Nihonkai Math. J.
Vol. 12(2001), 101-112

JACOBI VECTOR FIELDS ALONG GEODESICS IN GLUED RIEMANNIAN MANIFOLDS

NOBUHIRO INNAMI

ABSTRACT. Let M_α, $\alpha \in \Lambda$, be complete connected Riemannian manifolds which are glued at their boundary. We call such a manifold $M = \cup_{\alpha\in\Lambda} M_\alpha$ a *glued Riemannian manifold.* Geodesics in a glued Riemannian manifold M are by definition locally minimizing curves in M. The variation vector fields through geodesics satisfy the Jacobi equation in each component manifold. In this paper we find the equation which show how Jacobi vector fields change in passing across the boundary of a component manifold into the neighboring component. As an application we characterize glued Riemannian manifolds whose glued boundary separates conjugate points.

1. DEFINITIONS AND STATEMENTS

1.1. Glued Riemannian manifolds. Busemann and Phadke ([1]) have made glued G-surfaces M such that there exist points around which any geodesic circles are not convex in M. Glued surfaces are often used as intuitive examples in some papers and literature on Riemannian geometry of geodesics. The two sides of billiard tables and some collapsing Riemannian manifolds are considered to be a kind of glued Riemannian manifolds. The surface of things are often made up of smooth surfaces with boundary. Thinking those examples and ones in Section 4, we give the definition of glued Riemannian manifolds.

A complete connected one-dimensional *glued Riemannian manifold* M is by definition a piecewise smooth Riemannian manifold which is, therefore, isometric to a closed curve with suitable length or an interval in the real line.

We assume for the inductive method that *complete glued Riemannian manifolds* M with dimension $n-1$ are defined.

Let M be a complete connected topological manifold with dimension n and boundary B (possibly $B = \emptyset$). We say that (M, g) is a *complete glued Riemannian manifold* with boundary B having a decomposition $\Gamma : M = \cup_{\alpha\in\Lambda} M_\alpha$ if the decomposition Γ satisfies the following.

(1) Each (M_α, g_α), $g_\alpha = g|M_\alpha$, is a smooth complete Riemannian manifold with boundary B_α and dimension n.

1991 *Mathematics Subject Classification.* Primary 53C20;.

Key words and phrases. geodesic, Jacobi vector field, Riemannian manifold.

Partly supported by the Grants-in-Aid for Scientific Research, the Ministry of Education, Science and Culture, Japan

(2) (Int $M_\alpha) \cap M_\beta = \emptyset$ for $\alpha \neq \beta \in \Lambda$ where Int M_α is the interior of M_α.
(3) Each connected component of the boundary B_α of M_α with Riemannian metric $g_\alpha | B_\alpha$ is also a glued Riemannian manifold with dimension $n-1$ for any $\alpha \in \Lambda$.
(4) If M_α and M_β are glued at $p \in B_\alpha \cap B_\beta$ and there is a neighborhood U of p in M such that $U = (U \cap M_\alpha) \cup (U \cap M_\beta)$, then both B_α and B_β are differentiable at p as hypersurfases in M_α and M_β, respectively. $T_pB_\alpha = T_pB_\beta$ and $g_\alpha = g_\beta$ on T_pB_α.

1.2. The law of passage and reflection. Let M be a glued Riemannian manifold with boundary B and the decomposition $M = \cup_{\alpha\in\Lambda} M_\alpha$. Let $B^t = \cup_{\alpha\in\Lambda} B_\alpha$. Let N_α be the inward unit normal vector field to B_α in M_α for each $\alpha \in \Lambda$. Each N_α is defined on the set of points at which B_α is differentiable. For any point $p \in B^t$ where B^t is differentiable we are going to define the law Q of passage and reflection depending on whether $p \notin B$ or $p \in B$.

If $p \in B_\alpha \cap B_\beta$ for some $\alpha \neq \beta \in \Lambda$, then the map $Q_{\alpha\beta} : T_pM_\alpha \to T_pM_\beta$ is defined as

$$Q_{\alpha\beta}(X) = X - g_\alpha(X, N_\alpha)N_\alpha - g_\alpha(X, N_\alpha)N_\beta$$

for any tangent vector $X \in T_pM_\alpha$. We call it the *law of passage* at p from M_α to M_β.

If $p \in B$, and, hence, $p \in B_\alpha$ for a single $\alpha \in \Lambda$, then the map $Q_\alpha : T_pM_\alpha \to T_pM_\alpha$ is defined as

$$Q_\alpha(X) = X - 2g_\alpha(X, N_\alpha)N_\alpha$$

for any tangent vector $X \in T_pM_\alpha$. We call it the *law of reflection* at p to B. The law Q_α of reflection is considered to be a special case of the law $Q_{\alpha\alpha}$ of passage.

We may simply write Q without confusion instead of $Q_{\alpha\beta}$ and Q_α. The law Q comes from the condition of straightness and isometry, that is,

(1) $g_\beta(Q(X), Y) = g_\alpha(X, Y)$, and $g_\beta(Q(X), N_\beta)g_\alpha(X, N_\alpha) \leq 0$ for any $Y \in T_pB_\alpha = T_pB_\beta$ and $X \in T_pM_\alpha$,
(2) $g_\beta(Q(X), Q(X)) = g_\alpha(X, X)$ for any $X \in T_pM_\alpha$.

