Nihonkai Math. J.
Vol.12(2001), 165-195

Minimal singular compactifications of the affine plane

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Abstract. Let X be a minimal compactification of the complex affine plane $\mathbf{C}^2$. In this paper, we show that X is a log del Pezzo surface of rank one and determine the singularity type of X in the case where X has at most quotient singularities.

2000 Mathematical Subject Classification. Primary 14J26, Secondary 14J17.
Key words and phrases. Compactification, Log del Pezzo surface.

0 Introduction

A normal compact complex surface X is called a *compactification* of the complex affine plane $\mathbf{C}^2$ if there exists a closed subvariety Γ of X such that $X - \Gamma$ is biholomorphic to $\mathbf{C}^2$. We denote simply the compactification by the pair (X, Γ). A compactification (X, Γ) of $\mathbf{C}^2$ is said to be *minimal* if Γ is irreducible.

Remmert–Van de Ven [26] proved that if (X, Γ) is a minimal compactification of $\mathbf{C}^2$ and X is smooth then $(X, \Gamma) = (\mathbf{P}^2, \text{line})$. Brenton [3], Brenton–Drucker–Prins [4] and Miyanishi–Zhang [21] studied minimal compactifications of $\mathbf{C}^2$ with at most rational double points and proved the following results.

Theorem 0.1 (cf. [3], [4] and [21]) *If (X, Γ) is a minimal compactification of $\mathbf{C}^2$ and X has at most rational double points, then X is a log del Pezzo surface of rank one (for the definition, see Definition 2.1). Further, if $\operatorname{Sing} X \neq \emptyset$, then the singularity type of X is given as one of the following:*

$$A_1,\ A_1 + A_2,\ A_4,\ D_5,\ E_6,\ E_7,\ E_8.$$

Conversely, if X is a Gorenstein log del Pezzo surface of rank one such that the singularity type of X is given as one of the listed as above, then X is a minimal compactification of $\mathbf{C}^2$.

Theorem 0.2 (cf. [21, Theorem 2]) *Let X be a Gorenstein log del Pezzo surface of rank one. Then X is a minimal compactification of $\mathbf{C}^2$ if and only if $\pi_1(X - \mathrm{Sing}\, X) = (1)$.*

Recently, Furushima [7] classified minimal compactifications of $\mathbf{C}^2$ which are normal hypersurfaces of degree ≤ 4 in $\mathbf{P}^3$.

In the present article, we study minimal compactifications of $\mathbf{C}^2$ with at most quotient singular points (cf. [2]). Let X be a minimal compactification of $\mathbf{C}^2$ with at most quotient singular points. We prove that X is a log del Pezzo surface of rank one and determine the singularity type of X (see Theorem 1.1).

Here, we propose the following problems:

Problem 1 (Converse of Theorem 1.1) Let X be a log del Pezzo surface of rank one. Assume that the singularity type of X is given as one of the listed in Appendix C. Is then X a minimal compactification of $\mathbf{C}^2$?

Problem 2 (cf. [20]) Let X be a log del Pezzo surface of rank one. Assume that $\pi_1(X - \mathrm{Sing}\, X) = (1)$. Is then X a minimal compactification of $\mathbf{C}^2$?

In general, Problems 1 and 2 are false (see §§3 and 4). However, Theorems 0.1 and 0.2 imply that Problems 1 and 2 are true in the case where X has at most rational double points. Recently, the author [17] classified the log del Pezzo surfaces of rank one and of index two (see [17, Theorem 1]). By [17, Theorem 1], we know that Problems 1 and 2 are true if the index of X is equal to two. We prove that Problem 1 is true if the index of X is equal to three (Theorem 1.2).

In our forthcoming paper, we prove the following result.

With the same notation and assumptions as in Problem 1, assume further that X has a non-cyclic quotient singular point. Then X is a minimal compactification of $\mathbf{C}^2$.

TERMINOLOGY. A $(-n)$-curve is a nonsingular complete rational curve with self-intersection number $-n$. A reduced effective divisor D is called an NC-divisor (resp. an SNC-divisor) if D has only normal (resp. simple normal) crossings. We employ the following notation:

K_X : the canonical divisor on X.

$\overline{\kappa}(X - D)$: the logarithmic Kodaira dimension of an open surface $X - D$ (cf. [11], etc.).

$\rho(X)$: the Picard number of X.

$\mathbf{F}_n (n \geq 0)$: the Hirzebruch surface of degree n.

$M_n(n \geq 0)$: a minimal section of a fixed ruling on $\mathbf{F}_n$.

$\# D$: the number of all irreducible components in Supp D.

1 Results

We state the main results of the present article.

In §3, we prove the following result.

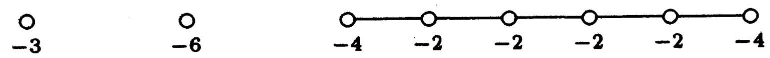
Theorem 1.1 *Let (X, Γ) be a minimal compactification of $\mathbf{C}^2$. Assume that X has at most quotient singular points and* Sing $X \neq \emptyset$. *Then the following assertions hold true:*

(1) *X is a log del Pezzo surface of rank one.*

(2) *Let $\pi : (V, D) \to X$ be the minimal resolution of X, where D is the reduced exceptional divisor, and let C be the proper transform of Γ on V. Then, $C \cong \mathbf{P}^1$, the divisor $C + D$ is an SNC-divisor and the weighted dual graph of $C + D$ is given as* (n) $(1 \leq n \leq 32)$ *in Appendix C. In particular,* $\#\mathrm{Sing}\, X \leq 2$.

In §5, we prove the following result.

Theorem 1.2 *Let X be a log del Pezzo surface of rank one. Assume that the index of X is equal to three, i.e.,* $\min\{n \in \mathbf{N} |\ nK_X$ *is Cartier*$\} = 3$. *Then X is a minimal compactification of $\mathbf{C}^2$ if and only if the singularity type of X is given as one of the following weighted dual graphs* (1) $\sim$ (11).



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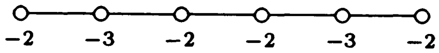

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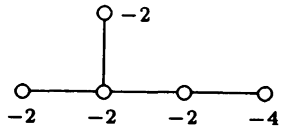

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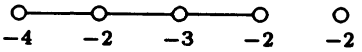

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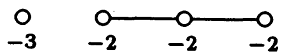

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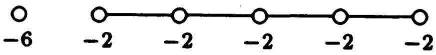

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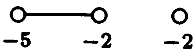

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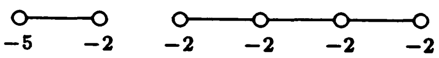

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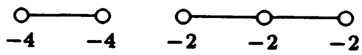

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2 Preliminary results

We recall some basic notions in the theory of peeling (cf. [19] and [22]). Let (X, D) be a pair of a nonsingular projective surface X and an SNC-divisor D on X. We call such a pair (X, D) an *SNC-pair*. A connected curve T consisting of irreducible components of D (a connected curve in D, for short) is a *twig* if the dual graph of T is a linear chain and T meets $D - T$ in a single point at one of the end components of T, the other end of T is called the *tip* of T. A connected curve R (resp. F) in B is a *rod* (resp. *fork*) if R (resp. F) is a connected component of D and the dual graph of R (resp. F) is a linear chain (resp. the dual graph of the exceptional curves of a minimal resolution of a non-cyclic quotient singularity (cf. [2])). A connected curve E in D is *rational* (resp. *admissible*) if each irreducible component of E is rational (resp. if there are no (-1)-curves in Supp E and the intersection matrix of E is negative definite). An admissible rational twig T in D is *maximal* if T is not extended to an admissible rational twig with more irreducible components of D. For the list of the weighted dual graphs of all admissible rational forks, see [22, pp. 55 $\sim$ 56] and [19, pp. 207 $\sim$ 208].

Now, let A be an admissible rational rod. Then the weighted dual graph of A is given as in Figure 1. Then we denote the admissible rational rod A by $[a_1, \ldots, a_r]$. We denote the determinant of A by $d(A)$ (cf. [22, p. 87], [6, (3.3)], etc.). The admissible rational rod $[a_r, \ldots, a_1]$ is called the *transposal* of A and denoted by ${}^tA$. We define also $\bar{A} = [a_2, \ldots, a_r]$ and $\underline{A} = [a_1, \ldots, a_{r-1}]$. We call $e(A) = d(\bar{A})/d(A)$ the *inductance* of A. By [6, Corollary (3.8)] (see also [5, Proposition A.5]), e defines a one-to-one correspondence from the set of all admissible rational rods to the set of rational numbers in the interval $(0, 1)$. Hence there exists uniquely an admissible rational rod A^* whose inductance is equal to $1 - e({}^tA)$. We call the admissible rational rod A^* the *adjoint* of A.

$(r \geq 1,\ a_i \geq 2\ (1 \leq i \leq r))$

Figure 1

We state some results concerning log del Pezzo surfaces of rank one wihch will be used in §§3 $\sim$ 5.

Definition 2.1 A log del Pezzo surface X is a normal projective surface satisfying the following two conditions:

(i) X is singular but has at most quotient singularities.

(ii) The anticanonical divosor $-K_X$ is ample.

X is said to have rank one if $\rho(X) = 1$.

