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SYMPLECTIC RIGIDITY AND WEAK COMMUTATIVITY

Franco Cardin and Simone Vazzoler

We present a new and simple proof of Eliashberg–Gromov's theorem based on the notion of C^0-commutativity introduced by Cardin and Viterbo in [**1**].

1. Introduction

The notion of C^0-commutativity has been introduced in [**1**] in connection with the problem of finding variational solutions of multi-time Hamilton–Jacobi equations. The proof of the main theorem is essentially based on the use of Viterbo's capacities. In that paper, it has been foreseen that from the C^0-commutativity framework Eliashberg–Gromov's theorem on symplectic rigidity could follow. An interesting proof of this fact has been recently worked out by Humilière (see [**4**]) using the concept of pseudo-representations.

The question of the C^0-closure of the group of symplectomorphisms is widely considered as the starting point of the study of symplectic topology; for this reason is important to enrich this particular area with new proofs, eventually trying to simplify the subject. This note presents a new proof of Eliashberg–Gromov's theorem, starting from the concept of C^0-commutativity and using simple algebraic arguments.

2. Weak commutativity and Eliashberg–Gromov's theorem

In the following, by a function of class $C^{1,1}$ we mean a C^1 function with Lipschitz derivative.

Definition 1. Let H, K be two autonomous Hamiltonians. We will say that H and K C^0-commute if there exist two sequences H_n, K_n of C^1 Hamiltonians C^0-converging to H and K, respectively, such that, in the C^0-topology:

$$\lim_{n\to\infty}\{H_n, K_n\} = 0. \tag{2.1}$$

The previous definition is a good extension of the standard Poisson brackets since the following theorem holds.

Theorem 2 (Cardin, Viterbo [1]). *Let H and K be two compactly supported Hamiltonians of class $C^{1,1}$. If they C^0-commute then $\{H, K\} = 0$ in the usual sense.*

The following lemma is the generalization of Theorem 2 to the affine at infinity case.

Lemma 3 (Humilière, [4]). *Let u, v two affine maps $\mathbb{R}^{2n} \to \mathbb{R}$ and H_n, K_n be compactly supported Hamiltonians, such that*

$$H_n \to H,\ K_n \to K,\ \{H_n + u, K_n + v\} \to 0.$$

Then $\{H + u, K + v\} = 0$.

In what follows, we will consider only sequences of compactly supported symplectomorphisms $\Phi^{(n)}$, more precisely such that $\operatorname{supp}(\Phi^{(n)} - \mathrm{Id})$ is compact.

Theorem 4 (Symplectic rigidity, [2, 3]). *The group of compactly supported symplectomorphisms is C^0-closed in the group of all diffeomorphisms of $\mathbb{R}^{2d}$.*

Proof. To fix the notations: $(q, p) = (q_1, \dots, q_d, p_1, \dots, p_d) \in \mathbb{R}^{2d}$ and denote by

$$(Q_1^{(n)}(q,p), \dots, Q_d^{(n)}(q,p), P_1^{(n)}(q,p), \dots, P_d^{(n)}(q,p)), \tag{2.2}$$

a sequence of symplectic transformations C^0-converging to

$$(Q_1(q,p), \dots, Q_d(q,p), P_1(q,p), \dots, P_d(q,p)).$$

Note that we have to prove only $\{Q_i, P_i\} = 1$. In fact, the other relations $\{Q_i, Q_j\} = 0 = \{P_i, P_j\}$, and $\{Q_i, P_j\} = 0$ for $i \neq j$, are automatically satisfied in view of Lemma 3. Now we define a new sequence (using the previous one)

$$\begin{cases} \widetilde{Q}_i^{(n)} = Q_i^{(n)} + \dfrac{1}{\sqrt{d}} \displaystyle\sum_{k=1}^{d} P_k^{(n)}, \\ \widetilde{P}_i^{(n)} = P_i^{(n)} + \dfrac{1}{\sqrt{d}} \displaystyle\sum_{k=1}^{d} Q_k^{(n)}, \end{cases} \tag{2.3}$$

that will C^0-converge to

$$\begin{cases} \widetilde{Q}_i = Q_i + \dfrac{1}{\sqrt{d}} \displaystyle\sum_{k=1}^{d} P_k, \\ \widetilde{P}_i = P_i + \dfrac{1}{\sqrt{d}} \displaystyle\sum_{k=1}^{d} Q_k. \end{cases} \tag{2.4}$$