We notice that $Q(N_\alpha) = -N_\beta$ and $Q(Y) = Y$ for any $Y \in T_pB_\alpha$.

1.3. Geodesics. Let $c : [a, b] \to M$ be a curve (any interval is possible as a domain) and $J_c = \{s \in [a, b] \mid c(s) \in B^t\}$. We say that a curve c is *regular* if J_c has no accumulation point in $[a, b]$. For a regular curve c let a map $T : J_c \to J_c$ be given by

$$T(s) = \min\{u \in P_c \mid s < u\}$$

for any $s \in J_c$. The map T is like a *ceiling function* in the billiard ball problems.

We say that a regular curve parametrized by arc-length $\gamma : [a, b] \to M$ is a *geodesic curve* in M if the following are satisfied for any $s \in J_\gamma$.

(1) $\gamma | [s, T(s)]$ is a geodesic curve in M_α in the usual sense for some $\alpha \in \Lambda$.

(2) If $\gamma(s) \in B_\alpha$ for some $\alpha \in \Lambda$, then B_α is differentiable at $\gamma(s)$ and $\dot{\gamma}(s-0) \notin T_{\gamma(s)}B_\alpha$.

(3) $\dot{\gamma}(s+0) = Q(\dot{\gamma}(s-0))$

1.4. Statements. Let M^{n+1} be a complete glued Riemannian manifold with $B^t \neq \emptyset$. A variation of a geodesic curve γ through geodesic curves yields a Jacobi vector field Y along γ in each component manifold M_α of M where γ is contained. The Jacobi vector field Y and its covariant derivative with respect to $\dot{\gamma}$ may not be continuous at a point in B^t. The purpose of the present note is to describe what happens to Y at those points. In Section 2 and 3 we prove some lemmas which tell us how Y changes when the geodesic γ passes across B^t into the neighboring component. Although we can prove those results by simple modification of notation in the corresponding proofs of theorems for billiard ball tables ([2]), we write them for convenience and completeness because they are fundamental and important formulas in the study of glued Riemannian manifolds. The formulas in Lemma 2.3 suggests us that many properties for usual Jacobi vector fields along geodesics in smooth Riemannian manifolds hold true in our case of glued Riemannian manifolds. Indeed, we can bring many theorems for usual smooth Riemannian manifolds in those ones. However, we introduce just one of them without proof, because the proofs are simple modifications.

Let T_1M be the unit tangent bundle of M and $\pi : T_1M \to M$ the natural projection. For a vector $v \in T_1M$ let γ_v be the geodesic with $\dot{\gamma}_v(0) = v$. If $\pi(v) \in B^t$, then $\dot{\gamma}_v(0)$ is considered either $\dot{\gamma}_v(+0)$ or $\dot{\gamma}_v(-0)$. The geodesics γ_v are defined on the whole real line $(-\infty, \infty)$ for almost all $v \in T_1M$. We denote the set of all such vectors by SM. We denote the set of all vectors $v \in SM$ with $q = \pi(v) \in B^t$ and $g_\alpha(v, N_\alpha(q)) > 0$ by $(B^t)_{in}$, assuming $v \in T_qM_\alpha$. Let T be the ceiling function on $(B^t)_{in}$, i.e., $T(v)$ is the first parameter such that $\gamma_v(T(v)) \in B^t$, $T(v) > 0$ (possibly $+\infty$). Let $F : (B^t)_{in} \to (B^t)_{in}$ be a map given by $F(v) = \dot{\gamma}_v(T(v) + 0)$ for any $v \in (B^t)_{in}$.

We say that $\gamma(t_1)$ is a *conjugate point* to $\gamma(t_0)$, $t_0 \neq t_1$, if there exists a nontrivial Jacobi vector field Y along γ with $Y(t_0) = Y(t_1) = 0$ such that Y satisfies (1)-(3) in Lemma 2.3. We will show a theorem in relation to the following property.

(P) We say that M is *with B^t isolated by conjugate points* if there exist positive measurable functions ν and μ on $(B^t)_{in}$ such that $\gamma_v(\nu(v))$ is the first conjugate point to $\gamma_v(-\mu(v))$ along γ_v and $T(v) \geq \nu(v) + \mu(F(v))$ for any $v \in (B^t)_{in}$.

In order to state our result we need a few terminologies more. Let dM and dB_α be the volume forms on M and B_α (resp.) induced from the Riemannian metric and let ${}^\alpha S$ be the second fundamental form of B_α at differentiable points with respect to N_α for any $\alpha \in \Lambda$.

Let $\lambda_{{}^\alpha S}$ denote the maximal eigenvalue function of ${}^\alpha S$, i.e., $\lambda_{{}^\alpha S}(q)$ is the maximal eigenvalue of ${}^\alpha S$ at $q \in B^t$ if we think $q \in B_\alpha$ for some $\alpha \in \Lambda$. In connection to the condition (P) and some theorems in [2] we introduce the following theorem as an application of our lemmas to be proved in this paper.