Let X be a log del Pezzo surface of rank one and let $\pi : (V, D) \to X$ be the minimal resolution of X, where D is the reduced exceptional divisor. Let $D = \sum_i D_i$ be the decomposition of D into irreducible components. Then there exists uniquely an effective $\mathbf{Q}$-divisor $D^\# = \sum_i \alpha_i D_i$ such that $D^\# + K_V$ is numerically equivalent to $\pi^* K_X$. In Lemmas 2.2 $\sim$ 2.6, we retain this situation.

Lemma 2.2 *With the same notation as above, we have:*

(1) $-(D^\# + K_V)$ *is nef and big. Moreover, for any irreducible curve* F, $-(D^\# + K_V \cdot F) = 0$ *if and only if* F *is a component of* D.

(2) *Every* $(-n)$*-curve with* $n \geq 2$ *is a component of* D.

(3) V *is a rational surface.*

Proof. See [27, Lemma 1.1].

Lemma 2.3 *There is no* (-1)*-curve* E *such that, after contracting* E *and consecutively (smoothly) contractible curves in* $E + D$*, the divisor* $E + D$ *becomes a union of admissible rational rods and forks.*

Proof. See [27, Lemma 1.4].

By Lemma 2.2 (1), we can find an irreducible curve M such that $-(M \cdot D^\# + K_V)$ attains the smallest positive value. In Lemmas 2.4 and 2.5, we fix such a curve M.

Lemma 2.4 *Suppose that* $|M + D + K_V| \neq \emptyset$ *and* X *has a singular point* P *which is not a rational double point. Then* P *is a cyclic quotient singular point and the other singular points on* X *are rational double points.*

Proof. By [27, Lemma 2.1], there exists a unique decomposition of D as a sum of effective integral divisors $D = D' + D''$ such that:

(i) $(M \cdot D_i) = (D'' \cdot D_i) = (K_V \cdot D_i) = 0$ for any component D_i of D'.

(ii) $M + D'' + K_V \sim 0$.

Then Supp $D' \cap$ Supp $D'' = \emptyset$ and each connected component of D' can be contracted to a rational double point. By the hypothesis, $D'' \neq 0$. Since $M + D'' + K_V \sim 0$, we know that $D'' = \pi^{-1}(P)$ and D'' is a linear chain of smooth rational curves. Q.E.D.

Suppose that $|M + D + K_V| = \emptyset$. The divisor $M + D$ is then an SNC-divisor, consisting of smooth rational curves and the dual graph of $M + D$ is a tree (see [27, Proof of Lemma 2.2]). Here we note the following lemma.

Lemma 2.5 *Suppose that* $|M+D+K_V| = \emptyset$*. Then either* (V, D) *is* $(\mathbf{F}_n, M_n)$*, where* $n = -(D^2) \geq 2$*, or we may assume that* M *is a* (-1)*-curve.*

Proof. See [27, Lemma 2.2] and [8, Proposition 3.6].

We recall the results in [15] concerning a classification of log del Pezzo surfaces of rank one with unique singular points.

Lemma 2.6 *Suppose that* $\#\mathrm{Sing}\, X = 1$. *Put* $P := \mathrm{Sing}\, X$. *Then the following assertions hold true:*

(1) *If* P *is a quotient singular point of type* E_n $(n = 6, 7, 8)$ (*cf.* [15] *and* [19, p. 208]), *then there exists a* (-1)*-curve* C *such that* $C + D$ *is an SNC-divisor and the weighted dual graph of* $C + D$ *is given as* (n) $(4 \leq n \leq 13)$ *in Appendix C. In particular,* X *is a minimal compactification of* $\mathbf{C}^2$.

(2) *Assume that* P *is a quotient singular point of type* D, *i.e., the weighted dual graph of* D *is given as in Figure 2, where* $r \geq 3$ *and* $a_i \geq 2$ *for* $i = 0, 3, \ldots, r$. *Then there exists a* (-1)*-curve* E *such that* $(E \cdot D) = (E \cdot D_t) = 1$, *where* $t = 1$ *or* 2.

(3) *Assume that* P *is a cyclic quotient singular point. Then there exists a* (-1)*-curve* C *such that* $(C \cdot D) = 1$. *Moreover,* $X - \{P\}$ *contains* $\mathbf{C}^* \times \mathbf{C}^*$, *where* $\mathbf{C}^* = \mathbf{C} - \{0\}$, *as a Zariski open subset.*

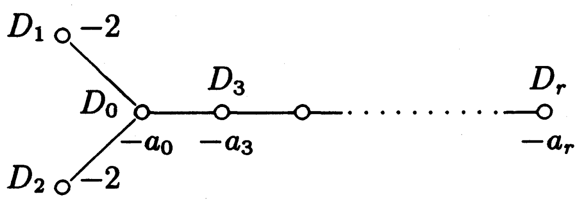


Figure 2

Proof. (1) See [15, Theorem 2.1].
(2) See [15, Theorem 3.1].
(3) See [15, Theorem 4.1].

We recall Morrow's result [24, Theorem 9] concerning minimal normal compactifications of $\mathbf{C}^2$.

Definition 2.7 Let S be a smooth complex affine surface and let (V, D) be a pair of a smooth projective surface V and an NC-divisor D on V. We call the pair (V, D) a *normal compactification* (resp. a *normal algebraic* *compactification*) of S if S is biholomorphic (resp. isomorphic) to $V - D$. A normal compactification (or a normal algebraic compactification) (V, D) of S is said to be *minimal* if $(E \cdot D - E) \geq 3$ for any (-1)-curve $E \subset \mathrm{Supp}\, D$.

Morrow [24] gave a list of all minimal normal compactifications of $\mathbf{C}^2$. In §3, we use the following result.

Lemma 2.8 *Let (V, D) be a minimal normal compactification of $\mathbf{C}^2$. Then D is an SNC-divisor, each irreducible component of D is a smooth rational curve and the dual graph of D is a linear chain. Moreover, if $\rho(V) \geq 3$, then D contains exactly two irreducible components, say D_1 and D_2, with non-negative self-intersection numbers and $(D_1 \cdot D_2) = 1$.*

Proof. See [24].

For the list of all boundary dual graphs of the minimal normal compactifications of $\mathbf{C}^2$, see [24, Theorem 9].

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Let (X, Γ) be a minimal compactification of $\mathbf{C}^2$. Assume that $\mathrm{Sing}\, X \neq \emptyset$ and X has only quotient singular points as singularities. Let $\pi : (V, D) \to X$ be the minimal resolution of X, where D is the reduced exceptional divisor, and let C be the proper transform of Γ on V.

Proof of the assertion (1). By [24, Theorems 9 $\sim$ 11] (see also [13]), V is a smooth projective rational surface and $V - D$ is isomorphic to $\mathbf{C}^2$ as an algebraic variety. So X is a normal projective rational surface by [1] and $\overline{\kappa}(X - \mathrm{Sing}\, X) = -\infty$. Since X is a minimal compactification of $\mathbf{C}^2$, we have $\rho(X) = 1$. Hence [27, Remark 1.2] and [19, Lemma 2.7] imply that X is a log del Pezzo surface of rank one. This proves the assertion (1).

Proof of the assertion (2). Let $D = \sum_i D_i$ be the decomposition of D into irreducible components. Let $\mu : \tilde{V} \to V$ be a composite of blowing-ups such that $\tilde{D} := \mu^*(C + D)_{\mathrm{red}}$ becomes an NC-divisor and μ is the shortest among such birational morphisms. From now on, we call such a birational morphism μ a *minimal NC-map* for the pair $(V, C + D)$. Let $\tilde{C}$ be the proper transform of C on $\tilde{V}$.

Lemma 3.1 *With the same notation as above, C is a rational curve with at most unibranch singular points.*

Proof. Since $\tilde{V} - \tilde{D} \cong \mathbf{C}^2$, we have $|\tilde{D} + K_{\tilde{V}}| = \emptyset$. By [18, Lemma I.2.1.3], each irreducible component of $\tilde{D}$ is a smooth rational curve, $\tilde{D}$ is an SNC-divisor and the dual graph of $\tilde{D}$ is a tree. Hence C is a rational curve and each singular point of C is a unibranch singular point. Q.E.D.

Lemma 3.2 *With the same notation as above, we have* $\#\mathrm{Sing}\, X \leq 2$.

Proof. Suppose to the contrary that $\#\mathrm{Sing}\, X \geq 3$. Since $\mathrm{Sing}\, X \subset \Gamma$, we have

$$(\tilde{C} \cdot \tilde{D} - \tilde{C}) \geq \#\mathrm{Sing}\, X \geq 3.$$

Since μ is a minimal NC-map for the pair $(V, C+D)$ and $\tilde{D}$ is an SNC-divisor (cf. the proof of Lemma 3.1), we have

$$(E \cdot \tilde{D} - E) \geq 3$$

for every (-1)-curve $E \subset \mathrm{Supp}\,(\tilde{D} - \tilde{C})$. So the pair $(\tilde{V}, \tilde{D})$ is a minimal normal compactification of $\mathbf{C}^2$ (see Definition 2.7). This contradicts Lemma 2.8 because the dual graph of $\tilde{D}$ is not linear by the hypothesis. Q.E.D.

