Degree of symmetry of a certain product manifold

By

Tsuyoshi WATABE

(Received October 31, 1974)

Introduction

The degree of symmetry $N(M)$ of a compact connected differentiable manifold M is the maximum of the dimensions of the compact connected Lie groups which can act almost effectively and differentiably on M.

In this note we shall prove the following theorems.

THEOREM 1. $N(S^k \times CP_n) = N(S^k) + N(CP_n) = \dfrac{k(k+1)}{2} + n^2 + 2n.$

THEOREM 2. $N(CP_n \times CP_k) = N(CP_n) + N(CP_k) = n^2 + 2n + k^2 + 2k.$

Here S^k denotes k-dimensional sphere and CP_n n-dimensional complex projective space. In the following all actions are assumed to be differentiable.

1. Statement of results

We shall write $X \underset{Q}{\sim} Y$ if X and Y have isomorphic Q-cohomology ring, where Q denotes the field of rational numbers.

Let X be an orientable closed $(2n+k)$-manifold such that $X \underset{Q}{\sim} CP_n \times S^k$. Assume that $N(X) \geqq \dim SU(n+1) + \dim SO(k+1)$ and X has no 2 torsion.

We shall prove the following proposition in section 5.

PROPOSITION A *Let X be as above. If k is* 1 *or* 2, *then X is diffeomorphic to* $CP_n \times S^k$ *and* $N(X) = N(CP_n) + N(S^k)$.

We consider the case in which k is greater than 2. It is easily seen that $N(X) > \dfrac{1}{8}$ $(\dim X + 7) \dim X$. Let G be a compact connected Lie group of $\dim G = N(X)$ which acts almost effectively on X. We may assume that $G = T^r \times G_1 \times \cdots \times G_s$, where T^r is r-dimensional torus and G_i is a simple Lie group. By a result in [3], there is a normal subgroup of G, say G_1, with the following properties

(1) $\dim G_1+\dim N(H_1, G_1)/H_1>\frac{1}{8}(\dim X+7)\dim G_1/H_1$

and

(2) $\dim H_1>\dfrac{\dim X-9}{\dim X-1}\dim G_1$

where $H_1=(H\cap G_1)^0$ (H: a principal isotropy subgroup of G) and $N(H_1, G_1)$ is the normalizer of H_1 in G_1.

We shall consider the cace in which $2n+k\leqq 25$ and prove the same result as proposition A for this case in section 6. Assume $2n+k\geqq 26$. By the same arguments as in [5], the possible pair (G_1, H_1) is one of the followings: $(Sp(l), Sp(l-1)\times(Sp(1))(2l>k/2+n)$, $(SU(l). N(SU(l-1), SU(l)))\left(2l-2\geq\frac{k}{2}+n\right)$, $(SU(l), SU(l-1))(2l-2>k/2+n)$ and $(So(l), So(l-1))(2l-3>k/2+n)$.

Case 1. $(G_1, H_1)=(Sp(l), Sp(l-1)\times Sp(1))$.

It follows from the Vietoris-Begle theorem that the orbit map $\pi: X\longrightarrow X/G_1$ induces an isomorphism $\pi^*: H^i(X/G_1: Q)\longrightarrow H^i(X: Q)$ for $i\leqq 3$. Hence the generator a of $H^2(X: Q)$ is in image of π^*. Since $\dim X/G_1=k+2n-4l+4<2n$, we have a contradiction.

Case 2. $(G_1, H_1)=(SU(l-1), N(SU(l-1), SU(l)))$

Since $N(SU(l-1), SU(l))$ is maximal, we have $X=CP_{l-1}\times X^*$, where X^* is the orbit space. Let a be a generator of $H^2(X: Q)$ and p the projection $X\longrightarrow CP_{l-1}$. Then $a=p^*(b)$, where b is a generator of $H^2(CP_{l-1}: Q)$. It is not difficult to see that $l=n+1$. Let $G=G_1\times K$. From the following observation (see [3]) it follows that K acts on X^* almost effectively.

Observation Let (G, M) be a smooth action with H as a principal isotropy subgroup. Suppose K is an equivariant differentiable transformation group on M and the $G\times K$ action on M is almost effective, and K_0 is the ineffective kernel of the induced K-action on $M(H)/G$. Then K_0 is locally isomorphic to subgroup of $N(H, G)/H$.

From the fact that $\dim K\geqq\dim SO(k+1)$ and the fact that $\dim X^*=k$ it follows that $K=SO(k+1)$ and $X^*=S^k$. Thus we have $X=CP_n\times S^k$ and $N(X)=\dim SU(n+1)+\dim SO(k+1)$. Moreover we have proved that G acts transitively on X.

Case 3. $(G_1, H_1)=(SU(l), SU(l-1))$.

Subcase 1. There is no fixed point of $SU(l)$-action.

Put $N=N(SU(l-1), SU(l))$. Consider the case in which $X_{(N)}\neq\phi$. By the same arguments as in [1], there is continuous map $f: X\longrightarrow CP_{l-1}$ such that $f^*: H^*(CP_{l-1}: Q)\longrightarrow H^*(X: Q)$ is injective. It follows that $l=n+1$. Let $Y=X(H)/SU(l)$, where H denotes a principal isotropy subgroup of $SU(l)$-action. Then Y is a $(k-1)$-dimensional manifold on which $K=G/G_1$ acts with ineffective kernel N of dimension $\leqq 1$. It follows that dim $K/N \leqq \frac{k(k-1)}{2}$. Since dim $K \geq$ dim $SO(k+1)$, we have dim $K/N \geq$ dim $SO(k+1)-1$, which is a contradiction.

Next we consider the case in which $X_{(N)}=\phi$. Put $P=F(SU(l-1), X)$. It is known that $X=S^{2l-1}\underset{S^1}{\times} P$, and $X^*=P/S^1$(see [5]).

