; l_4^2 - \lambda_1 l_3 l_4 + \lambda_0 l_3^2 \neq 0, \right. \qquad \left. l_j \in \mathbb{C}, \quad j = 1, 2, 3, 4 \right\}. \tag{2.1}$$

We define an equivalence relation $\sim$ in $\mathbf{L}$ by

$$L = (a_1 R + a_2)/(a_3 R + a_4) \sim M = (b_1 R + b_2)/(b_3 R + b_4) \in \mathbf{L},$$

if

$$\begin{cases} \lambda_0(a_1 b_3 - b_1 a_3) = a_2 b_4 - b_2 a_4 \\ \lambda_1(a_1 b_3 - b_1 a_3) = a_1 b_4 - b_1 a_4 + a_2 b_3 - a_3 b_2. \end{cases} \tag{2.2}$$

Proposition 2.2.

(i) *If $L = (a_1 R + a_2)/(a_3 R + a_4) \in \mathbf{L}$, then $L \sim L^* = A_1 R + A_2$, where*

$$A_1 = \frac{-a_2 a_3 + a_1 a_4}{\lambda_0 a_3^2 - \lambda_1 a_3 a_4 + a_4^2}, \quad A_2 = \frac{\lambda_0 a_1 a_3 - \lambda_1 a_2 a_3 + a_2 a_4}{\lambda_0 a_3^2 - \lambda_1 a_3 a_4 + a_4^2}.$$

(ii) *If $L = a_1 R + a_2 \sim M = b_1 R + b_2$, then $a_1 = b_1$ and $a_2 = b_2$.*

By Proposition 2.2, we can take, for each equivalent class in $\mathbf{L}$, a unique representative which is an entire linear transformation. We denote by $\mathbf{L}^* = \mathbf{L}^*(\lambda_1, \lambda_0)$ the set of all such representatives. We define $aL + bM$ and LM as follows: For $a, b \in \mathbb{C}$, $L = a_1 R + a_2$, $M = b_1 R + b_2 \in \mathbf{L}^*$,

$$aL + bM = (aa_1 + bb_1)R + aa_2 + bb_2, \tag{2.3}$$

$$LM = (a_1 b_2 + a_2 b_1 - \lambda_1 a_1 b_1)R + (a_2 b_2 - \lambda_0 a_1 b_1). \tag{2.4}$$

Let $L = a_1 R + a_2$, $M = b_1 R + b_2$ be two elements of $\mathbf{L}^*$. We say that L and M are independent, if $a_1 b_2 - a_2 b_1 \neq 0$. We can easily obtain the following propositions:

Proposition 2.3. *Let L and M be elements of $\mathbf{L}^*$. If L and M are independent, then for any $N \in \mathbf{L}^*$, there exist τ_1, τ_2 such that $N = \tau_1 L + \tau_2 M$.*

Proposition 2.4. *Let L and M be elements of $\mathbf{L}^*$. If L and M are independent, then for any $N = aR + b \in \mathbf{L}^*$ with $\lambda_0 a^2 - \lambda_1 ab + b^2 \neq 0$, NL and NM are also independent.*

Let $f(z)$ be a transcendental meromorphic function. Let all functions $\alpha_1(z)$, ..., $\alpha_4(z)$, $\beta_1(z)$, ..., $\beta_4(z)$, $\gamma_1(z)$, ..., $\gamma_4(z)$, $\lambda_1(z)$, $\lambda_0(z)$ be small functions with respect to $f(z)$ satisfying

$$
\begin{aligned}
&\Lambda(z) := \lambda_1(z)^2 - 4\lambda_0(z) \not\equiv 0, \\
&\tilde{\alpha}(z) := \alpha_4(z)^2 - \lambda_1(z)\alpha_3(z)\alpha_4(z) + \lambda_0(z)\alpha_3(z)^2 \not\equiv 0, \\
&\tilde{\beta}(z) := \beta_4(z)^2 - \lambda_1(z)\beta_3(z)\beta_4(z) + \lambda_0(z)\beta_3(z)^2 \not\equiv 0, \\
&\tilde{\gamma}(z) := \gamma_4(z)^2 - \lambda_1(z)\gamma_3(z)\gamma_4(z) + \lambda_0(z)\gamma_3(z)^2 \not\equiv 0.
\end{aligned} \tag{2.5}
$$

Let z_0 be a simple pole of $f(z)$. We say that z_0 is *representable in the second sense* by $\alpha_1(z)$, ..., $\alpha_4(z)$, $\beta_1(z)$, ..., $\beta_4(z)$, $\gamma_1(z)$, ..., $\gamma_4(z)$, $\lambda_1(z)$ and $\lambda_0(z)$, if

$$
\begin{aligned}
f(z) = \frac{R}{z - z_0} + \alpha + \beta(z - z_0) + \gamma(z - z_0)^2 + \delta(z - z_0)^3 \\
+O(z - z_0)^4, \qquad \text{as } z \to z_0
\end{aligned} \tag{2.6}
$$

in a neighbourhood of z_0, and

$$
R^2 + \lambda_1(z_0)R + \lambda_0(z_0) = 0, \qquad \Lambda(z_0) \neq 0, \tag{2.7}
$$

$$
\begin{aligned}
&\alpha = \frac{\alpha_1(z_0)R + \alpha_2(z_0)}{\alpha_3(z_0)R + \alpha_4(z_0)}, \qquad \beta = \frac{\beta_1(z_0)R + \beta_2(z_0)}{\beta_3(z_0)R + \beta_4(z_0)}, \\
&\gamma = \frac{\gamma_1(z_0)R + \gamma_2(z_0)}{\gamma_3(z_0)R + \gamma_4(z_0)}, \qquad \tilde{\alpha}(z_0) \neq 0,\ \tilde{\beta}(z_0) \neq 0,\ \tilde{\gamma}(z_0) \neq 0.
\end{aligned} \tag{2.8}
$$

For the sake of brevity, we call such a simple pole an S2-*pole*. Now we define the stronger and weaker conditions than S2-pole. In addition to the condition (2.5), let $\delta_1(z)$, ..., $\delta_4(z)$ be small functions with respect to $w(z)$ so that

$$
\tilde{\delta}(z) := \delta_4(z)^2 - \lambda_1(z)\delta_3(z)\delta_4(z) + \lambda_0(z)\delta_3(z)^2 \not\equiv 0. \tag{2.9}
$$

Let z_0 be a simple pole of $f(z)$. We say that z_0 is *strongly representable in the second sense* by $\alpha_1(z)$, ..., $\alpha_4(z)$, $\beta_1(z)$, ..., $\beta_4(z)$, $\gamma_1(z)$, ..., $\gamma_4(z)$, $\delta_1(z)$, ..., $\delta_4(z)$, $\lambda_1(z)$ and $\lambda_0(z)$, if $f(z)$ is written as in (2.6), R satisfies (2.7), and α, β, γ, are represented as in (2.8), and

$$
\delta = \frac{\delta_1(z_0)R + \delta_2(z_0)}{\delta_3(z_0)R + \delta_4(z_0)}, \qquad \tilde{\delta}(z_0) \neq 0. \tag{2.10}
$$

For the sake of brevity, we call such a simple pole an SS2-*pole*. We say that z_0 is *weakly representable in the second sense* by $\alpha_1(z), \ldots, \alpha_4(z), \beta_1(z), \ldots, \beta_4(z)$, $\lambda_1(z)$ and $\lambda_0(z)$, if the conditions with respect to α and β in (2.5)–(2.8) hold. For the sake of brevity, we call such a simple pole an WS2-*pole*.