Remark 3.3 By [28], there exist a log del Pezzo surface X of rank one such that $\pi_1(X - \mathrm{Sing}\, X) = (1)$ and $\#\mathrm{Sing}\, X \geq 3$. By Lemma 3.2, such a surface is not a minimal compactification of $\mathbf{C}^2$. So Problem 2 is false.

Lemma 3.4 *The divisor $C + D$ is an SNC-divisor. Namely,* $\mu = \mathrm{id}$.

Proof. By Lemma 2.8, it suffices to show that the divisor $C + D$ is an NC-divisor. Suppose to the contrary that there exists a point $P \in \mathrm{Supp}\,(C + D)$ such that the divisor $C + D$ is not normal crossing at P. Since D is an SNC-divisor, μ is a composite of blowing-ups of infinitely near points on C. We note that the weighted dual graph of $\tilde{D}$ is a tree because $|\tilde{D} + K_{\tilde{V}}| = \emptyset$. Let E be a (-1)-curve which is exceptional with respect to μ. By the minimality of μ, $(E \cdot \tilde{D} - E) \geq 3$, i.e., $\tilde{D}$ is not linear. By Lemma 2.8, $(\tilde{V}, \tilde{D})$ is not a minimal normal compactification of $\mathbf{C}^2$. Hence there exisits a (-1)-curve

$H \subset \operatorname{Supp} \tilde{D}$ such that $(H \cdot \tilde{D} - H) \leq 2$. It then follows from the minimality of μ that $H = \tilde{C}$.

Let $f : \tilde{V} \to W$ be a sequence of contractions of (-1)-curves and subsequently contractible curves in Supp $\tilde{D}$, starting with the contraction of $\tilde{C}$, such that $\tilde{D}_W := f_*(\tilde{D})$ is an NC-divisor and has no (-1)-curves F with $(F \cdot \tilde{D}_W - F) \leq 2$, i.e., the pair $(W, \tilde{D}_W)$ is a minimal normal compactification of $\mathbf{C}^2$. Note that $f_*(E) \neq 0$ because E is a (-1)-curve. Since $(E \cdot \tilde{D} - E) \geq 3$ and the dual graph of $\tilde{D}$ is a tree, we know that the number of connected components of $\tilde{D}_W - f_*(E) \geq 2$ and if the equality holds then $(f_*(E)^2) \geq 0$ and every irreducible component of $\tilde{D}_W - f_*(E)$ has self-intersection number ≤ -1. This contradicts Lemma 2.8 because $\rho(W) \geq 3$. Q.E.D.

Lemma 3.5 *Assume that* $(C^2) \neq -1$. *Then* $V = \mathbf{F}_n$ $(n \geq 2)$, $D = M_n$ *and* C *is a fiber of the ruling on* V. *Namely, the weighted dual graph of* $C + D$ *is given as* (1) *in Appendix C.*

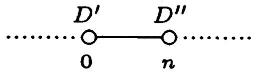
Proof. If C is not a (-1)-curve, then $(V, C + D)$ is a minimal normal compactification of $\mathbf{C}^2$ by Lemma 3.4. Since every irreducible component of D has self-intersection number ≤ -2, the assertion follows from [24, Theorem 9] (see also Lemma 2.8). Q.E.D.

In the subsequent arguments, we assume that C is a (-1)-curve. Note that $(V, C+D)$ is then not a minimal normal compactification of $\mathbf{C}^2$ because $(C \cdot D) \leq 2$ by Lemmas 3.2 and 3.4. Let $\nu : V \to W$ be a sequence of contractions of (-1)-curves and subsequently contractible curves in Supp$(C+D)$, starting with the contraction of C, such that (W, D_W), where $D_W = \nu_*(C + D)$, becomes a minimal normal compactification of $\mathbf{C}^2$.

Lemma 3.6 *With the same notation and assumptions as above,* $(W, D_W) = (\mathbf{P}^2, H)$ *or* $(\mathbf{F}_n, M_n + \ell)$, *where* H *is a line on* $\mathbf{P}^2$ *and* ℓ *is a fiber of a fixed ruling on* $\mathbf{F}_n$.

Proof. Put $Q := \nu(C)$. We note that Q is a unique fundamental point of ν because C is a unique (-1)-curve in Supp $(C + D)$.

Suppose that (W, D_W) is isomorphic to neither $(\mathbf{P}^2, \text{line})$ nor $(\mathbf{F}_n, M_n+$(a fiber of fixed ruling on $\mathbf{F}_n$)). Then, by [24, Theorem 9] (see also Lemma 2.8), D_W contains two components D' and D'' such that $(D'^2) = 0$, $(D''^2) = n > 0$ and $(D' \cdot D'') = 1$ (see Figure 3).



$(n > 0)$

Figure 3

Since Q is a unique fundamental point of ν, we know that $(\nu'(D')^2) \geq -1$ or $(\nu'(D'')^2) \geq -1$. This is a contradiction because C is a (-1)-curve and every irreducible component of D has self-intersection number ≤ -2.

Q.E.D.

We put $V = \mathrm{dil}_{Q_\ell} \circ \cdots \circ \mathrm{dil}_{Q_1}(W)$ and $\nu_i = \mathrm{dil}_{Q_i} \circ \cdots \circ \mathrm{dil}_{Q_1}$, where $\ell \geq 1$ and dil_{Q_i} is a blowing-up with center Q_i $(i = 1, \ldots, \ell)$. Put $E_i := \mathrm{dil}_{Q_i}^{-1}(Q_i)$. Then Lemma 3.6 implies that $(E_i \cdot \nu_i^*(D_W)_{\mathrm{red}} - E_i) = 1$ or 2. Hence we obtain the following:

Lemma 3.7 *With the same notation and assumptions as above, assume further that* $\#\mathrm{Sing}\, X = 2$. *Let* $D = D^{(1)} + D^{(2)}$ *be the decomposition of* D *into connected components. Then we have:*

(1) *One of* $D^{(1)}$ *and* $D^{(2)}$ *is a rod.*

(2) *Assume that* $D^{(1)}$ *and* $D^{(2)}$ *are rods. Then* C *meets a terminal component of* $D^{(1)}$ *or* $D^{(2)}$.

(3) *Assume that* $D^{(1)}$ *is a fork* ($D^{(2)}$ *is then a rod by the assertion* (1)). *Let* $D_0^{(1)}$ *be the branching component of* $D^{(1)}$, *i.e.,* $(D_0^{(1)} \cdot D^{(1)} - D_0^{(1)}) = 3$. *Then* $(C \cdot D_0^{(1)}) = 0$ *and* C *meets a terminal component of* $D^{(2)}$.

Now we determine the weighted dual graph of $C + D$ in the case where C is a (-1)-curve. We consider the following two cases separately.

Case 1: $\#\mathrm{Sing}\, X = 1$. Put $P := \mathrm{Sing}\, X$. We consider the following three subcases 1-1 $\sim$ 1-3 separately.

Subcase 1-1: P is a cyclic quotient singular point. Note that, by taking a suitable birational morphism ν, we may assume that $W = \mathbf{F}_n$ $(n \geq 2)$, $D_W = M_n + \ell$, where ℓ is a fiber of the ruling on $\mathbf{F}_n$, and that $Q := \nu(C) \not\in M_n$. Since D is a rod, the weighted dual graph of $\nu^*(\ell)_{\mathrm{red}}$ is given as in Figure 4. In Figure 4, the subgraph denoted by the encircled A is given as in Figure 1 and the subgraph denoted by the encircled A^* is the weighted dual graph of the adjoint of A (cf. §2), where we consider A as an admissible rational rod whose weighted dual graph is given as in Figure 1. Hence the weighted dual graph of $C + D$ is given as (2) in Appendix C.

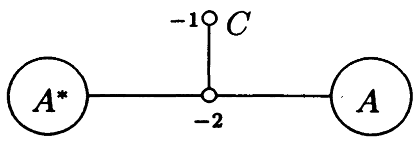


Figure 4

Subcase 1-2: P is a quotient singular point of type D. Let $D = \sum_{i=0}^{r} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 2. It then follows form Lemma 3.6 and the argument before Lemma 3.7 that C meets D_1 or D_2 (see also [15, §3]). Hence we know that the weighted dual graph of $C + D$ is given as (3) in Appendix C.

Subcase 1-3: P is a quotient singular point of type E_n ($n = 6, 7, 8$). Since C is a (-1)-curve and $(C \cdot D) = 1$, by using the same argument as in the proof of [15, Theorem 2.1], we know that the weighted dual graph of $C + D$ is given as (n) ($4 \leq n \leq 13$) in Appendix C.

Case 2: $\#\mathrm{Sing}\, X = 2$. Let $D = D^{(1)} + D^{(2)}$ be the decomposition of D into connected components. By Lemma 3.7 (1), we may assume that $D^{(2)}$ is a rod. We consider the following two subcases separately.

Subcase 2-1: $D^{(1)}$ is a rod. By Lemma 3.7 (2), we may assume that C meets a terminal component of $D^{(2)}$. If C meets a terminal component of $D^{(1)}$ then, by Lemma 3.6, we know that the weighted dual graph of $C + D$ is given as (14) in Appendix C. Assume that C meets a component $D_i^{(1)}$ of $D^{(1)}$, which is not a terminal component of $D^{(1)}$. Then, by the argument before Lemma 3.7, $D_i^{(1)} + C + D^{(2)}$ can be contracted to a smooth point. So $D^{(2)}$ consists entirely of (-2)-curves and $(D_i^{(1)})^2 = -2 - \#D^{(2)}$. Thus, we know that the weighted dual graph of $C + D$ is given as (15) in Appendix C.

