

BEYOND THE HIT PROBLEM: MINIMAL PRESENTATIONS OF
ODD-PRIMARY STEENROD MODULES, WITH APPLICATION
TO $CP(\infty)$ AND BU

DAVID J. PENGELLEY AND FRANK WILLIAMS

(communicated by Donald M. Davis)

Abstract

We describe a minimal unstable module presentation over the Steenrod algebra for the odd-primary cohomology of infinite-dimensional complex projective space and apply it to obtain a minimal algebra presentation for the cohomology of the classifying space of the infinite unitary group. We also show that there is a unique Steenrod module structure on any unstable cyclic module that has dimension one in each complex degree (half the topological degree) with a fixed alpha-number (sum of ‘digits’) and is zero in other degrees.

1. Introduction

The projective spaces $RP(\infty)$ and $CP(\infty)$ play a pivotal role in algebraic topology, and have an amazing combination of features. As Eilenberg-MacLane spaces they represent key cohomology groups. Contrastingly, in the past two decades we have learned that their cohomologies are unstable injective modules over the Steenrod algebra $\mathcal{A}$ (at $p = 2$ for $RP(\infty)$, and at odd primes p for $CP(\infty)$ when considering “complex” (i.e., evenly) graded modules) [1, 2, 6, 14]. This surprising feature has been key to solving famous problems like the Segal and Sullivan conjectures.

We might even imagine that by now we understand their cohomologies $H^*(RP(\infty); \mathbb{F}_2)$ and $H^*(CP(\infty); \mathbb{F}_p)$ very well. As an algebra each is polynomial on a single generator with $\mathcal{A}$ -action determined by extremely simple formulas. And as an unstable $\mathcal{A}$ -algebra each is free on one generator. What could be simpler?

But how well do we understand their $\mathcal{A}$ -module structures, which are key to what they actually tell us about other spaces? In exactly what way are these remarkable $\mathcal{A}$ -modules built from generators and relations in order to produce all the amazing properties of projective spaces delineated above? If we ask first for minimal $\mathcal{A}$ -generators, this is the classical hit problem (i.e., which elements are not hit by the $\mathcal{A}$ -action), and it is not hard to answer this. We shall describe a minimal set of $\mathcal{A}$ -module generators $\{u(s) | s \geq 0\}$ for $H^*(CP(\infty); \mathbb{F}_p)$. In fact, for each integer

Received December 9, 2006, revised September 3, 2007; published on October 23, 2007.

2000 Mathematics Subject Classification: 55R40, 55R45, 55S05, 55S10.

Key words and phrases: Steenrod algebra, unstable, Kudo-Araki-May algebra, complex projective space, BU .

Copyright © 2007, International Press. Permission to copy for private use granted.

$s \geq 0$ the complex degree of the generator $u(s)$ is the least integer d for which $\alpha(d) = s$. (Here, and throughout this paper, “complex degree” will refer to one-half the topological degree, and the “alpha-number” of a nonnegative integer n , $\alpha(n)$, will mean the sum of the p -ary digits of n .)

Going beyond the hit problem, the question of which $\mathcal{A}$ -relations are then necessary among these minimal $\mathcal{A}$ -generators in order to glue together precisely the cohomology of projective space is extremely delicate. Pleasantly, we find that they are not that great in number, are essentially unique, and can be written down explicitly.

We accomplished this for $H^*(RP(\infty); \mathbb{F}_2)$ in [9], and will now do so for $H^*(CP(\infty); \mathbb{F}_p)$ (with p odd), where the answer has some fascinating extra twists but is still tractable. In so doing, we will analyze the cyclic subquotients of the $\mathcal{A}$ -filtration of $H^*(CP(\infty); \mathbb{F}_p)$ given by alpha-number of complex degree, and determine their minimal $\mathcal{A}$ -relations. The modules in this composition series are simple modules in the category $\mathcal{U}'_p/\mathcal{N}il$ of evenly graded unstable $\mathcal{A}$ -modules modulo nilpotence described in [1, 5]. We also show that each of them is uniquely characterized as a cyclic unstable $\mathcal{A}$ -module just by having dimension one in each even degree with a fixed alpha-number for the complex degree, and dimension zero in all other degrees. The generators of these cyclic modules are just the images in the filtered quotients of the elements $u(s)$.

There are considerable similarities between the odd-primary and the mod two cases. However, there are differences that are quite interesting. The filtered quotients in the mod two case are fairly well-known $\mathcal{A}$ -modules; i.e., they are isomorphic to the free unstable modules on single generators $t_{2^{n-1}-1}$ in degree $2^{n-1} - 1$ subject to the $\mathcal{A}$ -relations $Sq^{2^k} t_{2^{n-1}-1} = 0$ for $0 \leq k \leq n - 3$ [9]. In the odd-primary case, since the Steenrod algebra is concentrated in complex degrees divisible by $p - 1$, any $\mathcal{A}$ -module splits as a direct sum of $p - 1$ $\mathcal{A}$ -modules, each of these in degrees with fixed residue mod $(p - 1)$.

We shall see that the filtered quotients of $H^*(CP(\infty); \mathbb{F}_p)$ are isomorphic to certain modules $\mathcal{M}_{n,a}$ on generators $t_{ap^{n-1}-1}$ in (complex) degrees of the form $ap^{n-1} - 1$, where $1 \leq a \leq p - 1$, with $a - 1$ labeling the mod $(p - 1)$ residue summand, and n further reflecting the filtration by alpha-number within this summand. If we let $s = \alpha(ap^{n-1} - 1) = (a - 1) + (n - 1)(p - 1)$, then $ap^{n-1} - 1$ is the smallest integer with alpha-number s . Hence the generator of $\mathcal{M}_{n,a}$ is in the lowest degree with its alpha-number. In this case the minimal $\mathcal{A}$ -relations include the expected $\mathcal{P}^{p^k} t_{ap^{n-1}-1} = 0$ for $0 \leq k \leq n - 3$, but also include either one or two additional relations that depend on a . These modules $\mathcal{M}_{n,a}$ are quite interesting: Since they have dimension one in degrees with alpha-number equal to that of $ap^{n-1} - 1$, are zero in other degrees, and this uniquely characterizes them as cyclic $\mathcal{A}$ -modules, they may be regarded as basic building blocks of structures having to do with alpha-number. Moreover, they are analogs to an interesting phenomenon at the prime 2.

At $p = 2$, Franjou and Schwartz [3, 13, 14] considered the category $\mathcal{V}_{n-1}/\mathcal{V}_{n-2}$. ($\mathcal{V}_{n-1}$ is the full subcategory of $\mathcal{U}/\mathcal{N}il$, i.e., modulo nilpotence, with objects the unstable $\mathcal{A}$ -modules of weight $n - 1$, i.e., trivial in degrees with alpha-number

greater than $n - 1$.) They showed that $\mathcal{V}_{n-1}/\mathcal{V}_{n-2}$ is equivalent to the category of right modules over the group ring $\mathbb{F}_2[\Sigma_{n-1}]$ on the symmetric group. The cyclic unstable $\mathcal{A}$ -module we described above, on $t_{2^{n-1}-1}$ with relations $Sq^{2^k} t_{2^{n-1}-1} = 0$ for $0 \leq k \leq n - 3$, is of dimension one in each degree with alpha-number $n - 1$, and dimension zero in all other degrees, and thus corresponds under this equivalence of categories to the unique nontrivial rank one module over $\mathbb{F}_2[\Sigma_{n-1}]$.

Our $\mathcal{A}$ -modules $\mathcal{M}_{n,a}$ are odd-primary versions of these mod 2 $\mathcal{A}$ -modules, occupying a similar spot in the odd-primary analogue (see [1, p. 395] and [5]) of the theory in [3] of reduced unstable mod 2 $\mathcal{A}$ -modules. One of our theorems produces bases for the $\mathcal{M}_{n,a}$ that show that they are reduced and have weight exactly $\alpha = \alpha(ap^{n-1} - 1)$, i.e., lie in $\mathcal{V}'_\alpha - \mathcal{V}'_{\alpha-1}$ in the filtration of $\mathcal{U}'_p/\mathcal{N}il$. It would be interesting to find a direct proof, without using our basis theorem, that $\mathcal{M}_{n,a}$ is reduced and lies in $\mathcal{V}'_\alpha$. Such information might lead to an alternate proof of some of our results by invoking an odd-primary version of 2-primary results in [3].

Finally, in roughly the same way that our minimal $\mathcal{A}$ -presentation of $H^*(RP(\infty); \mathbb{F}_2)$ led to a minimal unstable $\mathcal{A}$ -algebra presentation of the symmetric algebra $H^*(BO; \mathbb{F}_2)$ [9], our minimal unstable $\mathcal{A}$ -module presentation for $H^*(CP(\infty); \mathbb{F}_p)$ will lead to a minimal unstable $\mathcal{A}$ -algebra presentation of the symmetric algebra $H^*(BU; \mathbb{F}_p)$.

Many of our methods will be the same ones we used in [7, 9, 10] to determine minimal relations for unstable $\mathcal{A}$ -modules and $\mathcal{A}$ -algebras. A fundamental element of our computations is the odd-primary even topological Kudo-Araki-May algebra, $\mathcal{K}$, whose definition and properties we developed in [11], and which is well-suited to studying unstable $\mathcal{A}$ -modules. We summarize necessary ingredients from this material in an appendix to the present paper.

In the following section we shall list the principal results of this paper.

2. Definitions and principal results

To set the stage, we work with coefficients in the field $\mathbb{F}_p$, for p an odd prime. We consider only evenly graded modules over the Steenrod algebra $\mathcal{A}$ (with no Bocksteins), and generally use the complex degree (half the topological degree) throughout to describe the grading. **N.B:** Every $\mathcal{A}$ -module in this paper will be assumed **unstable** without further mention.

The hit problem for $H^*(CP(\infty)) = \mathbb{F}_p[u]$ is easily solved. The formula for the action of the Steenrod algebra

$$\mathcal{P}^k u^m = \binom{m}{k} u^{m+(p-1)k}$$

quickly yields

Proposition 2.1. *A minimal set of $\mathcal{A}$ -module generators for $H^*(CP(\infty))$ is given by the set*

$$\{u^{ap^{n-1}-1} \mid 1 \leq a \leq p-1 \text{ and } n \geq 1\}.$$

As explained above, we can uniquely index these elements by the alpha-number of

their degrees, as $\{u(s) \mid s \geq 0\}$, where $s = \alpha(ap^{n-1} - 1) = (a-1) + (n-1)(p-1)$, and the degree $ap^{n-1} - 1$ of each is the smallest integer with its alpha-number.

Our main theorem is the following, whose proof will occupy most of this paper.

Theorem 2.2 (proven in Section 5). *A minimal set of $\mathcal{A}$ -module relations on the minimal module generators $\{u^{ap^{n-1}-1} \mid 1 \leq a \leq p-1 \text{ and } n \geq 1\}$ for $H^*(CP(\infty))$ is given by the following equations:*

For $0 \leq l \leq n-3$,

$$\mathcal{P}^{p^l} u^{ap^{n-1}-1} = \mathcal{P}^{ap^{n-2}} \mathcal{P}^{p^l} u^{ap^{n-2}-1},$$

and for $a \geq 2$,

$$(a-1)\mathcal{P}^{(p-a+1)p^{n-2}} u^{ap^{n-1}-1} = \binom{p-1}{a-2} \mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}} u^{ap^{n-2}-1}$$

and

$$(a-1)\mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}} u^{ap^{n-1}-1} = a\mathcal{P}^{p^{n-1}+p^{n-2}} u^{ap^{n-1}-1},$$

while for $a = 1$

$$2\mathcal{P}^{p^{n-1}+p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} u^{ap^{n-1}-1} = \mathcal{P}^{p^{n-1}+p^{n-2}+p^{n-3}} \mathcal{P}^{p^{n-2}} u^{ap^{n-1}-1}.$$

A sketch of the proof is as follows. It is straightforward to check

Proposition 2.3. *Our claimed set of relations is satisfied in $H^*(CP(\infty))$.*

Proof. The proof is left to the reader. $\square$

Thus the proof of the presentation will primarily involve showing that this set of relations is sufficient, i.e., that $H^*(CP(\infty))$ has the same graded rank as the quotient $\mathcal{M}$ of the free (unstable) $\mathcal{A}$ -module on generators in the specified degrees by the sub- $\mathcal{A}$ -module generated by the set of relations. To analyze this, we shall filter the two modules compatibly over $\mathcal{A}$, in a fashion related to the alpha-number of (complex) degree. We shall first describe a basis for each filtered quotient of $H^*(CP(\infty))$, using monomials from the Kudo-Araki-May algebra $\mathcal{K}$ applied to a generating class. Although any element described using $\mathcal{K}$ can in principle also be described using $\mathcal{A}$ (and vice-versa) by iterating the conversion formula

$$(-1)^j d_j u = \mathcal{P}^{q-j} u,$$

where u is a module class of (complex) degree q , in practice it seems that $\mathcal{K}$, whose algebra structure is dramatically different from that of $\mathcal{A}$, is often much more transparent for describing bases of unstable $\mathcal{A}$ -modules. In this case, we will provide a basis for $H^*(CP(\infty))$ expressed in terms of $\mathcal{K}$ -monomials we call “chosen”. Then we show that these same chosen monomials produce a basis for the corresponding filtered quotient of $\mathcal{M}$, by showing that every element of $\mathcal{M}$ can be expressed in terms of chosen monomials. This determination that chosen monomials suffice to span $\mathcal{M}$, based on its abstract defining relations, is the lengthy part of the proof. We refer the reader to our appendix for a summary of relevant information about $\mathcal{K}$. Finally, the proof will also verify that the relations we give are all necessary.

We begin the process of describing our “chosen” monomials in $\mathcal{K}$ with some definitions. The fact that these particular monomials will have something special to do with the alpha-number of degrees is not at all obvious, and will emerge in our proofs.