Suppose that the fibre bundle $\xi: S^1 \longrightarrow S^{2l-1}\times P \longrightarrow X$ is trivial. We may asssume that $k \geqq 2l-1$. In fact, if k$<2l-1$, then dim $X^*<2n$. By the same arguments as in case 1, we can show a contradiction. Moreover we can prove that $k=2l-1$.

Suppose $k>2l-1$. Since $H^*(S^{2l-1}\times P: Q)\simeq H^*(X\times S^1: Q)$, we have $H^i(P:Q)\simeq H^i(X\times S^1: Q)$ for $i<2l-1$. From the assumption $k>2l-1$, it follows that $k>2l-1>k/2+$n$+1$ and hence $k>2n+4$. Hence we have $H^i(X\times S^1: Q)=0$ for $2n+1<i<k$. It follows that $H^{2n+k-2l+2}(P: Q)=0$, which is a contradiction because P is an orientable closed $(2n+k-2l+2)$−manifold. Put $G=K\times G_1$. Then dim $SO(k+1)+$dim $SU(n+1)-$dim $SU\left(\frac{k-1}{2}\right)>2n^2+6n+4$, which is seen to be a contradiction by the same arguments as above. Thus we have shown the fibre bundle ξ is not trivial. Consider the following commutative diagram:

$$\begin{array}{ccc} S^1 & = & S^1 \\ \downarrow & & \downarrow \\ S^{2l-1}\times P & \xrightarrow{pr} & S^{2l-1} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & CP_{l-1} \end{array}.$$

where f is induced by pr: $S^{2l-1}\times P \longrightarrow S^{2l-1}$. Let $e\in H^2(X: Q)$ be the rational Euler class of the bundle ξ. We may assume e is a generator of $H^2(X: Q)$ (Note $e^n\neq 0$, $e^{n+1}=0$). Then we have $f^*(b)=e$, where b is a generator of $H^2(CP_{l-1}: Q)$. It is not difficult to see that $l=n+1$ and hence dim $G/G_1 \geqq$ dim $SO(k+1)$. Since dim $X^*=k-3$, we can show a contradiction.

Subcase 2. There is at least one fixed point.

Let U be a closed invariant tubular nbhd. of $F=F(SU(l), X)$. It is known that $X=\partial(D^{2l}\times P)/S^1$ and $X^*=P/S^1 \cup \partial P/S^1\times[0, 1]$, $\partial P/S^1=$F, where $P=F(SU(l-1), X-\text{int } U)$. Note that U can be chosen to be invariant under $K=G/G_1$ and hence P is also invariant under K. Then $G=G_1\times K$ acts on $\partial(D^{2l}\times P)/S^1$ by $(g, h)[v, x]=[gv, hx]$, where $(g, h)\varepsilon G_1\times K$ and $v\varepsilon D^{2l}$, $x\varepsilon P$. This implies that K-action on P is almost effective and hence K acts on ∂P almost effectively.

In section 2, we shall prove the following

PROPOSITION 1. $\partial P/S^1 \underset{Q}{\sim} CP_n \times S^{k-2l}$, $S^{2n+1}\times CP_{\frac{k-2l-1}{2}}$ $(k\geqq 2n+1,\ k-2l<2n+1)$, $S^k \times CP_{n-l}$ $CP_{\frac{k-1}{2}}\times S^{2k-2l+1}$ or $CP_{\frac{2n+k-2l}{2}}$ $(k=2n+2-2l)$. *Moreover* $\partial P/S^1$ *has no 2-torsion.*

Since the situations for four cases are parallel, we consider only the case $\partial P/S^1 \underset{Q}{\sim} S^{2n+1} \times CP_{\frac{k-2l-1}{2}}$ $(k>2n,\ k-2l<2n+1)$. Let N be the ineffective kernel of K-action on $\partial P/S^1$. Then N is a group of bundle automorphisms of the bundle $S^1 \longrightarrow \partial P \longrightarrow \partial P/S^1$. Since the action of N on ∂P is almost effective, we have $\dim N\leqq 1$. Since $\dim K \geqq \dim SO(k+1) + \dim SU(n+1) - (\dim SU(l)$, $k>2n$ and $4l>4n+4$, we have $\dim K/N > \dim SO(2n+2) + \dim SU\left(\frac{k-2l+1}{2}\right)$. By induction, it follows that $\dim K/N = \dim SO(2n+2) + \dim SU\left(\frac{k-2l+1}{2}\right)$, which is clearly impossible.

Case 4. $(G_1, H_1) = (SO(l), SO(l-1))$.

We may assume that $k>l-1$.

Subcase 1. There is no fixed point of $SO(l)$-action.

It follows from the fact $H^1(X:Z_2)=0$ that the $SO(l)$-action has a unique conjugacy class $(SO(l-1))$ of isotropy subgroups, and hence the orbit map $X \longrightarrow X^*$ is an S^{l-1} bundle with Z_2 as structural group. It is not difficult to see that $H^1(X^*:Z_2)=0$ and hence the fibre bundle is trivial, i,e. $X \approx S^{l-1}\times X^*$.

Suppose that $l-1<k$. Then $h=\dim X^* = 2n+k-(l-1)>2n$. If $h<k$, then $H^h(X^*\times S^{l-1}:Q)\neq 0$ (Note that X^* is an orientable closed h-dimensional manifold). On the other hand $H^h(S^k\times CPn:Q)=0$, because $2n<h<k$. This is a contradiction. Thus we have shown that $h\geq k$, and hence $l-1\leqq 2n$. Comparing the dimension of $H^{l-1}(S^{l-1}\times X^*:Q)$ and $H^{l-1}(S^k\times CP_n:Q)$, we can show a contradiction. Thus we have shown that $l-1=k$, In other words, $G\sim SO(k+1)$, $X=S^k\times X^*$ and hence $H^*(X^*:Q)=H^*(CP_n:Q)$ (as rings). Now $K=G/G_1$ acts almost effectively on X^* and $\dim K\geqq \dim SU(n+1)$.