Clearly $\{\widetilde{Q}_i^{(n)}, \widetilde{P}_i^{(n)}\} = 0$, in fact

$$\begin{aligned} \{\widetilde{Q}_i^{(n)}, \widetilde{P}_i^{(n)}\} = \{Q_i^{(n)}, P_i^{(n)}\} + \frac{1}{\sqrt{d}} \sum_{k=1}^{d} (\{P_k^{(n)}, P_i^{(n)}\} + \{Q_i^{(n)}, Q_k^{(n)}\}) \\ + \frac{1}{d} \sum_{k=1}^{d} \{P_k^{(n)}, Q_k^{(n)}\} = 1 - 1 = 0. \end{aligned}$$

Again using Lemma 3, we obtain $\{\widetilde{Q}_i, \widetilde{P}_i\} = 0$: passing to the limit,

$$\begin{aligned} \{\widetilde{Q}_i, \widetilde{P}_i\} &= \{Q_i, P_i\} + \frac{1}{\sqrt{d}} \sum_{k=1}^{d} (\{P_k, P_i\} + \{Q_i, Q_k\}) + \frac{1}{d} \sum_{k=1}^{d} \{P_k, Q_k\} \\ &= \{Q_i, P_i\} + \frac{1}{d} \sum_{k=1}^{d} \{P_k, Q_k\} = 0. \end{aligned}$$

Define (just to simplify the notations) $C_i(q,p) = \{Q_i, P_i\}$. For every fixed $(q,p) \in \mathbb{R}^{2d}$, the last homogeneous linear system reads

$$\begin{pmatrix} d-1 & -1 & \dots & -1 \\ -1 & d-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & d-1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.5}$$

This system has $C_1 = C_2 = \cdots = C_d$ as solution; in fact, the $d \times d$ matrix has determinant equal to zero: if we sum the last $d-1$ rows we get the opposite of the first row; in particular the rank of the matrix is $d-1$, so the subspace of solutions has dimension 1 and is exactly the space of equal components vectors. Recalling the Jacobi identity

$$\{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\} = 0, \tag{2.6}$$

we obtain, considering terms like $\{Q_i, \{Q_j, P_j\}\}$ for $i \neq j$,

$$0 = \{Q_i, \{Q_j, P_j\}\} + \{Q_j, \{P_j, Q_i\}\} + \{P_j, \{Q_i, Q_j\}\} \tag{2.7}$$

and since $\{Q_i, P_j\} = \{Q_i, Q_j\} = 0$, we get

$$\{Q_i, \{Q_j, P_j\}\} = 0. \tag{2.8}$$

In the same way, considering $\{P_i, \{Q_j, P_j\}\}$ we obtain

$$\{P_i, \{Q_j, P_j\}\} = 0. \tag{2.9}$$

Once we have set $C_1(q,p) = C_2(q,p) = \cdots = C_d(q,p) = C(q,p)$, using the previous relations, we have

$$\begin{cases} \{Q_1, C\} = 0 \\ \{Q_2, C\} = 0 \\ \quad\vdots \\ \{P_{d-1}, C\} = 0 \\ \{P_d, C\} = 0, \end{cases} \tag{2.10}$$

which is a homogeneous linear system of the type $A \cdot DC = 0$

$$\begin{pmatrix} -Q_{1,p_1} & \dots & -Q_{1,p_d} & Q_{1,q_1} & \dots & Q_{1,q_d} \\ -Q_{2,p_1} & \dots & -Q_{2,p_d} & Q_{2,q_1} & \dots & Q_{2,q_d} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -P_{d,p_1} & \dots & -P_{d,p_d} & P_{d,q_1} & \dots & P_{d,q_d} \end{pmatrix} \begin{pmatrix} C_{,q_1} \\ C_{,q_2} \\ \vdots \\ C_{,p_d} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.11}$$

From the fact that $\Phi : (q,p) \mapsto (Q_1, \dots, Q_d, P_1, \dots, P_d)$ is a diffeomorphism we have $\det A \neq 0$ (because $A = D\Phi \cdot \mathbb{E}$ where $\mathbb{E}$ is the symplectic matrix) and so $C(q,p) = C$, a constant. It remains to show that $C = 1$. This comes from the fact that outside a compact set of $\mathbb{R}^{2d}$ we have $(Q_1, \dots, Q_d, P_1, \dots, P_d) = (q_1, \dots, q_d, p_1, \dots, p_d)$ and so $\{q_i, q_j\} = \{p_i, p_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$ outside this compact set. From the fact that the Poisson brackets are (at least) continuous, it follows that we must have $C = 1$. $\square$

References

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Università degli Studi di Padova
Dipartimento di Matematica Pura e Applicata
Via Trieste, 63
35121 Padova
Italy
E-mail address: cardin@math.unipd.it
E-mail address: svazzole@gmail.com
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