

# Self-duality for mathematical programming in complex space

By

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## 1. Introduction

Duality theorem of mathematical programming problems in complex space have been given in [1] and [5] for linear programs, in [2], [4] and [6] for quadratic programs, in [3], [7], [8] and [9] for nonlinear programs. Self-dual problems, that is, problems whose primal and dual formulations are equivalent, have been investigated by Duffin [10], Dorn [11], Hanson [13], Cottle [14] in real case, and by Mond and Hanson [4] in complex case.

In this paper, it will be shown that the self-duality theorem for quadratic programming can be extended to constraints involving polyhedral cone in complex space. Moreover, we extended the duality theorem [12] to the case of complex nonlinear programming and the self-dual program will be given as its special case.

## 2. Duality in complex quadratic programming

By Abrams and Ben-Israel, the duality theorem for complex quadratic programming [2] was extended to constraints involving polyhedral cone.

For  $x \in C^n$ ,  $y \in C^n$ ,  $(x, y) \equiv y^H x$  denotes the inner product of  $x$  and  $y$  in complex space. And for any nonempty subset  $S \subset C^n$ , let

$$S^* \equiv \{y \in C^n : x \in S \rightarrow \operatorname{Re}[(x, y)] \geq 0\}$$

denotes the polar of  $S$ . Also,  $S \subset C^n$  is a polyhedral cone if for some positive integer  $k$  and  $A \in C^{n \times k}$ ,

$$S = \{Ax : x \geq 0\}.$$

For any  $A \in C^{m \times n}$ ,  $A^T$  denotes the transpose of  $A$  and  $A^H$  denotes the conjugate transpose of  $A$ .

Let  $B \in C^{n \times n}$  be a positive definite Hermitian matrix,  $A \in C^{m \times n}$ ,  $b \in C^m$ ,  $c \in C^n$ , and let  $S \subset C^n$ ,  $T \subset C^m$  be polyhedral cones.

Consider the following quadratic programming problems in complex space.

(P1) minimize

$$f(x) \equiv \operatorname{Re}\left(\frac{1}{2}x^H Bx + c^H x\right) \quad (1)$$

subject to

$$Ax - b \in T \quad (2)$$

$$x \in S \quad (3)$$

(D1) maximize

$$g(y, z) \equiv \operatorname{Re}\left(-\frac{1}{2}y^H B y + b^H z\right) \quad (4)$$

subject to

$$c + B y - A^H z \in S^* \quad (5)$$

$$z \in T^*, \quad (6)$$

Duality theorem for this programming problem is given by Abrams and Ben-Israel [6].

THEOREM 1.

(a) If (P1) has an optimal solution  $x_0$ , there exists a vector  $z_0$  such that  $(x_0, z_0)$  is an optimal solution of (D1) and  $f(x_0) = g(x_0, z_0)$ .

(b) If (D1) has an optimal solution  $(y_0, z_0)$ , there exists a vector  $x_0$ , such that  $Bx_0 = By_0$ , which is an optimal solution of (P1) and  $g(y_0, z_0) = f(x_0)$ .

PROOF. The proof is given in [6].

A special case of this problem, which is self-dual, will be considered in the next section.

### 3. Self-dual complex quadratic programming

The concept of self-dual program was given by Dorn [11]. If constraints are added to, or subtracted from, a program in such a way that the solution (both the optimal value of objective function and the optimal values of the variables) is unchanged, the new program thus constructed is called equivalent to the original program. A program is called self-dual if it is equivalent to its dual.

Consider the following complex quadratic programming problem:

(P2) minimize

$$f(x) \equiv \operatorname{Re}(x^H A x + p^H x) \quad (7)$$

subject to

$$Ax + p \in S^* \quad (8)$$

$$x \in S \quad (9)$$

where  $A \in C^{n \times n}$  is an Hermitian positive definite matrix,  $p \in C^n$ ,  $x \in C^n$ , and  $S \subset C^n$  is a polyhedral cone.

**THEOREM 2.** *If the problem (P2) is feasible, then it is self-dual. Moreover, the minimum value of  $f(x)$  is zero.*

**PROOF.** If  $B=2A$ ,  $c=-b=p$ ,  $T=S^*$  in (P1), we have then the problem (P2). Therefore, the dual problem of (P2) is the following:

(D2) maximize

$$g(y, z) \equiv \operatorname{Re}(-y^H A y - p^H z) \quad (10)$$

subject to

$$p + 2Ay - A^H z \in S^* \quad (11)$$

$$z \in S. \quad (12)$$

Let  $x$  be a feasible solution of (P2). From (8) and (9),

$$\operatorname{Re}[(Ax + p)^H x] \geq 0. \quad (13)$$

Since then  $f(x) = \operatorname{Re}(x^H Ax + p^H x) \geq 0$  is bounded below, therefore (P2) has an optimal solution  $x_0$ . By Theorem 1, there exists a  $z_0$  such that  $(x_0, z_0)$  is optimal for problem (D2) and such that

$$\operatorname{Re}(x_0^H A x_0 + p^H x_0) = \operatorname{Re}(-y_0^H A y_0 - p^H z_0). \quad (14)$$