It is known that $K\sim SU(n+1)$ and $M^*=CP_n$ (see [3], [5]).

Thus we have proved that $\dim G = \dim SO(k+1) + \mathit{dim}\ SU(n+1)$ and $X=CP_n\times S^k$.

Subcase 2. There is at least one fixed point.

Let U be an invariant closed tubular nbhd of $F=F(SO(l), X)$. Since $H^1(X-\text{int}\ U:Z_2)=0$, X-int U is an S^{l-1} bundle over $X^*-F\times(0,1)\approx X^*$ with Z_2 as structure group. Since $H^1(X^*:Z_2)=0$, this fibre bundle is triual, i,e. X-int $U\approx S^{l-1}\times X^*$ and hence $X=\partial(D^l\times X^*)$.

In section 3, we shall prove the following.

PROPOSITION 2. *$\partial X^* \sim CP_n \times S^{k-l-1}$ and ∂X^* has no 2-torsion.*

Since $K = G/G_1$ acts almost effectively ∂X^* and dim $K \geq$ dim $SO(k+1)$+dim $SU(n+1)$ $-$dim $SO(l) \geq$ dim $SU(n+1)$+dim $SO(k-l+1)$, the induction argument shows that dim K= dim $SU(n+1)$+dim $SO(k-l)$, which is easily seen to be a contradiction.

Thus we have proved the following.

PROPOSITION B *Let X be an orientable closed $(2n+k)$-manifold $(k>2)$ such that $X \sim CP_n \times S^k$, with no 2-torsion. Assume $N(X) \geqq$ dim $SU(n+1)$+dim $SO(k+1)$. Then X is diffeomorphic to $CP_n \times S^k$ and $N(X)$ = dim $SU(n+1)$+dim $SO(k+1)$.*

Theorem 1 in the Introduction follows immeadiately from this proposition B.

Next we shall prove the following proposition modulo some lemmas.

PROPOSITION C. *Let X be an orientable closed $(2n+2k)$-manifold such that $X \underset{Q}{\sim} CP_n \times CP_k (k \geq n)$. Assume that $N(X) \geqq$ dim $SU(n+1)$+dim $SU(k+1)$. Then X is diffeomorphic to $CP_n \times CP_k$ and $N(X)$ = dim $SU(n+1)$+dim $SU(k+1)$.*

We shall prove proposition C for the case $n+k \leqq 12$ in section 6.

Assume $n+k \geqq 13$. Consider a compact connected Lie group G with dim $G = N(X)$ which acts almost effectively on X. Then there exists a simple normal subgroup G_1 of G such that

$$\text{(i)} \qquad \dim G_1 + \dim N(H_1 : G_1)/H_1 > \frac{1}{8}(2n+2k+7)\dim G_1/H_1,$$

and

$$\text{(ii)} \qquad \dim H_1 > \frac{2n+2k-9}{2n+2k-1}\dim G,$$

where $H_1 = (H \cap G_1)^0$ (H =a principal isotropy subgroup of G-action. Possible pairs (G_1, H_1) are proved to be the followings: $(SO(l), (SO(l-1))(2l>n+k)$, $(Sp(l), Sp(l-1) \times Sp(1))$ $(2l>n+k)$ $(SU(l), N(SU(l-1))(2l-2 \geq n+k)$ and $(SU(l), SU(l-1))(2l-2>n+k)$. It is not difficult to see that cases of $(SO(l), SO(l-1))$ and $(Sp(l), Sp(l-1) \times Sp(1))$ are impossible.

Consider the case of $(SU(l), N(SU(l-1))$. It is known that $X \approx CP_{l-1} \times X^*$. Let f be the projection $X \longrightarrow CP_{l-1}$, a_1, a_2 generators of $H^2(X:Q)$ such that $H^*(X:Q) = Q[a_1]/(a_1{}^{n+1}) \otimes Q[a_2]/(a_2{}^{k+1})$ and b generator of $H^2(CP_{l-1}:Q)$. Put $f^*(b) = \alpha a_1 + \beta a_2$.

Assume that $\alpha \neq 0$ and $\beta \neq 0$. Then $(\alpha a_1 + \beta a_2)^{n+k} \neq 0$ and hence $b^{n+k} \neq 0$. This is clearly a contradiction From the assumption that $k \geqq n$, it follows that $\beta \neq 0$, $\alpha = 0$ and $l = k+1$, and hence $X = CP_k \times X^*$. It is clear that $X^* \underset{Q}{\sim} CP_n$. Since $K = G/G_1$ acts on X^* almost effectively and dim $K \geqq$ dim $SU(n+1)$, $X^* = CP_n$ and $K \sim SU(n+1)$.

Next consider the case of $(SU(l), SU(l-1))$. Put $N = N(SU(l-1))$. If $X_{(N)} = \phi$, we can easily show a contradiction. Asssume that $X_{(N)} \neq \phi$ and $F = F(SU(l), X) = \phi$. Then it is known that $X = (S^{2l-1} \times P)/S^1$, where $P = F(SU(l-1), X)$. In section 4, it is shown that $P \underset{Q}{\sim} CP_n$. Since $K=G/G_1$ acts on P almost effectively and $\dim K \geqq \dim SU(n+1)$, we have $K \sim SU(n+1)$ and $P=CP_n$. Then a principal isotropy subgroup H of G-action contains $SU(k)\times N(SU(n-1))$, which is proved to be a contradiction by dimensional arguments.