Let z_0 be a pole of $f(z)$ such that $\Lambda(z_0) \neq 0$. We denote by $\mathbf{L}[z_0; R]$ the set of linear transformations of R as in (2.1):

$$\text{(2.1}'\text{)}\quad \begin{aligned} \mathbf{L}[z_0; R] = \mathbf{L}_{(\lambda_1(z),\lambda_0(z))}(z_0) = \Big\{ L = \frac{l_1(z_0)R + l_2(z_0)}{l_3(z_0)R + l_4(z_0)} \Big| & \\ l_j(z),\ j = 1, 2, 3, 4, \text{ small for } f(z), & \\ \text{with } l_4(z_0)^2 - \lambda_1(z_0)l_3(z_0)l_4(z_0) + \lambda_0(z_0)l_3(z_0)^2 \neq 0 \Big\}. & \end{aligned}$$

Let R_1 and R_2 be the roots of (2.7) for a fixed z_0. Since $\Lambda(z_0) \neq 0$, we have $R_1 \neq R_2$. By simple calculation, $L = (a_1(z_0)R + a_2(z_0))/(a_3(z_0)R + a_4(z_0))$, $M = (b_1(z_0)R + b_2(z_0))/(b_3(z_0)R + b_4(z_0)) \in \mathbf{L}(z_0)$, satisfying $L_{|R=R_j} = M_{|R=R_j}$ $j = 1, 2$ if and only if

$$\text{(2.2}'\text{)}\quad \begin{cases} \lambda_0(z_0)(a_1(z_0)b_3(z_0) - b_1(z_0)a_3(z_0)) = a_2(z_0)b_4(z_0) - b_2(z_0)a_4(z_0), \\ \lambda_1(z_0)(a_1(z_0)b_3(z_0) - b_1(z_0)a_3(z_0)) = a_1(z_0)b_4(z_0) - b_1(z_0)a_4(z_0) + \\ \qquad\qquad + a_2(z_0)b_3(z_0) - a_3(z_0)b_2(z_0). \end{cases}$$

Hence, the following (A) and (B) are equivalent to each other:

(A) $L, M \in \mathbf{L}[z_0; R]$, $L \sim M$,

(B) $L, M \in \mathbf{L}[z_0; R]$, $L = M$ under the condition (2.7).

The conditions in (2.5) imply that $\alpha, \beta, \gamma \in \mathbf{L}[z_0; R]$, while (2.9) implies $\delta \in \mathbf{L}[z_0; R]$. In other words, the conditions (2.5) and (2.9) are the criteria for α, β, γ and δ to be resonances or not, see e.g. [1, 718–720], [7, 334–340]. By Proposition 2.2, for any $L \in \mathbf{L}[z_0; R]$, we have a unique entire form $L^* \in \mathbf{L}^*[z_0; R]$ such that $L_{|R=R_j} = L^*_{|R=R_j}$, $j = 1, 2$. From now on, under the condition (2.7), we write $L = (a_1(z_0)R + a_2(z_0))/(a_3(z_0)R + a_4(z_0))$, in the form $A_1(z_0)R + A_2(z_0)$, where $A_1(z)$ and $A_2(z)$ are defined as in Proposition 2.2 (i). We can ascertain that the operations (2.3) and (2.4) in $\mathbf{L}^*[z_0; R]$ are well defined under the condition (2.7). Hence Propositions 2.3 and 2.4 hold for the elements of $\mathbf{L}^*[z_0; R]$. Let $[R]$ be a root of (2.7) for a fixed z_0, where $\lambda_1(z_0)^2 - 4\lambda_0(z_0) \neq 0$. We denote by $[\mathbf{L}]^*[z_0; R]$ the set of values of the elements of $\mathbf{L}^*[z_0; R]$ for $R = [R]$. We obtained the result below, see [4, Lemma 2.1], [5, Lemma 2.4].

Lemma 2.5. *Let $w(z)$ be a transcendental meromorphic function and let $\alpha_1(z)$, $\ldots, \alpha_4(z), \beta_1(z), \ldots, \beta_4(z), \gamma_1(z), \ldots, \gamma_4(z), \delta_1(z), \ldots, \delta_4(z), \lambda_1(z)$ and $\lambda_0(z)$ be small functions with respect to $w(z)$. We denote by $n_{\langle S2\rangle}(r, w)$, $n_{\langle SS2\rangle}(r, w)$ and $n_{\langle WS2\rangle}(r, w)$ the numbers of the S2-poles, the SS2-poles and the WS2-poles of $w(z)$ in $|z| \leqq r$, respectively. The integrated counting function $N_{\langle S2\rangle}(r, w)$,*

$N_{\langle SS2\rangle}(r,w)$ and $N_{\langle WS2\rangle}(r,w)$ are defined in terms of $n_{\langle S2\rangle}(r,w)$, $n_{\langle SS2\rangle}(r,w)$ and $n_{\langle WS2\rangle}(r,w)$ in the usual way, respectively. If

$$m(r,w)+(N(r,w)-N_{\langle S2\rangle}(r,w))=S(r,w), \tag{2.10}$$

then either $w(z)$ satisfies a differential equation of the form

$$c(z)(w')^2+B(z,w)w'+A(z,w)=0, \tag{2.11}$$

where $c(z)$ is a small function with respect to $w(z)$, and $B(z,w)$, $A(z,w)$ are polynomials in w having small coefficients with respect to $w(z)$, or $w(z)$ satisfies a differential equation of second order

$$w''=\tilde{P}(z,w)w'+\tilde{Q}(z,w), \tag{2.12}$$

where $\tilde{P}(z,w)$, $\tilde{Q}(z,w)$ are polynomials in w having small coefficients with respect to $w(z)$. If

$$m(r,w)+(N(r,w)-N_{\langle SS2\rangle}(r,w))=S(r,w), \tag{2.13}$$

then $w(z)$ satisfies a differential equation of the form (2.11). If

$$m(r,w)+(N(r,w)-N_{\langle WS2\rangle}(r,w))=S(r,w), \tag{2.14}$$

then either $w(z)$ satisfies a differential equation of the form (2.11), $w(z)$ satisfies a differential equation of second order (2.12), or $w(z)$ satisfies a differential equation of third order

$$w'''=(\sigma_1(z)w+\sigma_0(z))w''+\sigma_1(z)(w')^2+E(z,w)w'+F(z,w), \tag{2.15}$$

where $\sigma_0(z)$, $\sigma_1(z)$ are small functions with respect to $w(z)$, and $E(z,w)$, $F(z,w)$ are polynomials in w having small coefficients with respect to $w(z)$ with $\deg_w E(z,w)\leqq 2$, $\deg_w F(z,w)\leqq 4$. In particular, if $\lambda_1(z)\equiv 0$, then $\sigma_1(z)\equiv 0$ in (2.15).