Notation 2.4. For an integer $j = \sum j_l p^l$, in p -ary representation, we shall write this representation as $j = (\dots, j_l, \dots, j_0)$.

Definition 2.5. A generator d_j of $\mathcal{K}$ is called *chosen* if the p -ary digits of j are nondecreasing from left to right; i.e., if $j = (\dots, j_l, \dots, j_0)$, then $j_{l+1} \leq j_l$, for $l \geq 0$.

Definition 2.6. A 2-fold monomial $d_i d_j \in \mathcal{K}$ is called *chosen* provided that d_i and d_j are both chosen, that each digit of i is less than or equal to the corresponding digit of j , and that if $j_l \neq p-1$, then $i_{l+1} = 0$ (in these circumstances the last condition is equivalent to $i < p^{\nu_p(j+1)+1}$, where ν_p is the exponent of p -divisibility).

Definition 2.7. An arbitrary monomial $d_I = d_{i_1} d_{i_2} \cdots d_{i_k}$ (for $k \geq 0$) is called *chosen* provided that each d_{i_l} is chosen and for each l , $1 \leq l \leq k-1$, the monomial $d_{i_l} d_{i_{l+1}}$ is chosen.

We also recall from the appendix the following definition in $\mathcal{K}$.

Definition 2.8. A monomial $d_I = d_{i_1} d_{i_2} \cdots d_{i_k}$ is called *admissible* provided that for each l , $i_l \leq i_{l+1}$.

We recall that the admissibles are a basis for $\mathcal{K}$, and note that every chosen monomial is admissible. Also notice that the definition of chosen refers only to monomials in $\mathcal{K}$, having nothing directly to do with the degree of a class of application in a module.

We also recall here that a basis for the free unstable $\mathcal{A}$ -module on a class in degree m consists of applying all admissibles in $\mathcal{A}$ of (complex) excess less than or equal to m , or, equivalently, admissibles $d_I \in \mathcal{K}$ with final index less than m .

We are almost ready to describe $H^*(CP(\infty))$ in terms of the monomials we have labeled as chosen.

Notation 2.9. Let $\mathcal{F}_s H^*(CP(\infty))$ denote the direct sum of $H^{2k}(CP(\infty))$ for all k such that $\alpha(k) \leq s$, where $\alpha(k)$ denotes the alpha-number of the integer k . One can check from the equation at the beginning of this section that this is a filtration by sub- $\mathcal{A}$ -modules.

The subquotient module $\mathcal{F}_s H^*(CP(\infty)) / \mathcal{F}_{s-1} H^*(CP(\infty))$ is concentrated in, and of rank one, in precisely those (complex) degrees with alpha-number s . As we noted above, if we write $s = q(p-1) + r$, where $0 \leq r \leq p-2$, then the smallest degree with this alpha-number has p -ary representation $(r, p-1, \dots, p-1)$, and is $(r+1)p^q - 1$, which is precisely the degree of a minimal $\mathcal{A}$ -module generator already noted above for $H^*(CP(\infty))$. So there is a correspondence between the filtered quotients and the minimal generators. It is not surprising that the quotients turn out to be cyclic modules over $\mathcal{A}$ (equivalently $\mathcal{K}$) on these generators. The important content of the following theorem is the explicit systematic identification of a single

chosen monomial in $\mathcal{K}$ representing a basis element for each degree with a given alpha-number. The theorem does this by melding the definition of chosen monomials in $\mathcal{K}$ with the unstable conditions of an $\mathcal{A}$ -module, obeyed by the cohomology of any space, that if x is a class of complex degree m , then $d_i x = 0$ for $i > m$, and $d_m x = (-1)^m x$, so that $d_i x$ represents new elements only when $i < m$. For this reason we make the following definition.

Definition 2.10. We call the application of a monomial d_I to a class x of degree m *unstable* if the right-hand factor d_l of d_I satisfies $l < m$.

Theorem 2.11 (proven in Section 3). *The filtered quotient*

$$\mathcal{F}_s H^*(CP(\infty)) / \mathcal{F}_{s-1} H^*(CP(\infty)) \simeq \bigoplus_{\alpha(k)=s} H^{2k}(CP(\infty))$$

has as a vector space basis the set of all unstable $d_I u^{(r+1)p^q-1}$ where d_I is chosen.

Remark 2.12. In particular, this theorem tells us that the chosen monomials in $\mathcal{K}$, applied unstably to any element in a degree of the form $ap^{n-1} - 1$ (for $a \leq p-1$), land in precisely one-to-one fashion in all degrees with the same alpha-number. This will be true in any module; the theorem indicates that in $H^*(CP(\infty))$ they all represent nonzero elements as well.

Now we define the abstract module we claim presents $H^*(CP(\infty))$.

Notation 2.13. Let F be the free $\mathcal{A}$ -module on abstract classes $t_{ap^{n-1}-1}$, for $1 \leq a \leq p-1$ and $n \geq 1$ (where subscripts indicate the complex degree of each class, here and in the future). Let J be the sub- $\mathcal{A}$ -module generated by the following relations:

For $0 \leq l \leq n-3$,

$$\mathcal{P}^{p^l} t_{ap^{n-1}-1} = \mathcal{P}^{ap^{n-2}} \mathcal{P}^{p^l} t_{ap^{n-2}-1},$$

and for $a \geq 2$,

$$(a-1) \mathcal{P}^{(p-a+1)p^{n-2}} t_{ap^{n-1}-1} = \binom{p-1}{a-2} \mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}} t_{ap^{n-2}-1}$$

and

$$(a-1) \mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}} t_{ap^{n-1}-1} = a \mathcal{P}^{p^{n-1}+p^{n-2}} t_{ap^{n-1}-1},$$

while for $a = 1$

$$2 \mathcal{P}^{p^{n-1}+p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} t_{ap^{n-1}-1} = \mathcal{P}^{p^{n-1}+p^{n-2}+p^{n-3}} \mathcal{P}^{p^{n-2}} t_{ap^{n-1}-1}.$$

Define $\mathcal{M}$ to be the quotient $\mathcal{A}$ -module F/J .

Next we wish to filter $\mathcal{M}$ in a fashion compatible with the filtration of $H^*(CP(\infty))$. We again use the correspondence between natural numbers s and the smallest degree with s for its alpha-number, described by writing $s = q(p-1) + r$ (for $0 \leq r \leq p-2$), yielding the smallest degree $(r+1)p^q - 1 = (r, p-1, \dots, p-1)$ with this alpha-number. Notice that the correspondence is monotonic.

Notation 2.14. Let $\mathcal{F}_s\mathcal{M}$ denote the sub- $\mathcal{A}$ -module of $\mathcal{M}$ generated by all t 's of degree less than or equal to the degree $(r+1)p^q - 1$ with alpha-number s . Let $\mathcal{M}_{n,a}$ denote the cyclic subquotient module of this filtration with generator in degree $ap^{n-1} - 1$ (obtained by letting $r = a - 1$ and $q = n - 1$ determine s , and defining $\mathcal{M}_{n,a} = \mathcal{F}_s\mathcal{M}/\mathcal{F}_{s-1}\mathcal{M}$).

As we did for $H^*(CP(\infty))$, we now wish to analyze the filtered quotients $\mathcal{M}_{n,a}$, and ultimately to see that they agree with those of $H^*(CP(\infty))$.

Remark 2.15. We note that $\mathcal{M}_{n,a}$ has a single $\mathcal{A}$ -generator $t_{ap^{n-1}-1}$, subject to the following $\mathcal{A}$ -relations:

For $0 \leq l \leq n - 3$,

$$\mathcal{P}^{p^l} t_{ap^{n-1}-1} = 0,$$

and for $a \geq 2$,

$$\mathcal{P}^{(p-a+1)p^{n-2}} t_{ap^{n-1}-1} = 0$$

and

$$(a-1)\mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}} t_{ap^{n-1}-1} = a\mathcal{P}^{p^{n-1}+p^{n-2}} t_{ap^{n-1}-1},$$

while for $a = 1$

$$2\mathcal{P}^{p^{n-1}+p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} t_{ap^{n-1}-1} = \mathcal{P}^{p^{n-1}+p^{n-2}+p^{n-3}} \mathcal{P}^{p^{n-2}} t_{ap^{n-1}-1}.$$

We define a map from F to $H^*(CP(\infty))$ by taking each $t_{ap^{n-1}-1}$ to $u^{ap^{n-1}-1}$. By Proposition 2.3, this map carries the submodule J to zero, so there is an induced map $\mathcal{M} \rightarrow H^*(CP(\infty))$. This is the map we shall show is an isomorphism.

Remark 2.16. From our earlier results and discussion it is clear that this map is well-defined, respects the filtrations on $\mathcal{M}$ and $H^*(CP(\infty))$, and is an epimorphism.

Theorem 2.2 will result from seeing that this map induces $\mathcal{A}$ -isomorphisms on the filtered quotients. This is ensured by the next theorem, whose proof occupies the bulk of the paper, determining a basis for $\mathcal{M}_{n,a}$ analogous to that of Theorem 2.11 for the filtered quotients of $H^*(CP(\infty))$.

Theorem 2.17 (proven in Section 4). *A vector space basis for $\mathcal{M}_{n,a}$ consists of the set of all unstable $d_I t_{ap^{n-1}-1}$ where d_I is chosen.*

Corollary 2.18. *From this theorem and Theorem 2.11, the map $\mathcal{M} \rightarrow H^*(CP(\infty))$ defined above induces isomorphisms*

$$\mathcal{M}_{n,a} = \mathcal{F}_s\mathcal{M}/\mathcal{F}_{s-1}\mathcal{M} \simeq \mathcal{F}_s H^*(CP(\infty))/\mathcal{F}_{s-1} H^*(CP(\infty))$$

where $s = \alpha(ap^{n-1} - 1)$. In particular, $\mathcal{M}_{n,a}$ is therefore concentrated in degrees with alpha-number s , and always has rank one there.

The presentation of $H^*(CP(\infty))$ in Theorem 2.2 is now essentially immediate. We make this explicit, and verify the minimality of the relations, in Section 5.

We established above that the abstract modules $\mathcal{M}_{n,a}$ are cyclic unstable $\mathcal{A}$ -modules that have dimension one in degrees with alpha-number equal to that

of $ap^{n-1} - 1$ and are zero in other degrees. Their importance is underscored by the fact that this characterizes them uniquely:

Theorem 2.19 (proven in Section 6). *Let $\alpha_0 \geq 1$. Let M be a cyclic (unstable) module over the Steenrod algebra $\mathcal{A}$, p odd, such that $\dim(M_l) = 1$ if $\alpha(l) = \alpha_0$ and $\dim(M_l) = 0$ if $\alpha(l) \neq \alpha_0$. Then, as $\mathcal{A}$ -modules, $M \cong \mathcal{M}_{n,a}$, where $\alpha(ap^{n-1} - 1) = \alpha_0$.*

We now move to our minimal presentation of $H^*(BU)$ as an unstable $\mathcal{A}$ -algebra. Our intention is to identify a minimal sub- $\mathcal{A}$ -module (itself minimally presented as an $\mathcal{A}$ -module) that generates $H^*(BU)$ as an $\mathcal{A}$ -algebra, form the free unstable $\mathcal{A}$ -algebra on this module, and then impose a minimal set of $\mathcal{A}$ -algebra relations to obtain $H^*(BU)$.

Remark 2.20. There is a map $S^2 \wedge CP(\infty)_+ \longrightarrow BU$ that induces an epimorphism on integral cohomology. (Here $CP(\infty)_+$ denotes the union of $CP(\infty)$ with a disjoint basepoint.) The map classifies the virtual bundle $(\eta_1 - 1) \otimes (\eta_\infty \cup 0)$, where η_1 and η_∞ denote the canonical line bundles over $S^2 = CP(1)$ and $CP(\infty)$, respectively. A computation using the Chern character verifies that this map induces the desired epimorphism on integral cohomology. For details, see, e.g., the monograph [4, p. 73]. As all products vanish in the cohomology of $S^2 \wedge CP(\infty)_+$ since it is a suspension, the induced map on indecomposables is an isomorphism. (An alternative way to see that $QH^*(BU)$ is $\mathcal{A}$ -isomorphic to $\Sigma^2 H^*(CP(\infty))$ is via the mod p Wu formulas in $H^*(BU)$; see [12, 15].)

Since the indecomposable quotient $QH^*(BU)$ is isomorphic as an $\mathcal{A}$ -module to the double (topological) suspension of $H^*(CP(\infty))$, then by Proposition 2.1 the set of Chern classes $\{c_{ap^{n-1}} \mid 1 \leq a \leq p-1 \text{ and } n \geq 1\}$ is a minimal set of $\mathcal{A}$ -algebra generators for $H^*(BU)$. Hence up to algebra decomposables we will find in $H^*(BU)$ the double suspension of the relations in the module $\mathcal{M}$ that minimally presents $H^*(CP(\infty))$. We list these analogous relations in $H^*(BU)$:

Remark 2.21. Since the relations in Theorem 2.2 were verified in Proposition 2.3 to hold in $H^*(CP(\infty))$, we have the following relations in $H^*(BU)$:

For $0 \leq l \leq n-3$,

$$\mathcal{P}^{p^l} c_{ap^{n-1}} = \mathcal{P}^{ap^{n-2}} \mathcal{P}^{p^l} c_{ap^{n-2}} + D_1(a, n, l),$$

and for $a \geq 2$,

$$(a-1) \mathcal{P}^{(p-a+1)p^{n-2}} c_{ap^{n-1}} = \binom{p-1}{a-2} \mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}} c_{ap^{n-2}} + D_2(a, n)$$

and

$$(a-1) \mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}} c_{ap^{n-1}} = a \mathcal{P}^{p^{n-1}+p^{n-2}} c_{ap^{n-1}} + D_3(a, n),$$

while for $a = 1$

$$2 \mathcal{P}^{p^{n-1}+p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} c_{ap^{n-1}} = \mathcal{P}^{p^{n-1}+p^{n-2}+p^{n-3}} \mathcal{P}^{p^{n-2}} c_{ap^{n-1}} + D_4(a, n).$$

Here $D_1(a, n, l), \dots, D_4(a, n)$ are decomposable polynomials in the elements

$$\{\mathcal{P}^J c_{ap^{k-1}} \mid 1 \leq a \leq p-1 \text{ and } k \geq 1\}$$

that may, in principle, be computed using the mod p Wu formulas [12, 15].