Subcase 2-2: $D^{(1)}$ is a fork. Let $D_0^{(1)}$ be the branching component of $D^{(1)}$. Then Lemma 3.7 (3) implies that C meets a terminal component of $D^{(2)}$ and does not meet $D_0^{(1)}$. Let $T^{(1)}$ be the maximal (admissible rational) twig of $D^{(1)}$ meeting C. By the argument before Lemma 3.7, we know that $D_0^{(1)} + T^{(1)} + C + D^{(1)}$ can be contracted to a smooth point. So we can determine the weighted dual graph of $D_0^{(1)} + T^{(1)} + C + D^{(1)}$. Hence, by using Lemma 3.6, we know that the weighted dual graph of $C + D$ is given as (n) $(16 \leq n \leq 32)$ in Appendix C. For example, if the weighted dual graph of D is given as in Figure 2 then, by virtue of Lemma 3.6, we know that $T^{(1)}$ is a (-2)-curve. Hence the weighted dual graph of $C + D$ is given as (16) in Appendix C.

The proof of Theorem 1.1 is thus completed.

4 Counterexamples

In this section, we give counterexamples to Problems 1 and 2 (see Examples 4.2 and 4.3).

Counterexample to Problem 1. Let X be a log del Pezzo surface of rank one and let $\pi : (V, D) \to X$ be the minimal resolution of X, where D is the reduced exceptional divisor. In Example 4.1 (resp. 4.2) below, we shall construct a log del Pezzo surface X of rank one such that the weighted dual graph of D is given as in Figure 5 and X is a compactification of $\mathbf{C}^2$ (resp. not a compactification of $\mathbf{C}^2$).

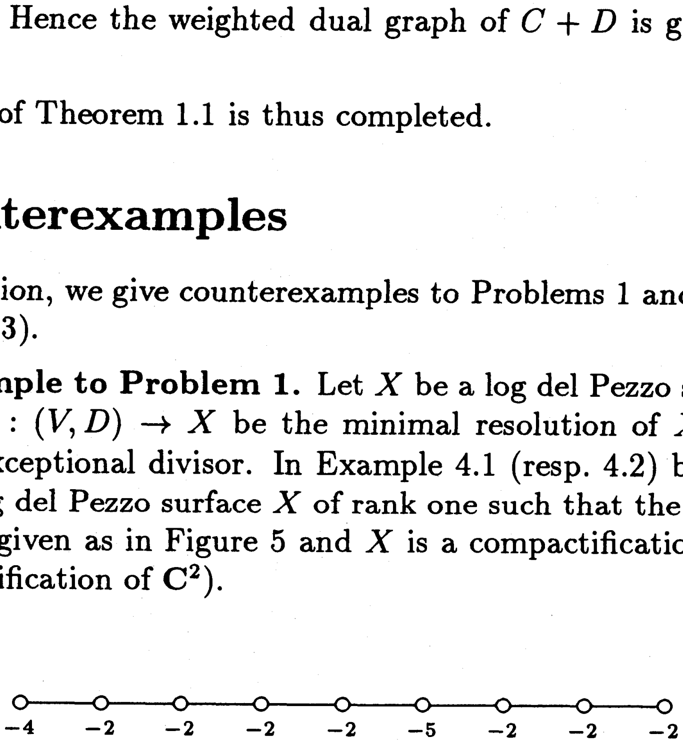


Figure 5

Example 4.1 Let ℓ be a fiber of the ruling on $\mathbf{F}_2$ (see Figure 6-(i)). Let $\mu : V \to \mathbf{F}_2$ be a birational morphism such that the configuration of $\mu^{-1}(M_2 + \ell)$ is shown as in Figure 6-(ii), where C is the last exceptional curve in the process of μ. Put $D := \mu^*(M_2 + \ell)_{\mathrm{red}} - C$. Then the weighted dual graph of D is given as in Figure 5. Let $\nu : V \to X$ be the contraction of D and put $\Gamma := \nu_*(C)$. It is then clear that (X, Γ) is a minimal compactification of $\mathbf{C}^2$.

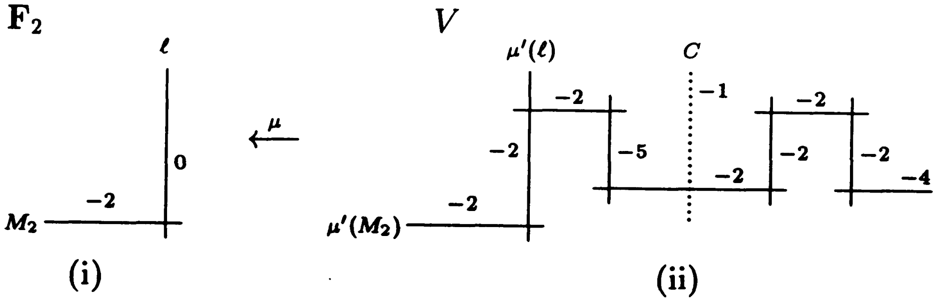


Figure 6

Example 4.2 Let ℓ_1 and ℓ_2 be fibers of the ruling on $\mathbf{F}_5$ (see Figure 7-(i)). Let $\mu : V \to \mathbf{F}_5$ be a birational morphism such that the configuration of $\mu^{-1}(M_5 + \ell_1 + \ell_2)$ is shown as in Figure 7-(ii). Put $D := \mu^*(M_5 + \ell_1 + \ell_2)_{\mathrm{red}} - (C_1 + C_2)$. Then the weighted dual graph of D is given as in Figure 5. A divisor $\mu^*(\ell_1)$ defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|\mu^*(\ell_1)|} : V \to \mathbf{P}^1$ and $\varphi := \Phi|_{V-D} : V - D \to \mathbf{P}^1$ is an $\mathbf{A}^1$-fibration (i.e., a general fiber of φ is isomorphic to the affine line $\mathbf{A}^1$) onto $\mathbf{P}^1$. So $\overline{\kappa}(V - D) = -\infty$ (cf. [18, Chapter I]). Let $\nu : V \to X$ be the contraction of D. Then $\rho(X) = 1$. By [27, Remark 1.2] and [19, Lemma 2.7], X is then a log del Pezzo surface of rank one.

Now we calculate the fundamental group of $V - D = X - \mathrm{Sing}\, X$. Since Φ has just two singular fibers $\mu^*(\ell_1)$ and $\mu^*(\ell_2)$ and the multiplicity of C_1 (resp. C_2) in $\mu^*(\ell_1)$ (resp. $\mu^*(\ell_2)$) is equal to two (resp. four), we know that every fiber of φ is irreducible and φ has just two multiple fibers $m_1\Gamma_1$ and $m_2\Gamma_2$ with $\{m_1, m_2\} = \{2, 4\}$. By [6, Proposition (4.9)], $\pi_1(V - D)$ is generated by σ_1 and σ_2 with the relation $\sigma_1\sigma_2 = \sigma_1^2 = \sigma_2^4 = 1$. Hence $\pi_1(V - D) \cong \mathbf{Z}/2\mathbf{Z}$. Since $\pi_1(X - \mathrm{Sing}\, X) \neq (1)$, we know that X is not a compactification of $\mathbf{C}^2$.

Counterexample to Problem 2. In Remark 3.3, we note that there exists a log del Pezzo surface X of rank one such that $\pi_1(X - \mathrm{Sing}\, X) = (1)$ and $\#\mathrm{Sing}\, X \geq 3$. Hence, by Theorem 1.1, Problem 2 is false. In Example 4.3 below, we give an example of a log del Pezzo surface X of rank one such that $\pi_1(X - \mathrm{Sing}\, X) = (1)$, $\#\mathrm{Sing}\, X = 1$ and X is not a compactification of $\mathbf{C}^2$.

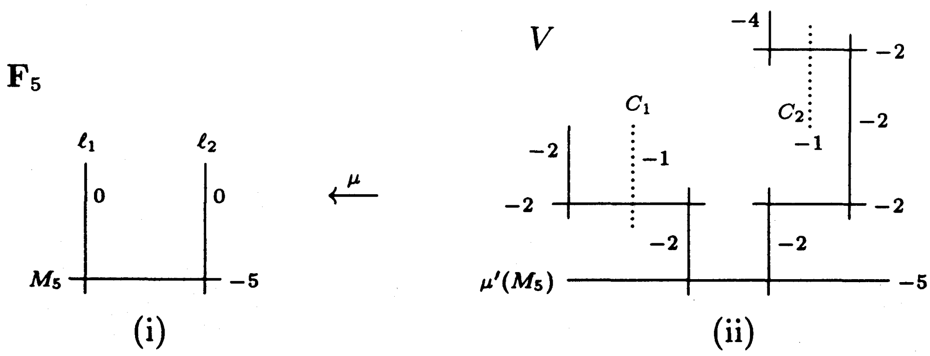


Figure 7

Example 4.3 Let ℓ_1 and ℓ_2 be fibers of the ruling on $\mathbf{F}_n$ $(n \geq 2)$. See Figure 8-(i). Let $\mu : V \to \mathbf{F}_n$ be a birational morphism such that the configuration of $\mu^{-1}(M_n + \ell_1 + \ell_2)$ is shown as in Figure 8-(ii). Put $D := \mu^*(M_n + \ell_1 + \ell_2) - (C_1 + C_2)$. Let $\nu : V \to X$ be the contraction of D. Similarly to Example 4.2, we know that X is a log del Pezzo surface of rank one with $\#\mathrm{Sing}\, X = 1$, and $\pi_1(X - \mathrm{Sing}\, X) = \pi_1(V - D) = (1)$. However, X is not a compactification of $\mathbf{C}^2$ by Theorem 1.1.

