Reflexivities with modifiers and conditions for reflexivity of Banach spaces

By

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1. Reflexivities with modifiers

Denote by X' and X'' the first conjugate spece and the second one of a normed space X respectively. And for a subset A of X' let ${}^{\circ}A$ and A° denote the annihilator of A in X and in X'' respectively. We shall use briefly the symbol $A^{\circ\circ}$ in place of $(A^{\circ})^{\circ}$.

Let F be a subset of X' and consider the linear operator $\pi_F=\mu\cdot\pi:X\longrightarrow X''/F^{\circ}$ where $\pi:X\longrightarrow X''$ is the canonical imbedding and $\mu: X''\longrightarrow X''/F^{\circ}$ is the natural operator, X''/F° being the quotient normed space defined in the usual way. It is a problem to find out a subset F of X' such that the operator π_F is surjective, namely $\pi_F(X)=X''/F^{\circ}$. When such a subset F exists in X', we shall call, for a time, X to be *reflexive with the modifier* F°, or shortly *reflexive (mod.* F°*)*. A trivial example of such a subset is the set (0) consisting of the zero element 0 of X' alone, for $(0)^{\circ}=X''$. If F_1 and F_2 are two subsets of X' such that $F_1\supset F_2$, we have $F_1{}^{\circ}\subset F_2{}^{\circ}$, hence if X is reflexive (mod. $F_1{}^{\circ}$) it is so also with the modifier $F_2{}^{\circ}$. A maximal subset F of X' for which X is reflexive (mod. F°) may be, in general, a proper subset of X'. Since $(X')^{\circ}=(0)$, X is reflexive (mod. $(X')^{\circ}$) if and only if X is reflexive in the usual sense. It is easy to prove the following

THEOREM 1. *X' is reflexive (mod. $(\overline{\pi(X)})^{\circ}$), where $\overline{\pi(X)}$ means the closure with respect to the norm topology of X''. And $X'''=\pi_1(X')\oplus(\overline{\pi(X)})^{\circ}$ where $\pi_1: X'\longrightarrow X'''$ is the canonical imbedding.*

PROOF. For any $x'''\in X'''$ define $x'\in X'$ by $x'x=x'''(\pi x)$ for each $x\in X$. Then we have $x'''-\pi_1x'\in(\pi(X))^{\circ}$. Since $(\overline{\pi(X)})^{\circ}=(\pi(X))^{\circ}$ and $\pi_1(X')\cap(\overline{\pi(X)})^{\circ}=(0)$, the proof is complete.

This theorem implies, as a special case, that X' is reflexive if X is reflexive. J. Dixmier [2, Theorem 15] has shown that $X'''=\pi_1(X')\oplus(\pi(X))^{\circ}$ where X is a Banach space. This is also a special case of our Theorem 1, for $\pi(X)$ is closed if X is complete.

Since $(\mathrm{v}F)^{\circ}=F^{\circ}$, where $\mathrm{v}F$ means the closed linear subspace generated by F, we may assume, in this problem, that F is a closed linear subspace of X', without loss of generality.

The following theorem follows from the second part of Theorem 1.

THEOREM 2. *We cannot find out in X'' any closed linear subspace G which contains $\overline{\pi(X)}$ (if X is complete, $\pi(X)$) as a proper subspace and relative to which X' is reflexive (mod. G°).*

PROOF. $(\overline{\pi(X)})^\circ - G^\circ$ is non-empty, for otherwise we have $\overline{\pi(X)} = G$.
This theorem implies that if X' is reflexive (consequently, reflexive (mod. $(X'')^\circ$), $\overline{\pi(X)} = X''$ (if X is complete, $\pi(X) = X''$, i. e. X is reflexive also).