We shall prove that these form a minimal set of relations for $H^*(BU)$ as an unstable algebra over the Steenrod algebra.

Since there are decomposables to contend with amongst the equations connecting these Chern classes, we do not immediately impose $\mathcal{A}$ -module relations on our abstract generators imitating the generating Chern classes.

Notation 2.22. Let $\mathcal{N}$ be the free (unstable) $\mathcal{A}$ -module on abstract classes $\tau_{ap^{n-1}}$, for $1 \leq a \leq p-1$ and $n \geq 1$.

Let $U(\mathcal{N})$ be the free unstable $\mathcal{A}$ -algebra on the $\mathcal{A}$ -module $\mathcal{N}$ (the odd-primary analogue of [16, pp. 28–29]). Let $\mathcal{I}$ be the $\mathcal{A}$ -ideal in $U(\mathcal{N})$ generated by the following relations:

For $0 \leq l \leq n-3$,

$$\mathcal{P}^{p^l} \tau_{ap^{n-1}} = \mathcal{P}^{ap^{n-2}} \mathcal{P}^{p^l} \tau_{ap^{n-2}} + D_1(a, n, l),$$

and for $a \geq 2$,

$$(a-1)\mathcal{P}^{(p-a+1)p^{n-2}} \tau_{ap^{n-1}} = \binom{p-1}{a-2} \mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}} \tau_{ap^{n-2}} + D_2(a, n)$$

and

$$(a-1)\mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}} \tau_{ap^{n-1}} = a\mathcal{P}^{p^{n-1}+p^{n-2}} \tau_{ap^{n-1}} + D_3(a, n),$$

while for $a = 1$

$$2\mathcal{P}^{p^{n-1}+p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} \tau_{ap^{n-1}} = \mathcal{P}^{p^{n-1}+p^{n-2}+p^{n-3}} \mathcal{P}^{p^{n-2}} \tau_{ap^{n-1}} + D_4(a, n).$$

(Here $D_1, \dots, D_4$ are the polynomials in the preceding remark with the Chern classes $c_{ap^{n-1}}$ replaced by the elements $\tau_{ap^{n-1}}$.) Finally, let $\mathcal{G} = U(\mathcal{N})/\mathcal{I}$, a quotient $\mathcal{A}$ -algebra of $U(\mathcal{N})$.

We define a map $U(\mathcal{N}) \rightarrow H^*(BU)$ by taking $\tau_{ap^{n-1}}$ to the Chern class $c_{ap^{n-1}}$. Since the ideal $\mathcal{I}$ is taken to zero by this map, we obtain a map $\phi: \mathcal{G} \rightarrow H^*(BU)$. We shall check that the induced map $Q\phi$ on indecomposable quotients is an isomorphism. Since $H^*(BU)$ is a polynomial algebra, we obtain our presentation for $H^*(BU)$.

Theorem 2.23 (proven in Section 6). *The map $\phi: \mathcal{G} \rightarrow H^*(BU)$ is an isomorphism of unstable $\mathcal{A}$ -algebras.*

Theorem 2.24. *Our presentation for $H^*(BU)$ is minimal, in the sense that the module $\mathcal{N}$ injects into $H^*(BU)$ and our set of relations is minimal.*

It is clear from previous results that our set of relations imposed on $\mathcal{G}$ is minimal, so Theorem 2.24 is an immediate consequence of the following theorem, which shows

that in $H^*(BU)$ there are no $\mathcal{A}$ -module relations among any Chern classes (even though for injectivity of $\mathcal{N}$ we only need this for those in degrees ap^{n-1}).

Theorem 2.25 (proven in Section 6). *Let $\mathcal{R}$ denote the free $\mathcal{A}$ -module on abstract classes t_m , for $m \geq 1$. Then the map $\mathcal{R} \rightarrow H^*(BU)$ defined by taking each τ_m to the corresponding Chern class c_m is a monomorphism.*

3. Proof of Theorem 2.11

Fix the nonnegative integer s . Let k_0 be the smallest nonnegative integer for which $\alpha(k_0) = s$. We shall define a bijection between the set of chosen monomials $d_i \cdots d_l$ with $l < k_0$ and a basis for $\bigoplus_{\alpha(k)=s} H^{2k}(CP(\infty))$, by assigning to $d_i \cdots d_l$ the element $d_i \cdots d_l x^{k_0}$. Of course we will need to show that this assignment is valid, nonzero, and creates a one-to-one correspondence between the chosen monomials and the degrees k with $\alpha(k) = s$.

We begin with a short calculation showing that the assignment always lands in the correct degrees. Let $d_i d_j \cdots d_l$ be a chosen monomial with $l < k_0$. There exist a (with $1 \leq a \leq p-1$) and n such that

$$k_0 = ap^{n-1} - 1 = (a-1, p-1, \dots, p-1).$$

Write $i = (i_{q-1}, \dots, i_0)$, where $i_{q-1} \neq 0$ and $q \leq n$. Since $d_i d_j \cdots d_l$ is a chosen monomial, then $j, \dots, l$ are of the form

$$j = (j_t, \dots, j_{q-1}, p-1, \dots, p-1).$$

If k is the degree of $d_j \cdots d_l x^{k_0}$, then by iteration we have k of the form (see the appendix to recall calculation of degrees involving $\mathcal{K}$)

$$k = (k_s, \dots, k_q, j_{q-1}, p-1, \dots, p-1).$$

If we set $m = pk - (p-1)i$, then m is the degree of $d_i \cdots d_l x^{k_0}$ and we calculate

$$\begin{aligned} m &= (m_{s+1}, \dots, m_0) \\ &= k_s p^{s+1} + \cdots + (k_{q-1} - i_{q-1})p^q + [(p-1) - i_{q-2} + i_{q-1}]p^{q-1} \\ &\quad + \cdots + [(p-1) - i_0 + i_1]p + i_0. \end{aligned}$$

An induction based on this formula yields the following lemma.

Lemma 3.1. *If $d_i d_j \cdots d_l$ is a chosen monomial with $l < k_0$ and m is the degree of $d_i \cdots d_l x^{k_0}$, then $\alpha(m) = s$.*

Next we will reverse this calculation, beginning solely with a degree m with $\alpha(m) = s$, and finding a chosen monomial $d_i d_j \cdots d_l$ that produces a basis element in $H^{2m}(CP(\infty))$ when applied to x^{k_0} . The formulas above will be our guide. We

note that in the formula for m we had

$$\sum_{r=0}^{q-1} m_r = (q-1)(p-1) + i_{q-1} > (q-1)(p-1),$$

and

$$\sum_{r=0}^q m_r = (q-1)(p-1) + k_{q-1} \leq q(p-1).$$

This motivates us to define q from m as follows:

Definition 3.2. Let $q \geq 0$ be the least integer such that $\sum_{r=0}^q m_r \leq q(p-1)$.

With q in hand we can continue to solve from the equations above for k and i .

Definition 3.3. Let q be as just defined, and set

$$k_r = \begin{cases} p-1, & \text{for } 0 \leq r \leq q-2; \\ \sum_{l=0}^q m_l - (q-1)(p-1), & \text{for } r = q-1; \\ m_{r+1}, & \text{for } r \geq q. \end{cases}$$

Note that from the definition of q , for $r \geq 0$ we have $0 \leq k_r \leq p-1$, and that for $0 \leq r \leq q-1$ we have

$$0 < m_0 + \cdots + m_r - r(p-1) \leq p-1.$$

Set

$$i = \sum_{r=0}^{q-1} [m_0 + \cdots + m_r - r(p-1)]p^r.$$

Note that if $m = ap^{n-1} - 1$, then $i = k = m$, and that otherwise $i < k$.

Next we check that for the i we have defined, x^m is actually in the image of d_i . In $H^*(CP(\infty))$, we have, for any i and k , the formula

$$d_i x^k = (-1)^i \binom{k}{i} x^{pk-(p-1)i}.$$

Using this, we compute that for our defined m , k , and i ,

$$m = pk - (p-1)i \text{ and } d_i x^k = (\text{unit}) \cdot x^m.$$

Now we continue this prescription backwards to produce a chosen monomial connecting x^{k_0} to x^m . We note that the digits of i are nondecreasing, so it is chosen. Further, we note that if we start with k as just defined in place of m and iterate the process to find a j by the same recipe we used to define i , we obtain

$$j_r = p-1, \text{ for } r \leq q-2$$

and

$$j_{q-1} = i_{q-1} + m_q \geq i_{q-1}.$$

So, inductively, we can start with an integer m such that $\alpha(m) = s$, and produce a

chosen admissible $d_i d_j \cdots d_l$, with $l < k_0$, such that

$$x^m = (\text{unit}) d_i d_j \cdots d_l x^{k_0},$$

where k_0 is the least positive integer such that $\alpha(k_0) = s$. This shows that assigning to a monomial $d_i d_j \cdots d_l$ with $l < k_0$ the element $d_i d_j \cdots d_l x^{k_0}$ is surjective from the set of such monomials to a basis for $\bigoplus_{\alpha(k)=s} H^{2k}(CP(\infty))$.

It remains only to see that there is a unique such monomial for each degree m with $\alpha(m) = s$. But it is clear from our displayed formulas that began this section that if we start with a chosen monomial $d_i d_j \cdots d_l$ with $l < k_0$, and consider only the degree m of the element $d_i d_j \cdots d_l x^{k_0}$, then apply our backwards algorithm above to find a new value for i , that the algorithm produces the same value for i that we began with in the chosen monomial.

Summing up, we have shown that the set of elements $d_i d_j \cdots d_l x^{k_0}$, ranging over the chosen monomials $d_i d_j \cdots d_l$ with $l < k_0$, forms a basis for

$$\bigoplus_{\alpha(k)=s} H^{2k}(CP(\infty)).$$

This proves Theorem 2.11.

4. Proof of Theorem 2.17

From Theorem 2.11 and the remark following it, the map of filtered quotients from $\mathcal{M}_{n,a}$ to the corresponding filtered quotient of $H^*(CP(\infty))$ provides a nonzero representation of all the chosen unstable monomials on $x^{ap^{n-1}-1}$ in $H^*(CP(\infty))$, with exactly one chosen in each degree with the same alpha-number as $ap^{n-1} - 1$, and none elsewhere. Thus to prove that the chosen monomials applied unstably to the generator of $\mathcal{M}_{n,a}$ provide a basis for the module, we need only show that they span $\mathcal{M}_{n,a}$. Our strategy will be to show that each admissible monomial that applies unstably to the generator u_m of $\mathcal{M}_{n,a}$ can be expressed in $\mathcal{M}_{n,a}$ as a multiple of a chosen unstable monomial on u_m . Here $m = ap^{n-1} - 1$ is the degree of the generator, with $1 \leq a \leq p - 1$.

We will frequently and often without mention use the Adem relations in the Steenrod algebra:

$$\mathcal{P}^a \mathcal{P}^b = \sum_t (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} \mathcal{P}^{a+b-t} \mathcal{P}^t,$$

with $\mathcal{P}^t$ in complex degree $(p-1)t$. We will also often use without mention the fact that the initial relations in $\mathcal{M}_{n,a}$ (in Remark 2.15) clearly make any terms $\mathcal{P}^t u_m = 0$ for $0 < t < p^{n-2}$. The other relation in $\mathcal{M}_{n,a}$ involving a single Steenrod operation will be used in the proof for length one monomials, and the relations involving 2-fold monomials will be used in Lemmas 4.19 and 4.21.

4.1. Length one monomials

We shall prove Theorem 2.17 in stages. We note that the set of monomials $d_J u_m$ in $\mathcal{M}_{n,a}$ is filtered by the length of $J = (j_1, \dots, j_r)$, $d_J u_m = d_{j_1} \cdots d_{j_r} u_m$. In this

subsection we shall deal with monomials of length one. Our first stage is the following lemma.

Lemma 4.1. *Fix n and a . In the module $\mathcal{M}_{n,a}$, with generator u_m in complex degree $ap^{n-1} - 1$, if d_j is unchosen, then $d_j u_m = 0$.*

Proof. By definition, d_j is chosen if the digits of j are nondecreasing. That is, if we write

$$j = (j_{n-1}, \dots, j_0),$$

then this says that $j_{n-1} \leq \dots \leq j_0$. Now write $\mathcal{P}^t u_m = (-1)^j d_j u_m$. We shall call $\mathcal{P}^t u_m$ *chosen* if d_j is chosen. Now

$$\deg(u_m) = (a-1, p-1, \dots, p-1).$$

So for $\mathcal{P}^t u_m$ to be chosen, we need, writing

$$\begin{aligned} t &= (t_{n-1}, \dots, t_0) \\ &= (a-1-j_{n-1}, p-1-j_{n-2}, \dots, p-1-j_0), \end{aligned}$$

that

$$t_{n-1} + p - a \geq t_{n-2} \geq t_{n-3} \geq \dots \geq t_0.$$

Thus if $\mathcal{P}^t u_m$ is unchosen, we must have $t_{n-1} < t_{n-2} - (p-a)$ or $t_r < t_{r-1}$ for some $1 \leq r \leq n-2$.