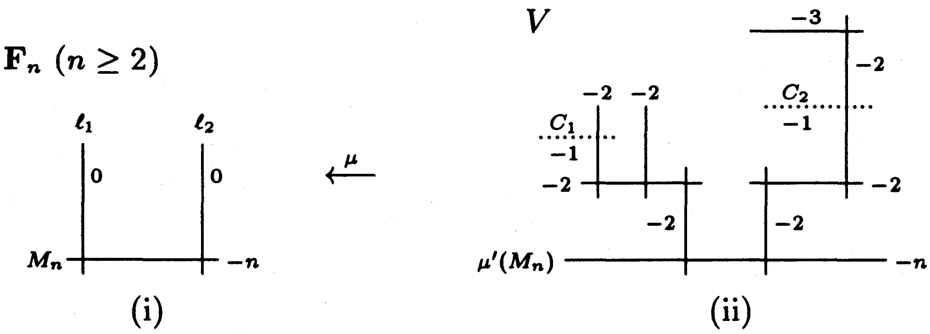


Figure 8

We propose the following problem:

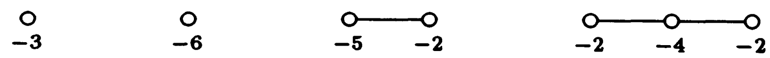
Problem 3 Let X be a log del Pezzo surface of rank one. Assume that $\pi_1(X - \mathrm{Sing}\, X) = (1)$ and the singularity type of X is given as one of the listed in Appendix C. Is then X a minimal compactification of $\mathbf{C}^2$?

5 Proof of Theorem 1.2

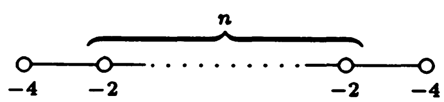
In this section, we prove Theorem 1.2

Let X be a log del Pezzo surface of rank one and of index three and let $\pi : (V, D) \to X$ be the minimal resolution of X, where D is the reduced exceptional divisor. Since the index of X is equal to three, each singular point of X is either a rational double point or a quotient singular point of index three. It is clear that X has at least one quotient singular points of index three.

Lemma 5.1 *Let $P \in X$ be a quotient singular point of index three. Then the singularity type of P is given as the following weighted dual graph* (n) $(1 \le n \le 9)$. *In particular, P is a cyclic quotient singular point or of type D.*

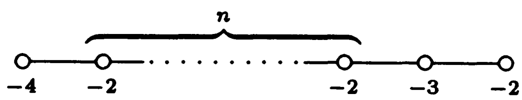


(1) (2) (3) (4)



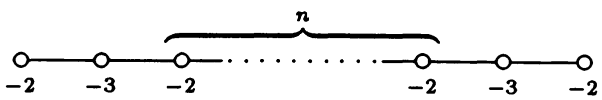
$(n \ge 0)$

(5)



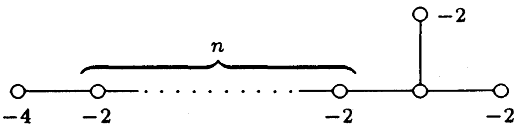
$(n \ge 0)$

(6)



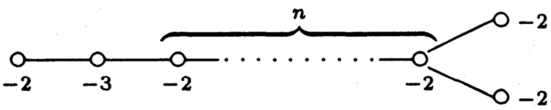
$(n \ge 0)$

(7)



$(n \geq 0)$

(8)



$(n \geq 0)$

(9)

Proof. See [29, Proposition 6.1].

By using Theorem 1.1 and Lemma 5.1, we can prove the following:

Lemma 5.2 *Assume that X is a minimal compactification of $\mathbf{C}^2$. Then the weighted dual graph of D is given as* (n) $(1 \leq n \leq 11)$ *in Theorem* 1.2.

Proof. Since X is a minimal compactification of $\mathbf{C}^2$, $\#\mathrm{Sing}\, X \leq 2$.

We first treat the case $\#\mathrm{Sing}\, X = 1$. Put $P := \mathrm{Sing}\, X$. Then P is a quotient singular point of index three. If P is a cyclic quotient singular point then, by Theorem 1.1, the weighted dual graph of D looks like (1) or (2) in Appendix C. So it follows from Lemma 5.1 that the weighted dual graph of D is given as one of (1) $\sim$ (4) in Theorem 1.2. If P is not a cyclic quotient singular point, then P is of type D and the weighted dual graph of D looks like (3) in Appendix C. So it follows from Lemma 5.1 that the weighted dual graph of D is given as (5) in Theorem 1.2.

We next treat the case $\#\mathrm{Sing}\, X = 2$. Assume that X has a non-cyclic quotient singular point, say P. Theorem 1.1 then implies that P is not a rational double point. So P is of type D by Lemma 5.1 and hence the weighted dual graph of D looks like (16) in Appendix C. By using Lemma 5.1 again, we know that the index of P is then not equal to three. This is a contradiction. Hence we know that all singular points of X are cyclic quotient singular points. Then the weighted dual graph of D looks like (14) or (15) in Appendix C. Hence, by using Lemma 5.1, we know that the weighted dual graph of D is given as (n) $(6 \leq n \leq 11)$ in Theorem 1.2. Q.E.D.

We prove that if the singularity type of X is given as (n) $(1 \leq n \leq 11)$ in Theorem 1.2 then X contains $\mathbf{C}^2$ as a Zariski open subset. We treat the cases (3), (5) and (10) (see Theorem 1.2) only. The other cases can be treated similarly.

Case (3). Let $D = \sum_{i=1}^{6} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 9. Lemma 2.6 (3) implies that there exists a (-1)-curve C such that $(C \cdot D) = 1$. By Lemma 2.3, we may assume that $(C \cdot D) = (C \cdot D_i) = 1$, $i = 2$ or 3.

Assume that $i = 3$. Then, a divisor $F = 2(C + D_3) + D_2 + D_4$ defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbf{P}^1$, D_1 and D_5 are sections of Φ and D_6 is contained in a singular fiber of Φ, say G. Since D_6 is a (-4)-curve, we have $\#G \geq 5$. So we have

$$\rho(V) = 7 \geq 2 + (\#F - 1) + (\#G - 1) \geq 9,$$

which is a contradiction. Hence, $i = 2$.

Now, a divisor $F = 4(C+D_2)+3D_3+2D_4+D_1+D_5$ defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbf{P}^1$ and D_6 is a section of Φ. Since $\rho(V) = 7 = 2+(\#F-1)$, Φ has no singular fibers other than F. So $V - (C + D) \cong \mathbf{C}^2$ and hence X becomes a minimal compactification of $\mathbf{C}^2$.

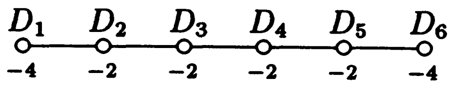


Figure 9

Case (5). Let $D = \sum_{i=0}^{4} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 2, where we put $r = 4$, $a_0 = a_3 = 2$ and $a_4 = 4$. Lemma 2.6 (2) implies that there exists a (-1)-curve C such that $(C \cdot D) = (C \cdot D_i) = 1$, $i = 1$ or 2. We may assume that $i = 1$. Then, a divisor $F = 2(C + D_1 + D_0) + D_2 + D_3$ defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbf{P}^1$ and D_4 is a section of Φ. Since $\rho(V) = 6 = 2 + (\#F - 1)$, Φ has no singular fibers other than F. So $V - (C + D) \cong \mathbf{C}^2$ and hence X becomes a minimal compactification of $\mathbf{C}^2$.

Case (10). Let $D = \sum_{i=1}^{6} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 10.

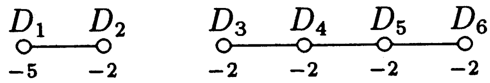


Figure 10

We note that $\rho(V) = \#D + 1 = 7$ and

$$D^{\#} = \frac{2}{3}D_1 + \frac{1}{3}D_2$$

(for the definition of $D^{\#}$, see §2). Let M be an irreducible curve on V such that $-(M \cdot D^{\#} + K_V)$ attains the smallest positive value (cf. §2).

Suppose that $|M+D+K_V| \neq \emptyset$. Then Lemma 2.4 implies that $(M \cdot D_1) = (M \cdot D_2) = 1$ and $M + D_1 + D_2 + K_V \sim 0$. We have

$$(M^2) = (D_1 + D_2 + K_V)^2 = 4$$

and

$$(M \cdot K_V) = (M \cdot -M - D_1 - D_2) = -6.$$

Hence,

$$-(M \cdot D^{\#} + K_V) = 5.$$

On the other hand, since $\rho(V) = 7$, there exists a (-1)-curve E on V. Then we have

$$-(E \cdot D^{\#} + K_V) = 1 - (E \cdot D^{\#}) \leq 1 < -(M \cdot D^{\#} + K_V),$$

which is a contradiction. Hence we know that $|M + D + K_V| = \emptyset$.