THEOREM 3. *Let a normed space X be reflexive (mod. F°). Then every closed linear subspace Y of X such that $Y^\circ \subset F$ is reflexive (mod. G°) where*

$$G = {}^\circ((I'')^{-1}(F^\circ)),$$

$I'' : Y'' \longrightarrow Y^{\circ\circ}$ being the congruence operator (isometrical linear isomorphism) induced by the second adjoint operator of the inclusion operator $I : Y \longrightarrow X$.

REMARK. When X is reflexive in the usual sense, every closed linear subspace Y of X is also reflexive. This result is a special case of Theorem 3. For, in this case, we may put $F = X'$ as we noted before, and the relation $Y^\circ \subset F$ is true without any assumption on Y.

PROOF. of THEOREM. 3. Let $I' : X'/Y^\circ \longrightarrow Y'$ be the congruence operator induced by the first adjoint operator of I. Namely we have

$$(I'[x'])y = x'(Iy), \quad (I''y'')x' = y''(I'[x'])$$

for any $x' \in X'$, $y \in Y$ and $y'' \in Y''$ where $[x']$ means the element of X'/Y° represented by x'. Let $\mu : X'' \longrightarrow X''/F^\circ$ and $\nu : Y'' \longrightarrow Y''/(I'')^{-1}(F^\circ)$ be the natural operators. Consider the diagram

$$\begin{array}{ccccc}
Y & \xrightarrow{\ \tau\ } & Y'' & \xrightarrow{\ \nu\ } & Y''/(I'')^{-1}(F^\circ) \\
\downarrow I & & \downarrow I'' & & \downarrow J \\
X & \xrightarrow{\ \pi\ } & X'' & \xrightarrow{\ \mu\ } & X''/F^\circ
\end{array}$$

where τ is the canonical imbedding and J is the injection induced naturally by I''. The commutativity relations $\pi I = I''\tau$ and $\mu I'' = J\nu$ are verified easily. Consequently we have

$$\mu\pi I = J\nu\tau. \tag{1}$$

Moreover

$$J(Y''/(I'')^{-1}(F^\circ)) = Y^{\circ\circ}/F^\circ. \tag{2}$$

It follows from our assumption that for an arbitrary element $x'' \in Y^{\circ\circ}$ there exists an element $x_0 \in X$ such that $x'' - \pi x_0 \in F^\circ \subset Y^{\circ\circ}$. Therefore $\pi x_0 \in Y^{\circ\circ}$. Since Y is closed we have ${}^\circ(Y^\circ) = Y$. Hence $\pi(X) \cap Y^{\circ\circ} = \pi I(Y)$ and therefore x_0 is an element of Y. This result and the injectiveness ef J and the relations (1), (2) shoew us the surjectiveness of $\nu\tau$. Now it follows from the relation $G^\circ \supset (I'')^{-1}(F^\circ)$ that $\tau_G : Y \longrightarrow$

Y''/G° is surjective. The proof is complete.

2. Operator $\mathfrak{F}$ and Conditions for reflexivity

Let a normed space X be reflexive (mod. F°). Since $\pi({}^\circ F)=\pi(X)\cap F^\circ$ the operator π_F induces a linear isomorphism

$$\mathfrak{G}: X/{}^\circ F \longrightarrow X''/F^\circ.$$

And for any element $[x] \in X/{}^\circ F$ which is represented by $x\in X$,

$$\|\mathfrak{G}[x]\|=\inf_{y''\in F^\circ}\|\pi x+y''\|\leqq \inf_{y''\in\pi({}^\circ F)}\|\pi x+y''\|=\|[x]\|.$$

Thus we have $\|\mathfrak{G}\|\leqq 1$. Let $\mathfrak{F}$ be the inverse linear operator of $\mathfrak{G}$. From $1 \leqq \|\mathfrak{G}\|\ \|\mathfrak{F}\|$ we have $\|\mathfrak{F}\|\geqq 1$.

THEOREM 4. *If X is reflexive Banacn space, then* $\|\mathfrak{F}\|=1$.

PROOF. Since $\pi(X)=X''$ we have $\pi({}^\circ F)=F^\circ$ and $\|\mathfrak{G}[x]\|=\|[x]\|$.