Using (11) and (12)

$$\operatorname{Re}[(p + 2Ay - A^H z)^H z] \geq 0$$

or

$$\operatorname{Re}(-z^H A z + 2y^H A z) \geq \operatorname{Re}(-p^H z).$$

Then, we have

$$\begin{aligned} g(y, z) &= \operatorname{Re}(-y^H A y - p^H z) \leq \operatorname{Re}(-y^H A y - z^H A z + 2y^H A z) \\ &= \operatorname{Re}[-(z - y)^H A (z - y)] \leq 0 \end{aligned}$$

where the last inequality results from the assumption of matrix  $A$ . It follows that

$$\text{maximum } g(y, z) \leq 0.$$

From (13), however,

$$\text{minimum } f(x) \geq 0.$$

Combining these with (14) then

$$\begin{aligned} 0 &\leq \text{minimum } f(x) = \text{Re}(x^H A x + p^H x) \\ &= \text{maximum } g(y, z) = \text{Re}(-y^H A y - p^H z) \leq 0. \end{aligned}$$

Since  $A$  is an Hermitian positive definite matrix

$$\text{maximum } g(y, z) = \text{Re}(-y^H A y - p^H z) = 0$$

if and only if  $y=z$ . The constraint  $y=z$  may, therefore, be added to (11) and (12) and an equivalent problem is obtained. The equivalent problem so obtained is

(D2') maximize

$$g(y, z) \equiv \text{Re}(-y^H A y - p^H z)$$

subject to

$$p + 2A y - A^H z \in S^*, \quad z \in S, \quad \text{and } y = z.$$

This equivalent problem may be reduced by eliminating  $z$  to:

(D2'') maximize

$$g(y) \equiv \text{Re}(-y^H A y - p^H y)$$

subject to

$$A y + p \in S^* \quad \text{and } y \in S,$$

which is precisely the original problem. This completes the proof of the theorem.

#### 4. Duality in complex nonlinear programming

The analytic function  $f: C^n \times C^n \rightarrow C$  will be said to be convex in a domain  $A$  if for all  $z_1, z_2 \in A$

$$\text{Re}[f(z_2, \bar{z}_2) - f(z_1, \bar{z}_1) - (z_2 - z_1)^T \nabla_1 f(z_1, \bar{z}_1) - (z_2 - z_1)^H \nabla_2 f(z_1, \bar{z}_1)] \geq 0, \quad (15)$$

where  $\nabla$  denoting the gradient vector, that is,

$$\begin{aligned} \nabla_1 f(z, \bar{z}) = \nabla_z f(z, \bar{z}) &\equiv \left( \frac{\partial f(z, \bar{z})}{\partial z_i} \right) & i=1, \dots, n \\ \nabla_2 f(z, \bar{z}) = \nabla_{\bar{z}} f(z, \bar{z}) &\equiv \left( \frac{\partial f(z, \bar{z})}{\partial \bar{z}_i} \right) & i=1, \dots, n. \end{aligned}$$

A function  $f$  will be called concave if  $-f$  is convex. Similarly, for an analytic function  $g: C^n \times C^n \rightarrow C^m$  is convex in a domain  $A$  if for any  $z_1, z_2 \in A$

$$\text{Re}[g(z_2, \bar{z}_2) - g(z_1, \bar{z}_1) - D_1 g(z_1, \bar{z}_1)(z_2 - z_1) - D_2 g(z_1, \bar{z}_1)(\bar{z}_2 - \bar{z}_1)] \geq 0$$

where

$$D_1 g(z, \bar{z}) = D_z g(z, \bar{z}) \equiv \left( \frac{\partial g_i(z, \bar{z})}{\partial z_j} \right) \quad i=1, \dots, m, \quad j=1, \dots, n$$

$$D_2 g(z, \bar{z}) = D_{\bar{z}} g(z, \bar{z}) \equiv \left( \frac{\partial g_i(z, \bar{z})}{\partial \bar{z}_j} \right) \quad i=1, \dots, m, j=1, \dots, n.$$

We assume throughout that  $f(z, \bar{z}): C^n \times C^n \rightarrow C$  is convex,  $g(z, \bar{z}): C^n \times C^n \rightarrow C^m$  is concave with respect to  $S$  [8],  $S \cap R^m \subset R^m_+$ , and both be analytic in a neighborhood of a qualified point [7]. Consider the following two programs.