Thus we may assume that there exists at least one fixed point. Let F be the fixed point set, U an invariant closed tubular nbhd. of F and $P=F(SU(l-1), X-\mathrm{int}\ U)$. In section 3, we shall prove that $\partial P/S^1 \underset{Q}{\sim} CP_{k-l}\times CP_n$. Now $K=G/G_1$ acts almost effectively on $\partial P/S^1$ and $\dim K \geqq \dim SU(n+1)+\dim SU(k+1)-\dim SU(l) > \dim SU(n+1)+\dim SU(k-l+1)$. By induction K is locally isomorphic to $SU(n+1)\times SU(k-l+1)$, which is clearly a contradiction.

In the following sections, unless it is stated to the contrary, the field Q of rational numbers is used as coeffiients of homology and cohomology.

2. An $SU(l)$-action on $X \underset{Q}{\sim} CP_n \times S^k$

In this section let X be an orientable closed $(2n+k)$-manifold such that $X \underset{Q}{\sim} CP_n\times S^k$ $(k\geq 3)$ on which $SU(l)\left(2l-2>n+\frac{k}{2}\right)$ acts with $SU(l-1)$ as a principal isotropy subgroup at least one fixed point and non-empty $X_{(N)}$ $(N=N(SU(l-1), SU(l)))$. Let U be a closed invariant tubular nbhd of the fixed point set $F=F(SU(l), X)$, and P the submanifold $F(SU(l-1), X\text{-int}\ U)$. We may assume that the restricted action of $SU(l)$ on U has just two types of orbits; principal arbit and fixed point. It is known that $X=\partial[D^{2l}\times P]/S^1$, $X^*=P/S^1\cup\partial P/S^1\times[0,1]$ attached along $\partial P/S^1$ and $P/S^1\times\{0\}$ and $\partial P/S^1\approx F$. Put $Y=D^{2l}\times P$.

We shall fiirst consider the case in which the fibre bundle $\xi: S^1 \longrightarrow \partial Y \longrightarrow X$ is trivial. Then we have $\partial Y=S^1\times X \underset{Q}{\sim} S^1\times CP_n\times S^k$. Consider the Q-cohomology exact sequence of the pair $(Y, \partial Y)$

$$\begin{tikzcd} \longrightarrow H^h(Y,\partial Y) \arrow[r] & H^h(Y) \arrow[r, "*i"] & H^h(\partial Y) \arrow[r, "\delta"] & H^{h+1}(Y,\partial Y) \longrightarrow \\ & H^h(P) \arrow[u, "p^*"] \arrow[ur, "f^*"'] & & \end{tikzcd}$$

where p^* is induced by the projection $p: Y \longrightarrow P$ and f^* is induced by the composition $f^*: \partial Y \xrightarrow{i} Y \xrightarrow{p} P$

Since f has a cross section $P \xrightarrow{j} \partial Y \longrightarrow Y \longrightarrow P$, i^* injective. Hence we have the following short exact sequence;

$$\begin{tikzcd} 0 \arrow[r] & H^h(Y) \arrow[r, "i^*"] \arrow[d, phantom, "\wr"] & H^h(\partial Y) \arrow[r] & H^{h+1}(Y,\partial Y) \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\ & H^h(P) & & H_{2n+k-h+1}(P) & \end{tikzcd} \tag{2.1}$$

Let $i_1 : \partial P \longrightarrow P$ be inclusion, $\tilde{c}$ a generator of $H^1(\partial Y)$ and $\tilde{a}$ generator of $H^2\ (\partial Y)$. Since dim $P = 2n+k-2l+2 < 2l-1$ and $H^{2n+k-2l+2}(P) = 0$, it follows from (2. 1) that $2n+1 < 2n+k-2l+2 < k$, and hence $P \underset{Q}{\sim} S^1 \times CP_n$. Thus we have $f^*(\tilde{c}) = c \epsilon H^1(P)$ is non-zero. It follows that $c_1 = i_1^*(c) \neq 0$. Suppose that $i_1^*(f^*(a)) = a_1 = 0$. Since $H^*(P)$ is generated by c and $a = f^*(\tilde{a})$, $i_1^* : H^h(P) \longrightarrow H^h(\partial P)$ is zero for $h \geqq 2$.

Hence we have the following exact sequence:

(2. 2) $\quad 0 \longrightarrow H^h(\partial P) \longrightarrow H^{h+1}(P, \partial P) \longrightarrow H^{h+1}(P) \longrightarrow 0 \quad (h \geq 2).$

Since $H^3(P) \neq 0$, we have $H_{2n+k-2l-1}(P) \approx H^3(P, \partial P) \neq 0$ and hence $2n+k-2l-1 \leq 2n+1$, which implies that $k-2l \leq 2$. It is not difficult to see that the cases of $k-2l$ and of $k-2l = 2$ are impossible. When $k-2l=1$, it is shown that $\partial P \underset{Q}{\sim} S^1 \times S^{2n+1}$. Since the fibre bundle $S^1 \longrightarrow \partial P \longrightarrow \partial P/S^1$ is also trivial, we have $\partial P \approx S^1 \times \partial P/S^1$. It is clear that $\partial P/S^1 \underset{Q}{\sim} S^{2n+1}$.

Suppose next that $i_1^*(f^*(a)) = a_1 \neq 0$. Let m be the largest integer such that $a_1^m \neq 0$. It is not difficult to see that $m = n$. Hence we have $\partial P \underset{Q}{\sim} S^1 \times CP_n \times S^{k-2l}$ or $S^1 \times CP_{\frac{k-2l-1}{2}} \times S^{2n+1}$ which implies $\partial P/S^1 \sim CP_n \times S^{k-2l}$, or $CP_{\frac{k-1l-1}{2}} \times S^{2n+1}$.