The second part in this section is devoted to an exhibition of lemmas on a representable double poles. Let $f(z)$ be a transcendental meromorphic function and let $r_1(z)$, $r_2(z)$, $a_0(z)$, $a_1(z)$, ... , $a_5(z)$ be small functions with respect to $f(z)$. Let z_0 be a double pole of $f(z)$. We call z_0 *representable double pole in the first sense of $f(z)$ by $r_1(z)$, $r_2(z)$, $a_0(z)$, $a_1(z)$, ... , $a_3(z)$*, if $w(z)$ is written in a neighbourhood of z_0 as

$$\begin{aligned} f(z)=\frac{r_2(z_0)}{(z-z_0)^2}+\frac{r_1(z_0)}{z-z_0}+a_0(z_0)+a_1(z_0)(z-z_0)\\ +a_2(z_0)(z-z_0)^2+a_3(z_0)(z-z_0)^3+O(z-z_0)^4,\\ \text{as } z\to z_0. \end{aligned} \tag{2.16}$$

For the sake of simplicity, we abbreviate it D1-*pole*. Further we define the stronger and the weaker conditions than D1-pole. We say that z_0 is ***strongly representable double pole of $f(z)$ in the first sense by $r_1(z)$, $r_2(z)$, $a_0(z)$, $a_1(z)$, ..., $a_5(z)$***, if the coefficients of the first eight terms of the Laurent series of $f(z)$ in a neighbourhood of z_0 are written in terms of the small functions, say, $r_1(z)$, $r_2(z)$, $a_0(z)$, $a_1(z)$, ..., $a_5(z)$, respectively. We further define the notion of the double pole that is ***weakly representable in the first sense by $r_1(z)$, $r_2(z)$, $a_0(z)$, $a_1(z)$***, if the coefficients of the first four terms of the Laurent series of $f(z)$ in a neighbourhood of z_0 are written in terms of the small functions, say, $r_1(z)$, $r_2(z)$, $a_0(z)$, $a_1(z)$, respectively. We simply call them SD1-*pole* and WD1-*pole*, respectively.

We denote by $n_{\langle \mathrm{D1} \rangle}(r, f)$, $n_{\langle \mathrm{SD1} \rangle}(r, f)$ and $n_{\langle \mathrm{WD1} \rangle}(r, f)$ the number of the D1-poles, SD1-poles and WD1-poles of $f(z)$ in $|z| \leqq r$, respectively. The integrated counting function $N_{\langle \mathrm{D1} \rangle}(r, f)$, $N_{\langle \mathrm{SD1} \rangle}(r, f)$ and $N_{\langle \mathrm{WD1} \rangle}(r, f)$ are defined in terms of $n_{\langle \mathrm{D1} \rangle}(r, f)$, $n_{\langle \mathrm{SD1} \rangle}(r, f)$ and $n_{\langle \mathrm{WD1} \rangle}(r, f)$ in the usual way, respectively.

Lemma 2.6. *Let $w(z)$ be a transcendental meromorphic function and let $r_1(z)$, $r_2(z)$, $a_0(z)$, $a_1(z)$, ..., $a_5(z)$ be small functions with respect to $w(z)$. If*

$$m(r, w) + (N(r, w) - N_{\langle \mathrm{D1} \rangle}(r, w)) = S(r, w), \tag{2.17}$$

then $w(z)$ satisfies a differential equation of the form (2.11) or satisfies a differential equation of second order of the form (2.12). If

$$m(r, w) + (N(r, w) - N_{\langle \mathrm{SD1} \rangle}(r, w)) = S(r, w), \tag{2.18}$$

then $w(z)$ satisfies a differential equation of the form (2.11). If

$$m(r, w) + (N(r, w) - N_{\langle \mathrm{WD1} \rangle}(r, w)) = S(r, w), \tag{2.19}$$

then $w(z)$ satisfies a differential equation of the form (2.11), $w(z)$ satisfies a differential equation of the form (2.12) with $\deg_w \tilde{P}(z, w) = 0$, or $w(z)$ satisfies a differential equation of third order of the form (2.15) with $\sigma_1(z) \equiv 0$, say,

$$w''' = \sigma(z) w'' + E(z, w) w' + F(z, w), \tag{2.21}$$

where $E(z, w)$, $F(z, w)$ are polynomials in w having small coefficients with respect to $w(z)$.

Proof of Lemma 2.6. In the proofs of Lemma 2.6 the term "small function" means small meromorphic function with respect $w(z)$. First we consider the case where $w(z)$ satisfies the condition (2.17). Let z_0 be a D1-pole of $w(z)$. We see from (2.16) that the principal parts of the Laurent series at z_0 of the functions $w'(z)^2$, $w(z)^3$, $w'(z)w(z)$, $w(z)^2$, $w''(z)$, $w'(z)$, $w(z)$ are written in terms of small functions directly. Hence there exist small functions $\sigma_2(z)$, $\sigma_3(z)$, ..., $\sigma_6(z)$ and $\tau_2(z)$, $\tau_3(z)$, ..., $\tau_4(z)$, $\kappa_1(z)$, $\kappa_2(z)$ in a neighbourhood of z_0

$$U_1(z, w(z), w'(z)) = \frac{\kappa_1(z_0)}{z - z_0} + O(1), \quad \text{as } z \to z_0, \tag{2.21}$$

where

$$U_1(z,w,w') = (w')^2 + \sigma_6(z)w^3 + \sigma_5(z)w'w + \sigma_4(z)w^2 + \sigma_3(z)w' + \sigma_2(z)w$$

and

$$(2.22) \qquad U_2(z,w(z),w'(z),w''(z)) = \frac{\kappa_2(z_0)}{z-z_0} + O(1), \quad \text{as } z \to z_0,$$

where

$$U_2(z,w,w',w'') = w'' + \tau_4(z)w^2 + \tau_3(z)w' + \tau_2(z)w$$

From (2.21) and (2.22), there exist small functions $\mu_1(z)$, $\mu_2(z)$, which satisfy $|\mu_1| + |\mu_2| \not\equiv 0$ such that

$$U(z,w,w',w'') = \mu_1(z)U_1(z,w,w') + \mu_2(z)U_2(z,w,w',w'')$$

is regular at z_0. Thus, by (2.17) we have that $N(r,U) = S(r,w)$, where $U(z) = U(z,w(z),w'(z),w''(z))$. By (2.17) and the theorem on the logarithmic derivative,

$$m(r,U) \leqq 4m(r,w) + S(r,w) \leqq S(r,w).$$

It follows that $U(z)$ is a small function. Thus the function $w(z)$ satisfies a differential equation of the form (2.11) or satisfies a differential equation of second order of the form (2.12).

Secondly we consider the case where $w(z)$ satisfies the condition (2.18). Let z_0 be a SD1-pole of $w(z)$. It is easy to see that the principal parts of the Laurent series at z_0 of the functions $w(z)^4$, $w'(z)^2w(z)$, $w'(z)w(z)^2$, $w'(z)^2$, $w(z)^3$, $w'(z)w(z)$, $w(z)^2$, $w'(z)$, $w(z)$ are written in terms of small functions directly. Hence there exist small functions $\tilde{\sigma}_2(z)$, $\tilde{\sigma}_3(z)$, ... , $\tilde{\sigma}_8(z)$ and $\tilde{\tau}_2(z)$, $\tilde{\tau}_3(z)$, ... , $\tilde{\tau}_6(z)$, $\tilde{\kappa}_1(z)$, $\tilde{\kappa}_2(z)$ in a neighbourhood of z_0

$$(2.23) \qquad \tilde{U}_1(z,w(z),w'(z)) = \frac{\tilde{\kappa}_1(z_0)}{z-z_0} + O(1), \quad \text{as } z \to z_0,$$

where

$$\begin{aligned}\tilde{U}_1(z,w,w') =& w^4 + \tilde{\sigma}_8(z)(w')^2w + \tilde{\sigma}_7(z)w'w^2 + \tilde{\sigma}_6(z)w^3 + \tilde{\sigma}_5(z)w'w \\ &+ \tilde{\sigma}_4(z)w^2 + \tilde{\sigma}_3(z)w' + \tilde{\sigma}_2(z)w\end{aligned}$$

and

$$(2.24) \qquad \tilde{U}_2(z,w(z),w'(z)) = \frac{\tilde{\kappa}_2(z_0)}{z-z_0} + O(1), \quad \text{as } z \to z_0,$$

where

$$\tilde{U}_2(z,w,w') = w^3 + \tilde{\tau}_6(z)(w')^2 + \tilde{\tau}_5(z)w'w + \tilde{\tau}_4(z)w^2 + \tilde{\tau}_3(z)w' + \tilde{\tau}_2(z)w$$