By Lemma 2.5, we may assume that M is a (-1)-curve. Note that $(M \cdot D) = 1$ or 2 and $(M \cdot D_1 + D_2) = 0$ or 1 (see §2). We consider the following three subcases (10)-(i) $\sim$ (10)-(iii) separately.

Subcase (10)-(i): $(M \cdot D_1 + D_2) = 0$. Then Lemma 2.3 implies that $(M \cdot D) = (M \cdot D_i) = 1$, $i = 4$ or 5. We may assume that $i = 4$. A divisor $F = 2(M + D_4) + D_3 + D_5$ defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbf{P}^1$, D_6

is a section of Φ and $D_1 + D_2$ is contained in a singular fiber of Φ, say G. Since $\operatorname{Supp} G$ contains D_1 which is a (-5)-curve, we have $\#G \geq 5$. Then

$$\rho(V) = 7 \geq 2 + (\#F - 1) + (\#G - 1) \geq 9,$$

which is a contradiction. Hence this subcase does not take place.

Subcase (10)-(ii): $(M \cdot D_1 + D_2) = (M \cdot D_2) = 1$. Then we have

$$-(M \cdot D^{\#} + K_V) = \frac{2}{3}.$$

Lemma 2.3 implies that $(M \cdot D_3 + D_4 + D_5 + D_6) = 1$. We may assume that $(M \cdot D_3) = 1$ or $(M \cdot D_4) = 1$.

Assume that $(M \cdot D_3) = 1$. Then, a divisor $F = 2M + D_2 + D_3$ defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbf{P}^1$, D_1 and D_4 are sections of Φ and $D_5 + D_6$ is contained in a singular fiber of Φ, say G. Since D_1 is a section of Φ, $\operatorname{Supp} G$ contains an irreducible curve E with $(E \cdot D_1) = 1$. By Lemma 2.2 (2), E is a (-1)-curve. Then we have

$$-(E \cdot D^{\#} + K_V) \leq \frac{1}{3} < -(M \cdot D^{\#} + K_V),$$

which is a contradiction. Similarly, we have a contradiction if $(M \cdot D_4) = 1$. Hence this subcase does not take place.

Subcase (10)-(iii): $(M \cdot D_1 + D_2) = (M \cdot D_1) = 1$. By Lemma 2.3, $(M \cdot D_3 + D_4 + D_5 + D_6) = 1$. If $(M \cdot D_3) = 1$ or $(M \cdot D_6) = 1$, then we can easily see that $V - (M + D) \cong \mathbf{C}^2$ (cf. Cases (3) and (5)). Hence X becomes a minimal compactification of $\mathbf{C}^2$.

Suppose that $(M \cdot D_4) = 1$ or $(M \cdot D_5) = 1$. We may assume that $(M \cdot D_4) = 1$. Then, a divisor $F = 2(M + D_4) + D_3 + D_5$ defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbf{P}^1$, D_6 is a section of Φ, D_1 is a 2-section of Φ, and D_2 is contained in a singular fiber of Φ, say G. By Lemma 2.2 (2) and $\rho(V) = 7$, we know that $G = E_1 + D_2 + E_2$, where E_1 and E_2 are (-1)-curves, $(E_1 \cdot D_2) = (E_2 \cdot D_2) = 1$ and $(E_1 \cdot E_2) = 0$. Since D_1 is a 2-section of Φ and the multiplicity of D_2 in G is equal to one, we may assume that E_1 meets D_1. Then

$$-(E_1 \cdot D^{\#} + K_V) = 1 - \frac{2}{3}(E_1 \cdot D_1) - \frac{1}{3}(E_1 \cdot D_2) \leq 0,$$

which contradicts Lemma 2.2 (1).

Theorem 1.2 is thus verified.

Appendix

A Fundamental groups of some open rational surfaces with $\overline{\kappa} = -\infty$

Let X be a normal projective rational surface defined over $\mathbf{C}$ with unique singular point. Assume that the singular point of X is a quotient singular point. In [10], Gurjar and Zhang proved the following result.

Theorem A.1 *With the same notation and assumptions as above, assume further that $\overline{\kappa}(X - \operatorname{Sing} X) \leq 1$. Then $\pi_1(X - \operatorname{Sing} X)$ is a finite group.*

In this section, we prove the following result by using the results in [15].

Proposition A.2 *With the same notation and assumptions as above, assume further that $\overline{\kappa}(X - \operatorname{Sing} X) = -\infty$. Then $\pi_1(X - \operatorname{Sing} X)$ is a finite abelian group.*

Proof. By [10, Lemma 1], it suffices to show that $\pi_1(X - \operatorname{Sing} X)$ is abelian.

Assume that X is not log relatively minimal, i.e., there exists an irreducible curve E on X such that $(E^2) < 0$ and $(E \cdot K_X) < 0$ (cf. [22, Chapter II, §4]). Let $f : X \to X'$ be the contraction of E. Since $\#\operatorname{Sing} X = 1$, it follows from [22, Capter II, §4] (see also [14]) that X' has at most one quotient singular point and $\overline{\kappa}(X' - \operatorname{Sing} X') = \overline{\kappa}(X - \operatorname{Sing} X) = -\infty$. It is clear that $\pi_1(X - \operatorname{Sing} X)$ is a subgroup of $\pi_1(X' - \operatorname{Sing} X')$. Thus, to prove Proposition A.2, we may assume that X is log relatively minimal.

Since $\overline{\kappa}(X - \operatorname{Sing} X) = -\infty$ and X is log relatively minimal, one of the following two cases takes place by [19, Lemma 2.7] and [14, Theorem 1.1].

(i) There exists a $\mathbf{P}^1$-fibration $h : X \to \mathbf{P}^1$ such that every fiber of h is irreducible and h has only one multiple fiber F.

(ii) X is a log del Pezzo surface of rank one.

We consider the above two cases separately.

Case (i). By virtue of [14, Theorem 1.1], $\mathrm{Sing}\, X \in \mathrm{Supp}\, F$. Then $X - \mathrm{Supp}\, F \cong \mathbf{P}^1 \times \mathbf{A}^1$ and hence $\pi_1(X - \mathrm{Sing}\, X) = (1)$. In this case the assertion holds.

Case (ii). Put $P := \mathrm{Sing} X$. If P is of type E_n $(n = 6, 7, 8)$, then $\pi_1(X - P) = (1)$ because X is a minimal compactification of $\mathbf{C}^2$ by Lemma 2.6 (1). If P is a cyclic quotient singular point, then $\pi_1(X - P)$ is abelian by Lemma 2.6 (3).

Assume that P is of type D. Let $\pi : (V, D) \to X$ be the minimal resolution of X and let $D = \sum_{i=0}^{r} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 2. Then Lemma 2.6 (2) implies that there exisits a (-1)-curve E such that $(E \cdot D) = (E \cdot D_i) = 1$, where $i = 1$ or 2. We may assume that $i = 1$. Put $F := 2(E + D_1 + D_0) + D_2 + D_3$. By Lemma 2.3, $a_0 = a_3 = 2$. So F defines a $\mathbf{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbf{P}^1$, D_4 is a section of Φ and $D_5, \ldots, D_r$ are contained in a fiber G of Φ if $r \geq 5$. Here we note that $r \geq 4$ and if $r = 4$ then $\pi_1(V - D) = (1)$. Assume that $r \geq 5$. Then, since $\rho(V) = \#D + 1 = r + 2$, G contains a unique (-1)-curve E' and $(G)_{\mathrm{red}} = D_5 + \cdots + D_r + E'$. Let m be the multiplicity of E' in G. By using the same argument as in Example 4.2, we know that

$$\pi_1(V - D) = \begin{cases} (1) & \text{if } m \text{ is odd,} \\ \mathbf{Z}/2\mathbf{Z} & \text{if } m \text{ is even.} \end{cases}$$

In particular, $\pi_1(V - D) = \pi_1(X - P)$ is abelian. Q.E.D.

Remark A.3 In Case (ii), we know that $\pi_1(X - \mathrm{Sing}\, X)$ is finite by virtue of [8] and [9].

B A proof of a result of Ramanujam

Let k be an algebraically closed field of arbitrary characteristic, which we fix as the ground field throughout the present section. Let S be a smooth affine algebraic surface defined over k. Let (V, D) be a pair of a smooth projective surface V and a reduced normal crossing divisor D on V. We call (V, D) a *normal algebraic compactification* of S if S is isomrophic to $V - D$ (cf. Definition 2.7). A normal algebraic compactification (V, D) of S is said to be *minimal* if $(E \cdot D - E) \geq 3$ for any (-1)-curve $E \subset D$. Note that minimal normal algebraic compactifications of S exist since S is an affine algebraic surface.

When $S = \mathbf{C}^2$, Morrow [24, Theorem 9] gave a classification of minimal normal algebraic compactifications (V, D) of S. His argument depended heavily on the following theorem which is the main result of Ramanujam [25] (see also [23]).