The requirement that $\mathcal{P}^{p^i} u_m = 0$ for $0 \leq i \leq n-3$ yields immediately that $\mathcal{P}^t u_m = 0$ for all $0 < t < p^{n-2}$, i.e., when $t_{n-1} = t_{n-2} = 0$. Further, if $t_{n-1} = 0$, then the requirement that $\mathcal{P}^{(p-a+1)p^{n-2}} u_m = 0$ tells us that $\mathcal{P}^t u_m = 0$ for all t such that $(p-a+1)p^{n-2} \leq t < p^{n-1}$.

Henceforth assume that either t_{n-1} or t_{n-2} is nonzero. We shall induct on the degree of $\mathcal{P}^t u_m$. Fix a value of t and suppose that all unchosen $\mathcal{P}^s u_m = 0$ for $s < t$.

Case 1. Suppose $\mathcal{P}^t u_m$ is unchosen and that there is an $1 \leq r \leq n-2$ for which $t_r < t_{r-1}$. (Note that this must happen if $a = 1$.) Then one can check that $\mathcal{P}^{t-p^r} u_m$ is unchosen and in lower topological degree than $\mathcal{P}^t u_m$, so we may assume inductively that $\mathcal{P}^{t-p^r} u_m = 0$. Suppose there exists a least integer $0 \leq s \leq r-2$ such that $t_s \neq 0$. Then $\mathcal{P}^{t-p^s} u_m$ is unchosen and

$$0 = \mathcal{P}^{p^s} \mathcal{P}^{t-p^s} u_m = (\text{unit}) \mathcal{P}^t u_m.$$

So, without loss of generality, we may assume that $t_{r-2} = \dots = t_0 = 0$. We have

$$0 = \mathcal{P}^{p^r} \mathcal{P}^{t-p^r} u_m = -(t_{r-1} - t_r) \mathcal{P}^t u_m.$$

If $a = 1$, then this is the only possible case; this completes the proof when $a = 1$.

Case 2. Suppose that $t_{n-1} < t_{n-2} - (p-a)$. (Then $a > 1$.) Without loss of generality, as above, we may take $t_{n-3} = \dots = t_0 = 0$, so that $t = t_{n-1}p^{n-1} + t_{n-2}p^{n-2}$. Noting that in this case $\mathcal{P}^{p^{n-2}} \mathcal{P}^{t-p^{n-2}} u_m = (t_{n-2}) \mathcal{P}^t u_m$, we may assume that $t_{n-2} = t_{n-1} + (p-a) + 1$. When $t_{n-1} = 0$, we have that $\mathcal{P}^t u_m = 0$ is one of the defining relations for our module.

There remains only to consider $t_{n-1} > 0$. Using Adem relations, we get the following formulas, using the inductive hypothesis on degree to eliminate needing to write down many terms:

$$\mathcal{P}^{p^{n-1}} \mathcal{P}^{t-p^{n-1}} u_m = -(p-a+1) \mathcal{P}^t u_m + \mathcal{P}^{t-p^{n-2}} \mathcal{P}^{p^{n-2}} u_m,$$

$$\begin{aligned} \mathcal{P}^{p^{n-1}+p^{n-2}} \mathcal{P}^{t-p^{n-1}-p^{n-2}} u_m &= (p-a)(p-t_{n-2}) \mathcal{P}^t u_m \\ &\quad - (p-t_{n-2}+1) \mathcal{P}^{t-p^{n-2}} \mathcal{P}^{p^{n-2}} u_m \\ &\quad + \mathcal{P}^{t-p^{n-2}-p^{n-3}} \mathcal{P}^{p^{n-2}+p^{n-3}} u_m, \end{aligned}$$

$$\begin{aligned} \mathcal{P}^{p^{n-1}+\dots+p^{n-s}} \mathcal{P}^{t-p^{n-1}-\dots-p^{n-s}} u_m &= -\mathcal{P}^{t-p^{n-2}-\dots-p^{n-s}} \mathcal{P}^{p^{n-2}+\dots+p^{n-s}} u_m \\ &\quad + \mathcal{P}^{t-p^{n-2}-\dots-p^{n-s-1}} \mathcal{P}^{p^{n-2}+\dots+p^{n-s-1}} u_m \\ &\quad \text{for } 3 \leq s \leq n-1, \end{aligned}$$

and

$$\mathcal{P}^{p^{n-1}+\dots+p+1} \mathcal{P}^{t-p^{n-1}-\dots-p-1} u_m = -\mathcal{P}^{t-p^{n-2}-\dots-p-1} \mathcal{P}^{p^{n-2}+\dots+p+1} u_m.$$

Since $\mathcal{P}^{t-p^{n-1}} u_m$ through $\mathcal{P}^{t-p^{n-1}-\dots-p-1} u_m$ are unchosen and of lower topological degree than $\mathcal{P}^t u_m$, then inductively they are zero. We have a matrix equation $MX = 0$, where

$$M = \begin{bmatrix} -(p-a+1) & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ (p-a)(p-t_{n-2}) & -(p-t_{n-2}+1) & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 \end{bmatrix},$$

and

$$X = \begin{bmatrix} \mathcal{P}^t u_m \\ \mathcal{P}^{t-p^{n-2}} \mathcal{P}^{p^{n-2}} u_m \\ \vdots \\ \mathcal{P}^{t-p^{n-2}-\dots-p^{n-s}} \mathcal{P}^{p^{n-2}+\dots+p^{n-s}} u_m \\ \vdots \\ \mathcal{P}^{t-p^{n-2}-\dots-p-1} \mathcal{P}^{p^{n-2}+\dots+p+1} u_m \end{bmatrix}.$$

This matrix is nonsingular since its determinant mod p is $(-a+1-t_{n-2})(-1)^n$, which is nonzero since $t_{n-1} > 0$. Hence all entries of X are zero. In particular $\mathcal{P}^t u_m = 0$. $\square$

4.2. Length two monomials

The proof for length one monomials proceeded by induction on topological degree. The remainder of the proof that the chosen monomials span $\mathcal{M}_{n,a}$ will continue this way. Within each topological degree we also order the admissible unstable monomials on u_m as follows:

Definition 4.2. If d_I and d_J are admissibles that are unstable and in the same topological degree when applied to u_m , then we define $d_I u_m$ to be lower in order than $d_J u_m$ provided that d_I has shorter length than d_J . Further, if they have the same length, then we define $d_I u_m$ to be lower in order than $d_J u_m$ provided that I is lower than J in lexicographical ordering, starting from the left.

Remark 4.3. Note that our ordering is only defined for admissible monomials that are unstable when applied to u_m (i.e., their final subscript is less than m). If an admissible is not unstable when applied to u_m , then it will collapse to zero or to an admissible unstable monomial of shorter length when applied to u_m .

Remark 4.4. Although our ordering is only defined for admissible monomials, note that a $\mathcal{K}$ -Adem relation (see appendix) applied to an inadmissible always produces admissible terms of lower lexicographic order than the inadmissible. We may use this without mention in calculations.

Remark 4.5. In a fixed topological and length degree, the admissible unstable monomials applied to u_m are finite in number.

Inductive Assumption. Assume inductively that in topological degrees less than a given one, every unchosen admissible unstable monomial $d_I u_m$ of $\mathcal{M}_{n,a}$ is a sum of admissible monomials of lower order than itself. (Hence, in those lower degrees, $\mathcal{M}_{n,a}$ has the chosen unstable monomials on u_m as a basis and is isomorphic to the corresponding filtered quotient of $H^*(CP(\infty))$.)

We proceed to the proof of the theorem for length-two unstable admissibles on u_m in the given degree. We shall fix an unstable admissible $d_i d_j u_m$ and inductively assume also, within its topological degree, that every unchosen unstable admissible $d_k d_l u_m$ of lower order than $d_i d_j u_m$ is a sum of admissibles of lesser order than itself. Our goal will be to show that if $d_i d_j$ is unchosen, then $d_i d_j u_m$ is a sum of 2-fold (or 1-fold) admissibles of lower order than itself.

Definition 4.6. If d_j is chosen, let $\bar{i}(j)$ denote the largest i for which $d_i d_j$ is chosen.

Remark 4.7. Suppose that r is an integer for which $j_l = p - 1$ for $l < r$ and that $j_r \neq p - 1$. Then

$$\bar{i}(j) = (0, \dots, 0, j_r, p - 1, \dots, p - 1).$$

Remark 4.8. A very useful formula in the Kudo-Araki-May algebra is the following, for $i \leq j$:

$$d_i d_j = d_{pj-(p-1)i} d_i + \sum_{t \geq 1} (-1)^{t-1} \binom{(p-1)(j-i-t)-1}{pt} d_{i-pt} d_{j+t}.$$

For $i < j$ this is simply a rewriting of the Adem relation for the inadmissible $d_{pj-(p-1)i} d_i$ (see appendix), but it switches the element d_i between appearance on

the right and left in a 2-fold monomial, and extracts an expression for the admissible of highest order in the relation.

This allows us to prove immediately that a large class of unchosen admissibles are sums of chosens.

Lemma 4.9. *If the admissible $d_i d_j$ is unchosen, and either $i \leq \bar{i}(j)$, or d_i or d_j is unchosen, then $d_i d_j u_m$ can be expressed as a sum of lower order terms.*

Proof. Checking the requirements of the definition for chosenness, we see that if $i \leq \bar{i}(j)$, then the only way $d_i d_j$ can be unchosen is for either d_i or d_j to be unchosen; thus by the preceding remark and the result above for 1-folds, the lemma follows. $\square$

So we only need to consider cases in which

- the monomial $d_i d_j$ is admissible, $j < m$, and
- both d_i and d_j are chosen, and
- the index $i > \bar{i}(j)$.

Our general strategy now is to find $d_i d_j u_m$ in the image of a Steenrod operation from a lesser topological degree, and to apply the inductive assumptions to see that it is a sum of terms of lower order than itself. The numerous cases that will need individual consideration stem from the fact that the Steenrod operation required depends on the p -ary representations of i and j . In particular, for the rest of the proof we will let r denote the greatest integer for which $i_l = j_l = p - 1$ for $0 \leq l < r$. We note that if $r = n - 1$ ($r = n - 2$ if $a = 1$), then $d_i d_j u_m$ is automatically chosen, so we may also assume henceforth that

- the index r satisfies $0 \leq r \leq n - 2$.

The lemmas in the rest of this section have as their common goal to show that if $d_i d_j$ is unchosen and satisfies the four bulleted restrictions above, then $d_i d_j u_m$ can be expressed as a sum of lower order terms. We thus combine the bullets as a set of common hypotheses for all lemmas that follow, and together the succeeding lemmas will cover all possibilities for the p -ary representations of i and j subject to these hypotheses.

Common hypotheses for all succeeding lemmas: We assume that $d_i d_j$ is admissible and unchosen, $j < m$, both d_i and d_j are chosen, and $i > \bar{i}(j)$. Also, with r denoting the greatest integer for which $i_l = j_l = p - 1$ for $0 \leq l < r$, we assume that $0 \leq r \leq n - 2$.

Before beginning to cover particular cases, we pause for two preparatory lemmas for showing that many types of terms in $\mathcal{M}_{n,a}$ are actually zero by climbing up inductively from lower degrees in which we already know that $\mathcal{M}_{n,a}$ is zero. These two lemmas do not depend on the common hypotheses.

Lemma 4.10. *Suppose that $\alpha(|d_i d_j u_m|) = \alpha(m)$. Then for $0 \leq l \leq r - 1$,*

$$\alpha(|d_i d_j u_m| - (p - 1)p^l) < \alpha(m).$$

Hence, by the inductive assumption, all terms in degree $|d_i d_j u_m| - (p - 1)p^l$ in $\mathcal{M}_{n,a}$ are zero.

Proof. We have

$$\begin{aligned} |d_i d_j u_m| &= p^2(m-j) + p(j-i) + i \\ &= (*, \dots, *, i_r, p-1, \dots, p-1). \end{aligned}$$

The result follows immediately. $\square$

Lemma 4.11. *Let $0 < s < p^k$. For any $q \geq 0$, $\mathcal{P}^{qp^k+s}$ is in the right ideal of $\mathcal{A}$ generated by $\{\mathcal{P}^1, \mathcal{P}^p, \dots, \mathcal{P}^{p^{k-1}}\}$.*

Proof. This is presumably well-known. It follows immediately from the Adem relations. $\square$

Notice how these two lemmas can work together to show that a term is zero. If $\alpha(|d_i d_j u_m|) = \alpha(m)$, and if we can see that $d_i d_j u_m$ is in the image of $\mathcal{P}^{qp^k+s}$, where $0 < s < p^k$ and $k \leq r$ (i.e., it is in the image of $\mathcal{P}^l$ where $l \not\equiv 0 \pmod{p^r}$), then it is in the image of zero.

Now we begin covering particular cases of the form of the p -ary representations of i and j . Throughout we will continue frequently to use Adem relations on inadmissibles without explicit mention, as well as the basic relations in $\mathcal{M}_{n,a}$ that $\mathcal{P}^t u_m = 0$ for $0 < t < p^{n-2}$, the fact that 1-folds are of lower order than 2-folds, and our result already proven for 1-folds.

Lemma 4.12. *Along with the common hypotheses above, suppose that $j_r \neq i_r$. Then $d_i d_j u_m$ can be expressed as a sum of lower order terms.*

Proof.

Part 1. We have

$$\begin{aligned} i &= (i_{n-1}, \dots, i_{r+1}, i_r, p-1, \dots, p-1) \\ &= Ip^{r+1} + (i_r + 1)p^r - 1, \end{aligned}$$

$$\begin{aligned} j &= (j_{n-1}, \dots, j_{r+1}, j_r, p-1, \dots, p-1) \\ &= Jp^{r+1} + (j_r + 1)p^r - 1, \end{aligned}$$

and

$$\begin{aligned} m &= (a-1, p-1, \dots, p-1) \\ &= Ap^{r+1} - 1. \end{aligned}$$

Then

$$d_j u_m = (-1)^j \mathcal{P}^\delta u_m,$$

where $\delta = (A - J)p^{r+1} - (j_r + 1)p^r$.