From (2.23) and (2.24), there exist small functions $\tilde{\mu}_1(z)$, $\tilde{\mu}_2(z)$, which satisfy $|\tilde{\mu}_1| + |\tilde{\mu}_2| \not\equiv 0$ such that

$$\tilde{U}(z, w, w') = \tilde{\mu}_1(z)\tilde{U}_1(z, w, w') + \tilde{\mu}_2(z)\tilde{U}_2(z, w, w')$$

is regular at z_0. Hence using the similar reasoning in the first case, we conclude that $\tilde{U}(z) = \tilde{U}(z, w(z), w'(z))$ is a small function. Thus the function $w(z)$ satisfies a differential equation of the form (2.11).

Finally we consider the case where $w(z)$ satisfies the condition (2.19). Let z_0 be a WD1-pole of $w(z)$. We see that the principal parts of the Laurent series at z_0 of the functions $w'(z)w(z)$, $w(z)^2$, $w'(z)$, $w(z)$, $w''(z)$ $w'''(z)$ are written in terms of small functions directly. Hence there exist small functions $\hat{\sigma}_2(z)$, $\hat{\sigma}_3(z)$, $\dots$, $\hat{\sigma}_5(z)$ and $\hat{\tau}_2(z)$, $\hat{\tau}_3(z)$, $\hat{\tau}_4(z)$, $\hat{\kappa}_1(z)$, $\hat{\kappa}_2(z)$ in a neighbourhood of z_0

$$\hat{U}_1(z, w(z), w'(z), w''(z), w'''(z)) = \frac{\hat{\kappa}_1(z_0)}{z - z_0} + O(1), \quad \text{as } z \to z_0, \tag{2.25}$$

where

$$\hat{U}_1(z, w, w', w'', w''') = w''' + \hat{\sigma}_5(z)w'w + \hat{\sigma}_4(z)w'' + \hat{\sigma}_3(z)w' + \hat{\sigma}_2(z)w$$

and

$$\hat{U}_2(z, w(z), w'(z), w''(z)) = \frac{\hat{\kappa}_2(z_0)}{z - z_0} + O(1), \quad \text{as } z \to z_0, \tag{2.26}$$

where

$$\hat{U}_2(z, w, w', w'') = w^2 + \hat{\tau}_4(z)w'' + \hat{\tau}_3(z)w' + \hat{\tau}_2(z)w.$$

From (2.25) and (2.26), there exist small functions $\hat{\mu}_1(z)$, $\hat{\mu}_2(z)$, $|\hat{\mu}_1| + |\hat{\mu}_2| \not\equiv 0$ such that

$$\hat{U}(z, w, w', w'', w''') = \hat{\mu}_1(z)\hat{U}_1(z, w, w', w'', w''') + \hat{\mu}_2(z)\hat{U}_2(z, w, w', w''')$$

is regular at z_0. It follows from the similar reasoning in the first and the second cases that that $U(z) = \hat{U}(z, w(z), w'(z))$ is a small function. This implies our assertion. □

Furthermore, we state a preliminary lemma relating with representable poles.

Lemma 2.7. *Let $w(z)$ and $H(z)$ be transcendental meromorphic functions such that $m(r, w) + m(r, H) = S(r, w)$, and let $\lambda_0(z)$, $\lambda_1(z)$, $\alpha_1(z)$, $\dots$, $\alpha_4(z)$, $p(z)$ be small functions with respect to $w(z)$. Suppose that*

$$\begin{aligned} &\Lambda(z) := \lambda_1(z)^2 - 4\lambda_0(z) \not\equiv 0 \\ &\tilde{\alpha}(z) := \alpha_4(z)^2 - \lambda_1(z)\alpha_3(z)\alpha_4(z) + \lambda_0(z)\alpha_3(z)^2 \not\equiv 0, \end{aligned} \tag{2.27}$$

and suppose that almost all poles of $H(z)$ are poles of $w(z)$ and for almost all poles z_0 of $w(z)$, we can write $H(z)$ and $w(z)$ in a neighborhood of z_0 as

$$H(z) = \frac{p(z_0)}{z - z_0} + O(1), \qquad \text{as } z \to z_0, \tag{2.28}$$

$$w(z) = \frac{R}{z - z_0} + \alpha + O(z - z_0), \qquad \text{as } z \to z_0, \tag{2.29}$$

with

$$R^2 + \lambda_1(z_0)R + \lambda_0(z_0) = 0, \tag{2.30}$$

$$\alpha = \frac{\alpha_1(z_0)R + \alpha_2(z_0)}{\alpha_3(z_0)R + \alpha_4(z_0)}, \tag{2.31}$$

Then there exist small functions $\eta_0(z)$, $\eta_1(z)$ and $h(z)$ with respect to $w(z)$ such that

$$w^2 - \lambda_1(z)w' - \lambda_0(z)\tilde{H}'(z) + \eta_0(z)w + \eta_1(z)\tilde{H}(z) + h(z) = 0, \tag{2.32}$$

where $\tilde{H}(z) = H(z)/p(z)$.

Proof of Lemma 2.7. We may assume that $\alpha \in [\mathbf{L}]^*[z_0; R]$ and the coefficients of the principle parts of the Laurent expansions of the functions $w(z)$, $w'(z)$, $\tilde{H}'(z)$ belong to $[\mathbf{L}]^*[z_0; R]$ by (2.27)–(2.28). Put $F(z, w, w') = w^2 - \lambda_1(z)w' - \lambda_0(z)\tilde{H}'(z)$. Then by our assumption, we can write $F(z) := F(z, w(z), w'(z))$ in a neighborhood of z_0 as

$$F(z) = \frac{L_1}{z - z_0} + O(1), \qquad \text{as } z \to z_0, \qquad L_1 \in [\mathbf{L}]^*[z_0; R].$$

By Proposition 2.3 there exist small functions $\eta_1(z)$, $\eta_0(z)$ with respect to $w(z)$ such that

$$-h(z) := F(z) + \eta_1(z)w(z) + \eta_0(z)\tilde{H}(z)$$

is regular at z_0. Thus, by our assumption, we get $N(r, h) = S(r, w)$. By virtue of the lemma on the logarithmic derivative,

$$m(r, h) \leqq 2m(r, w) + m(r, \tilde{H}) + S(r, w) + S(r, \tilde{H}) \leqq S(r, w) + S(r, \tilde{H}).$$

By our assumption, we have $T(r, \tilde{H}) = T(r, w) + S(r, w)$, which implies that $S(r, \tilde{H}) = S(r, w)$. Hence $h(z)$ is a small function with respect to $w(z)$, from which the assertion (2.32) follows. □