Next we shall consider the case in which the fibre bundle ξ is not trivial. Let $e \epsilon H^2(H : Q)$ be the rational Euler class. Since $k > 2$, we have $e^n \neq 0$ and $e^{n+1} = 0$. From the Gysin sequence of ξ, it follows that Q-cohomelogy groups of ∂Y are:

$$H^0 = H^{2n+k+1} = H^{2n+1} = H^k = Q$$

$$H^i = 0 \qquad \text{otherwise}$$

As before, we have the following exact sequence:

$$0 \longrightarrow H^h(P) \longrightarrow H^h(\partial Y) \longrightarrow \underset{\underset{H_{2n+k+1-h}(P)}{\wr\wr}}{H^{h+1}(Y, \partial Y)} \longrightarrow 0$$

If $k > 2n$, then $2l-2 > \dfrac{k}{2} + n > 2n$ and hence we have dim $P = 2n+k-2l+2 < k$. Since $H^{2n+1}(\partial Y) = 0$, dim P must be greater than or equal to $2n+1$. This implies that $P \underset{Q}{\sim} S^{2n+1}$.

From the cohomolgy exact sequence of rair $(P, \partial P)$, it follows that $H^i\ (\partial P) \overset{\delta}{\approx} H^{i+1}(P, \partial P) \approx H_{2n+k-2l+1-i}(P)$ for $0 < i < 2n$ and hence we have

$$H^0(\partial P) = H^{k-2l}(\partial P) = H^{2n+1}(\partial P) = H^{2n+1+2l}(\partial P) = Q$$

$$H^i(\partial P) = 0 \qquad \text{for i; otherwise.}$$

Let a be a generator of $H^{2n+1}\ (\partial P)$ and b a generator of $H^{k-2l}\ (\partial P)$. Denote $\tilde{a}$ be the element of $H^{2n+1}(P)$ such that $i^*(\tilde{a}) = a$ ($i : \partial P \longrightarrow P$ inclusion) and $a \epsilon H_{2n+1}(P)$ the dual of $\tilde{a}$.

Then we may assume $\delta b = D^{-1}\alpha$ where $D: H^{k-2l+1}(P, \partial P) \longrightarrow H_{2n+1}(P)$ is Poincare duality. It can be shown that $b \cup a \neq 0$. In fact we have $<[P, \partial P], \tilde{a} \cup D^{-1}\alpha> = <[P, \partial P] \cap D^{-1}\alpha, \tilde{a}> = <\alpha, \tilde{a}> \neq 0$, and hence we have $D\delta(a \cup b) = D\delta(b \cup i^*(\tilde{a})) = D(D^{-1}\alpha \cup \tilde{a}) = <[P, \partial P], D^{-1}\alpha\tilde{a}> \neq 0$, which implies $a \cup b \neq 0$. These arguments imply that $\partial P \underset{Q}{\sim} S^{k-2l} \times S^{2n+1}$ or $S^{2n+k-2l+1} (k = 2n-2l+1)$. By similar arguments we can show that when $k \leq 2n$, $\partial P \underset{Q}{\sim} S^k \times S^{2n-2l+1}$ or $S^{2n+k-2l+1} (k = 2n-2l+2)$. It follows from the following proposition that $\partial P/S^1 \underset{Q}{\sim} CP_n \times S^{k-2l}$, $CP_{\frac{k-2l-1}{2}} \times S^{2n+1}$, $CP_{n-l} \times S^k$ $CP_{\frac{k-1}{2}}{}^2 \times S^{2n-2l+1}$, or $CP_{2n+k-2l} (k = 2n-2l+2)$.

PROPOSITION (2. 3) Let X be an orientable closed $(m+n)$-manifold such that $X \underset{Q}{\sim} S^m \times S^n (m, n \geqq 2)$, where at least one of m and n is odd. If a circle group S^1 acts on X on freely, then the orbit space X^* has the Q-cohomology ring of one of the followings;

$$CP_{\frac{m-1}{2}} \times S^n, \text{ or } CP_{\frac{n-1}{2}} \times S^m.$$

PROOF. Let $e \varepsilon H^2 (X^* : Q)$ be the Euler class. We shall consider only the case in which $m = 2m'$, $n = 2n'+1$. From the Gysin sequence;

$$\longrightarrow H^i(X) \longrightarrow H^{i-1}(X^*) \xrightarrow{\cup e} H^{i+1}(X^*) \longrightarrow H^{i+1}(X) \longrightarrow,$$

it follows that $H^{i-1}(X^*) \approx H^{i+1}(X^*)$ for $i < 2m'-1$, for $2n'+1 < i < 2n'+2m'$ and for $2m' < i < 2n'$. Let h be the largest integer such that $e^h \neq 0$. It is easy to see that dim $H^{2m'}(X^*) = 2$ and hence dim $H^{2n'}(X^*) = 2$. This implies that $X^* \sim S^{2m'} \times CP_n$. Q.E.D.

Now we shall prove the last part of proposition 1. Consider the case in which the fibre bundle $S^1 \longrightarrow \partial(D^{2l} \times P) \longrightarrow X$ is trivial and hence the bundle $S^1 \longrightarrow S^{2l-1} \times P \longrightarrow X$ $-$int U is also trivial. Note that when the fixed point set F is empty, the argument is varid. Then int U has no 2-torsion. Since $H^i(X - \text{int } U: Z) = H_{2n+k-i}(X, U, : Z)$, U has no 2-torsion.

By similar argumets, it is proved that $\partial P/S^1$ has no when the bundle is not trivial.

Thus we have completed the proof of Proposition 1.