Theorem. *If M^{n+1} is compact, of nonpositive curvature and with B^t isolated by conjugate points, then*

$$\sum_{\alpha\in\Lambda}\int_{B_\alpha} \lambda_{{}^\alpha S}\, dB_\alpha \geq \frac{\left(\sum_{\alpha\in\Lambda}\mathrm{vol}(B_\alpha)\right)^2}{(n+1)\mathrm{vol}(M)}$$

and the equality sign is true only if all M_α, $\alpha \in \Lambda$, are isometric to a spherical domain of radius r with flat metric where $r = (\lambda_{{}^\alpha S})^{-1}$ is constant.

The theorem is a generarization of Theorem D in [2]. Example 4.4 shows that the equality sign does not hold true in general even if $n = 1$.

2. Variation vector fields

Let M be a glued Riemannian manifold with boundary B and let $q \in B_\alpha \cap B_\beta$ (possibly $\alpha = \beta$) be a point satisfying the condition (4) in Subsection 1.1. Let $X_\beta \in T_qM_\beta$. We define a map $P_\beta : {X_\beta}^\perp \to T_qB_\beta$ as

$$P_\beta(v) = v - \frac{g_\beta(v, N_\beta)}{g_\beta(X_\beta, N_\beta)}X_\beta,$$

where ${X_\beta}^\perp = \{w \in T_qM_\beta \,|\, g_\beta(w, X_\beta) = 0\}$. Let ${}^\xi S$ be the second fundamental form with respect to the unit normal vector field N_ξ to B_ξ which satisfies by definition that

$${}^\xi\nabla_Z N_\xi = -{}^\xi S_q(Z)$$

for any tangent vector $Z \in T_qB_\alpha$ where $\xi = \alpha, \beta$, and ${}^\xi\nabla$ is the Levi-Civita connection with respect to g_ξ. Notice that ${}^\xi S(Z) \in T_qB_\xi$ and ${}^\xi S$ is a symmetric linear transformation of $T_qB_\alpha = T_qB_\beta$. We define a map $A(X_\beta) : {X_\beta}^\perp \to {X_\beta}^\perp$ as

$$A(X_\beta)(v) = g_\beta(X_\beta, N_\beta)({}^\alpha S + {}^\beta S)\circ P_\beta(v) - g_\beta(X_\beta, ({}^\alpha S + {}^\beta S)\circ P_\beta(v))N_\beta$$

for any tangent vector $v \in {X_\beta}^\perp$.

Lemma 2.1. *The map $A(X_\beta)$ is symmetric.*

Proof. Let $v, w \in {X_\beta}^\perp$. Then, we have that

$$\begin{aligned}
&g_\beta(A(X_\beta)(v), w)\\
=&g_\beta\Bigg(g_\beta(X_\beta, N_\beta)({}^\alpha S + {}^\beta S)\circ P_\beta(v) - g_\beta(X_\beta, ({}^\alpha S + {}^\beta S)\circ P_\beta(v))N_\beta,\\
&\qquad\qquad\qquad\qquad P_\beta(w) + \frac{g_\beta(w, N_\beta)}{g_\beta(X_\beta, N_\beta)}X_\beta\Bigg)\\
=&g_\beta(X_\beta, N_\beta)g_\beta(({}^\alpha S + {}^\beta S)\circ P_\beta(v), P_\beta(w))\\
=&g_\beta(X_\beta, N_\beta)g_\beta(P_\beta(v), ({}^\alpha S + {}^\beta S)\circ P_\beta(w))\\
=&g_\beta(v, A(X_\beta)(w))
\end{aligned}$$

This completes the proof.

We will see the reason why A is defined as above. We first observe that the difference between ${}^{\alpha}\nabla$ and ${}^{\beta}\nabla$ around $q \in B_{\alpha} \cap B_{\beta}$.

Lemma 2.2. *Let $Y \in T_q B_{\alpha}$ and X a tangent vector field to M_{α} defined around q in B_{α}. Then, we get the equation*

$$
\begin{aligned}
&{}^{\beta}\nabla_Y Q(X) - Q({}^{\alpha}\nabla_Y X) \\
=&g_{\beta}(Q(X), ({}^{\alpha}S + {}^{\beta}S)(Y))N_{\beta} - g_{\beta}(Q(X), N_{\beta})({}^{\alpha}S + {}^{\beta}S)(Y).
\end{aligned}
$$

Proof. If Z is a tangent vector field to B_{α}, then we have that

$$
{}^{\beta}\nabla_Y Z = {}^{\alpha}\nabla_Y Z + g_{\beta}(Z, {}^{\beta}S(Y))N_{\beta} - g_{\alpha}(Z, {}^{\alpha}S(Y))N_{\alpha},
$$

because the induced connection from ${}^{\alpha}\nabla$ is the same as the one from ${}^{\beta}\nabla$ around q in $B_{\alpha} \cap B_{\beta}$. Since $X - g_{\alpha}(X, N_{\alpha})N_{\alpha}$ is tangent to B_{α}, we get the eqation

$$
\begin{aligned}
&{}^{\beta}\nabla_Y Q(X) = {}^{\beta}\nabla_Y (X - g_{\alpha}(X, N_{\alpha})N_{\alpha}) - {}^{\beta}\nabla_Y (g_{\alpha}(X, N_{\alpha})N_{\beta}) \\
=&Q({}^{\alpha}\nabla_Y X) + g_{\alpha}(X, ({}^{\alpha}S + {}^{\beta}S)(Y))N_{\beta} + g_{\alpha}(X, N_{\alpha})({}^{\alpha}S + {}^{\beta}S)(Y) \\
=&Q({}^{\alpha}\nabla_Y X) + g_{\beta}(Q(X), ({}^{\alpha}S + {}^{\beta}S)(Y))N_{\beta} - g_{\beta}(Q(X), N_{\beta})({}^{\alpha}S + {}^{\beta}S)(Y)
\end{aligned}
$$

This completes the proof.

