A note on invariant measures for the Galton-Watson process with state-dependent immigration

By

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1. Introduction

Consider the Galton-Watson branching process with state-dependent immigration, where immigration is allowed in a generation iff the previous generation was empty (Pakes (1971) [3]).

Let $A(x)=\sum_{j=0}^{\infty} a_j x^j$ and $B(x)=\sum_{j=0}^{\infty} b_j x^j$ $(|x|\leqq 1)$ be the probability generating functions of the offspring and immigration distributions respectively. We shall assume that

1) $0<a_0,\quad a_0+a_1,\quad b_0<1$, and

2) $\alpha=A'(1-)<\infty$.

Denote the size of the n-th generation by X_n $(n=0, 1, \cdots)$.

Now we discuss the problem of the existence and uniqueness of invariant measure of the Markov chain $\{X_n\}$, that is, a non-negative sequence $\{\mu_i\}$ $(i=0, 1, \cdots; \mu_i>0$ for some $i)$ such that

$$\mu_j=\sum \mu_i p_{ij} \qquad (j=0, 1, \cdots),$$

where p_{ij} is the one-step transition probability from state i to j.

The following results are given by Pakes (1971) [3].

LEMMA A. *Suppose an invariant measure, $\{\mu_i\}$, of the Markov chain $\{X_n\}$* exists. Then $U(x)=\sum_{i=0}^{\infty}\mu_i x^i$ *converges for $x\in[0, q)$ and satisfies the functional equation*

$$U[A(x)]=U(x)+\mu_0(1-B(x)), \qquad \mu_0>0 \tag{1}$$

for $x\in[0, q)$, where q is the least positive solution of $x=A(x)$, so that $q=1$ if $\alpha\leqq 1$ and *$0<q<1$ if $\alpha>1$.*

THEOREM B. *When $\alpha\leqq 1$, the Markov chain, $\{X_n\}$, possesses a unique (up to a constant multiplier) invariant measure. And we obtain*

(2) $$U(x)=1+\sum_{n=0}^{\infty}\{B(A_n(x))-B(A_n(0))\}$$

as the unique solution of (1) *on* (0, 1) *chosen so that* $U(0)=1$, *where* $A_{n+1}(x)=A(A_n(x))$ *and* $A_0(x)=x$.

In this paper we consider the existence and uniqueness of invariant measure of the Markov chain $\{X_n\}$ in the case that $\alpha>1$.

2. Preparation

Considering the ordinary Galton-Watson process $\{Z_n\}$ generated by $A(s)$, an invariant measure $\{\pi_i\}$ ($i=1,2,\cdots;\ \pi_i>0$ for some i) is equivalent to a solution $\pi(s)=\sum_{i=0}^{\infty}\pi_i s^i$, convergent in $[0,q)$ and whose coefficients are of appropriate form, to the functional equation

(3) $$\pi(A(s))=\pi(s)+1,\qquad s\in[0,q).$$

Such an invariant measure for the process always exists (see theorem 11. 1 in Harris (1963) [1]), and is known to be unique (up to a constant multiplier) when $\alpha=1$. However, if $\alpha\neq1$, as shown by Kingman (1065) [2], uniqueness no longer holds in general.

From lemma A, it clearly suffices to demonstrate the existence and uniqueness of a regular function, which has non-negative coefficients, and which satisfies the equation

(4) $$U[A(x)]=U(x)+(1-B(x))\qquad(0\leqq x<q),$$

in which case $\mu_0=1$.

It is easily seen that the problem of finding a solution of the right form to (4) (in general) is equivalent to finding a solution of the same nature to

(5) $$\mathfrak{P}\left(\frac{A(qy)}{q}\right)=\mathfrak{P}(y)+(1-B(qy)),\qquad 0\leqq y<1,$$

where we have put $\mathfrak{P}(y)=U(qy)$.

Since in (5) $B(qy)$ generates a defective distribution if $q<1$, and $A(qy)/q$ generates a non-supercritical distribution ($0<A'(q)\leqq1$; $A'(q)<1$ iff $\alpha\neq1$), the general problem of seeking solution to (5) is subsumed by that of investigating appropriate solutions to the system

(6) $$\mathfrak{P}(A(y))=\mathfrak{P}(y)+(1-B(y)),\ y\in[0,1),$$

where $B(y)$ and $A(y)$ satisfy our basic assumption, but with the additional restriction on $A(y)$ that $A'(1-)=\alpha\leqq1$, and allowing for the possibility that $B(y)$ may generate a defective distribution, i.e. $B(1-)\leqq1$.

Thus (6) as a whole corresponds to a non-supercritical process with state-dependent immigration which may be defective.

3. Theorem and the proof

THEOREM. *Under the noted assumptions on* (6), *a solution, of correct form, to* (6) *always exists. It is unique if* $B(1-)=1$; *and in general non-unique if* $B(1-)<1$ *and* $\alpha<1$.

Note. Although we are unable to answer at the moment the question of uniqueness if $B(1-)<1$ and $\alpha=1$, this problem does not actually occur in the narrower context of (5) which is our primary concern.

Proof. The case that $B(1)=1$ follows from theorem B.

Let us note from this that even if $B(1)<1$,

$$\mathfrak{P}_1(y)=1+\frac{1}{B(1)}\sum_{n=0}^{\infty}\{B(A_n(y))-B(A_n(0))\}, \qquad 0\leqq y<1,$$

is convergent, since in fact it generates the (unique) invariant measure for the process with offspring p. g. f. $A(s)$ and (proper) immigration p. g. f. $B(s)/B(1)$.

Hence, we obtain the fact that $\sum_{n=0}^{\infty}\{B(A_n(y))-B(A_n(0))\}$ is convergent for $y\in[0, 1)$ and has non-negative coefficients.

It is seen without difficulty that

$$\mathfrak{P}(y)=1+(1-B(1))\,\pi(y)+\sum_{n=0}^{\infty}\{(B(A_n(y))-B(A_n(0))\} \tag{7}$$

solves (6). Futhermore, since $(1-B(1))>0$, it follows that (7) generates a non-negative term series (terms not all zero) of the correct sort.

Now if $B(1)<1$ and if $\alpha<1$, it follows from Kingman's result that sometimes distinct $\pi(y)$'s may be substituted into (7) giving distinct $\mathfrak{P}(y)$'s, and hence leading to lack of uniqueness, in general.

The proof of the theorem is complete.

From this theorem, we concluse that when $\alpha>1$, invariant measure of $\{X_n\}$ always exists, but it is in general non-unique.

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