3. An application to a third order differential equation

In this section, we consider the differential equation of third order

$$w''' = \lambda(z)w'' + R(z,w)w' = \lambda(z)w'' + \frac{P(z,w)}{Q(z,w)}w', \tag{3.1}$$

where $\lambda(z)$ is a meromorphic function and $P(z,w)$ and $Q(z,w)$ are polynomials in w with meromorphic coefficients with $\deg_w P(z,w) = p$ and $\deg_w Q(z,w) = q$, respectively:

$$\begin{cases} P(z,w) = \xi_p(z)w^p + \xi_{p-1}(z)w^{p-1} + \cdots + \xi_0(z), & \xi_p(z) \not\equiv 0 \\ Q(z,w) = \eta_q(z)w^q + \eta_{q-1}(z)w^{q-1} + \cdots + \eta_0(z), & \eta_q(z) \not\equiv 0, \end{cases} \tag{3.2}$$

where $\xi_j(z)$, $j = 0, 1, \ldots, p$, $\eta_k(z)$, $k = 0, 1, \ldots, q$, are meromorphic functions. We suppose that $P(z,w)$ and $Q(z,w)$ are relatively prime. Sometimes we call $\xi_j(z)/\eta_q(z)$, $\eta_k(z)/\eta_q(z)$ as the reduced coefficients of $R(z,w)$. Put $\max(p,q) = \deg_w R(z,w) = d$. We are concerned with the determination of the equation (3.1) that admits a meromorphic solution, and we will treat the equation (3.1) from the function theoretic point of view. Applying the lemmas and the theorems in Section 1.2 and 3.1–3.3 to the equation (3.1), we try to obtain Malmquist–Yosida–Steinmetz type theorems to the equation (3.1), say, we consider the problem: Under what conditions the admissible solution of (3.1) satisfies some lower order differential equation. Recalling the results of second order equation, we know the articles, for instance, Ince [3], Ishizaki [1], v. Rieth [11] and Steinmetz [13]–[18] treated the second order differential equation of the form

$$w'' = \tilde{L}(z,w)(w')^2 + \tilde{M}(z,w)w' + \tilde{N}(z,w), \tag{3.3}$$

where $\tilde{L}(z,w)$, $\tilde{M}(z,w)$ and $\tilde{N}(z,w)$ are rational functions in z and w.
Here we prove the following theorem.

Theorem 3.1. *Suppose that the equation* (3.1) *possesses an admissible solution* $w(z)$. *Then either* $w(z)$ *satisfies a first order differential equation of the form* (2.11) *or the equation* (3.1) *is of the following forms:*

$$w''' = \lambda(z)w'' + (\xi_2(z)w^2 + \xi_1(z)w + \xi_0(z))w', \tag{3.4}$$

$$w''' = \lambda(z)w'' + (\xi_1(z)w + \xi_0(z))w', \tag{3.5}$$

$$w''' = \lambda(z)w'' + \xi_0(z)w'. \tag{3.6}$$

To prove Theorem 3.1, we need Lemmas 3.2–3.6 below, which imply Theorem 3.1.

Lemma 3.2. *Suppose that the equation* (3.1) *possesses an admissible solution* $w(z)$. *Then we have*

$$dT(r,w) \leqq 2\overline{N}(r,w) + N\left(r, \frac{1}{w'}\right) + S(r,w). \tag{3.7}$$

Proof of Lemma 3.2. We may write (3.1) as

$$\frac{w'''}{w'} - \lambda(z)\frac{w''}{w'} = R(z,w) = \frac{P(z,w)}{Q(z,w)}. \tag{3.8}$$

If z_0 is an admissible pole of $w(z)$, then $w'''(z)/w'(z) - \lambda(z)w''(z)/w'(z)$ has a double pole at z_0. Hence

$$N\left(r, \frac{w'''}{w'} - \lambda\frac{w''}{w'}\right) \leqq 2\overline{N}(r,w) + N\left(r, \frac{1}{w'}\right) + S(r,w). \tag{3.9}$$

From (3.9) and by the Valiron–Mokhon'ko theorem [9] and the theorem on the logarithmic derivative

(3.10)

$$\begin{aligned} dT(r,w) &= T(r,R) + S(r,w) = m\left(r, \frac{w'''}{w'} - \lambda\frac{w''}{w'}\right) \\ &+ N\left(r, \frac{w'''}{w'} - \lambda\frac{w''}{w'}\right) + S(r,w) = N\left(r, \frac{w'''}{w'} - \lambda\frac{w''}{w'}\right) + S(r,w). \end{aligned}$$

Thus from (3.9) and (3.10), we obtain (3.7). □

Lemma 3.3. *Suppose that* $q \geqq p$ *in the equation* (3.1), *and suppose that the equation* (3.1) *possesses an admissible solution* $w(z)$. *Then either* (3.1) *is of the form* (3.6), *or* $w(z)$ *satisfies a Riccati equation.*

Proof of Lemma 3.3. Suppose that $w(z)$ has an admissible pole z_0. Since $q \geqq p$, $R(z) := R(z, w(z))$ is regular at z_0. While $w'''(z)/w'(z) - \lambda(z)w''(z)/w'(z)$ has a double pole at z_0, which is a contradiction. Thus, we have $\overline{N}(r,w) = S(r,w)$. Hence, by Lemma 3.2, the theorem on the logarithmic derivative and the first fundamental theorem

(3.11)

$$\begin{aligned} dT(r,w) &\leqq N\left(r, \frac{1}{w'}\right) + S(r,w) \leqq m(r,w') + N(r,w') + S(r,w) \\ &\leqq m\left(r, \frac{w'}{w}\right) + m(r,w) + N(r,w) + \overline{N}(r,w) + S(r,w) \\ &\leqq T(r,w) + S(r,w). \end{aligned}$$

This implies that $d \leqq 1$. If $d = 0$, then (3.1) is of the form (3.6). It remains to treat the case $d = 1$. We write (3.1) as

$$\frac{w'''}{w'} - \lambda(z)\frac{w''}{w'} = \frac{\tilde{\xi}_1(z)w + \tilde{\xi}_0(z)}{w - \eta(z)} = \tilde{\xi}(z) + \frac{\tilde{\xi}_1(z)\eta(z) + \tilde{\xi}_0(z)}{w - \eta(z)}. \tag{3.12}$$

From (3.12), if z_1 is an admissible zero of $w(z) - \eta(z)$, then z_1 is a zero of $w'(z)$ and $\omega(z_1, 1/(w - \eta)) \leqq \omega(z_1, 1/w')$. From (3.12), we get $\overline{N}(r, w) = S(r, w)$. We define $\sigma(z) := w'(z)/(w(z) - \eta(z))$. Then $N(r, \sigma) = S(r, w)$ and by (3.12) and the theorem on the logarithmic derivative

$$m(r, \sigma) \leqq m\left(r, \frac{w' - \eta'}{w - \eta}\right) + m\left(r, \frac{\eta'}{w - \eta}\right) + S(r, w) \leqq S(r, w).$$

Hence, $\sigma(z)$ is a small function with respect to $w(z)$. Therefore, $w(z)$ satisfies a first order differential equation, i.e., $w' = \sigma(z)w - \sigma(z)\eta(z)$. □

For a meromorphic function $f(z)$, we define $\omega(z_0, f)$ as follows: if z_0 is a pole of multiplicity μ ($\geqq 1$) for $f(z)$, then $\omega(z_0, f) = \mu$; if $f(z_0) \neq \infty$, then $\omega(z_0, f) = 0$. We sometimes write $\omega(z_0, 1/(f - a))$ as $\omega(z_0, a; f)$.