Let $\gamma : [a, b] \to M$ be a geodesic curve with $\gamma(t_0) = q$, $\gamma([a, t_0 - 0]) \subset M_{\alpha}$, $\gamma([t_0 + 0, b]) \subset M_{\beta}$. We write $X_{\alpha}(t) = \dot{\gamma}(t)$ for $a \le t \le t_0 - 0$ and $X_{\beta}(t) = \dot{\gamma}(t)$ for $t_0 + 0 \le t \le b$. Consider a variation $\varphi : [a, b] \times (-\varepsilon, \varepsilon) \to M_{\alpha} \cup M_{\beta}$ such that $\varphi(t, 0) = \gamma(t)$ and $\varphi_s = \varphi(\cdot, s)$ is a geodesic curve for each s and the function $t_0(s)$ of the parameters at which the geodesic curves pass across or reflect is smooth for s. Let $Y_{\alpha}(t)$ be the variation vector field for $a \le t \le t_0 - 0$ and $Y_{\beta}(t)$ for $t_0 + 0 \le t \le b$. Then, we prove the following.

Lemma 2.3.

$$
{}^{\xi}\nabla_{X_{\xi}} {}^{\xi}\nabla_{X_{\xi}} Y_{\xi} + R_{\xi}(Y_{\xi}, X_{\xi})X_{\xi} = 0, \tag{1}
$$

$$
Q(Y_{\alpha}(t_0)) = Y_{\beta}(t_0), \tag{2}
$$

$$
Q({}^{\alpha}\nabla_{X_{\alpha}} Y_{\alpha}(t_0)) - {}^{\beta}\nabla_{X_{\beta}} Y_{\beta}(t_0) = A(X_{\beta}(t_0))(Y_{\beta}{}^{o}(t_0)), \tag{3}
$$

where $\xi = \alpha, \beta$ and R_{ξ} is the Riemannian curvature tensor and $Y_{\beta}{}^{o}$ is the perpendicular component of Y_{β} to $X_{\beta}(t_0)$. Further, if $g_{\alpha}(Y_{\alpha}(a), X_{\alpha}(a)) = 0$, then

$$
X_{\xi} \perp Y_{\xi} \quad \textit{for} \quad \xi = \alpha, \beta.
$$

Proof. (1): Since φ is a variation through geodesic curves, Y_ξ is a Jacobi vector field along γ, and, hence, satisfies (1).

(2): Differentiating both sides of $\varphi(t_0(s)-0,s)=\varphi(t_0(s)+0,s)$ at $s=0$, we have

$$t_0'(0)X_\alpha(t_0)+Y_\alpha(t_0)=t_0'(0)X_\beta(t_0)+Y_\beta(t_0),$$

and, hence,

$$\begin{aligned}Y_\beta(t_0)&=Y_\alpha(t_0)+t_0'(0)(X_\alpha(t_0)-X_\beta(t_0))\\&=Y_\alpha(t_0)+t_0'(0)(g_\alpha(X_\alpha(t_0),N_\alpha)N_\alpha+g_\alpha(X_\alpha(t_0),N_\alpha)N_\beta),\end{aligned}$$

since $Q(X_\alpha)=X_\beta$. We also have

$$t_0'(0)=-\frac{g_\alpha(Y_\alpha(t_0),N_\alpha)}{g_\alpha(X_\alpha(t_0),N_\alpha)},$$

since $t_0'(0)X_\alpha(t_0)+Y_\alpha(t_0)\in T_{\gamma(t_0)}B_\alpha$. Therefore, we get the equation

$$Y_\beta(t_0)=Y_\alpha(t_0)-(g_\alpha(Y_\alpha(t_0),N_\alpha)N_\alpha+g_\alpha(Y_\alpha(t_0),N_\alpha)N_\beta)=Q(Y_\alpha(t_0)).$$