(P3) minimize

$$\operatorname{Re} f(z, \bar{z})$$

subject to

$$g(z, \bar{z}) \in S \tag{16}$$

$$z \in T \tag{17}$$

where  $S \subset C^m$  and  $T \subset C^n$  are polyhedral cones.

(D3) maximize

$$H(z, u) = \operatorname{Re} [ f(z, \bar{z}) - (g(z, \bar{z}), u) - \overline{(\nabla_1 f(z, \bar{z}))} + \nabla_2 f(z, \bar{z}), z ] \\ + (D_1^H g(z, \bar{z})u + D_2^T g(z, \bar{z})\bar{u}, z)$$

subject to

$$\overline{\nabla_1 f(z, \bar{z})} + \nabla_2 f(z, \bar{z}) - D_1^H g(z, \bar{z})u - D_2^T g(z, \bar{z})\bar{u} \in T^* \tag{18}$$

$$u \in S^* \tag{19}$$

where

$$D_1^T g(z, \bar{z}) = \left( \frac{\partial g_i(z, \bar{z})}{\partial z_j} \right)^T \quad i=1, \dots, m, j=1, \dots, n$$

$$D_2^H g(z, \bar{z}) = \left( \frac{\partial g_i(z, \bar{z})}{\partial \bar{z}_j} \right)^H \quad i=1, \dots, m, j=1, \dots, n.$$

**THEOREM 3.** *If  $(z_0, \bar{z}_0)$  is an optimal solution of (P3), then there exists a vector  $u_0 \in S^*$  such that  $(z_0, u_0)$  is an optimal solution for (D3), and the extreme values of the two objective functions are equal.*

**PROOF.** Let  $(z_0, \bar{z}_0)$  be an optimal solution of (P3). The necessary conditions for  $(z_0, \bar{z}_0)$  to be an optimal solution of (P3) are given by Abrams and Ben-Israel; there exist a  $u_0 \in S^*$  such that

$$\overline{\nabla_1 f(z_0, \bar{z}_0)} + \nabla_2 f(z_0, \bar{z}_0) = D_1^H g(z_0, \bar{z}_0)u_0 + D_2^T g(z_0, \bar{z}_0)\bar{u}_0 \tag{20}$$

and

$$\operatorname{Re}(g(z_0, \bar{z}_0), u_0) = 0. \tag{21}$$

Since  $T$  is a polyhedral cone,  $0 \in T^*$ . Therefore,  $(z_0, u_0)$  lies in the set of feasible solution of (D3). We now show that it is also optimal.

Let  $(z, u)$  be any other feasible solution of (D3). We have then, by (20) and (21)

$$\begin{aligned}
& H(z_0, u_0) - H(z, u) \\
&= \text{Re}[f(z_0, \bar{z}_0) - (g(z_0, \bar{z}_0), u_0) - (\overline{\nabla_1 f(z_0, \bar{z}_0)} + \nabla_2 f(z_0, \bar{z}_0), z_0) \\
&+ (D_1^H g(z_0, \bar{z}_0)u_0 + D_2^T g(z_0, \bar{z}_0)\bar{u}_0, z_0) - f(z, \bar{z}) + (g(z, \bar{z}), u) \\
&+ (\overline{\nabla_1 f(z, \bar{z})} + \nabla_2 f(z, \bar{z}), z) - (D_1^H g(z, \bar{z})u + D_2^T g(z, \bar{z})\bar{u}, z)] \\
&= \text{Re}[f(z_0, \bar{z}_0) - f(z, \bar{z}) + (g(z, \bar{z}), u) + (\overline{\nabla_1 f(z, \bar{z})} + \nabla_2 f(z, \bar{z}), z) \\
&- D_1^H g(z, \bar{z})u + D_2^T g(z, \bar{z})\bar{u}, z).
\end{aligned}$$

The definition of convexity of  $f(z, \bar{z})$  implies

$$\text{Re}[f(z_0, \bar{z}_0) - f(z, \bar{z})] \geq \text{Re}[(z_0 - z)^T \nabla_1 f(z, \bar{z}) + (z_0 - z)^H \nabla_2 f(z, \bar{z})].$$

We have then,

$$\begin{aligned}
& H(z_0, u_0) - H(z, u) \\
&\geq \text{Re}[(z_0 - z)^T \nabla_1 f(z, \bar{z}) + (z_0 - z)^H \nabla_2 f(z, \bar{z}) + (g(z, \bar{z}), u) \\
&+ (\overline{\nabla_1 f(z, \bar{z})} + \nabla_2 f(z, \bar{z}), z) - (D_1^H g(z, \bar{z})u + D_2^T g(z, \bar{z})\bar{u}, z) \\
&= \text{Re}[(\nabla_1 f(z, \bar{z}) + \nabla_2 f(z, \bar{z}), z_0) + (g(z, \bar{z}), u) - (D_1^H g(z, \bar{z})u \\
&+ D_2^T g(z, \bar{z})\bar{u}, z)].
\end{aligned}$$