3. An SO (L)-action on $\mathbf{X} \underset{\mathbf{Q}}{\sim} \mathbf{CP}_n \times \mathbf{S}^k$

In this section we shall consider an $SO(l)$-action on an orientable $(2n+k)$-mainfold $X\left(2l > \frac{k}{2} + n + 3\right)$ such that $X \underset{Q}{\sim} S^k \times CP_n (k \geq 3)$ with no 2-torsion, with $SO(l-1)$ as a principal isotropy subgroup and non-empty fixed point set F. We have proved that $X = \partial(D^l \times X^*)$ and $X^* \underset{Q}{\sim} CP_n$. We shall prove that $\partial X^* \underset{Q}{\sim} CP_n \times S^{k-l}$ if $k - l \geq 1$. We may assume that $k > l-1$.

Case 1. $k=l$.

From the exact sequence of the pair $(X^*, \partial X^*)$, it follows that $H^{2i}(\partial X^* : Q) \cong H^{2i}(X^*, Q)+H^{2i+1}(X^*, \partial X^* : Q)$ for $0 \leqq i \leqq n$.

Let $a \in H^2(X^* : Q)$ be a generator of $H^*(X^* : Q)$ and $b \in H^1(X^*, \partial X^* : Q)$ the element such that Db is the dual of a^n, where $D: H^1(X^*, \partial X_*) \longrightarrow H_{2n}(X^*)$ is Poincare duality.

Let $a_2=i^*(a)$ and $b_0 \in H^0(X^* : Q)$ the element such that $\delta b_0 = b$.

Lemma (3. 1) $\delta(b_0 \cup a_2{}^h) \neq 0 \quad i \leq h \leq n$

Proof Since $<[X^*, \partial X^*] \cap (\delta(b_0 \cup a_2{}^h)), a^{n-h}> = <[X^*, \partial X^*] \cap (\delta b_0 \cup a^h), a^{n-h}> = <([X^*, \partial X^*] \cap \delta b_0) \cap a^h, a^{n-h}> = <[X^*, \partial X^*] \cap \delta b_0, a^n> \neq 0$, we have $\delta b_0 \cup a^h \neq 0$, where $[X^*, \partial X^*]$ is the fundametal class of X^*. Q.E.D.

Case 2. $k-l \geq 1$.

For this case, it is not difficult to show that $\partial X^* \underset{Q}{\sim} CP_n \times S^{k-l}$ and with no 2-torsion.

4. An SU(L)-action on $\mathbf{X} \underset{\mathbf{Q}}{\sim} \mathbf{CP}_n \times \mathbf{CP}_k$

In this section, we shall consider an $SU(l)$-action on an orientable $(2n+2k)$-mainfold $X(2l-2>n+k)$ such tca $X \underset{Q}{\sim} CP_n \times CP_k((k \geq n)$ with $SU(l-1)$ as a principal isotorpy subgroup, non-empty $X_{(N)}$ $(N=N(SU(l-1), SU(l))$ and non-empty fixed point set F. It known that $X=\partial(D^{2l} \times P)/S^1$ where $P=F(SU(l-1), X\text{-int } U)$. It is not difficult to see that $H^i(P : Q) \approx H^i(\partial Y : Q)$ for $i<2l-1$. Note that $k \geq l$. Suppose the fibre bundle $S^1 \longrightarrow \partial Y \longrightarrow X$ is trivial. Then we have $\partial Y = X \times S^1$, and hence we have $H^{2n+2k-2l+2}(\partial Y : Q) \neq 0$. Since $2n+2k-2l+2<2l-1$, we have $H^{2n+2k-2l+2}(P : Q) \neq 0$, which is a contradiction because P is a manifold with non empty boundary. Thus the fibre bundle $S^1 \longrightarrow \partial Y \longrightarrow X$ is not trivial. Let $e \in H^2(X : Q)$ be its Eular class. From the Gysin sequence it follows that there exists the following exact sequence:

$$(4.\ 1) \quad 0 \longrightarrow H^{2i+1}(\partial Y) \longrightarrow H^{2i}(X) \longrightarrow H^{2i+2} \longrightarrow H^{2i+2} \longrightarrow (\partial Y) \longrightarrow 0.$$

Let α and β generators of $H^*(X : Q)$ i,e. $H^*(X : Q)=Q[\alpha]/(\alpha^{n+1}) \otimes Q[\beta]/(\beta^{k+1})$. We may assume that $e=A\alpha+B\beta$, where A, B is 0 or 1. Suppose $e=\alpha+\beta$. Then put $c=P^*(\alpha-\beta)$ where $P: \partial Y \longrightarrow X$ is projection. It follows from (4. 1) that cohomology groups of ∂Y are:

$H^*(\partial Y) : Q)$ is generated by	c for dimension $\leqq 2n$
$H^{2i+1}(\partial Y : Q) \approx 0$	for $k \geq i \geq 1$
$H^{2i}(\partial Y : Q)=0$	for $2k<2i<2n+2k+1$.

Thus we have proved that $P \underset{Q}{\sim} CP_n$. For the cases of $e=\alpha$ and $e=\beta$, it can be shown that $P \underset{Q}{\sim} CP_n$. We shall calculate cohomology ring of ∂P.

Case 1. $k=l$.

Consider the following exact sequence:

$$0 \longrightarrow H^1(\partial P) \longrightarrow \underset{\substack{\wr\wr \\ H_{2n}(P).}}{H^2(P, \partial P)} \xrightarrow{j^*} H^2(P) \xrightarrow{i^*} H^2(\partial P) \longrightarrow 0$$

Suppose $H^1(\partial P)=0$. Then we have $i^*(a)=0$ where a is a generater of $H^2(P)$. Thus we have $H^i(\partial P)=0$ for $i \leqq 2n$, and hence $\partial P \underset{Q}{\sim} S^{2n+1}$.

Suppose $H^1(\partial P) \neq 0$. It is not difficult to see that $\partial P \underset{Q}{\sim} S^1 \times CP_n$.

Case 2. $k>l$.