(3): Let $\psi:[a,b]\times(-\epsilon,\epsilon)\to M$ be a reparametrization of φ such that $\psi(\bar t(t,s),s)=\varphi(t,s)$, $\bar t(t_0(s),s)=t_0$ and $\|\bar X_\alpha(t_0,s)\|_\alpha=\|\bar X_\beta(t_0,s)\|_\beta$ where $\bar X_\xi(\bar t,s)=\dfrac{\partial\psi}{\partial\bar t}(\bar t,s)$ for $a\le\bar t\le t_0-0$ if $\xi=\alpha$ and $t_0+0\le\bar t\le b$ if $\xi=\beta$. Let $\bar Y_\xi(\bar t,s)=\dfrac{\partial\psi}{\partial s}(\bar t,s)$ be the variation vector field for $\xi=\alpha,\beta$ as before. Then, $\bar Y_\alpha(t_0,s)=\bar Y_\beta(t_0,s)$ for all s. We see from Lemma 2.2 that

$$\begin{aligned}&({}^\beta\nabla_{\bar X_\beta}\bar Y_\beta)(t_0)=({}^\beta\nabla_{\bar Y_\beta}\bar X_\beta)(t_0)={}^\beta\nabla_{\bar Y_\alpha}(Q(\bar X_\alpha))\\&=Q\left({}^\alpha\nabla_{\bar X_\alpha}\bar Y_\alpha(t_0)\right)+g_\beta(\bar X_\beta,({}^\alpha S+{}^\beta S)(\bar Y_\beta))N_\beta-g_\beta(\bar X_\beta,N_\beta)({}^\alpha S+{}^\beta S)(\bar Y_\beta).\end{aligned}$$

It should be noted that

$$g_\beta(\bar X_\beta,({}^\alpha S+{}^\beta S)(\bar Y_\beta))N_\beta-g_\beta(\bar X_\beta,N_\beta)({}^\alpha S+{}^\beta S)(\bar Y_\beta)\in X_\beta(t_0)^\perp,$$

$\bar X_\xi$ can change to X_ξ because of the linear property of ${}^\xi\nabla$, and

$$Y_\xi{}^o:=Y_\xi-g_\xi(Y_\xi,X_\xi)X_\xi=\bar Y_\xi-g_\xi(\bar Y_\xi,X_\xi)X_\xi$$

for $\xi=\alpha,\beta$. Since φ is the variation through unit speed geodesics, we can see that $g_\xi(Y_\xi,X_\xi)=\mathit{const.}$, and, hence, ${}^\xi\nabla_{X_\xi}Y_\xi={}^\xi\nabla_{X_\xi}Y_\xi{}^o$. Moreover, we have

$$\begin{aligned}{}^\beta\nabla_{X_\beta}Y_\beta{}^o&={}^\beta\nabla_{X_\beta}(\bar Y_\beta-g_\beta(\bar Y_\beta,X_\beta)X_\beta)\\&=Q({}^\alpha\nabla_{X_\alpha}\bar Y_\alpha)+g_\beta(X_\beta,({}^\alpha S+{}^\beta S)(\bar Y_\beta))N_\beta-g_\beta(X_\beta,N_\beta)({}^\alpha S+{}^\beta S)(\bar Y_\beta)\\&\qquad-g_\beta(Q({}^\alpha\nabla_{X_\alpha}\bar Y_\alpha),X_\beta)X_\beta.\end{aligned}$$

Since $P_\beta(Y_\beta{}^o(t_0)) = \bar{Y}_\beta(t_0)$ and

$$g_\beta(Q({}^\alpha\nabla_{X_\alpha}\bar{Y}_\alpha), X_\beta)X_\beta = Q({}^\alpha\nabla_{X_\alpha} g_\alpha(\bar{Y}_\alpha, X_\alpha)X_\alpha),$$

we see that

$${}^\beta\nabla_{X_\beta}Y_\beta{}^o = Q({}^\alpha\nabla_{X_\alpha}Y_\alpha{}^o) - A(X_\beta(t_0))(Y_\beta{}^o(t_0))$$

and, therefore, (3) is proved.

(4): Since φ is a variation through unit speed geodesic curves, the length of each geodesic curve is

$$t - a = \int_a^t \|X_\alpha(t,s)\|_\alpha \, dt,$$

if $t \in [a, t_0(s) - 0]$. Differentiating at $s = 0$, we have

$$0 = g_\alpha(Y_\alpha(t), X_\alpha(t)) - g_\alpha(Y_\alpha(a), X_\alpha(a))$$

if $a \le t \le t_0 - 0$. If $t \in [t_0(s) + 0, b]$, then we have that

$$t - a = \int_a^{t_0(s)} \|X_\alpha(t,s)\|_\alpha \, dt + \int_{t_0(s)}^t \|X_\beta(t,s)\|_\beta \, dt.$$

Differentiating at $s = 0$, we get the equation

$$\begin{aligned} 0 = g_\alpha(Y_\alpha(t_0), X_\alpha(t_0)) &- g_\alpha(Y_\alpha(a), X_\alpha(a)) \\ &+ g_\beta(Y_\beta(t), X_\beta(t)) - g_\beta(Y_\beta(t_0), X_\beta(t_0)) \\ &+ t_0{}'(0)\,(\|X_\alpha(t_0)\|_\alpha - \|X_\beta(t_0)\|_\beta) \end{aligned}$$

if $t_0 + 0 \le t \le b$. It follows from the first equation and the assumption that $g_\alpha(Y_\alpha(t), X_\alpha(t)) = 0$ for $a \le t \le t_0 - 0$. Since $Y_\beta(t_0) = Q(Y_\alpha(t_0))$, $X_\beta(t_0) = Q(X_\alpha(t_0))$ and $\|X_\alpha(t_0)\|_\alpha = \|X_\beta(t_0)\|_\beta = 1$, we also have that $g_\beta(Y_\beta(t), X_\beta(t)) = 0$ for $t_0 + 0 \le t \le b$. This completes the proof of Lemma 2.3.

