On imbedding closed 4-dimensional manifolds in Euclidean space

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1. Introduction

It is an open question to give an imbedding of an oriented closed differentiable 4-manifold in 7-dimensional Euclidean space space R^7 [1]. Recently, M. Hirsch has proved that such a manifold can be imbedded in R^7 piecewise linearly [4]. A closed n-dimensionaly manifold M^n will be said to be *almost differentiably imbeddable* in R^m if $M^n - x$, where x is a point of M, is differentiably imbeddable in R^m. It is known that a closed differentiable 4-manifold is almost differentiably imbeddable in R^7 [3].

In what follows, all manifolds are understood to be differentiable and compact. Differentiable will always mean of class C^∞. The notation R^n will be used for the n-dimensional Euclidean space. We write $M_1 \approx M_2$ if M_1 and M_2 are diffeomorphic The notation $\#$ will mean of the connected sum defined in [7].

In this paper, we shall prove the following

THEOREM 1. *All 4-dimensional closed π-manifolds are imbeddable in R^7.*

THEOREM 2. *All simply connected closed 4-dimensional π-manifolds are imbeddable in R^6.*

THEOREM 3. *All homotopy 4-spheres are imbeddable in R^5.*

The result of Theorem 3 has been obtained by S. Smale (unpublished).

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2. Imbedding of homotopy spheres

It is known that all homotopy n-spheres are imbeddable in R^{n+k}, where $n<2k-2$ [2]. It is easy to show that a homotopy n-sphere is imbeddable in R^{n+1} if and only if it is h-cobordant to the standard n-sphere S^n. Hence if n is greater than 4 the standard n-sphere is the only homotopy n-sphere which is imbeddable in R^{n+1}.

According to a result of [3], we can prove the following (we assume $n>5$)

THEOREM. *For odd n, a homotopy n-sphere is imbeddable in R^{n+2} if and only if it bounds a π-manifold.*

In fact, it follows from Theorem 4.1 in [3] that a homotopy n-sphere which is boundary of a π-manifold is imbeddable in R^{n+2}. The converse follows from the following result which includes the case n is even, which is due to M. Kervaire.

A simply connected closed n-manifold M^n which is imbeddable in R^{n+2} bounds a π-manifold.

For even n, we have

THEOREM. *Any homotopy n-sphere which is not standard n-sphere S^n is not imbeddable in R^{n+2}.*

In fact a homotopy n-sphere which is imbeddable in R^{n+2} bounds a π-manifold. Then, by Theorem 5.1 in [5], such a homotopy n-sphere is diffeomorphic to the standard n-sphere.

Many results on imbedding of homotopy sphere have been obtained in [6].

To obtain an imbedding of homotopy 4-sphere in R^5, we need the following lemmas.

LEMMA 1. [Theorem 1, 7]

Let M^n be a closed n-manifold. Then the following two statements are equivalent.

(1) *There exists a closed n-manifold N^n such that*

$$M^n \# N^n \approx S^n$$

(2) $M^n \# R^n \approx R^n$

Notes that $M^n \# R^n$ is diffeomorphic to $M^n - x$, for some point of M^n.

Since, for a homotopy n-sphere M^n, $M^n \# (-M^n)$, where $-M^n$ denotes the manifold with the orientation reversed, is h-cobordant to the standard n-sphere, and for $n \geqslant 5$, by the result in [9], $M^n \# (-M^n)$ is diffeomorphic to S^n.

we have

LEMMA 2. *For $n \geqslant 5$, any homotopy n-sphere is almost diffentiable imbeddable in R^n.*
Moreover we need the following lemma

LEMMA 3. [Lemma 2.3, 5]

Let M^n be a simply connected closed n-manifold. Then M^n is h-cobordant to the standard n-sphere S^n if and only if M^n bounds a contractible manifold.

Now the proof of Theorem 3 is as follows.

The fact θ_4 (=the group of homotopy 4-spheres) is trivial, and lemma 3 implies that any homotopy 4-sphere Σ bounds contractible 5-manifold V. Let $\tilde{V}$ be the manifold obtained from the union of two copies of V by identifying the common boundary. Lemma 2 implies $\tilde{V}$ is almost diffentiabele imbeddable in R^5, since $\tilde{V}$ is a homotopy 5-sphere. Since V is imbeddable in $\tilde{V} - x$, for some point x of $\tilde{V}$, we have an imbedding of Σ in R^5. This completes the proof of Theorem 3.

3. The proof of Theorem 1.

In this section M denotes a closed 4-dimensional π-manifold. Let M be imbedded in R^{4+N}, where N is sufficiently large, with a normal N-frames, and $t(M, F)$ the element of $\pi_{4+N}(S^N)$ defined by Thom contruction. Since the stable homotopy group $\pi_{4+N}(S^N)$ vanishes, M bounds a 5-dimensional π-manifold V. By a sequence of spherical modifications, we may assume that V is a simply connected π-manifold.