Lemma 3.4. *Suppose that $q < p$ in the equation (3.1), and suppose that the equation (3.1) possesses an admissible solution $w(z)$. Then $m(r, w) = S(r, w)$ and there exists an admissible pole. Let z_0 be an admissible pole of $w(z)$. Then we have $\omega(z_0, w) = 2$ or $\omega(z_0, w) = 1$. Further, we have*

(a) $p - q = 1$ *if* $\omega(z_0, w) = 2$.

(b) $p - q = 2$ *if* $\omega(z_0, w) = 1$.

Proof of Lemma 3.4. Write (3.1) as

$$\{\xi_p(z)w\}w^{p-1} = \left(\frac{w'''}{w'} - \lambda(z)\frac{w''}{w'}\right)Q(z, w) - \sum_{j=0}^{p-1} \xi_j(z)w^j. \tag{3.13}$$

We regard $\eta_k(z)(w'''/w' - \lambda(z)w''/w')$, $k = 0, \ldots, q$ as coefficients. Since we assume that $p > q$, the degree of the right side of (3.13) is at most $p - 1$. Thus by the Clunie lemma, we have $m(r, w) = S(r, w)$. Further, from (3.13) and by Theorem 2 (i) in [6], we have $N_{(M}(r, w) = S(r, w)$ for some M. Hence we may assume that $w(z)$ has an admissible pole. Let z_0 be an admissible pole of $w(z)$ and set $\omega(z_0, w) = \mu$. From (3.8), $2 = \omega(z_0, w'''/w' - \lambda w''/w') = \omega(z_0, R) = (p - q)\mu$. This gives the assertion of Lemma 3.4. □

Lemma 3.5. *Suppose that $p - q = 1$ in the equation (3.1), and suppose that the equation (3.1) possesses an admissible solution $w(z)$. Then either (3.1) is of the form (3.5) or $w(z)$ satisfies a first order differential equation of the form (2.11).*

Proof of Lemma 3.5. In view of Lemma 3.4, we see that almost all poles of $w(z)$ are of order 2. Hence, we have

$$T(r, w) = N(r, w) + m(r, w) = 2\overline{N}(r, w) + S(r, w). \tag{3.14}$$

By Lemma 3.2 and the first fundamental theorem,

(3.15)

$$dT(r,w) \leqq 2\overline{N}(r,w) + N\left(r, \frac{1}{w'}\right) + S(r,w) \leqq N(r,w) + N(r,w') + m(r,w') + S(r,w) \leqq 2N(r,w) + \overline{N}(r,w) + S(r,w).$$

Combing (3.14) and (3.15), we get

$$dT(r,w) \leqq \frac{5}{2}T(r,w) + S(r,w),$$

hence $d \leqq 5/2 < 3$. Thus we may consider the cases $p = 1$ and $p = 2$. In case $p = 1$, then $q = 0$, which implies that (3.1) is of the form (3.5). If $p = 2$, then we may suppose that (3.1) is of the form

$$\frac{w'''}{w'} - \lambda(z)\frac{w''}{w'} = \tilde{\xi}_1(z)w + \tilde{\xi}_0(z) + \frac{\xi(z)}{w - \eta(z)}, \quad \tilde{\xi}_1(z) \not\equiv 0, \tag{3.16}$$

where $\tilde{\xi}_1(z)$, $\tilde{\xi}_0(z)$, $\xi(z)$ and $\eta(z)$ are small functions with respect to $w(z)$. Let z_0 be an admissible pole of $w(z)$. Write $w(z)$ in a neighbourhood of z_0 as

$$w(z) = \frac{R_2}{(z-z_0)^2} + \frac{R_1}{z-z_0} + O(1), \quad \text{as } z \to z_0,\ R_2 \neq 0. \tag{3.17}$$

From (3.16) and (3.17), we get

$$R_2 = \frac{12}{\tilde{\xi}_1(z_0)} \quad \text{and} \quad R_1 = \frac{1}{5}\left(-\frac{48\tilde{\xi}_1'(z_0)}{\tilde{\xi}_1(z_0)^2} + \frac{12\lambda(z_0)}{\tilde{\xi}_1(z_0)}\right). \tag{3.18}$$

From (3.18), we can write R_2 and R_1 in terms of small functions. Define

$$R_2(z) := \frac{12}{\tilde{\xi}_1(z)} \quad \text{and} \quad R_1(z) := \frac{1}{5}\left(-\frac{48\tilde{\xi}_1'(z)}{\tilde{\xi}_1(z)^2} + \frac{12\lambda(z)}{\tilde{\xi}_1(z)}\right),$$

and define $\varphi(z) := w'(z)/(w(z) - \eta(z))$. Then by (3.16), Lemma 3.4 and the theorem on the logarithmic derivative,

$$m(r,\varphi) \leqq m\left(r, \frac{w' - \eta'}{w - \eta}\right) + m\left(r, \frac{\eta'}{w-\eta}\right) + S(r,w) \leqq S(r,w). \tag{3.19}$$

From (3.16), if z_1 is an admissible zero of $w(z) - \eta(z)$, then z_1 is a zero of $w'(z)$ and $\omega(z_1, 1/(w-\eta)) \leqq \omega(z_1, 1/w')$. Hence z_1 cannot be a pole of $\varphi(z)$. Thus if z_0 is a pole of $\varphi(z)$ and neither a zero nor a pole of the coefficients, then we may assume that z_0 is an admissible pole of $w(z)$. Using (3.18), we write $\varphi(z)$ in a neighbourhood of z_0 as

$$\varphi(z) = \frac{-2}{z-z_0} + \frac{R_1(z_0)}{R_2(z_0)} + O(z-z_0), \quad \text{as } z \to z_0. \tag{3.20}$$

Put $\sigma(z) = w(z) - \mu_1(z)\varphi^2(z) + \mu_2(z)\varphi(z)$, where $\mu_1(z) = R_2(z)/4$ and $\mu_2(z) = -(2R_1(z) + R_2'(z))/2$. Then from (3.17), (3.18) and (3.20), $\sigma(z)$ is regular at z_0. This implies that $N(r,\sigma) = S(r,w)$. From (3.19),

$$m(r,\sigma) \leqq m(r,w) + 2m(r,\varphi) + S(r,w) \leqq S(r,w),$$

hence $\sigma(z)$ is a small function with respect to $w(z)$. Therefore, we conclude that $w(z)$ satisfies a first order equation of the form (2.11). □

Lemma 3.6. *Suppose that $p - q = 2$ in the equation (3.1), and suppose that the equation (3.1) possesses an admissible solution $w(z)$. Then either (3.1) is of the form (3.4) or $w(z)$ satisfies a first order differential equation of the form (2.11).*

Proof of Lemma 3.6. By Lemma 3.2 and the second fundamental theorem,

$$dT(r,w) \leqq 2\overline{N}(r,w) + N\left(r, \frac{1}{w'}\right) + S(r,w) \leqq 4T(r,w) + S(r,w),$$

which gives that $d \leqq 4$. Thus we consider the case $p = 2$, 3 and 4. If $p = 2$, then $q = 0$, which implies that (3.1) is of the form (3.4). It remains to treat the cases $p = 3$ and 4.