It is not difficult to see that $\partial P \underset{Q}{\sim} S^{2n-2l+1} \times CP_n$. It this case we can prove that $H^1(\partial P: Z_2)=0$. In fact since $H^1(X: Z_2)=0$, it follows from Z_2-Gysin sequence of the bundle $S^1 \longrightarrow \partial Y \longrightarrow X$ that $H^1(\partial Y: Z_2)=0$, which implies that $H^1(P: Z^2)=0$. It follows immeadiately that $H^1(\partial P: Z_2)=0$.

5. The proof of Proposition A

In this section we shall prove Proposition A. Since situations of the cases of $k=1$ and $k=2$ are almost parallel, we shall consider only the case of $k=1$. Then we have $N(X) > \frac{1}{4}(2n+1)(2n+3)$. Assume $n \geq 6$. The arguments which are completely analogous to that of section 1 show that it is sufficient to consider an action of $SU(l)$ with $N=N(SU(l-1))$ as principal isotrogy subgroup $(2l+2>2n+3)$ and action of $SU(l)$ with $SU(l-1$ as principal isotrogy subgroup, non-empty $X_{(N)}$ and non-empty fixed point set F $(2l+2) > 2n+3)$. For the first action, it is not dificult to show that $X=CP_n \times S^1$ and $N(X)=\dim SU(n+1)+1$. For the second action, dimensional considerations show a contradiction. Consider the cases in which $n \leqq 5$. Let $G=T^r \times G_1 \times G_1 \times \cdots \times G_s (T^r$: r-dim torus, G_i : simple) be a compact connected Lie group with $\dim G=N(X)$ which acts on X almost effectively

Case 1. n=5

Since $N(X) \geqq 36 > 3 \dim X$, there exists a simple normal subgroup, say G_1, of G such that $\dim G_1 + \dim N(H_1, G_1)/H_1 > 3 \dim G_1/H_1$. All pairs (G_1, H_1) except $(SU(6), N(SU(5), SU(6)))$ are proved to be impossible. For the case of $(SU(6), N(SU(5), SU(6))$ it is not difficult to show that $G=SU(6) \times SO(2)$ and $X=CP_6 \times S^1$.

Case 2 n=4

Case 3 n=3

We shall omit the proof since the proof is similar to case 1,

Case4 n=2

Since $9 \leqq N(X) \leqq 15$, G_i is one of G_2(exceptional Lie group of rank 2) $SO(6)$, $SO(5)$, $SU(4)$, $SU(3)$, or $SU(2)$. If some G_i is G_2, dim $H_1 \geq 14-5 \geq 9$ which is impossible. The same argument shows that no G_i is $SO(6)$. Suppose some G_i is $SU(4)$. Then dim $H_1 \geq 10$, and hence $H_1 \sim Sp(2)$, which implies that $X = SU(4)/Sp(2)$. This is a contradiction because $\pi_i(SU(4)/Sp(2)) = 0$ for $0 \leq i \leq 2$. If some G_i is $SO(5)$, then dim $H_1 \geq 5$, which implies that $H_1 \sim SO(4)$. For this case by Vietoris-Begle theorem we can show a contradiction. If some G_i is $SU(3)$ then dim $H_1 \geq 3$, $H_1 \sim Sp(2)$ or $N(SU(2), SU(3))$. In this case it is shown that $X = S^5$ or $CP_2 \times S^1$ and $N(X) = 9$. Thus we have shown that to complete our arguments it is sufficient to consider only following cases: $G = T^6 \times SU(2)$, $T^3 \times SU(2) \times SU(2)$, and $SU(2) \times SU(2)$. For the first and second cases, we can easily deduce a contradiction. Consider the case in which $G = SU(2) \times SU(2) \times SU(2)$ acts on X. Then dim $H \geq 4$. Let $p_i : G \longrightarrow SU(2)$ be the projection onto the i-th factor. Put $G = G_1 \times G_2 \times G_3$.

Case a dim H=4

In this case $X = G/H$.

Subcase 1 dim $\mathbf{p_3(H)=0}$. Then $G_3/p_3(H) \underset{Q}{\sim} S^3$.

Since dim $H \cap (G_1 \times G_2) = 4$, $G_1 \times G_2 / H \cap (G_1 \times G_2) \underset{Q}{\sim} S^2$ or pt. This contradicts to the structure of cohomology ring of X because X is a fibre space over $G_3/p_3(H)$ with $G_1 \times G_2 / H \cap (G_1 \times G_2)$ as fibre.

Subcase 2 dim $\mathbf{p_3(H)=1}$.

We have dim $H \cap (G_1 \times G_2) = 3$ and hence $X^* = X/G_1 \times G_2$ is a 2-dimensional manifold. Put $H_1 = H \cap (G_1 \times G_2)$. Suppose there is a 4-dimensional isotropy subgroup K. Since $p_2(K) \approx K/K \cap G_1$ and dim $K \cap G_1 = 0$ or 3, we have $K = G_1 \times K_2$ or $K_1 \times G_2$. Suppose $K = G_1 \times K_2$. Since $H_1 \leq K$, we have $p_2 (H_1 \leq p_2 (K) = K_2$ and hence dim $p_2(H_1) = 0$ or 1. If dim $p_2(H^1) = 0$, then $H_1 \cap G_1 = G$ which contradicts to the almost effectivity. It is clearly impossible that dim $p_2(H_1) = 1$. Similary we can show that $K \neq K_1 \times G_2$. Thus we have shown that there is no 4-dimensional isotropy subgroup. Hence possible isotropy subgroup of $G_1 \times G_2$-action are principal isotropy subgroup, exceptional isotropy subgroup and $G_1 \times G_2$. Therefore the Vietoris Begle mapping theorem shows that the orbit map $\pi : X \longrightarrow X^*$ induces isomorphisms $\pi^* : H^i(X^*) \longrightarrow H^i(X)$ for $i \leq 2$, which leads a contradiction.