Further,

$$\begin{aligned} |d_j u_m| &= (p-1)\delta + m \\ &= Np^{r+1} + (j_r + 1)p^r - 1. \end{aligned}$$

Subtracting i , we get

$$\gamma = Mp^{r+1} + (j_r - i_r)p^r,$$

so that

$$d_i d_j u_m = (-1)^{i+j} \mathcal{P}^\gamma \mathcal{P}^\delta u_m.$$

We compute:

$$\begin{aligned} \mathcal{P}^{p^r} d_{i+p^r} (d_j u_m) &= (-1)^{i+j+1} \mathcal{P}^{p^r} \mathcal{P}^{\gamma-p^r} (\mathcal{P}^\delta u_m) \\ &= (-1)^{i+j+1} \sum_{t=0}^{p^r-1} (-1)^{t+1} \\ &\quad \cdot \binom{(p-1)[Mp^{r+1} + (j_r - i_r - 1)p^r - t] - 1}{p^r - pt} \mathcal{P}^{\gamma-t} \mathcal{P}^t (\mathcal{P}^\delta u_m) \\ &= -(j_r - i_r) d_i d_j u_m + \sum_{t=1}^{p^r-1} \epsilon_t d_{i+pt} d_{j-t} u_m, \text{ for some } \epsilon_t \in \mathbb{F}_p. \end{aligned}$$

For $1 \leq t \leq p^{r-1} - 1$, $d_{j-t} u_m$ is unchosen, and hence zero. The remaining two parts of the proof will analyze the orders of $d_{i+p^r} d_{j-p^{r-1}} u_m$ and $\mathcal{P}^{p^r} d_{i+p^r} d_j u_m$.

Part 2. Note that

$$d_{i+p^r} d_{j-p^{r-1}} u_m = (-1)^{i+j} \mathcal{P}^{\gamma-p^{r-1}} \mathcal{P}^{\delta+p^{r-1}} u_m.$$

We aim to hit this with $\mathcal{P}^{p^{r-1}}$ from

$$\mathcal{P}^{\gamma-2p^{r-1}} \mathcal{P}^{\delta+p^{r-1}} u_m = (-1)^{i+j+1} d_{i+p^r+p^{r-1}} d_{j-p^{r-1}} u_m.$$

Now

$$i + p^r + p^{r-1} = (i_{n-1}, \dots, i_{r+1}, i_r + 2, 0, p-1, \dots, p-1)$$

(with possible abuse of notation since $i_r + 2$ may not be a valid digit), and

$$j - p^{r-1} = (j_{n-1}, \dots, j_{r+1}, j_r, p-2, p-1, \dots, p-1).$$

Write $K = (i_{n-1}, \dots, i_{r+1}, i_r + 2)$. The only possible chosen 2-fold in the topological degree of $d_{i+p^r+p^{r-1}} d_{j-p^{r-1}} u_m$ is $d_{i+p^r+p^{r-1}-Kp^r} d_{j-p^{r-1}+Kp^{r-1}} u_m$, i.e., if $d_{j-p^{r-1}+Kp^{r-1}} u_m$ is chosen. Since $K \geq 3$, we can compute that the image of this term under $\mathcal{P}^{p^{r-1}}$ is of lower order than $d_i d_j u_m$. Thus the image under $\mathcal{P}^{p^{r-1}}$ of $d_{i+p^r+p^{r-1}} d_{j-p^{r-1}} u_m$ is of lower order than $d_i d_j u_m$.

We compute

$$\begin{aligned} \mathcal{P}^{p^{r-1}} \mathcal{P}^{\gamma-2p^{r-1}} \mathcal{P}^{\delta+p^{r-1}} u_m &= (-1)^{i+j+1} d_{i+p^r} d_{j-p^{r-1}} u_m \\ &\quad + (-1)^{i+j} d_{i+p^r+p^{r-1}} d_{j-p^{r-1}-p^{r-2}} u_m \\ &\quad + \text{terms that are zero since } d_{j-p^{r-1}-t} u_m \text{ is} \\ &\quad \text{unchosen for values of } t \text{ between } 0 \text{ and } p^{r-2}. \end{aligned}$$

Iterating this analysis creates a downward induction which allows us to conclude that $d_{i+p^r} d_{j-p^{r-1}} u_m$ is a sum of terms of lower order than $d_i d_j u_m$.

Part 3. Finally we consider $d_{i+p^r} d_j u_m$. It is unchosen, so it is the sum of lower order terms. What can they be? Well,

$$i + p^r = (i_{n-1}, \dots, i_{r+1}, i_r + 1, p - 1, \dots, p - 1)$$

(as above, $i_r + 1$ might not be a valid digit), and

$$j = (j_{n-1}, \dots, j_{r+1}, j_r, p - 1, \dots, p - 1).$$

For $0 < t \leq p^{r-1}$, $d_{i+p^r-pt} d_{j+t}$ is unchosen (since d_{j+t} is unchosen); thus $d_{i+p^r} d_j u_m$ is a sum of terms of order lower than $d_i d_{j+p^{r-1}} u_m$. Hence by calculation similar to Part 1, $\mathcal{P}^{p^r} d_{i+p^r} d_j u_m$ is the sum of terms of order lower than $d_i d_j u_m$. $\square$

Lemma 4.13. *Along with the common hypotheses, suppose that $i_r = j_r$, and either $j_r \neq p - 2$, or $i > \bar{i}(j + p^r)$. Then $d_i d_j u_m$ can be expressed as a sum of terms of lower order than itself.*

Proof. As above, write

$$\begin{aligned} i &= Ip^{r+1} + (i_r + 1)p^r - 1, \\ j &= Jp^{r+1} + (j_r + 1)p^r - 1, \end{aligned}$$

and

$$m = Ap^{r+1} - 1.$$

As above, we may write

$$d_i d_j u_m = (-1)^{i+j} \mathcal{P}^\gamma \mathcal{P}^\delta u_m,$$

where

$$\delta = (A - J)p^{r+1} - (j_r + 1)p^r$$

and

$$\gamma = Mp^{r+1} + (j_r - i_r)p^r.$$

We compute

$$\begin{aligned} \mathcal{P}^{p^{r+1}} d_i d_{j+p^r} u_m &= (-1)^{i+j+1} \mathcal{P}^{p^{r+1}} \mathcal{P}^{\gamma+p^r-p^{r+1}} \mathcal{P}^{\delta-p^r} u_m \\ &= \text{lower order terms} + (-1)^{j+1} d_i \mathcal{P}^{p^r} \mathcal{P}^{(A-J)p^{r+1}-(j_r+2)p^r} u_m \\ &= \text{lower order terms} + (j_r + 1) d_i d_j u_m. \end{aligned}$$

So, since $i_r = j_r$, by the definition of r we have $j_r \neq p - 1$. Hence $d_i d_j u_m$ is equal to a unit multiple of $\mathcal{P}^{p^{r+1}} d_i d_{j+p^r} u_m$ plus lower order terms.

Now consider $\mathcal{P}^{p^{r+1}} d_i d_{j+p^r} u_m$. We have

$$j + p^r = (j_{n-1}, \dots, j_{r+1}, j_r + 1, p - 1, \dots, p - 1).$$

If $i > \bar{i}(j + p^r)$, then inductively $d_i d_{j+p^r} u_m$ is a sum of terms of lower order than itself and hence, using basic module relations and Adem relations, $\mathcal{P}^{p^{r+1}} d_i d_{j+p^r} u_m$ is a sum of terms of lower order than $d_i d_j u_m$.

So the only values of i to consider are $\bar{i}(j) < i \leq \bar{i}(j + p^r)$ and $j_r \neq p - 2$. Here, we have $j_r \neq i_r$, so the preceding lemma applies. $\square$

Remark 4.14. At this point, the cases still to check are those in which $i_r = j_r = p - 2$ and $i \leq \bar{i}(j + p^r)$.

Lemma 4.15. *Along with the common hypotheses, suppose that $r \leq n - 3$. Suppose also that there is an integer s such that $r < s \leq n - 2$ and $j_l = p - 2$ for $s > l \geq r$, and that $j_s \neq p - 2$. Then $d_i d_j u_m$ can be expressed as a sum of lower order terms.*

Proof. We may write

$$j = Jp^{s+1} + (j_s + 1)p^s - p^{s-1} - \dots - p^r - 1,$$

and

$$m = Ap^{s+1} - 1.$$

Then

$$d_j u_m = (-1)^j \mathcal{P}^\delta u_m,$$

where

$$\delta = (A - J)p^{s+1} - (j_s + 1)p^s + p^{s-1} + \dots + p^r.$$

We compute

$$\mathcal{P}^{p^{s+1}} d_i d_{j+p^s} u_m = \text{lower order terms} + d_i \mathcal{P}^{p^s} d_{j+p^s} u_m.$$

Further calculation with the Adem relations gives

$$d_i \mathcal{P}^{p^s} d_{j+p^s} u_m = (j_s + 2) d_i d_j u_m;$$

whence

$$\mathcal{P}^{p^{s+1}} d_i d_{j+p^s} u_m = (j_s + 2) d_i d_j u_m + \text{lower order terms}.$$

What about $d_i d_{j+p^s}$? Well,

$$j + p^s = (j_{n-1}, \dots, j_s + 1, p - 2, \dots, p - 2, p - 1, \dots, p - 1),$$

so $\bar{i}(j + p^s) = \bar{i}(j)$. Hence $d_i d_{j+p^s} u_m$ is unchosen. It is in lower topological degree, so it is a sum of terms of lower order than itself. Hence $\mathcal{P}^{p^{s+1}} d_i d_{j+p^s} u_m$ is a sum of terms of lower order than $d_i d_j u_m$. $\square$

Lemma 4.16. *Along with the common hypotheses, suppose that $r \leq n - 3$. Also suppose that $\bar{i}(j + p^r) \geq i > \bar{i}(j)$. Suppose that $i_r = p - 2$, that $j_r = \dots = j_{n-2} =$*

$p - 2$, and that $j_{n-1} \neq a - 1$. Then $d_i d_j u_m$ can be expressed as a sum of lower order terms.

Proof. We have, using the basic module relations,

$$\mathcal{P}^{p^n} d_i d_{j+p^{n-1}} u_m = \{\text{terms of lower order than } d_i d_j\} + d_i \mathcal{P}^{p^{n-1}} d_{j+p^{n-1}} u_m$$

and

$$j = (j_{n-1}, p-2, \dots, p-2, p-1, \dots, p-1).$$

By our inductive assumption, $\mathcal{M}_{n,a}$ is isomorphic to the corresponding filtered quotient of $H^*(\mathbb{C}P(\infty))$ in topological degrees below that of $d_i d_j u_m$. Hence we may calculate $\mathcal{P}^{p^{n-1}} d_{j+p^{n-1}} u_m$ in $H^*(\mathbb{C}P(\infty))$, where we label the generator x . We have

$$\mathcal{P}^{p^{n-1}} d_{j+p^{n-1}} x^m = (-1)^{j+1} \binom{pm - (p-1)j - p^n + p^{n-1}}{p^{n-1}} \binom{m}{j+p^{n-1}} x^M$$

and

$$d_j x^m = (-1)^j \binom{m}{j} x^M,$$

where $M = pm - (p-1)j$. We calculate that $\binom{pm - (p-1)j - p^n + p^{n-1}}{p^{n-1}} = j_{n-1} + 2$, that $\binom{m}{j+p^{n-1}} = (-1)^{n-r-1} \binom{a-1}{j_{n-1}+1}$, and that $\binom{m}{j} = (-1)^{n-r-1} \binom{a-1}{j_{n-1}}$. So

$$\mathcal{P}^{p^n} d_i d_{j+p^{n-1}} u_m = (\text{a unit}) \cdot d_i d_j u_m + \text{lower order terms.}$$

Since $d_i d_{j+p^{n-1}}$ is unchosen, we have by the inductive assumption that $d_i d_{j+p^{n-1}} u_m$ is a scalar multiple of the chosen in its degree, namely $d_{i-p^{r+1}} d_{j+p^{n-1}+p^r} u_m$. Hence $\mathcal{P}^{p^n} d_i d_{j+p^{n-1}} u_m$ is a multiple of $\mathcal{P}^{p^n} d_{i-p^{r+1}} d_{j+p^{n-1}+p^r} u_m$ which we may check, using the inductive assumption, is the sum of terms of lower order than $d_i d_j u_m$. $\square$

Lemma 4.17. *Along with the common hypotheses, suppose that $r \leq n-3$. Also suppose that $\bar{i}(j+p^r) \geq i > \bar{i}(j)$, and $i_r = p-2$, and that $j_r = \dots = j_{n-2} = p-2$, that $j_{n-1} = a-1$, and that $i_{r+1} \neq p=2$. Then $d_i d_j u_m$ can be expressed as a sum of lower order terms.*

Proof. We have

$$\begin{aligned} i &= (0, \dots, 0, i_{r+1}, p-2, p-1, \dots, p-1) \\ &= (i_{r+1} + 1)p^{r+1} - p^r - 1 \end{aligned}$$

and

$$j = (a-1, p-2, \dots, p-2, p-1, \dots, p-1).$$

So

$$d_i d_j u_m = (-1)^{i+j} \mathcal{P}^\gamma \mathcal{P}^\delta u_m,$$

where

$$\gamma = (a+1)p^{n-1} - (i_{r+1} + 1)p^{r+1}$$

and

$$\delta = p^{n-2} + p^{n-3} + \cdots + p^r.$$

We have

$$i + p^{r+1} = (0, \dots, i_{r+1} + 1, p - 2, p - 1, \dots, p - 1).$$

Consider $d_{i+p^{r+1}}d_j u_m$. There is a chosen monomial in this degree; it is $d_i d_{j+p^r} u_m$. Using the inductive assumption, we compute that

$$d_{i+p^{r+1}}d_j u_m = \left(\frac{-1}{i_{r+1} + 1} \right) d_i d_{j+p^r} u_m.$$