Consider now the inverse problem of this theorem. Let X be a Banach space which is reflexive (mod. F°) and the operator $\mathfrak{F}$ has the norm $\|\mathfrak{F}\|=1$. In this case $\|\mathfrak{G}\|=1$ and $\mathfrak{G}$ is a congrueuce operator. Let $C: X \longrightarrow X/{}^\circ F$ be the natural operator, $C'\ : (X/{}^\circ F)' \longrightarrow ({}^\circ F)^\circ$ be the congruence operator induced by the first adjoint operator of C, and C'': $(({}^\circ F)^\circ)' \longrightarrow (X/{}^\circ F)''$ be the adjoint operator of C'. On the other hand, a theorem due to M. Krein and V. Smulian [4] gives us a congruence operator s: $X''/({}^\circ F)^{\circ\circ} \longrightarrow (({}^\circ F)^\circ)'$, defined by $s\,[x'']=x''_1|_{({}^\circ F)^\circ}$ for any $x''_1 \in [x'']$ where $[x''] \in X''/({}^\circ F)^{\circ\circ}$ is the element represented by $x'' \in X''$ and $x''_1|_{({}^\circ F)^\circ}$ means the restriction of x''_1 to $({}^\circ F)^\circ$. Since $F^\circ \supset ({}^\circ F)^{\circ\circ}$ we can consider the natural operator $N\ : X''/({}^\circ F)^{\circ\circ} \longrightarrow X''/F^\circ$. By these relations we have the following diagram

$$X/{}^\circ F \xrightarrow{\ \mathfrak{G}\ } X''/F^\circ \xleftarrow{\ N\ } X''/({}^\circ F)^{\circ\circ} \xrightarrow{\ s\ } (({}^\circ F)^\circ)' \xrightarrow{\ C''\ } (X/{}^\circ F)''.$$

Now let $\sigma\ : X/{}^\circ F \longrightarrow (X/{}^\circ F)''$ be the canonical imbedding. Then we have, for any $[x] \in X/{}^\circ F$ and $\xi \in (X/{}^\circ F)'$,

$$(\sigma[x])\xi=\xi[x]=\xi(Cx)=(C'\xi)x=(\pi x)(C'\xi)=(s[\pi x])(C'\xi)=(C''(s[\pi x]))\xi,$$

where $[\pi x] \in X''/({}^\circ F)^{\circ\circ}$ is the element represented by $\pi x \in X''$. (Note that $[x_1]=[x_2]$ implies $[\pi x_1]=[\pi x_2]$ and vice versa, for $\pi({}^\circ F)=\pi(X)\cap({}^\circ F)^{\circ\circ}$.) Thus we have

$$\sigma[x]=C''(s[\pi x]) \text{ and } \mathfrak{G}[x]=Ns^{-1}(C'')^{-1}\sigma[x].$$

Therefore $X/{}^\circ F$ is reflexve if and only if the operator N is a congruence, that is $F^\circ=({}^\circ F)^{\circ\circ}$. Since F is closed subspace, this condition is equivalent to the regular closedness of F. If this condition is satisfied, X is reflexive provided that ${}^\circ F$ is reflexive [1]. Thus we have the following

THEOREM 5. *Let X be a Banach space, reflexive (mod. F°). If F is regularly closed* *and the operator* $\mathfrak{H}$ *has the norm* $\|\mathfrak{H}\|=1$ *and* $^{\circ}F$ *is reflexive, then* X *is reflexive.*

THEOREM 6. *Let* X *be a Banach space, reflexive (mod.* F°*). If* F *is total over* X *and the operator* $\mathfrak{H}$ *has the norm* $\|\mathfrak{H}\|=1$, *then* X *and* F' *are canonically equivalent* [5] *(*X *is* F*-reflexive* [6]*).*

PROOF. From the assumption $\|\mathfrak{H}\|=1$, $\mathfrak{G}$ is a congruence. Since $^{\circ}F=(0)\subset X$, $\mathfrak{G}=\pi_F$. X''/F° and F' are congruent by the operator $k: X''/F^{\circ}\longrightarrow F'$ defined by $k[x'']=x''|_F$ for any $[x'']\in X''/F^{\circ}$. Put $\varphi=k\cdot\pi_F : X\longrightarrow F'$. Then $(\varphi(x))f=(\pi x)f=f(x)$ for any $x\in X$ and $f\in F$. Thus φ is the *canonical mapping* [5].

