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TOWARD EQUIVARIANT IWASAWA THEORY, IV

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(*communicated by J.F. Jardine*)

Abstract

Let l be an odd prime number and K_∞/k a Galois extension of totally real number fields, with $k/\mathbb{Q}$ and K_∞/k_∞ finite, where k_∞ is the cyclotomic $\mathbb{Z}_l$-extension of k. In [RW2] a "main conjecture" of equivariant Iwasawa theory is formulated which for pro-l groups G_∞ is reduced in [RW3] to a property of the Iwasawa L-function of K_∞/k. In this paper we extend this reduction for arbitrary G_∞ to l-elementary groups $G_\infty = \langle s\rangle \times U$, with $\langle s\rangle$ a finite cyclic group of order prime to l and U a pro-l group. We also give first nonabelian examples of groups G_∞ for which the conjecture holds.

Dedicated to Victor Snaith on the occasion of his 60-th birthday.

Let l be a fixed odd prime number and K_∞/k a Galois extension of totally real number fields with $[k:\mathbb{Q}]$ finite and k_∞, the cyclotomic l-extension of k, contained in K_∞ with $[K_\infty : k_\infty]$ also finite. The respective Galois groups are $G_\infty = G_{K_\infty/k}$, $H = G_{K_\infty/k_\infty}$, $\Gamma_k = G_{k_\infty/k}$. We also fix a finite set S of primes of k containing l, ∞ and all primes which ramify in K_∞ [1].

In [RW2,§4] we formulated an equivariant refinement of the Main Conjecture of (classical) Iwasawa theory [Wi]. The main point of this paper is to reduce this "main conjecture" to a conjectural property of the Iwasawa L-function $L_{K_\infty/k,S}$ of K_∞/k.

Theorem (A). *The "main conjecture" of equivariant Iwasawa theory for K_∞/k is, up to its uniqueness assertion, equivalent to $L_{K_\infty/k,S}$ belonging to* $\mathrm{Det}\, K_1(\Lambda(G_\infty)_{\wedge})$.

The Iwasawa L-function $L_{K_\infty/k} (= L_{K_\infty/k,S})$ incorporates all the l-adic (S-truncated) Artin L-functions of K_∞/k by assigning to each l-adic character χ of G_∞ the Iwasawa power series of the corresponding L-function. This $L_{K_\infty/k}$ is a homomorphism from the character ring $R_l(G_\infty)$ to the units of the "Iwasawa algebra" $\Lambda^c_{\wedge}(\Gamma_k)$ of k, which is Galois equivariant, compatible with W-twisting, and

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[1] The reference to S is normally suppressed.

which satisfies the congruences $L_{K_\infty/k}(\chi)^l \equiv \Psi(L_{K_\infty/k}(\psi_l\chi)) \mod l\Lambda^{\mathrm{c}}_{\wedge}(\Gamma_k)$. These properties of $L_{K_\infty/k}$ are the foundation of the proof of Theorem A. For the notation we refer to the introductory §1 which also contains the map Det : $K_1(\Lambda(G_\infty)_{\wedge}) \to \mathrm{HOM}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}_{\wedge}(\Gamma_k)^\times)$.

The technical core of the proof of Theorem A is

Theorem (B). $\quad \mathrm{Det}\, K_1(\Lambda(G_\infty)_{\wedge}) \cap \mathrm{HOM}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times) \subset \mathrm{Det}\, K_1(\Lambda(G_\infty))$

When G_∞ is an l-group, equivalent theorems are stated in [RW3] with $_\bullet$ in place of $_\wedge$; for the proofs in [RW3] the $_\wedge$-form of Theorem B is however essential (see [RW3,§6]). We have emphasized here the $_\wedge$-form because this technical advantage persists (e.g. in Proposition 2).

The proof in [RW3,§1] that Theorem B implies Theorem A works not only for general groups G_∞ but also with $_\bullet$ replaced by $_\wedge$: In its fourth paragraph every $_\bullet$ needs to become $_\wedge$. Therefore it remains to use induction techniques to reduce Theorem B to the l-group case. These techniques are generalizations of those in [Ty, Fr] for finite groups to the setting of Iwasawa theory.

In the same way we obtain

Theorem (C). $L_{K_\infty/k} \in \mathrm{Det}\, K_1(\Lambda(G_\infty)_{\wedge})$ *if, and only if,* $L_{K'/k'} \in \mathrm{Det}\, K_1(\Lambda(G_{K'/k'})_{\wedge})$ *whenever* $G_{K'/k'}$ *is an* l*-elementary section of* G_∞*.*

Here $G_{K'/k'}$ is a *section* of G_∞, if $k \subset k' \subset K' \subset K_\infty$ is such that k'/k is finite and K_∞/K' finite Galois; a section $G_{K'/k'}$ is l*-elementary*, if $G_{K'/k'} = \langle s\rangle \times U$ for some finite cyclic subgroup $\langle s\rangle$ of order prime to l and some open l-subgroup U.