First we will treat the case $p = 4$. Let z_1 be an admissible zero of $Q(z) := Q(z, w(z))$. From (3.8) and by Theorem 1 (ii) in [6], z_1 must be a zero of $w'(z)$. Put $\sigma(z) = w'(z)/Q(z)$. Then $\sigma(z)$ is regular at z_1. Let z_0 be an admissible pole of $w(z)$. By Lemma 3.4 (b), $\omega(z_0, w) = 1$. Since $\omega(z_0, w') = 2$ and $\omega(z_0, Q) = 2$, $\sigma(z)$ is also regular at z_0. Therefore, $N(r,\sigma) = S(r,w)$. By the theorem on the logarithmic derivative,

$$m(r,\sigma) \leqq m\left(r, \frac{1}{Q}\right) + m\left(r, \frac{w'}{w}\right) + m(r,w) + S(r,w) \leqq S(r,w).$$

This means that $\sigma(z)$ is a small function with respect to $w(z)$. Hence, $w(z)$ satisfies a Riccati equation, i.e., $w' = \sigma(z)Q(z,w)$.

Finally we treat the case $p = 3$. In this case, we may write (3.1) as

$$\frac{w'''}{w'} - \lambda(z)\frac{w''}{w'} = \tilde{\xi}_2(z)w^2 + \tilde{\xi}_1(z)w + \tilde{\xi}_0(z) + \frac{\xi(z)}{w - \eta(z)}, \quad \tilde{\xi}_2(z) \not\equiv 0, \tag{3.21}$$

where $\tilde{\xi}_2(z)$, $\tilde{\xi}_1(z)$, $\tilde{\xi}_0(z)$, $\xi(z)$ and $\eta(z)$ are small functions with respect to $w(z)$. We define $\varphi(z) := w'(z)/(w(z) - \eta(z))$. Similarly to the case $p = 4$, if z_1 is an admissible zero of the $w(z)-\eta(z)$, then $\varphi(z)$ is regular at z_1. Thus if z_0 is a pole of $\varphi(z)$ neither a zero nor a pole of the coefficients, then z_0 is an admissible pole of $w(z)$. By Lemma 3.4 (b), almost all poles of $w(z)$ are simple poles. We write $w(z)$ in a neighbourhood of z_0 as

$$w(z)=\frac{R}{z-z_0}+\alpha+O(z-z_0), \quad \text{as } z\to z_0,\ R\neq 0. \tag{3.22}$$

From (3.21) and (3.22), we have

$$\tilde{\xi}_2(z_0)R^2-6=0, \tag{3.23}$$

$$2\tilde{\xi}_2(z_0)R\alpha=-\tilde{\xi}_2'(z_0)R^2-\tilde{\xi}_1(z_0)R+2\lambda(z_0). \tag{3.24}$$

We have

$$\varphi(z)=\frac{-1}{z-z_0}+O(1), \quad \text{as } z\to z_0. \tag{3.25}$$

Hence, by (3.23)–(3.25) and Lemma 2.7, $w(z)$ satisfies a differential equation of the form (2.11). Thus Lemma 3.6 is proved. □

By Theorem 3.1, under the assumption of the existence of an admissible solution, we can assume that (3.1) is of the form

$$w'''=\lambda(z)w''+\Xi(z,w)w', \tag{3.26}$$

where $\Xi(z,w)$ is a polynomials in w with meromorphic coefficients with $\deg_w \Xi(z,w)\leqq 2$. Moreover, we show the following theorem.

Theorem 3.7. *Suppose that $\lambda(z)\not\equiv 0$ in the equation (3.26) and suppose that the equation (3.26) possesses an admissible solution $w(z)$. Then $w(z)$ satisfies a linear differential equation, satisfies a differential equation of the form (2.11) or satisfies a differential equation of second order of the form (2.12) with $\deg_w \tilde{P}(z,w)=0$.*

Proof of Theorem 3.7. According to Lemma 3.4, we will divide the proof into two cases:

(a) $\deg_w \Xi(z,w)=2$, i.e., the equation (3.26) is of the form (3.4);
(b) $\deg_w \Xi(z,w)=1$, i.e., the equation (3.26) is of the form (3.5).

First we consider the case (a). It was said in Lemma 3.4 that almost all poles of $w(z)$ are simple poles and $m(r,w)=S(r,w)$. Let z_0 be an admissible simple pole of $w(z)$. We write $w(z)$ in a neighbourhood of z_0 as

$$w(z)=\frac{R}{z-z_0}+\alpha+\beta(z-z_0)+O(z-z_0)^2, \quad R\neq 0, \quad \text{as } z\to z_0. \tag{3.27}$$

Using Test-power text, we get

$$\xi_2(z_0)R^2 - 6 = 0, \tag{3.28}$$

$$(\xi_2(z_0)R)\alpha = P_1(R; z_0) \tag{3.29}$$

$$(\xi_2(z_0)R)\beta = P_2(R, \alpha; z_0) \tag{3.30}$$

where $P_j(\,.\,; z_0)$, $j = 1$, 2, are polynomials in the corresponding arguments with small coefficients. It follows from (3.28)–(3.30) that z_0 is a WS2-pole in terms of the fixed small functions. Hence we see by Lemma 2.5 that $w(z)$ satisfies a first order differential equation of the form (2.11) or satisfies a third order differential equation of the form (2.15) with $\sigma_1(z) \equiv 0$, i.e., (2.19). We may assume that $F(z, w) \not\equiv 0$ in (2.19). In fact, recalling the proof of Lemma 2.5 of the part of WS2-pole, see [5, proof of Lemma 4], $w(z)$ satisfies the equation

$$\phi(z) = \kappa_1(z)D_{11}(z, w, w', w'', w''') + \kappa_2(z)\bar{D}_{11}(z, w, w', w'') + \kappa_3(z)w,$$

where

$$\begin{aligned} D_{11}(z, w, w', w'', w''') =& 6w'w^2 - 2\lambda_1(z)(w''w + (w')^2) + \lambda_0(z)w''' \\ &+ \eta_{21}(z)ww' + \eta_{22}(z)w^3 + \nu_{11}(z)w' + \nu_{12}(z)w^2, \\ \bar{D}_{11}(z, w, w', w'') =& w'' + \bar{\eta}_{21}(z)ww' + \bar{\eta}_{22}(z)w^3 + \bar{\nu}_{11}(z)w' + \bar{\nu}_{12}(z)w^2. \end{aligned}$$

If $\bar{\kappa}_3(z) \not\equiv 0$, then we have nothing to prove. It is not difficult to see that we can admit the term $\bar{\eta}_{22}(z)w''$ in the construction of $\bar{D}_{11}(z, w, \dots, w''')$ in the place $\bar{\eta}_{22}(z)w^3$. Hence if $\bar{\kappa}_2(z) \not\equiv 0$, the assertion follows. So, we consider the case $\bar{\kappa}_2(z) \equiv 0$ and $\bar{\kappa}_3(z) \equiv 0$, which implies that $\bar{D}_{11}(z, w(z), \dots, w'''(z)) \equiv 0$. If $\bar{\eta}_{22}(z) \not\equiv 0$ or $\bar{\nu}_{12}(z) \not\equiv 0$, then the assertion is true. In case $\bar{\eta}_2(z) \equiv 0$ and $\bar{\nu}_{12}(z) \equiv 0$, then we see that $w(z)$ satisfies a third order differential equation of the form

$$w''' = \tilde{\Xi}(z, w)w',$$

where $\Xi(z, w)$ is a polynomial in w having small coefficients with respect to $w(z)$ with $\deg_w \Xi(z, w) \leqq 2$. Hence, we conclude that $w(z)$ satisfies a differential equation of the form (2.12) because of $\lambda(z) \not\equiv 0$.