I. Singer [6] proved the following corollary. This is also a corollary of our Thoerem 6.

COROLLARY. *Every conjugate space* X' *of a Banach space* X *is* $\pi(X)$*-reflexive.*

PROOF. Put $F=\pi(X)$. Then, it follows from Theorem 1 that X$'$ is reflexive (mod. F°). Moreover F is total over X'. The operator $\mathfrak{G}=(\pi_1)_F : X'\longrightarrow X'''/F^{\circ}$ is a congruence operator. For, when x$'\in X'$, we have

$$\|(\pi_1)_F(x')\|=\inf_{y'''\in F}\left\{\sup_{\substack{x''\in X''\\ \|x''\|=1}}|(\pi_1 x'+y''')x''|\right\}$$

$$\geqq\inf_{y'''\in F^{\circ}}\left\{\sup_{\substack{x''\in X''\\ \|x''\|=1}}|x''x'+y'''x''|\right\}=\sup_{\substack{x\in X\\ \|x\|=1}}|x'x|=\|x'\|$$

Thus we have $\|(\pi_1)_F\|\geqq 1$. Moreover $\|(\pi_1)_F(x')\|\leqq\|\pi_1 x'\|=\|x'\|$.
Therefore $\mathfrak{G}$ is a congruence operator. Thus the operator $\mathfrak{H}=\mathfrak{G}^{-1}=((\pi_1)_F)^{-1}$ has the norm $\|\mathfrak{H}\|=1$.

3. Other criterions

If π_F is surjective, we have $\pi(X)+F^{\circ}=X''$ and X is reflexive when $F^{\circ}\subset\pi(X)$.

Let M be the null manifold ef $x''\in$ F$^{\circ}$ that is $M=\{x'\in X' | x''x'=0\}$.
If x$''\neq 0$, M is a proper closed linear subspace of X'. From the Lemma in [3, V.3.10.], M° is a one-dimensional closed linear subspace of X'' generated by x''. Therefore $^{\circ}M$ is null- or one-dimensional, for $\pi(^{\circ}M)\subset M^{\circ}$. If $^{\circ}M$ is null-dimensional M is total over X and hence X is non-reflexive [5]. When $^{\circ}M$ is one-dimensional we have $\pi(^{\circ}M)=M^{\circ}\ni x''$, and therefore the next theorem follows.

THEOREM 7. *Let* X *be reflexive (mod.* F°*). If the null manifold of* x'' *is non-total over* X *for each* $x''\in F^{\circ}$ $(x''\neq 0)$, X *is reflexive.*

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References

1. P. CIVIN and B. YOOD, *Quisi-reflexive spaces*, Proc. Amer. Math. Soc., 8 (1957), 906–911.
2. J. DIXMIER, *Sur un théorème de Banach*, Duke Math. J., 15 (1948), 1057–1071.

3. M. DUNFORD and J. T. SCHWARTZ, Linear Operators, Part I, New York, 1958.

4. M. KREIN and V. ŠMULIAN, ***On regularly convex sets in the space conjugate to a Banach space***, Ann. of Math., 41 (1940), 556–583.

5. A. F. RUSTON, ***Conjugate Banach space***, Proc. Cambridge Phil. Soc., 53 (1957), 567–580.

6. I. SINGER, ***On a theorem of J. D. Weston***, J. London Math. Soc., 34 (1959), 320–324.