We now compute $(-1)^{i+j+1} \mathcal{P}^{p^{r+1}}$ on both sides of this equation. On the left-hand side, using the Adem relations, we obtain

$$\begin{aligned} (-1)^{i+j+1} \mathcal{P}^{p^{r+1}} d_{i+p^{r+1}} d_j u_m &= \mathcal{P}^{p^{r+1}} \mathcal{P}^{\gamma-p^{r+1}} \mathcal{P}^\delta u_m \\ &= -(i_{r+1} + 1) \mathcal{P}^\gamma \mathcal{P}^\delta u_m \\ &\quad + \sum_{s=0}^{r-1} \kappa_s \mathcal{P}^{\gamma-p^{r-1}-\cdots-p^s} \mathcal{P}^{\delta+p^{r-1}+\cdots+p^s} u_m \end{aligned}$$

for scalars κ_s . The terms in the summation can be shown to be zero by Lemmas 4.10 and 4.11 (note that $d_{i-p^{r+1}}d_{j+p^r}$ is chosen, so $\alpha(|d_i d_j u_m|) = \alpha(m)$ for applying Lemma 4.10). So we have

$$(-1)^{i+j+1} \mathcal{P}^{p^{r+1}} d_{i+p^{r+1}} d_j u_m = -(-1)^{i+j} (i_{r+1} + 1) d_i d_j u_m.$$

Next, we have, again using the Adem relations, that

$$\begin{aligned} (-1)^{i+j+1} \mathcal{P}^{p^{r+1}} d_i d_{j+p^r} u_m &= \mathcal{P}^{p^{r+1}} \mathcal{P}^{\gamma-p^{r+1}+p^r} \mathcal{P}^{\delta-p^r} u_m \\ &= -(i_{r+1} + 2) \mathcal{P}^{\gamma+p^r} \mathcal{P}^{\delta-p^r} u_m + \mathcal{P}^\gamma \mathcal{P}^\delta u_m \\ &= (-1)^{i+j} (-(i_{r+1} + 2) d_{i-p^{r+1}} d_{j+p^r} u_m + d_i d_j u_m). \end{aligned}$$

So

$$-(i_{r+1} + 1) d_i d_j u_m = \left(\frac{-1}{i_{r+1} + 1} \right) (-(i_{r+1} + 2) d_{i-p^{r+1}} d_{j+p^r} u_m + d_i d_j u_m).$$

Since $i_{r+1} \neq 0$ or $p - 2$, $(i_{r+1} + 1)^2 \neq 1$. Hence we see that $d_i d_j u_m$ is a multiple of $d_{i-p^{r+1}} d_{j+p^r} u_m$, a lower order term. $\square$

Lemma 4.18. *Along with the common hypotheses, suppose that $0 \leq r < n - 3$, and $\bar{i}(j + p^r) \geq i > \bar{i}(j)$. Also suppose that $i_r = j_r$, that $j_r = \cdots = j_{n-2} = p - 2$, that $j_{n-1} = a - 1$, and that $i_{r+1} = p - 2$. Then $d_i d_j u_m$ can be expressed as a sum of lower order terms.*

Proof. We have

$$i = (0, \dots, 0, p - 2, p - 2, p - 1, \dots, p - 1)$$

and

$$j = (a - 1, p - 2, \dots, p - 2, p - 1, \dots, p - 1).$$

So

$$d_i d_j u_m = (-1)^{i+j} \mathcal{P}^\gamma \mathcal{P}^\delta u_m,$$

where

$$\gamma = (a+1)p^{n-1} - p^{r+2} + p^{r+1}$$

and

$$\delta = p^{n-2} + p^{n-3} + \cdots + p^r.$$

We have

$$i + p^{r+2} = (0, \dots, 1, p-2, p-2, p-1, \dots, p-1).$$

Consider $d_{i+p^{r+2}} d_j u_m$. There is a chosen monomial in this degree; it is

$$d_{i-p^{r+1}} d_{j+p^{r+1}+p^r} u_m.$$

By the inductive assumption, $d_{i+p^{r+2}} d_j u_m = K d_{i-p^{r+1}} d_{j+p^{r+1}+p^r} u_m$, for some $K \in \mathbb{F}_p$. We now compute $\mathcal{P}^{p^{r+2}}$ on both sides of this equation. On the left-hand side, we use the hypothesis that $r < n-3$ and Lemmas 4.10 (note that $d_{i-p^{r+1}} d_{j+p^r}$ is chosen, so $\alpha(|d_i d_j u_m|) = \alpha(m)$) and 4.11 to obtain

$$\begin{aligned} \mathcal{P}^{p^{r+2}} d_{i+p^{r+2}} d_j u_m &= (-1)^{i+j+1} \mathcal{P}^{p^{r+2}} \mathcal{P}^{\gamma-p^{r+2}} \mathcal{P}^\delta u_m \\ &= 2(-1)^{i+j} \mathcal{P}^\gamma \mathcal{P}^\delta u_m \\ &= 2d_i d_j u_m. \end{aligned}$$

On the right-hand side, $\mathcal{P}^{p^{r+2}} d_{i-p^{r+1}} d_{j+p^{r+1}+p^r} u_m$ consists of terms of lower order than $d_i d_j u_m$. $\square$

Lemma 4.19. *Along with the common hypotheses, suppose that $r = n-3$, and $\bar{i}(j+p^r) \geq i > \bar{i}(j)$. Suppose also that $i_r = j_r$, that $j_r = \cdots = j_{n-2} = p-2$, that $j_{n-1} = a-1$, and that $i_{r+1} = p-2$. Then $d_i d_j u_m$ can be expressed as a sum of lower order terms.*

Proof. We have

$$j = (a-1, p-2, p-2, p-1, \dots, p-1)$$

and

$$i = (0, p-2, p-2, p-1, \dots, p-1).$$

If $a = 1$, then the conclusion is an immediate consequence of one of the defining relations for our module. So let $a > 1$.

Begin by noting that there is a chosen in the degree of $d_i d_j u_m$. It is

$$d_{i-p^{n-2}} d_{j+p^{n-3}} u_m.$$

Hence the alpha-number of this degree is $\alpha(m)$, so we may use Lemmas 4.10 and 4.11. Now consider $d_{i+p^{n-1}} d_j u_m$. Inductively it is a multiple of the chosen in this topological degree, which is a 1-fold, $d_{i-p^{n-2}} u_m$. We may then compute, as in the

preceding lemmas, that $\mathcal{P}^{p^{n-1}}d_{i+p^{n-1}}d_ju_m$ is of order lower than $d_id_ju_m$. We also compute

$$\begin{aligned}\mathcal{P}^{p^{n-1}}d_{i+p^{n-1}}d_ju_m &= (-1)^{i+j+1} \mathcal{P}^{p^{n-1}} \mathcal{P}^{(a-1)p^{n-1}+p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} u_m \\ &= -(a-1)d_id_ju_m + (-1)^{i+j+1} \mathcal{P}^{ap^{n-1}} \mathcal{P}^{2p^{n-2}+p^{n-3}} u_m.\end{aligned}$$

We shall show that the second term on the right is of order lower than $d_id_ju_m$. We have

$$\begin{aligned}\mathcal{P}^{p^{n-1}} \mathcal{P}^{(a-1)p^{n-1}} \mathcal{P}^{2p^{n-2}+p^{n-3}} u_m &= a \mathcal{P}^{ap^{n-1}} \mathcal{P}^{2p^{n-2}+p^{n-3}} u_m \\ &\quad + \mathcal{P}^{ap^{n-1}-p^{n-3}} \mathcal{P}^{2p^{n-2}+2p^{n-3}} u_m \\ &\quad + 2 \mathcal{P}^{ap^{n-1}-p^{n-3}-2p^{n-4}} \mathcal{P}^{2p^{n-2}+2p^{n-3}+2p^{n-4}} u_m \\ &\quad + \mathcal{P}^{ap^{n-1}-2p^{n-4}} \mathcal{P}^{2p^{n-2}+p^{n-3}+p^{n-4}} u_m \\ &\quad - 2 \mathcal{P}^{ap^{n-1}-p^{n-2}} \mathcal{P}^{3p^{n-2}+p^{n-3}} u_m.\end{aligned}$$

The left side is of lower order for the same reason that $\mathcal{P}^{p^{n-1}}d_{i+p^{n-1}}d_ju_m$ was above. The third and fourth terms on the right are zero by Lemmas 4.10 and 4.11. Further, note that

$$|d_id_ju_m| = (a, 1, p-2, p-2, p-1, \dots, p-1),$$

so the terms $\mathcal{P}^{ap^{n-1}-3p^{n-3}} \mathcal{P}^{2p^{n-2}+2p^{n-3}} u_m$ and $\mathcal{P}^{ap^{n-1}-3p^{n-2}} \mathcal{P}^{3p^{n-2}+p^{n-3}} u_m$ lie in degrees with lesser alpha-number and hence are zero by the inductive assumption. We compute that

$$\begin{aligned}0 &= \mathcal{P}^{2p^{n-3}} \mathcal{P}^{ap^{n-1}-3p^{n-3}} \mathcal{P}^{2p^{n-2}+2p^{n-3}} u_m = \mathcal{P}^{ap^{n-1}-p^{n-3}} \mathcal{P}^{2p^{n-2}+2p^{n-3}} u_m \\ &\quad + \text{terms that are zero by Lemmas 4.10 and 4.11.}\end{aligned}$$

If $p = 3$, $\mathcal{P}^{3p^{n-2}+p^{n-3}} u_m$ is unchosen. For $p \neq 3$, a calculation similar to the preceding one shows that

$$\begin{aligned}0 &= \mathcal{P}^{2p^{n-2}} \mathcal{P}^{ap^{n-1}-3p^{n-2}} \mathcal{P}^{3p^{n-2}+p^{n-3}} u_m = \mathcal{P}^{ap^{n-1}-p^{n-2}} \mathcal{P}^{3p^{n-2}+p^{n-3}} u_m \\ &\quad + \text{terms that are zero by Lemma 4.10 and 4.11.} \quad \square\end{aligned}$$

Remark 4.20. We have now completed all cases in which $0 \leq r \leq n-3$ or $a = 1$.

Lemma 4.21. *Along with the common hypotheses, suppose that $r = n-2$ and $a > 1$, that $i_{n-2} = j_{n-2} = p-2$, and that $j_{n-1} = a-1$. Then $i_{n-1} \neq 0$, and*

$$i_{n-1}d_id_ju_m = (-1)^{j+1} ad_{i-p^{n-1}}u_m,$$

so $d_id_ju_m$ can be expressed as a sum of lower order terms.

Proof. We have

$$\begin{aligned}i &= (i_{n-1}, p-2, p-1, \dots, p-1) \\ &= (i_{n-1}+1)p^{n-1} - p^{n-2} - 1\end{aligned}$$

and

$$\begin{aligned} j &= (a-1, p-2, p-1, \dots, p-1) \\ &= ap^{n-1} - p^{n-2} - 1. \end{aligned}$$

We have

$$d_i d_j u_m = (-1)^{i+j} \mathcal{P}^{(a-i_{n-1})p^{n-1}} \mathcal{P}^{p^{n-2}} u_m$$

and

$$d_{i-p^{n-1}} u_m = (-1)^{i-1} \mathcal{P}^{(a-i_{n-1})p^{n-1}+p^{n-2}} u_m.$$

Set $k = a - i_{n-1}$. Then we are to prove that

$$(a-k) \mathcal{P}^{kp^{n-1}} \mathcal{P}^{p^{n-2}} u_m = a \mathcal{P}^{kp^{n-1}+p^{n-2}} u_m.$$

Notice that $i_{n-1} \neq 0$ by unchosenness, and $i_{n-1} \leq a-1$ by admissibility, so we are considering $1 \leq k \leq a-1$.