Subcase 3 dim $\mathbf{p_3(H)=3}$.

Since dim $H_1=1$, $X=G_1\times G_2/H_1$. Put $\widetilde{X}=G_1\times G_2/T$. Then we have $H^*(\widetilde{X}:Q)=H^*(X:Q)$. From the Gysin sequence of $T\longrightarrow G_1\times G_2\longrightarrow\widetilde{X}$, it follows a contradiction.

Case b dim H$\geqq 5$,

It is not difficult to show that this is impossible.

Case 5 n$=1$

It is easy to show that $N(X)=4$.

6. Low dimensional cases

In this section we shall consider an orientable closed manifold X of dimension $m\leq 25$ such that $X\underset{Q}{\sim}CP_n\times S^k$ $(k\geq 3)$ or $CP_n\times CP_k$ $(k\geq n)$ and $H^1(X:Z_2)=0$. Assume $N(X)\geq$ dim $SU(n+1)+$dim $SO(n+1)$ or dim $SU(n+1)+$dim $SU(k+1)$. Since the situations of two cases are almost parallel, we shall consider only the case of $X\underset{Q}{\sim}CP_n\times S^k$.

It is easy to see that $N(X)\geqq 3$ dim X if $13\leqq$ dim $X\leqq 25$ or dim $X=12$ and $k\geqq 5$.

Let $C=T^r\times G_1\times\cdots\times G_s$ be a compact connected Lie group of dim $G=N(X)$ which acts almost effectively on X. There exists a simple factor, say G_1, with the following properties:

(6. 1) $\quad$ $\dim G_1+\dim N(H_1, G_1)/H_1\geq 3\dim G_1/H_1$

(6. 2) $\quad$ $\dim H_1\geq\frac{1}{2}\dim G_1$

and

(6. 3) $\quad$ $\dim H_1\geq\dim G_1-25$,

where H_1 denotes the identity component of a principal isotropy subgroup of G_1-action. i, e. $H_1=(H\cap G_1)^0$ ($H=$a principal isotropy subgroup of G-action).

By dimensional considerations, it is shown that G_1 is not E_8, E_7, E_6 or G_2. If $G_1=F_4$, H_1 must be Spin (9). Since dim $X/G_1=2n+k-16$, the Vietoris Begle theorem shows a contradiction when $k<16$. When $k\geq 16$, we have $N(X)\geq 4$ dim X and hence (6. 1) is replaced by

(6. 1)$'$ $\quad$ $\dim G_1+\dim N(H_1, G_1)/H_1\geq 4\dim G_1/H_1$

This inequality does not hold for $(F_4, Spin\ (9))$. Thus we have shown that G_1 must be classical.

Case 1 $G_1=SU(L)$

If $l\geq 9$, then dim $G_1/H_1\leq 25\leq\frac{1}{2}(l-1)^2$, and hence we have $(G, H_1)=(SU(l), SU(l$

$-1))$ or $(SU(l), N(SU(l-1), SU(l)))$. Considering subgroups of low dimensional $SU(l)$, we can also show that possible pair (G_1, H_1) is as above.

Case 2 $G_1 = SO(L)$ Note that $l \geq 5$)

In this case, we can also prove that possible pair (G_1, H_1) is $(SO(l), SO(l-1))$ but one exception of $(SO(7), G_2)$. Consider the exceptional case. It is sufficient to consider only the case of $CP_4 \times S^5$ or $CP_3 \times S^6$. Since G_2 is maximal in $SO(7)$, possible orbits are rational cohomology 7-sphere. Hence the orbit map $\pi : X \longrightarrow X/SO(7)$ induces isomorphisims $\pi^* : H^i(X/SO(7) : Q) \longrightarrow H^i(X : Q)$ for $i \leq 6$. Then the generator a of $H^2(X : Q)$ is in the image of π^*. Since dim $X^* = 6$, or 5, we have $a^4 = 0$ or $a^3 = 0$, which is a contradiction.

Case 3 $G_1 = S_p(L)$ $(l \geq 3)$

It is not difficult to see that this case is impossible. The same arguments as in section 2, 3, and 4 show that $X = CP_n \times S^k$ and $N(X) = \dim SU(n+1) + \dim SO(k+1)$. The details are omitted since they are tedioas.

There remains the following cases: $CP_4 \times S^4$, $\dim X = 11$, 10, 9, 8 and 7.

Case $CP_4 \times S^4$

We have $78 \geq N(X) \geq 34 > 2.8 \times 12$. There exists a simple normal subgroup G_1 of G with properties

(6. 4) $\dim G_1 + \dim N(H_1, G_1)/H_1 > 2.8 \dim G_1/H_1$

(6. 5) $\dim H_1 > \frac{4}{9} \dim G_1$

and

(6. 6) $\dim H_1 \geqq \dim G_1 - 12$.

It is easy to show that G_1 is not exceptional.

Subcase 1 $G_1 = SU(l)$.

Since $\dim G_1 \leq 78$, we have $l \leqq 8$. It follows from (6. 6) that $H_1 \sim SU(l-1)$, or $N(SU(l-1), SU(l))$. Moreover from (6. 4) and the tfact that $2l-1 \leq 12$ it follows that possible pair (G_1, H_1) is $(SU(5), N(SU(4), SU(5)))$ or $(SU(6), N(SU(5), SU(6)))$. Then it is easy to see that $X = CP_n \times S^4$ and $N(X) = \dim SU(5) + \dim SO(5)$.

Subcase 2 $G_1 = S_p(L)$, or $G_1 = SO(L)$

It is not difficult to see that this case is impossible. We shall omit the other cases since they are not difficult but tedious.

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