Theorem B.1 *If (V, D) is a minimal normal algebraic compactification of the affine plane $\mathbf{A}^2_k$, then the dual graph of D is linear.*

In this section, by using the similar argument to the proof of [16, Theorem 1.1], we give a new proof of Theorem B.1.

Let (V, D) be a minimal normal algebraic compactification of the affine plane $S := \mathbf{A}^2_k$. The following lemma is easy but useful.

Lemma B.2 (cf. [16, Lemma 2.2]) *There exists an irreducible linear pencil Λ on V such that the following conditions* (i) $\sim$ (iii) *are satisfied.*

(i) $\mathrm{Bs}\, \Lambda \subset D$ *and a general member of Λ is a rational curve.*

(ii) *The morphism $\varphi := \Phi_\Lambda|_S$ is an $\mathbf{A}^1_k$-fibration onto the affine line $\mathbf{A}^1_k$ without singular fibers.*

(iii) *Let $\mu : \tilde{V} \to V$ be a composition of blowing-ups with centers at the base points (including infinitely near base points) of Λ such that the proper transform $\tilde{\Lambda}$ of Λ by μ has no base points. Then $\tilde{\Lambda}$ gives rise to a $\mathbf{P}^1$-fibration $\Phi_{\tilde{\Lambda}}$ on $\tilde{V}$ over $\mathbf{P}^1$ and there exists a section of $\Phi_{\tilde{\Lambda}}$ in $\tilde{D} := \tilde{V} - \mu^{-1}(S)$.*

Proof. There exists a diagram

$$V \xleftarrow{f} W \xrightarrow{g} \mathbf{P}^2,$$

where f (resp. g) is a composition of blowing-ups with centers in D (resp. a line ℓ on $\mathbf{P}^2$) including infinitely near points. Let P_0 be a point on ℓ. Here we may assume that P_0 is blown up by g. Let Λ' be the irreducible linear pencil on $\mathbf{P}^2$ consisting of lines through P_0. Then the proper transform $g'(\Lambda')$ gives rise to a $\mathbf{P}^1$-fibration $\Phi_{g'(\Lambda')} : W \to \mathbf{P}^1$ and there exists a section of $\Phi_{g'(\Lambda')}$ in $W - g^{-1}(S)$. Moreover, $\Phi_{g'(\Lambda')}|_{g^{-1}(S)} : g^{-1}(S) \cong S \to \mathbf{A}^1_k$ is an $\mathbf{A}^1_k$-fibration onto $\mathbf{A}^1_k$ without singular fibers. Hence $\Lambda := f_*(g'(\Lambda'))$ becomes an irreducible linear pencil on V satisfying the conditions (i) $\sim$ (iii). Q.E.D.

Proof of Theorem B.1. Let Λ be an irreducible linear pencil satisfying the conditions (i) $\sim$ (iii) in Lemma B.2. If $\mathrm{Bs}\,\Lambda = \emptyset$, then it is clear that $\Phi_\Lambda : V \to \mathbf{P}^1$ is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$, i.e., V is a Hirzebruch surface, and D consists of a fiber of Φ_Λ and a section of Φ_Λ (cf. [12, Lemma 2.2]). So, in this case, the assertion holds.

Assume that $\mathrm{Bs}\,\Lambda \neq \emptyset$. Then $\#\mathrm{Bs}\,\Lambda = 1$, $\mathrm{Bs}\,\Lambda \in D$ and $P := \mathrm{Bs}\,\Lambda$ is a one-place point for a general member of Λ. Let $\mu : \tilde{V} \to V$ be the shortest composition of blowing-ups with center P (including infinitely near points of P) such that the proper transform $\tilde{\Lambda}$ of Λ by μ has no base points. Put $\tilde{D} := \mu^{-1}(D)$. Then $\tilde{V} - \tilde{D} = S$ and $\tilde{\Phi} := \Phi_{\tilde{\Lambda}} : \tilde{V} \to \mathbf{P}^1$ is a $\mathbf{P}^1$-fibration. Let $\tilde{D}_0$ be the last exceptional curve in the process μ. Then $\tilde{D}_0 \subset \tilde{D}$, $\tilde{D}_0$ is a section of $\tilde{\Phi}$ and the other components of $\tilde{D}$ are contained in fibers of $\tilde{\Phi}$. Let $D_1, \ldots, D_\ell$ be all components of D through P. Then $\ell = 1$ or 2 since D is an NC-divisor. By the minimality of the pair (V, D), we know that every component of $D - (D_1 + \cdots + D_\ell)$ has self-intersection number ≤ -2. Note that every irreducible component of D is a nonsingular rational curve and the dual graph of D is a tree because $\overline{\kappa}(S) = -\infty$ (cf. [18, Lemma I.2.1.3]).

Suppose to the contrary that the dual graph of D is not linear, i.e., there exists an irreducible component D' of D with $D'(D - D') \geq 3$. Let $D - D' = A_1 + \cdots + A_t$ be a decomposition of $D - D'$ into connected components. Since the dual graph of D is a tree, we have $t \geq 3$. So we may assume that $P \notin A_1 \cup A_2$. Let $\tilde{F}$ be a fiber of $\tilde{\Phi}$ containing $\mu'(D')$. Then $\mu'(A_1 + A_2) \subset \mathrm{Supp}\,(\tilde{F})$. Hence $\tilde{F}$ is a singular fiber.

Let $f : \tilde{V} \to \tilde{V}_1$ be a sequence of contractions of (-1)-curves and subsequently contractible curves in $\mathrm{Supp}\,(\tilde{F})$ such that $f(\mu'(D'))$ becomes a (-1)-curve. Note that such a birational morphism exists and $f(\tilde{D}_0)$ is a section of the $\mathbf{P}^1$-fibration $\tilde{\Phi} \circ f^{-1} : \tilde{V}_1 \to \mathbf{P}^1$. If $\mathrm{Supp}\,(\tilde{F}) \subset \tilde{D}$ then the weighted dual graph of $f_*(\mu'(A_i))$ $(i = 1, 2)$ is the same as that of A_i. Hence we have $(f_*(\mu'(D')) \cdot f_*(\tilde{F}_{\mathrm{red}} + \tilde{D}_0 - \mu'(D'))) \geq 3$, which is a contradiction.

Suppose that $\mathrm{Supp}\,(\tilde{F}) \not\subset \tilde{D}$. Let $\tilde{G}$ be a sum of irreducible components of $\tilde{F}_{\text{red}}$ which are not contained in $\tilde{D}$. Since $\tilde{F}|_S$ is a fiber of φ, we know that $\tilde{G}$ is irreducible and the multiplicity of $\tilde{G}$ in $\tilde{F}$ is equal to one. So we may assume that $\tilde{G}$ is not contracted in the process of f. Then the weighted dual graph of $f_*(\mu'(A_i))$ $(i = 1, 2)$ is the same as that of A_i. Hence, by using the same argument as in the case Supp $(\tilde{F}) \subset \tilde{D}$, we obtain a contradiction.

Q.E.D.

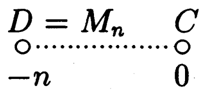
Remark B.3 (1) Recently, Kishimoto [12] gave an algebraic proof of [24, Theorem 9] without using Theorem B.1.

(2) In [24] and [25], Morrow and Ramanujam considered (minimal normal) "analytic" compactifications of $\mathbf{C}^2$ and proved that they are also algebraic compactifications of $\mathbf{C}^2$. In [6, Corollary (9.2)], Fujita proved the same result by using a different method.

C List of configurations

In the following list of configurations, the weight of the vertex corresponding to a (-2)-curve of D is omitted. In (2), (14) and (15), the subgraph denoted by the encircled A is given as in Figure 1 and the subgraph denoted by the encircled A^* is the weighted dual graph of the adjoint of A, where we consider A as an admissible rational rod whose weighted dual graph is given as in Figure 1. In (1), (2), (14), (15) and (16), $n \geq 2$. In (2) $\sim$ (32), C is a (-1)-curve. In (15), D consists of two rods.