We can show many properties of perpendicular Jacobi vector fields along a geodesic curve as were proved for ordinary ones.

3. The passage and mirror equation

Let $\gamma : [a, b] \to M$ be a geodesic curve and Y a vector field along γ. We call Y a *Jacobi vector field* along γ if it satisfies (1) – (3) in Lemma 2.3. Let $t_1 \in [a, b]$. We say that $\gamma(t_2)$ is a *conjugate point* to $\gamma(t_1)$ along γ if there is a nontrivial Jacobi vector field along γ with $Y(t_1) = 0$ and $Y(t_2) = 0$.

In this section we prove the passage and mirror equation and make the relation between $S = {}^\alpha S + {}^\beta S$ and A clear. Let λ_H denote the maximal eigenvalue of a symmetric linear transformation H.

Lemma 3.1 (The passage and mirror equation). *Let M_α and M_β be flat Riemannian manifolds with boundary B_α and B_β, respectively, such that M_α is glued to M_β around $q \in B_\alpha \cap B_\beta$ in $B_\alpha \cap B_\beta$. Let $\gamma : [0, t_0] \to M_\alpha \cup M_\beta$ be a geodesic curve passing across $B_\alpha \cap B_\beta$ at only one point $q = \gamma(a)$. Suppose γ meets at the angle θ to the tangent space $T_q B_\alpha$. If $\gamma(t_0)$ is the first conjugate point to $\gamma(0)$ along γ and $b = t_0 - a$, then we get*

$$\frac{\lambda_{S_q}}{\sin\theta} \geq \lambda_{A(\dot{\gamma}(a+0))} = \frac{1}{a} + \frac{1}{b},$$

where $S_q = {}^\alpha S_q + {}^\beta S_q$. The equality sign is true in the first inequality if and only if there are the eigenvectors of $A(\dot{\gamma}(a+0))$ and S_q with eigenvalues $\lambda_{A(\dot{\gamma}(a+0))}$ and λ_{S_q} in the subspace spanned by $\{N_\beta, \dot{\gamma}(a+0)\}$. In particular, the equality sign is always true if $n = 1$.

Proof. Let $X = \dot{\gamma}(a+0)$. Let $P(w) = w - \dfrac{g_\beta(w, N_\beta)}{g_\beta(X, N_\beta)} X$ for any $w \in X^\perp$. Then,

$$\begin{aligned} g_\beta(A(X)(w), w) &= g_\beta(X, N_\beta) g_\beta(S_q \circ P(w), P(w)) \\ &\leq \frac{1}{\sin\theta} \|w\|_\beta{}^2 g_\beta\left(S_q\left(\frac{P(w)}{\|P(w)\|_\beta}\right), \frac{P(w)}{\|P(w)\|_\beta}\right) \end{aligned}$$

for any $w \in X^\perp$, since

$$\|P(w)\|_\beta \leq \frac{1}{\sin\theta}\|w\|_\beta.$$

This proves the first inequality. In a flat glued Riemannian manifold $M = M_\alpha \cup M_\beta$ the matrix Jacobi field D along γ with $D(0) = 0$, and $D'(0) = I$ is written

$$D(t) = (t-a)(I - aA(X)) + aI$$

for $t \in [a, t_0]$ where I is the identity map. Hence, $D(t)$ is symmetric, $D(t_0) \geq 0$ and $\det D(t_0) = 0$ since $\gamma(t_0)$ is the first conjugate point to $\gamma(0)$. We see that

$$A(X) \leq \left(\frac{1}{a} + \frac{1}{b}\right) I$$

and

$$\lambda_{A(X)} = \frac{1}{a} + \frac{1}{b}.$$

This completes the proof.

We can also show the following lemmas which are straightforward modifications of Lemma 3.1.

Lemma 3.2. *If the flat Riemannian manifolds in Lemma 3.1 are replaced by the manifolds of constant curvature k^2 $(k > 0)$, then the a and b in Lemma 3.1 change to $\frac{1}{k}\tan ka$ and $\frac{1}{k}\tan kb$, respectively.*

Lemma 3.3. *If the flat Riemannian manifolds in Lemma 3.1 are replaced by the manifolds of constant curvature $-k^2$ $(k > 0)$, then the a and b in Lemma 3.1 change to $\frac{1}{k}\tanh ka$ and $\frac{1}{k}\tanh kb$, respectively.*

We show the relation between $S_q = {}^{\alpha}S_q + {}^{\beta}S_q$ and $A(X)$.