For $k=1$, this equation is one of the defining relations for our module. Inductively assume that the relation holds for a fixed value of k . We shall apply $\mathcal{P}^{p^{n-1}}$ to both sides of the equation and use Adem relations. On the left-hand side,

$$\begin{aligned} \mathcal{P}^{p^{n-1}} \mathcal{P}^{kp^{n-1}} \mathcal{P}^{p^{n-2}} u_m &= (k+1) \mathcal{P}^{(k+1)p^{n-1}} \mathcal{P}^{p^{n-2}} u_m \\ &\quad + \mathcal{P}^{(k+1)p^{n-1}-p^{n-3}} \mathcal{P}^{p^{n-2}+p^{n-3}} u_m \\ &\quad + 2 \mathcal{P}^{(k+1)p^{n-1}-p^{n-2}} \mathcal{P}^{2p^{n-2}} u_m. \end{aligned}$$

On the right-hand side,

$$\mathcal{P}^{p^{n-1}} \mathcal{P}^{kp^{n-1}+p^{n-2}} u_m = k \mathcal{P}^{(k+1)p^{n-1}+p^{n-2}} u_m + \mathcal{P}^{(k+1)p^{n-1}} \mathcal{P}^{p^{n-2}} u_m.$$

We shall prove that

$$\mathcal{P}^{(k+1)p^{n-1}-p^{n-3}} \mathcal{P}^{p^{n-2}+p^{n-3}} u_m = 0 = \mathcal{P}^{(k+1)p^{n-1}-p^{n-2}} \mathcal{P}^{2p^{n-2}} u_m.$$

Granting this, we compute

$$\begin{aligned} (a-k)(k+1) \mathcal{P}^{(k+1)p^{n-1}} \mathcal{P}^{p^{n-2}} u_m &= ak \mathcal{P}^{(k+1)p^{n-1}+p^{n-2}} u_m \\ &\quad + a \mathcal{P}^{(k+1)p^{n-1}} \mathcal{P}^{p^{n-2}} u_m. \end{aligned}$$

Combining terms, simplifying, and cancelling k , which is nonzero mod p , we obtain our goal:

$$(a-k-1) \mathcal{P}^{(k+1)p^{n-1}} \mathcal{P}^{p^{n-2}} u_m = a \mathcal{P}^{(k+1)p^{n-1}+p^{n-2}} u_m.$$

Next, we note that $\mathcal{P}^{(k+1)p^{n-1}-p^{n-3}} \mathcal{P}^{p^{n-2}+p^{n-3}} u_m = 0$ by Lemmas 4.10 and 4.11 (which apply in this degree because there is the chosen 1-fold $d_{i-p^{n-1}} u_m$ here). Finally, consider $\mathcal{P}^{(k+1)p^{n-1}-p^{n-2}} \mathcal{P}^{2p^{n-2}} u_m$. Well,

$$\begin{aligned} \mathcal{P}^{2p^{n-2}} \mathcal{P}^{(k+1)p^{n-1}-3p^{n-2}} \mathcal{P}^{2p^{n-2}} u_m &= \mathcal{P}^{(k+1)p^{n-1}-p^{n-2}} \mathcal{P}^{2p^{n-2}} u_m \\ &\quad + \text{terms that are zero by Lemmas 4.10 and 4.11.} \end{aligned}$$

The left-hand side is zero by alpha-number. This completes the proof of the lemma. $\square$

Lemma 4.22. *Along with the common hypotheses, suppose that $r = n - 2$ and $a > 1$, and let $i_r = j_r = p - 2$. Suppose that $j_{n-1} < a - 1$. Then $d_i d_j u_m$ can be expressed as a sum of lower order terms.*

Proof. We have

$$\begin{aligned} i &= (i_{n-1}, p-2, p-1, \dots, p-1) \\ &= (i_{n-1} + 1)p^{n-1} - p^{n-2} - 1 \end{aligned}$$

and

$$\begin{aligned} j &= (j_{n-1}, p-2, p-1, \dots, p-1) \\ &= (j_{n-1} + 1)p^{n-1} - p^{n-2} - 1. \end{aligned}$$

Define

$$\begin{aligned} k &= (a-1, p-2, p-1, \dots, p-1) \\ &= ap^{n-1} - p^{n-2} - 1. \end{aligned}$$

By the previous lemma, there is a unit A such that $Ad_j u_m = d_{j+p^{n-1}} d_k u_m$ and a unit B such that $d_i d_k u_m = Bd_{i-p^{n-1}} u_m$. So, using the preceding lemma and Remark 4.8, we compute

$$\begin{aligned} Ad_i d_j u_m &= d_i d_{j+p^{n-1}} d_k u_m \\ &= d_{i+p(j+p^{n-1}-i)} d_i d_k u_m + \sum_{1 \leq t \leq i/p} \epsilon_t d_{i-pt} d_{j+p^{n-1}+t} d_k u_m. \end{aligned}$$

Now for each term in this sum, $d_{j+p^{n-1}+t} d_k u_m$ is unchosen and in lower topological degree than $d_i d_j u_m$; hence inductively it can be expressed using terms of lower order than itself. But from our result for 1-folds, the only d with index larger than k that can act nontrivially on u_m is d_m , so $d_{j+p^{n-1}+t} d_k u_m$ collapses to a 1-fold; thus each term $d_{i-pt} d_{j+p^{n-1}+t} d_k u_m$ can be expressed as a sum of admissibles of lower order than $d_i d_j u_m$. So we may continue with

$$\begin{aligned} Ad_i d_j u_m &= d_{i+p(j+p^{n-1}-i)} d_i d_k u_m + \text{lower order terms than } d_i d_j u_m \\ &= Bd_{i+p(j+p^{n-1}-i)} d_{i-p^{n-1}} u_m + \text{lower order terms than } d_i d_j u_m \\ &= Bd_{i-p^{n-1}} d_{j+p^{n-2}} u_m + \text{lower order terms than } d_i d_j u_m \\ &= \text{lower order terms than } d_i d_j u_m. \end{aligned} \quad \square$$

4.3. Higher-fold monomials

We now move to the proof for three- and higher-folds. In this part, we consider $d_i d_j d_k d_L u_m$, admissible and unchosen, with L possibly empty, and with the last subscript of the entire monomial less than m . The goal is to show that $d_i d_j d_k d_L u_m$ can be expressed as a sum of lower order terms.

Assume inductively that every unchosen admissible of length degree less than that of $d_i d_j d_k d_L$, when applied to u_m , can be expressed in terms of lower order terms than itself. From our previous steps, we may assume that $d_j d_k d_L$ is chosen. Hence $d_i d_j$ is unchosen.

Lemma 4.23. *Under these hypotheses, $d_i d_j d_k d_L u_m$ is a sum of lower order terms.*

Proof. Let the symbol $\equiv$ represent congruence modulo terms of order lower than $d_i d_j d_k d_L$ applied to u_m . Using Adem relations (especially Remark 4.8 in both directions of the equality, and recalling that Adem relations on inadmissibles always reduce lexicographic order) and the inductive assumption, we obtain (where all monomial terms below have length no greater than that of $d_i d_j d_k d_L$).

$$\begin{aligned}
 d_i d_j d_k d_L u_m &= d_i d_{j+(k-j)p} d_j d_L u_m + \sum_{t>0} \varepsilon_t d_i d_{j-pt} d_{k+t} d_L u_m, \text{ for some } \varepsilon_t \in \mathbb{F}_p \\
 &\equiv d_i d_{j+(k-j)p} d_j d_L u_m \\
 &= d_{i+(j+(k-j)p-i)p} d_i d_j d_L u_m \\
 &\quad + \sum_{s>0} \eta_s d_{i-ps} d_{j+(k-j)p+s} d_j d_L u_m, \text{ for some } \eta_s \in \mathbb{F}_p \\
 &\equiv d_{i+(j+(k-j)p-i)p} d_i d_j d_L u_m \\
 &= d_{i+(j+(k-j)p-i)p} \cdot \sum (\text{terms of order lower than } d_i d_j d_L u_m) \\
 &\equiv d_{i+(j+(k-j)p-i)p} \cdot \sum_{i'<i} \kappa' d_{i'} \sum d_{M_{i'}} u_m \\
 &\quad + d_i d_{j+(k-j)p} \cdot \sum_{j'<j} \lambda' d_{j'} \sum d_{N_{j'}} u_m \\
 &\quad + d_i d_j d_k \cdot \sum_{P' \text{ lower order than } L} \mu' d_{P'} u_m, \text{ for scalars } \kappa', \lambda', \text{ and } \mu'.
 \end{aligned}$$

All terms in this summation can be expressed in terms of admissibles of order lower than $d_i d_j d_k d_L$ applied to u_m . $\square$

This completes the proof of Theorem 2.17.

5. Proof of Theorem 2.2

Proof of Theorem 2.2. Since the filtered $\mathcal{A}$ -map defined in the first section from $\mathcal{M}$ to $H^*(CP(\infty))$ induces isomorphisms on the filtered quotients (Corollary to Theorem 2.17), the map takes $\mathcal{M}$ isomorphically onto $H^*(CP(\infty))$.

Minimality of the relations in the presentation will follow from minimality of the induced relations in the filtered quotients $\mathcal{M}_{n,a}$ stated in Remark 2.15. While nonredundancy of the relations in $\mathcal{M}_{n,a}$ involving $\mathcal{P}^{p^l}$ and $\mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}}$ is obvious because they never occur on the right side of an Adem relation, the others are not as easy. But they will all succumb to reduction to a small value of n , as follows.

The Verschiebung $V: \mathcal{A} \rightarrow \mathcal{A}$ is a Hopf algebra map with $V(\mathcal{P}^k) = \mathcal{P}^{k/p}$ for all k , and it is easy to check that, due to the instability requirement, V induces a nontrivial homomorphism $\hat{V}: \mathcal{M}_{n,a} \rightarrow \mathcal{M}_{n-1,a}$ satisfying $\hat{V}(x t_{ap^{n-1}-1}) = V(x) t_{ap^{n-2}-1}$ for all $x \in \mathcal{A}$. Iterating $\hat{V}$ can confirm the nontriviality of all our relations.

For instance, for $a \geq 2$, the element $\mathcal{P}^{(p-a+1)p^{n-2}} t_{ap^{n-1}-1}$ iterates to

$$\mathcal{P}^{p-a+1} t_{ap-1} \in \mathcal{M}_{2,a}.$$

But $\mathcal{M}_{2,a}$ has no lower degree relation than $\mathcal{P}^{p-a+1} t_{ap-1} = 0$, which involves an unstable admissible basis element, since $a \geq 2$.

When $a = 1$, we can iterate $\hat{V}$ on the given special relation to its lowest full incarnation $2\mathcal{P}^{p^2+p+1}\mathcal{P}^p u_{p^2-1} = \mathcal{P}^{p^2+p}\mathcal{P}^{p+1} u_{p^2-1}$ in $\mathcal{M}_{3,1}$. In $\mathcal{M}_{3,1}$ the only other generating relation is $\mathcal{P}^1 u_{p^2-1}$. We thus study the two terms of this lowest special relation as elements in the quotient of the free unstable module on u_{p^2-1} by only the single relation $\mathcal{P}^1 u_{p^2-1}$. In fact we can choose to project these terms yet further to the free unstable module on u_{p+1} of degree $p+1$, subject to $\mathcal{P}^1 u_{p+1} = 0$, since there is a natural epimorphism from the free unstable module on u_{p^2-1} to the one on u_{p+1} . There $\mathcal{P}^{p^2+p+1}\mathcal{P}^p u_{p+1}$ is zero, since $\mathcal{P}^{p^2+p+1}\mathcal{P}^p$ has excess $2p+1$. On the other hand, $\mathcal{P}^{p^2+p}\mathcal{P}^{p+1} u_{p+1}$ is nonzero there, as the reader may easily verify: using the admissible basis for $\mathcal{A}$, one finds by direct calculation and the unstable admissible basis for the free module on u_{p+1} that the submodule $\mathcal{AP}^1 u_{p+1}$ of the free module contains no element with $\mathcal{P}^{p^2+p}\mathcal{P}^{p+1} u_{p+1}$ as a term. Thus the special relation cannot hold in $\mathcal{M}_{3,1}$, so all the $a = 1$ special relations are nonredundant. $\square$

6. Proofs of Theorems 2.19, 2.23, and 2.25

We begin this section by proving our uniqueness theorem for cyclic modules.

Proof of Theorem 2.19. We shall begin by checking that the defining relations for $\mathcal{M}_{n,a}$ are satisfied in M . Let $u \in M_{ap^{n-1}-1}$ be nonzero. We have

$$\alpha_0 = (n-1)(p-1) + (a-1),$$

since

$$ap^{n-1} - 1 = (a-1, p-1, \dots, p-1).$$

The next larger integer with its alpha-number is

$$(a, p-2, p-1, \dots, p-1) = (a+1)p^{n-1} - p^{n-2} - 1.$$

Hence $\mathcal{P}^{p^t} u = 0$, for $0 \leq t \leq n-3$.

Let $a > 1$. We have

$$\left| \mathcal{P}^{(p-a+1)p^{n-2}} u \right| = p^n + (a-1)p^{n-2} - 1,$$

whose alpha-number is $(n-2)(p-1) + a-1$. So $\mathcal{P}^{(p-a+1)p^{n-2}} u = 0$. In this case it remains to check the third relation.

Begin by noting that $\mathcal{P}^{ap^{n-1}+p^{n-2}} u = 0$, by unstability. The degree of this term is $ap^n + (p-1)p^{n-2} - 1$. The next smaller integer having alpha-number α_0 is $ap^n - p$. So by Lemma 4.11, $\mathcal{P}^{ap^{n-1}}\mathcal{P}^{p^{n-2}} u$ is the only possible nonzero admissible in its degree (note there are no possible nonzero admissibles of greater length in this degree). Since the alpha-number is α_0 , it is nonzero. Let $0 \leq l \leq a-2$. We shall induct on l . Assume that $l\mathcal{P}^{(a-l)p^{n-1}}\mathcal{P}^{p^{n-2}} u = a\mathcal{P}^{(a-l)p^{n-1}+p^{n-2}} u$ (this is true for

$l = 0$ from above; note that this induction goes downward in topological degree). As in the proof of Lemma 4.21, under the hypotheses of the present theorem, the Adem relations yield

$$\mathcal{P}^{p^{n-1}} \mathcal{P}^{(a-l-1)p^{n-1}+p^{n-2}} u = (a-l-1) \mathcal{P}^{(a-l)p^{n-1}+p^{n-2}} u + \mathcal{P}^{(a-l)p^{n-1}} \mathcal{P}^{p^{n-2}} u$$

and

$$\mathcal{P}^{p^{n-1}} \mathcal{P}^{(a-l-1)p^{n-1}} \mathcal{P}^{p^{n-2}} u = (a-l) \mathcal{P}^{(a-l)p^{n-1}} \mathcal{P}^{p^{n-2}} u.$$

The second equation, and the fact that $\mathcal{P}^{ap^{n-1}} \mathcal{P}^{p^{n-2}} u$ is nonzero from above, shows us inductively that in these degrees, left action by $\mathcal{P}^{p^{n-1}}$ is a (nonzero) isomorphism. We compute:

$$\begin{aligned} & \mathcal{P}^{p^{n-1}} (a \mathcal{P}^{(a-l-1)p^{n-1}+p^{n-2}} u - (l+1) \mathcal{P}^{(a-l-1)p^{n-1}} \mathcal{P}^{p^{n-2}} u) \\ &= a(a-l-1) \mathcal{P}^{(a-l)p^{n-1}+p^{n-2}} u \\ & \quad + a \mathcal{P}^{(a-l)p^{n-1}} \mathcal{P}^{p^{n-2}} u - (l+1)(a-l) \mathcal{P}^{(a-l)p^{n-1}} \mathcal{P}^{p^{n-2}} u \\ &= (a-l-1)(a \mathcal{P}^{(a-l)p^{n-1}+p^{n-2}} u - l \mathcal{P}^{(a-l)p^{n-1}} \mathcal{P}^{p^{n-2}} u) = 0. \end{aligned}$$

Hence

$$a \mathcal{P}^{(a-l-1)p^{n-1}+p^{n-2}} u - (l+1) \mathcal{P}^{(a-l-1)p^{n-1}} \mathcal{P}^{p^{n-2}} u \in \ker \mathcal{P}^{p^{n-1}} = 0;$$

so

$$a \mathcal{P}^{(a-l-1)p^{n-1}+p^{n-2}} u = (l+1) \mathcal{P}^{(a-l-1)p^{n-1}} \mathcal{P}^{p^{n-2}} u.$$

Taking $l = a - 2$, we obtain the desired relation.