(1)



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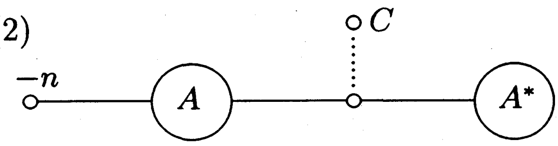

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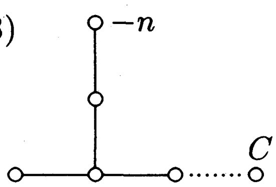

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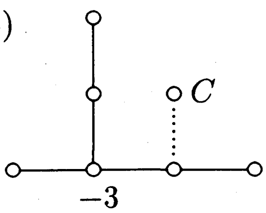

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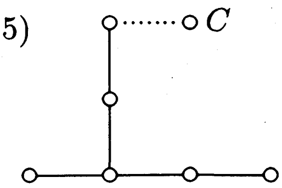

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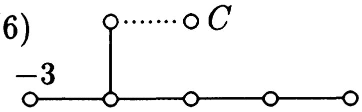

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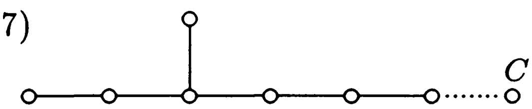

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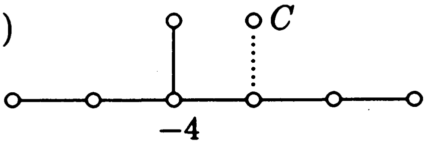

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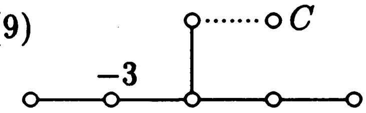

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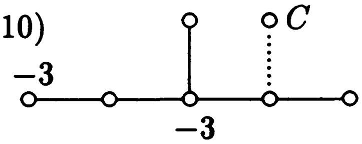

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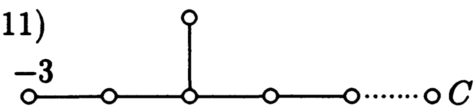

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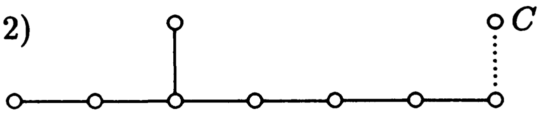

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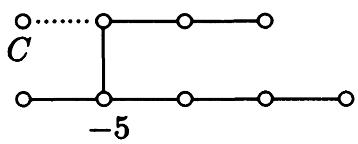

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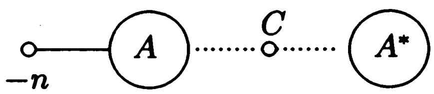

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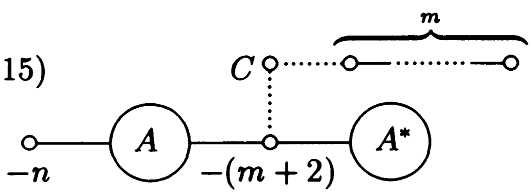
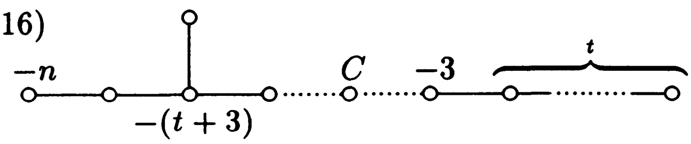


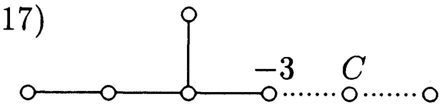
$(m \geq 1)$

(16)



$(t \geq 0)$

(17)



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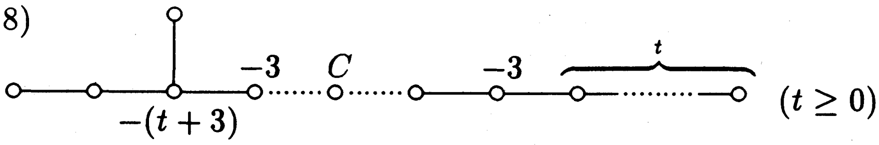

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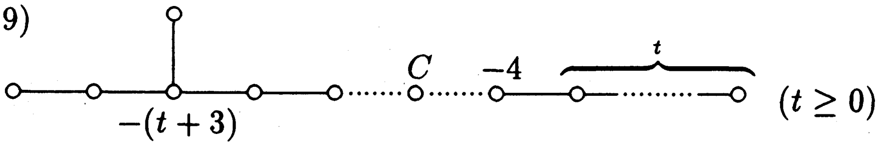

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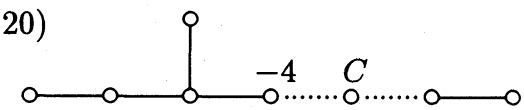

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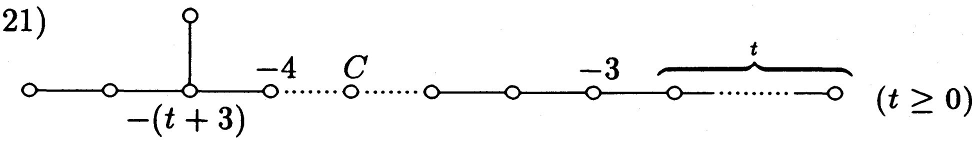

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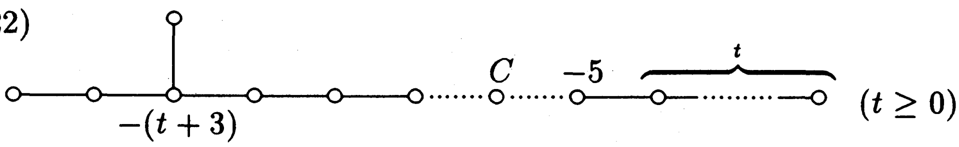

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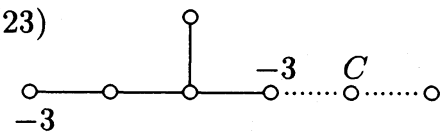

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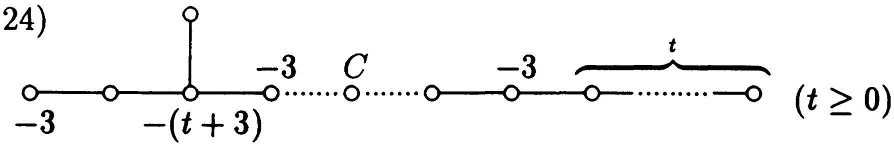

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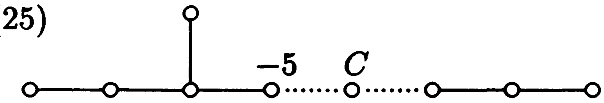

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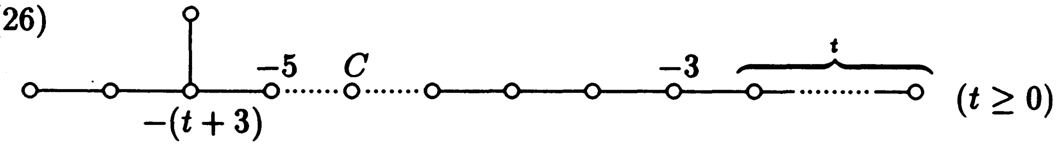

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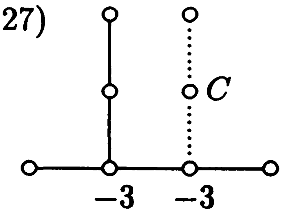

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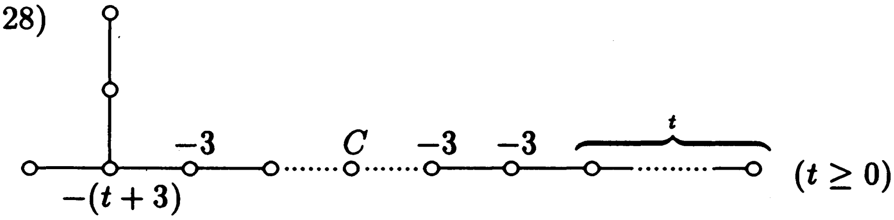

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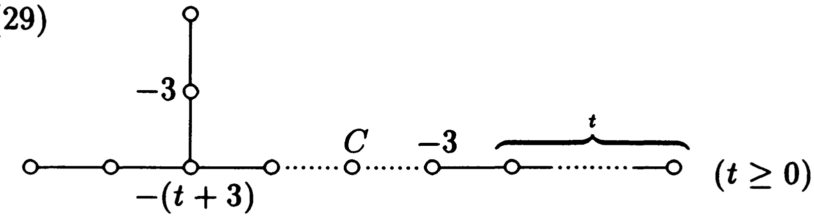

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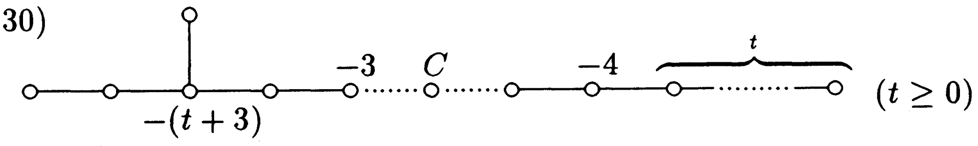

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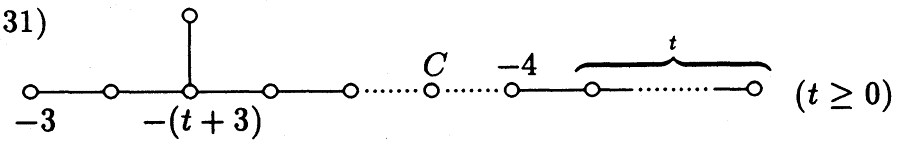

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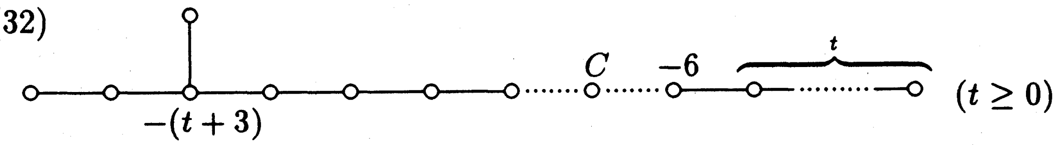

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Received October 8, 2001