Lemma 3.4. *Let M_α and M_β be (n+1)-dimensional Riemannian manifolds with boundary B_α and B_β, respectively, such that M_α is glued to M_β around $q \in B_\alpha \cap B_\beta$. Let $X \in T_qM_\beta$ and let X meet at the angle θ to T_qB_β. Then, the following are true.*

(1) *If the dimension of M is two, then $A(X) = \dfrac{\kappa_\alpha + \kappa_\beta}{\sin\theta}$, where κ_ξ is the geodesic curvature of B_ξ at q for $\xi = \alpha, \beta$.*
(2) *$S_q = 0$ if and only if $A(X) = 0$.*
(3) *If $S_q \leq 0$, then $A(X) \leq 0$ and $\mathrm{tr}A(X) \geq \dfrac{1}{\sin\theta}\mathrm{tr}S_q$.*
(4) *If $S_q \geq 0$, then $A(X) \geq 0$ and $\mathrm{tr}A(X) \leq \dfrac{1}{\sin\theta}\mathrm{tr}S_q$.*
(5) *If $S_q = \lambda I$, then $\mathrm{tr}A(X) = \dfrac{\lambda}{\sin\theta}(1 + (n-1)\sin^2\theta)$.*

Here tr S_q *is by definition the trace of S_q.*

Proof. Let $w_1, w_2 \in X^{\perp}$. We have that

$$g_\beta\left(A(X)(w_1), w_2\right) = g_\beta\left(X, N_\beta\right) g_\beta\left(S \circ P(w_1), P(w_2)\right)$$

Since P is surjective, the statement (2) and the first parts of (3) and (4) are clear.

In order to prove others we extend S_q, $A(X)$ and P linearly on T_qM by setting $S_q(N_\beta) = 0$, $A(X)(X) = 0$ and $P(X) = 0$. The traces of S_q and $A(X)$ do not change. Take an orthonormal basis $\{e_k\}$ such that $e_1, \cdots, e_n \in T_qB$ are eigenvectors of S_q with eigenvalues $\lambda_1, \cdots, \lambda_n$, respectively, and $e_{n+1} = N_\beta$. Then, we get

$$\begin{aligned}
trA(X) &= \sum_{k=1}^{n+1} g_\beta\left(A(X)(e_k), e_k\right) = g_\beta\left(X, N_\beta\right) \sum_{k=1}^{n+1} g_\beta\left(S_q \circ P(e_k), P(e_k)\right) \\
&= g_\beta\left(X, N_\beta\right) \left\{ \sum_{k=1}^{n} g_\beta\left(S_q(e_k), e_k\right) - \frac{1}{g_\beta\left(X, N_\beta\right)} g_\beta\left(S_q(X), N_\beta - \frac{X}{g_\beta\left(X, N_\beta\right)}\right) \right\} \\
&= \sin\theta \left\{ \sum_{k=1}^{n} \lambda_k + \frac{1}{\sin^2\theta} g_\beta\left(S_q(X), X\right) \right\}.
\end{aligned}$$

Since $S_q(X) = \sum_{k=1}^{n} \lambda_k g_\beta(X, e_k)\, e_k$, we have that

$$\mathrm{tr}A(X) = \frac{1}{\sin\theta} \sum_{k=1}^{n} \lambda_k (\sin^2\theta + g_\beta(X, e_k)^2).$$

Since $\sin\theta = g_\beta(X, e_{n+1})$, we see that $\sin^2\theta + g_\beta(X, e_k)^2 \le 1$ for each k, and the equality sign is true if $n = 1$. This completes the proof of (1), and

$$\mathrm{tr}A(X) \ge \frac{1}{\sin\theta}\mathrm{tr}S_q \quad \text{if } S_q \le 0,$$

$$\mathrm{tr}A(X) \le \frac{1}{\sin\theta}\mathrm{tr}S_q \quad \text{if } S_q \ge 0,$$

and

$$\mathrm{tr}A(X) = \frac{\lambda}{\sin\theta}(1 + (n-1)\sin^2\theta) \quad \text{if } S = \lambda I.$$

This completes the proof

Let both M_α and M_β be submanifolds in a Riemannian manifold $\tilde{M}$ of class C^∞. It is natural to ask what happens to $S = {}^\alpha S + {}^\beta S$ if $T_q M_\alpha = T_q M_\beta$ as the tangent spaces of submanifolds in $\tilde{M}$. The following lemma answers this question.

Lemma 3.5. *Let $\tilde{M}$ be a Riemannian manifold of class C^∞. Suppose a glued Riemannian manifold $M = \cup_{\alpha\in\Lambda} M_\alpha$ is immersed in $\tilde{M}$ and its component manifolds are of class C^∞ in $\tilde{M}$ as submanifolds. If $T_q M_\alpha = T_q M_\beta$ at any point $q \in B_\alpha \cap B_\beta$ at which B_α and B_β are differentiable, then we get the equation*

$${}^\alpha S_q + {}^\beta S_q = 0.$$

Therefore, $A(X) = 0$ for any tangent vector $X \in T_q M_\beta$ with $X \notin T_q B_\beta$.

Proof. From the assumption it follows that $N_\alpha + N_\beta = 0$. Let $\tilde{\nabla}$ be the Levi-Civita connection in $\tilde{M}$. We notice that the second fundamental form h_α is equal to h_β. Thus, we have that for any $Y \in T_q(B_\alpha \cap B_\beta)$,

$$\begin{aligned} 0 &= \tilde{\nabla}_Y (N_\alpha + N_\beta) \\ &= {}^\alpha\nabla_Y N_\alpha + {}^\beta\nabla_Y N_\beta + h_\alpha(N_\alpha + N_\beta, Y) \\ &= -({}^\alpha S + {}^\beta S)(Y), \end{aligned}$$

This completes the proof.