Since  $(z_0, \bar{z}_0)$  and  $(z, u)$  are feasible for (P3) and (D3) respectively,

$$\begin{aligned}
& z_0 \in T \\
& \overline{\nabla_1 f(z, \bar{z})} + \nabla_2 f(z, \bar{z}) - D_1^H g(z, \bar{z})u - D_2^T g(z, \bar{z})\bar{u} \in T^*.
\end{aligned}$$

Therefore, it follows that

$$\text{Re}[z_0^H (\overline{\nabla_1 f(z, \bar{z})} + \nabla_2 f(z, \bar{z}))] \geq \text{Re}[(D_1^H g(z, \bar{z})u + D_2^T g(z, \bar{z})\bar{u}, z_0)].$$

Thus from this inequality and concavity of  $g(z, \bar{z})$ , we obtain

$$\begin{aligned}
& H(z_0, u_0) - H(z, u) \\
&\geq \text{Re}[(D_1^H g(z, \bar{z})u + D_2^T g(z, \bar{z})\bar{u}, z_0 - z) + (g(z, \bar{z}), u)] \\
&= \text{Re}[(D_1 g(z, \bar{z}) + D_2 g(z, \bar{z}))(\bar{z}_0 - \bar{z}), u] + (g(z, \bar{z}), u) \\
&\geq \text{Re}[(g(z_0, \bar{z}_0), u)] \geq 0
\end{aligned}$$

where the last inequality follows from  $u \in S^*$  and  $g(z_0, \bar{z}_0) \in S$ . This establishes that  $(z_0, u_0)$  is an optimal solution of problem (D3).

Finally from (20) and (21), it follows that

$$H(z_0, u_0)$$

$$\begin{aligned}
&= \text{Re}[f(z_0, \bar{z}_0) - (g(z_0, \bar{z}_0), u_0) - (\overline{\nabla_1 f(z_0, \bar{z}_0)} + \nabla_2 f(z_0, \bar{z}_0), z_0) \\
&\quad + (D_1^H g(z_0, \bar{z}_0)u_0 + D_2^T g(z_0, \bar{z}_0)\bar{u}_0, z_0)] \\
&= \text{Re}f(z_0, \bar{z}_0)
\end{aligned}$$

verifying the equality of the two objective functions. This completes the proof of the theorem.

### 5. Self-duality in complex nonlinear programming

A special case of above problem, which is self-dual, will be considered in the following theorem.

Assume now that  $g: C^n \times C^n \rightarrow C^n$  is concave with respect to  $S$ ,  $S \cap R^n \subset R^n_+$  and  $(g, z)$  is convex. The complex program to be considered is the following.

(P4) minimize

$$\text{Re}[(g(z, \bar{z}), z)]$$

subject to

$$g(z, \bar{z}) \in S \tag{22}$$

$$z \in S^*. \tag{23}$$

**THEOREM 4.** *If there is a  $(z, \bar{z})$  satisfying (22) and (23), then the problem (P4) is self-dual and its optimal value is zero.*

**PROOF.** Let  $(z, \bar{z})$  be any feasible solution of (P4). From (22) and (23), we have

$$\text{Re}[(g(z, \bar{z}), z)] \geq 0.$$

Since minimum  $\text{Re}[(g(z, \bar{z}), z)] \geq 0$  is bounded below, therefore (P4) has an optimal solution. It is easy to see that if  $f(z, \bar{z}) = (g(z, \bar{z}), z)$  and  $T = S^*$ , then (P3) reduce to (P4). Therefore, the dual problem of the above program (P4) is the following.

(D4) maximize

$$\text{Re}[-(g(z, \bar{z}), z+u) - (D_1^H g(z, \bar{z})z + D_2^T g(z, \bar{z})\bar{z}, z-u)]$$

subject to

$$2g(z, \bar{z}) + D_1^H g(z, \bar{z})(z-u) + D_2^T g(z, \bar{z})(\bar{z}-\bar{u}) \in S^{**} \tag{24}$$

$$u \in S^*. \tag{25}$$

Since  $S$  is polyhedral cone,  $S^{**} = S$ . Therefore, from this formulation it is clear that if the constraints  $u=z$  are added to (24) and (25), the equivalent problem is obtained. Moreover, by Throrem 3, if  $(z_0, \bar{z}_0)$  is an optimal solution of (P4), then

$$\text{Re}[(g(z_0, \bar{z}_0), z_0)] = 0.$$

Hence, the optimal value of (P4) is zero, which proves the theorem.

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