In order to prove Theorem 1, we need the following lemma due to C. T. C. Wall [Theorem. p 567, 11].

LEMMA 4. *Suppose V has boundary ∂V, and that the pair $(V, \partial V)$ is r-connected, $r \leqq m-4$. If V immersed in R^s and $s \geqslant 2m-2r-1$, then V imbeds is R^s.*

Now it is straightforward to prove Theorem 1 by lemma 3. (Constructing the double of V and using Theorem 4.1 in [3], we can also prove Theorem 1).

4. The proof of Theorem 2

In this section, M denotes a simply connected closed 4-dimensional π-manifold. By the same argument as in Section 3, there is a 5-manifold V whose boundary is M. By Theorem 1 in [10], we may assume that V has the homotopy type of a bouquet of somo 2-spheres, and the second Stiefel-Whitney class of V vanishes. Let $\tilde{V}$ be the manifold obtained from the disjoint union of two copies of V by identifying the common boundary. It is known that $\tilde{V}$ is simply connected. Moreover we can show that $\tilde{V}$ is a π-manifold. In fact, it follows from the fact that the second Stiefel-Whitney class of V vanishes that $\tilde{V}$ has the vanishing second Stiefel-Whitney class. Consider the following cohomology exact sequense (Mayer-Vietoris sequence)

$$\rightarrow H^1(M) \rightarrow H^2(\tilde{V}) \xrightarrow{i^*} H^2(V) + H^2(V) \rightarrow$$

It is easy to see that $i^*(w_2(\tilde{V})) = w_2(V) + w_2(V)$. Since $w_2(V)=0$, and i^* is a monomorphism, we have $w_2(\tilde{V})=0$. Now, by obstruction theory, it follows that $\tilde{V}$ is a π-manifold (i. e. the normal frame bundle of an imbedding of $\tilde{V}$ in R^{11} has a cross section), using the fact $\pi_1(\tilde{V}) = 0$ ard $\overline{w_2}(\tilde{V}) = w_2(\tilde{V}) = 0$.

According Theorem A′ in [8], we have

$$\tilde{V} \approx (S^2 \times S^3) \# \cdots\cdots \# (S^2 \times S^3) \# M_{k_1} \# \cdots\cdots \# M_{k_r},$$

where M_{k_i} is a 5-manifold such that $H_2(M_{k_i}) = Z_{k_i} + Z_{k_i}$, $k_i > 1$. If $H_2(\tilde{V})$ is torsion free, $\tilde{V}$ is diffeomorpic to

$$(S^2\times S^3)\#\cdots\cdots\#(S^2\times S^3)$$

Then it is clear $\tilde{V}$ is imbeddable in R^6, and hence M imbeds in R^6. Thus to complete the proof of Theorem 2, it must be shown that H_2 $(\tilde{V})$ is torsion free. Consider the following cohomology exact sequence of the pair $(\tilde{V},\ V)$,

$$\rightarrow H^{q-1}(V)\xrightarrow{\partial}H^q(\tilde{V},V)\xrightarrow{h^*}H^q(\tilde{V})\xrightarrow{i^*}H^q(V)\rightarrow$$

Scince $H^q(\tilde{V},V)\approx H^q(V,M)$, we have an exact sequence

$$\rightarrow H^{q-1}(V)\xrightarrow{\delta'}H^q(V,M)\xrightarrow{j^*}H^q(\tilde{V})\xrightarrow{i^*}H^q(V)\rightarrow$$

We define a map k; $\tilde{V}\rightarrow V$ by $k(x)=x$, and $k(x')=x$, where x' is the element of a copy of V corresponding to x. Then we have $k\ i=$identity map of V, and hence the induced homomorphism

$$k^*;\ H^q(V)\rightarrow H^q(\tilde{V})$$

is a monomorphism, and

$$i^*;\ H^q(\tilde{V})\rightarrow H^q(V)$$

is an epimorphism. It follows that δ' is a trivial homomorphism. Thus we have an exact sequence

$$0\rightarrow H^q(V,M)\rightarrow H^q(\tilde{V})\rightarrow H^q(V)\rightarrow 0.$$

As a special case, we have an exact sequence

$$0\rightarrow H^3(V,M)\rightarrow H^3(\tilde{V})\rightarrow 0$$

Since H^3 (V,M) is isomorphic to $H_2(V)$, which is torsion free, H^3 $(\tilde{V})$ is also torsion free. Hence H_2 $(\tilde{V})$ is torsion free. This completes the proof of Theorem 2.

Added in proof. (1) The result of Theorem 3 is proved by M. Kervaire in his paper 'On Higher Dimensional Knots'. (A symposium honor of Marston Morse).

(2) Since this writing, I found a paper written by D. Barden which includes an imbedding of simply connected 5-dimensional π-manifold in R^6.

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