4. Examples

In this section we give some examples which help us in having the notion of glued Riemannian manifolds.

4.1. Surfaces of cylinders. Let $M = M_1 \cup M_2 \cup M_3$ be a union of the following three surfaces in the Euclidean space E^3:

$$\begin{aligned} M_1 &= \{(x, y, 0) \mid x^2 + y^2 \leq 1\}, \\ M_2 &= \{(x, y, z) \mid x^2 + y^2 = 1, 0 \leq z \leq 1\}, \\ M_3 &= \{(x, y, 1) \mid x^2 + y^2 \leq 1\}, \end{aligned}$$

and g_α, $\alpha = 1, 2, 3$, are induced Riemannian metrics from the natural Euclidean metric of E^3. Then, we see that

$$\begin{aligned} B_1 &= \{(x, y, 0) \mid x^2 + y^2 = 1\}, \\ B_3 &= \{(x, y, 1) \mid x^2 + y^2 = 1\}, \\ B_2 &= B_1 \cup B_3, \end{aligned}$$

and, hence, $B = \emptyset$, $B^t = B_1 \cup B_3 = B^0$. For any point $p \in B_1 \cap B_2$ we see that T_pM_1 is the xy-plane, T_pM_2 is the hyperplane through p and perpendicular to the vector from 0 to p, $N_1(p) = -p$, $N_2(p) = (0, 0, 1)$ which may be considered as a vector at p, and $T_p(B_1 \cap B_2) = T_pB_1$ is the tangent line to B_1 through p.

Let a unit vector $X_1 \in T_pM_1$ with $g_1(X_1, -N_1) = \sin\theta > 0$, Then, $Q(X_1)$ is a unit vector X_2 in T_pM_2 with $g_2(X_2, N_2) = \sin\theta > 0$. Since B_1 and B_3 are unit circles, we see that ${}^1S_p = I$ and ${}^3S_q = I$ for any point $p \in B_1$ and $q \in B_3$, respectively. Concerning B_2 we also see that ${}^2S = 0$. Further, $A(X_2) = \dfrac{1}{\sin\theta}I$, and $A(X_1) = \dfrac{1}{\sin\theta}I$ where I is the identity map.

4.2. Surfaces of cones. Let c be a positive and

$$\begin{aligned} M_1 &= \{(x, y, 0) \mid x^2 + y^2 \leq 1\} \\ M_2 &= \{(x, y, z) \mid x = t\cos\theta, y = t\sin\theta, z = (1-t)c, 0 \leq t \leq 1, 0 \leq \theta \leq \pi\} \\ M_3 &= \{(x, y, z) \mid x = t\cos\theta, y = t\sin\theta, z = (1-t)c, 0 \leq t \leq 1, \pi \leq \theta \leq 2\pi\}. \end{aligned}$$

with induced Riemannian metrics from the natural Euclidean metric of E^3. Then, B_1 is a unit circle, each of B_2 and B_3 consists of a half circle and two segments. M_2 is glued to M_3 at two segments of their boundary, and to M_1 at a half circle. M_1 is glued to M_2 and M_3 at their half circles.

Let $p = (0, 0, c) \in B_2 \cap B_3$. Both B_2 and B_3 are not differentiable at the vertex p, so that p must not be in the interior point of any geodesic curve.

If $p \in B_2 \cap B_3$ and $B_2 \cap B_3$ is differentiable at p, then ${}^2S_p = {}^3S_p = 0$. Hence, $A(X) = 0$ for any vector $X \in T_pM_2$ (and T_pM_3).

If $p \in B_1 \cap B_2$ and $B_1 \cap B_2$ is differentiable at p, then ${}^1S_p = I$, and ${}^2S_p = \frac{1}{\sqrt{1+c^2}}I$. Hence, $A(X) = \left(\frac{1}{\sqrt{1+c^2}} + 1\right)\frac{1}{\sin\theta}I$ where X meets at the angle θ to T_pB_1.

4.3. Tubular hypersurfaces. Let K be an imbedded submanifold in the Euclidean space E^{n+1} with boundary $\partial K \neq \emptyset$. Then, r-tubular hypersurfaces around K are considered to be glued hypersurfaces in E^{n+1} in which $A(X) = 0$ for any tangent vector at any point in glued boundary if $r > 0$ are sufficiently small.

4.4. Abstract glued surfaces. Let M_1 and M_2 be plane disks with radius $3r/2$ and M_3, M_4, M_5 with radius r. We glue M_1 to M_3, M_4, M_5 at three half circles of their boundary, and M_2 to them at the remainder part of their boundary. The glued surface is without boundary. By construction we know that ${}^\alpha S_p + {}^\beta S_p = kI$ for any point $p \in B^t$ where $k = 2/3r + 1/r$.

References

1. H.Busemann and B.B.Phadke, *Nonconvex spheres in G-spaces*, J. Indian Math. Soc. **44** (1979), 39–50.
2. N.Innami, *Integral formulas for polyhedral and spherical billiards*, J. Math. Soc. Japan **50,2** (1998), 339–357.

Department of Mathematics, Faculty of Science, Niigata University, Niigata, 950-2181, JAPAN

E-mail address: innami@math.sc.niigata-u.ac.jp

Received February 1, 2001