Secondly we treat the case (b). We also have by Lemma 3.4 that almost all poles of $w(z)$ are double poles and $m(r, w) = S(r, w)$. Let z_0 be an admissible double pole of $w(z)$. We write $w(z)$ in a neighbourhood of z_0 as

$$w(z) = \frac{R_2}{(z - z_0)^2} + \frac{R_1}{z - z_0} + a_0 + a_1(z - z_0) + O(z - z_0)^2, \quad R_2 \neq 0, \quad \text{as } z \to z_0. \tag{3.31}$$

Using Test-power test again, we get

$$\xi_1(z_0)R_2 - 12 = 0, \tag{3.32}$$

$$\left(3\xi_1(z_0) - \frac{6}{R_2}\right)R_1 = Q_1(R_2; z_0), \tag{3.33}$$

$$(\xi_1(z_0)R_2)a_0 = Q_2(R_1, R_2; z_0), \tag{3.34}$$

$$(\xi_1(z_0)R_2)a_1 = Q_3(R_1, R_2, a_0; z_0). \tag{3.35}$$

where $Q_j(\,.\, ; z_0)$, $j = 1, 2, 3$ are polynomials in the corresponding arguments with small coefficients. It follows from (3.32)–(3.35) that z_0 is a WD1-pole by the fixed small functions. Hence we see by Lemma 2.6 that $w(z)$ satisfies a first order differential equation of the form (2.11), satisfies a second order differential equation of the form (2.12), or satisfies a third order differential equation of the form (2.15). As in the case (a), we may assume that $F(z, w) \not\equiv 0$ in (2.15). In fact, we recall the proof of Lemma 2.6 of the part of WD1-pole. If $\hat{\mu}_2(z) \not\equiv 0$, then the assertion follows. Hence, we consider the case $\hat{\mu}_2(z) \equiv 0$, that is to say, $\hat{U}_1(z, w(z), \dots, w'''(z)) \equiv 0$. Similarly to the case (a), if $\hat{\sigma}_2(z) \equiv 0$, then $w(z)$ satisfies a differential equation of the form (2.15). In case $\hat{\sigma}_2(z) \not\equiv 0$, then obviously we have $F(z, w) \not\equiv 0$.

It remains to show that if $w(z)$ satisfies the third order differential equations (3.26) and (2.15) at the same time, then $w(z)$ satisfies a first order differential equation of the form (2.11) or satisfies a equation of the form (2.15). Actually, combining (3.26) and (2.15), we see that $w(z)$ satisfies the second order differential equation

$$(\sigma(z) - \lambda(z))w'' + (E(z, w) - \Xi(z, w))w' + F(z, w) = 0. \tag{3.36}$$

Since $F(z, w) \not\equiv 0$, we may assume that $\sigma(z) - \lambda(z) \not\equiv 0$, nevertheless $w(z)$ satisfies a Riccati equation by Theorem 2.1. From (3.26) and (3.36), we get, by simple computation,

$$A_w(z, w)(w')^2 + T(z, w)w' + U(z, w) = 0, \tag{3.37}$$

where

$$A(z, w) = \frac{E(z, w) - \Xi(z, w)}{\sigma(z) - \lambda(z)}, \quad B(z, w) = \frac{-F(z, w)}{\sigma(z) - \lambda(z)},$$
$$T(z, w) = A_z(z, w) + B_w(z, w) + \Xi(z, w) - A(z, w)(\lambda(z) + A(z, w)),$$
$$U(z, w) = B_z(z, w) - B(z, w)(\lambda(z) + A(z, w)).$$

From (3.37), if one of the polynomials $A_w(z, w)$, $T(z, w)$ and $U(z, w)$ does not vanish, then $w(z)$ satisfies a first order differential equation. So, we consider the case $A_w(z, w)$, $T(z, w)$ and $U(z, w)$ all vanish. From $A_w(z, w) \equiv 0$, we have $\deg_w A(z, w) = 0$. It follows from $T(z, w) \equiv 0$ that $\deg_w B_w(z, w) = \deg_w \Xi(z, w)$. This means that $\deg_w F(z, w) \leqq 3$. Hence, we have proved Theorem 3.2. $\square$

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References

[1] M. J. Ablowitz, A. Ramani and H. Segur, *A connection between nonlinear evolution equations and ordinary differential equations of P-type, I*, J. Math. Phys. **21** (1980), 715–721.

[2] W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[3] E. L. Ince, *Ordinary differential equations*, Dover, New York, 1956.

[4] K. Ishizaki, *Admissible solutions of second order differential equations*, Tôhoku Math. Jour. Math. **44** (1992), 305–325.

[5] ———, *On third order differential equations with admissible solutions*, Preprint.

[6] K. Ishizaki and N. Yanagihara, *On admissible solutions of algebraic differential equations*, Preprint.

[7] M. D. Kruskal, A. Ramani and B. Grammaticos, *Singularity analysis and its relation to complete partial and non-integrability*, Partially Integrable Evolutional Equations in Physics (R.Conte and N.Boccara, eds.) (1990), 321–372, Kluwer Academic Publ., Dordrecht–Boston–London.

[8] I. Laine, *Nevanlinna theory and complex differential equations*, W. Gruyter, Berlin–New York, 1992.

[9] A. Z. Mokhon'ko and V. D. Mokhon'ko, *Estimates for the Nevanlinna characteristic of some classes of meromorphic functions and their applications to differential equations*, Sib. Math. J. **15** (1974), 921–934.

[10] R. Nevanlinna, *Analytic Functions*, Springer-Verlag, Berlin–Heidelberg–New York, 1970.

[11] J. v. Rieth, *Untersuchungen gewisser Klassen gewöhnlicher Differentialgleichungen erster und zweiter Ordnung im Komplexen*, Doctoral Dissertation, Aachen, 1986.

[12] N. Steinmetz, *Ein Malmquistscher Satz für algebraische Differentialgleichungen erster Ordnung*, J. Reine Angew. Math. **316** (1980), 44–53.

[13] ———, *Zur Wertverteilung der Lösungen der vierten Painlevéschen Differentialgleichung*, Math. Z. **181** (1982), 553–561.

[14] ———, *Über eine Klasse von Painlevéschen Differentialgleichungen*, Arch. Math. **4** (1983), 261–266.

[15] ———, *Ein Malmquistscher Satz für algebraische Differentialgleichungen zweiter Ordnung*, Resultate Math. **10** (1986), 152–167.

[16] ———, *Meromorphic solutions of second-order algebraic differential equations*, Complex Variables Theory Appl. **13** (1989), 75–83.

[17] ——, *Meromorphe Löusungen der Differentialgleichungen* $Q(z,w)(d^2w/dz^2)^2 = P(z,w)(dw/dz)^2$, Complex Variables Theory Appl. **10** (1988), 31–41.

[18] ——, *Algebraische Differentialgleichungen erster und zweiter Ordung, Complex methods on partial differential equations, (Mathematical Research, edited by C. Withalm)*, vol. 53, Akademie–Verlag, Berlin, 1989, pp. 187–192.

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