Now take $a = 1$. We compute, using Lemma 4.11, that

$$\mathcal{P}^{(p-2)p^{n-2}} \mathcal{P}^{p^{n-1}+p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} u = -\mathcal{P}^{p^{n-1}+(p-1)p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} u$$

and that

$$\mathcal{P}^{(p-2)p^{n-2}} \mathcal{P}^{p^{n-1}+p^{n-2}+p^{n-3}} \mathcal{P}^{p^{n-2}} u = -2 \mathcal{P}^{p^{n-1}+(p-1)p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} u.$$

As above, this yields

$$\mathcal{P}^{p^{n-1}+p^{n-2}+p^{n-3}} \mathcal{P}^{p^{n-2}} u = 2 \mathcal{P}^{p^{n-1}+p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} u,$$

as desired, provided we know that $\mathcal{P}^{(p-2)p^{n-2}}$ is acting on the left as a monomorphism between these degrees. This follows by checking, from the basic relations and Lemma 4.11, that $\mathcal{P}^{p^{n-1}+(p-1)p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} u$ is the only possible nonzero admissible in its degree, and thus is nonzero by the hypothesis of the theorem.

In all cases, the defining relations for $\mathcal{M}_{n,a}$ are satisfied in M . Hence taking $t_{ap^{n-1}-1}$ to u defines an $\mathcal{A}$ -module map $\mathcal{M}_{n,a} \rightarrow M$. Since M is cyclic, this map is surjective. By Theorems 2.11 (and its remark), 2.17, and the hypothesis on M , it must be an isomorphism. $\square$

We next give the proof of the presentation for $H^*(BU)$.

Proof of Theorem 2.23. Recall the $\mathcal{A}$ -algebra map $\phi: \mathcal{G} \rightarrow H^*(BU)$ defined by taking $\tau_{ap^{n-1}}$ to the Chern class $c_{ap^{n-1}}$. By the definition of $\mathcal{G}$, the following relations are satisfied in the indecomposable quotient $Q\mathcal{G}$:

For $0 \leq l \leq n-3$,

$$\mathcal{P}^{p^l} \tau_{ap^{n-1}} = \mathcal{P}^{ap^{n-2}} \mathcal{P}^{p^l} \tau_{ap^{n-2}},$$

and for $a \geq 2$,

$$(a-1) \mathcal{P}^{(p-a+1)p^{n-2}} \tau_{ap^{n-1}} = \binom{p-1}{a-2} \mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}} \tau_{ap^{n-2}}$$

and

$$(a-1) \mathcal{P}^{p^{n-1}} \mathcal{P}^{p^{n-2}} \tau_{ap^{n-1}} = a \mathcal{P}^{p^{n-1}+p^{n-2}} \tau_{ap^{n-1}},$$

while for $a = 1$

$$2 \mathcal{P}^{p^{n-1}+p^{n-2}} \mathcal{P}^{p^{n-2}+p^{n-3}} \tau_{ap^{n-1}} = \mathcal{P}^{p^{n-1}+p^{n-2}+p^{n-3}} \mathcal{P}^{p^{n-2}} \tau_{ap^{n-1}}.$$

Since the analogous relations define the double (topological) suspension $\Sigma^2 \mathcal{M}$, there is a surjection of $\mathcal{A}$ -modules $\pi: \Sigma^2 \mathcal{M} \rightarrow Q\mathcal{G}$ given by taking $\Sigma^2 t_{ap^{n-1}-1}$ to $\tau_{ap^{n-1}}$. By Theorem 2.2 and Remark 2.20 the map $\Sigma^2 \mathcal{M} \rightarrow QH^*(BU)$ given by taking $\Sigma^2 t_{ap^{n-1}-1}$ to $c_{ap^{n-1}}$ is an isomorphism. Hence the induced $\mathcal{A}$ -module map $Q\phi: Q\mathcal{G} \rightarrow QH^*(BU)$ is an isomorphism. Since $H^*(BU)$ is a free commutative algebra, this guarantees that the algebra map ϕ is an isomorphism. $\square$

Finally, we prove that there are no $\mathcal{A}$ -relations amongst Chern classes.

Proof of Theorem 2.25. Since the symmetric algebra is an $\mathcal{A}$ -subalgebra of the polynomial algebra on variables $\{x_i\}$ of complex degree one, we shall work in the polynomial algebra. Identifying the Chern class c_m with the m -th elementary symmetric polynomial in the variables $\{x_i\}$, we see that c_m has a term $x_1 \cdots x_m$ as a summand. Applying any element of $\mathcal{K}$ to this term results in a polynomial having all summands of the form $\alpha x_1^{p^{e_1}} \cdots x_m^{p^{e_m}}$, $\alpha \in \mathbb{F}_p$. By symmetry, at least one of these terms satisfies the inequalities $e_1 \geq \cdots \geq e_m$. We call a monomial of this form a *basic monomial*. For such a term, if $1 \leq l \leq e_1 + 1$, let r_l be the number of occurrences of p^{e_1-l+1} as an exponent in $x_1^{p^{e_1}} \cdots x_m^{p^{e_m}}$. Then we shall refer to the tuple $(r_1, r_2, \dots, r_{e_1+1})$ as the *type* of $x_1^{p^{e_1}} \cdots x_m^{p^{e_m}}$. We say that a tuple $(m_1, \dots, m_a)$ has higher order than a tuple $(n_1, \dots, n_b)$ provided that $a > b$, or if $a = b$, provided that it is greater in lexicographic order, ordering from the left. Let $d_J = d_{j_1} \cdots d_{j_s}$, $j_s < m$, be an admissible monomial in $\mathcal{K}$. Then direct calculation reveals that $d_J c_m$ has a basic monomial summand whose type is $(m - j_s, j_s - j_{s-1}, \dots, j_2 - j_1, j_1)$. One can check that this term has the highest order of all basic monomials occurring in $d_J c_m$. Since we can recover m and J from this type, this proves that the free module $\mathcal{R}$ injects into $H^*(BU)$. $\square$

7. Appendix: The Kudo-Araki-May algebra $\mathcal{K}$

We recall here just the bare essentials about $\mathcal{K}$ needed to understand the proofs in this paper. We refer the reader to [11] for much more extensive information about $\mathcal{K}$.

The odd-primary (even) topological Kudo-Araki-May algebra $\mathcal{K}$ is the $\mathbb{F}_p$ -bialgebra (with identity) generated by elements $\{d_i : i \geq 0\}$ subject to homogeneous (Adem) relations

$$d_i d_j = \sum_l (-1)^{pl-i} \binom{(p-1)(l-j)-1}{pl-i-(p-1)j} d_{i+pj-pl} d_l \quad \text{for all } i, j \geq 0,$$

with coproduct ϕ determined by the formula

$$\phi(d_i) = \sum_{t=0}^i d_t \otimes d_{i-t}.$$

It is bigraded by length and complex topological degrees ($|d_i| = (p-1)i$), which behave skew-additively under multiplication: $|xy| = |x| + p|y|$. Moreover, $\mathcal{K}$ is finite in each bidegree. A monomial $d_{i_1} \cdots d_{i_n}$ is called *admissible* provided that $i_1 \leq \cdots \leq i_n$. The admissible monomials provide a vector space basis for $\mathcal{K}$.

The $\mathbb{F}_p$ -cohomology of any space concentrated in even dimensions is an unstable algebra over the Steenrod algebra $\mathcal{A}$ with no Bocksteins, and there is a correspondence between unstable $\mathcal{A}$ -algebras and unstable $\mathcal{K}$ -algebras, completely determined by iterating the conversion formulae:

$$(-1)^j d_j u_q = \mathcal{P}^{q-j} u_q, \text{ where } u_q \text{ is a cohomology class of (complex) degree } q.$$

Since the degree of the element is involved in the conversion, and this changes as operations are composed, the algebra structures of $\mathcal{A}$ and $\mathcal{K}$ are very different, and the skew additivity of the bigrading in $\mathcal{K}$ reflects this. Note for use in calculation that since $\mathcal{P}^{q-j}$ has complex degree $(p-1)(q-j)$, the complex degree of $d_j u_q$ is $pq - (p-1)j$. More generally, for a monomial $d_I = d_{i_1} d_{i_2} \cdots d_{i_l} \in \mathcal{K}$ of length l , degrees in a $\mathcal{K}$ -module and in $\mathcal{K}$ itself are related by

$$|d_I u_q| = p^l q - |d_I|.$$

The requirements for an unstable $\mathcal{K}$ -algebra, corresponding to the nature and requirements of an unstable $\mathcal{A}$ -algebra, are: On any element x_l of complex degree l ,

$$d_l x_l = (-1)^l x_l, \quad d_j x_l = 0 \text{ for } j > l, \quad \text{and } d_0 x_l = x_l^p.$$

Finally, and used in our proofs, the $\mathcal{K}$ -algebra structure obeys the (Cartan) formula according to the coproduct ϕ in $\mathcal{K}$:

$$d_i(xy) = \sum_{t=0}^i d_t(x) d_{i-t}(y).$$

References

- [1] D. Benson and V. Franjou, Séries de compositions de modules instables et injectivité de la cohomologie du groupe $\mathbb{Z}/2$, *Math. Zeit.* **208** (1991), 389–399.
- [2] G. Carlsson, G.B. Segal's Burnside ring conjecture for $(\mathbb{Z}/2)^k$, *Topology* **22** (1983), 83–103.

- [3] V. Franjou and L. Schwartz, Reduced unstable $\mathcal{A}$ -modules and the modular representation theory of the symmetric groups, *Ann. Sci. École Norm. Sup.* **23** (1990), 593–624.
- [4] A. Kono and D. Tamaki, *Generalized Cohomology*, Translations of Mathematical Monographs **230**, A. M. S., Providence, RI, 2006.
- [5] J. Lannes and S. Zarati, Sur les $\mathcal{U}$ -injectifs, *Ann. Sci. École Norm. Sup.* **19** (1986), 1–31.
- [6] H. Miller, The Sullivan conjecture on maps from classifying spaces, *Annals of Math.* **120** (1984), 39–87; and corrigendum, *Annals of Math.* **121** (1985), 605–609.
- [7] D. Pengelley, F. Peterson, and F. Williams, A global structure theorem for the mod 2 Dickson algebras, and unstable cyclic modules over the Steenrod and Kudo-Araki-May algebras, *Math. Proc. Cambridge Philos. Soc.* **129** (2000), 263–275.
- [8] D. Pengelley and F. Williams, Sheared algebra maps and operation bialgebras for mod 2 homology and cohomology, *Transactions of the American Mathematical Society* **352** (2000), 1453–1492.
- [9] D. Pengelley and F. Williams, Global structure of the mod two symmetric algebra, $H^*(BO; \mathbb{F}_2)$, over the Steenrod algebra, *Algebraic & Geometric Topology* **3** (2003), 1119–1138.
- [10] D. Pengelley and F. Williams, The global structure of odd-primary Dickson algebras as algebras over the Steenrod algebra, *Math. Proc. Cambridge Philos. Soc.* **129** (2000), 263–275.
- [11] D. Pengelley and F. Williams, The odd-primary Kudo-Araki-May algebra of algebraic Steenrod operations, and invariant theory, *Proceedings, International School and Conference in Algebraic Topology in Honor of Huynh Mui* (Vietnam National University, Hanoi, 2004), Geometry and Topology Publications Monograph Series, to appear.
- [12] F. P. Peterson, A mod p Wu formula, *Bol. Soc. Mat. Mexicana (2)* **20** (1975), no. 2, 56–58.
- [13] L. Schwartz, Unstable modules over the Steenrod algebra, functors, and the cohomology of spaces, in *Infinite Length Modules* (Bielefeld, 1998), 229–249, Trends in Mathematics, Birkhäuser, Basel, 2000.
- [14] L. Schwartz, *Unstable Modules over the Steenrod Algebra and Sullivan’s Fixed Point Set Conjecture*, University of Chicago Press, Chicago, IL, 1994.
- [15] P. Brian Shay, Mod p Wu formulas for the Steenrod algebra and the Dyer-Lashof algebra, *Proceedings of the American Mathematical Society* **63** (1977), 339–347.
- [16] N. E. Steenrod and D. B. A. Epstein, *Cohomology Operations*, *Annals of Mathematics Studies*, No. 50, Princeton Univ. Press, Princeton, NJ, 1962.

David J. Pengelley davidp@nmsu.edu www.math.nmsu.edu/~davidp
Frank Williams frank@nmsu.edu

Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003
USA

This article is available at <http://intlpress.com/HHA/v9/n2/a13>