If G_∞ is abelian, then the "main conjecture" holds by the Corollary to Theorem 9 in [RW3]. Theorem C provides first nonabelian examples of the "main conjecture". We expect more such examples to follow from the logarithmic methods of [RW3] for l-elementary groups. In more generality we know only that some l-power of $L_{K_\infty/k}$ is in Det $K_1(\Lambda(G_\infty)_{\wedge})$.

The paper is organized as follows. Its first section has some background material. In §2 we discuss $K_1(\Lambda(G_\infty))$ for $\mathbb{Q}_l$ - l - elementary groups G_∞ and deduce Theorems B and C for them. Then §3 is preliminary material on $\mathbb{Q}_l$ - q - elementary groups G_∞, with q a prime number different from l, which is used for the proof, in §4, of the full Theorems B and C. In §5 the examples appear.

We remark that because Theorems A and C are based on [RW3] they depend on the vanishing of Iwasawa's μ-invariant for k'_∞/k', for which we refer to [Ba].

1. Background

The Iwasawa L-function $L_{K_\infty/k,S}$ of K_∞/k is defined as follows (compare [RW2,§4]). Let χ be a $\mathbb{Q}_l{}^{\mathrm{c}}$-character of G_∞ with open kernel and write the l-adic S-truncated

Artin L-function $L_{l,S}(1-s,\chi)$, for $s \in \mathbb{Z}_l$, as the fraction $L_{l,S}(1-s,\chi) = \frac{G_{\chi,S}(u^s-1)}{H_\chi(u^s-1)}$ of the Deligne-Ribet power series $G_{\chi,S}(T)\,,\ H_\chi(T) \in \mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[[T]]$ associated to a generator γ_k of Γ_k [DR]. Above, $u \in 1 + l\mathbb{Z}_l$ describes the action of γ_k on the l-power roots of unity. Now set

$$L_{K_\infty/k,S}(\chi) = \frac{G_{\chi,S}(\gamma_k - 1)}{H_\chi(\gamma_k - 1)}$$

(which is independent of the choice of γ_k).

Recall that $\mathcal{Q}(G_\infty)$ is the total ring of fractions of the completed group ring

$$\Lambda(G_\infty) = \mathbb{Z}_l[[G_\infty]]$$

of G_∞ over $\mathbb{Z}_l$ (it is enough to invert the nonzero elements of $\Lambda(\Gamma)$ for a central open subgroup $\Gamma \simeq \mathbb{Z}_l$). The algebra $\mathcal{Q}(G_\infty)$ is a finite dimensional semisimple algebra over $\mathcal{Q}(\Gamma)$ with Γ, as before, central open in G_∞.

The map

$$\mathrm{Det}\,: K_1(\mathcal{Q}(G_\infty)) \to \mathrm{Hom}^*(R_l(G_\infty), \mathcal{Q}^{\mathrm{c}}(\Gamma_k)^\times)$$

is now defined as follows (compare [RW2,§3]).
If $[P,\alpha]$ represents an element in $K_1(\mathcal{Q}(G_\infty))$, with P a finitely generated projective $\mathcal{Q}(G_\infty)$-module and α an $\mathcal{Q}(G_\infty)$-automorphism of P, then

Det $[P,\alpha]$ is the function in Hom^* which takes the irreducible χ to

$$\det{}_{\mathcal{Q}^{\mathrm{c}}(\Gamma_k)}(\alpha \mid \mathrm{Hom}_{\mathbb{Q}_l{}^{\mathrm{c}}[H]}(V_\chi, \mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} P))\ .$$

Here, $\mathcal{Q}^{\mathrm{c}}(\Gamma_k) = \mathbb{Q}_l{}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} \mathcal{Q}(\Gamma_k)$, and V_χ is a $\mathbb{Q}_l{}^{\mathrm{c}}$-representation of G_∞ with character χ (always with open kernel). The $*$ on Hom requires $G_{\mathbb{Q}_l{}^{\mathrm{c}}/\mathbb{Q}_l}$-invariance and compatibility with W-twists; these properties are inherited from the representation theory of $\mathcal{Q}(G_\infty)$.

Restricting Det to $K_1(\Lambda(G_\infty))$, it takes values in $\mathrm{Hom}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times)$, with $\Lambda^{\mathrm{c}}(\Gamma_k) = \mathbb{Z}_l{}^{\mathrm{c}} \otimes_{\mathbb{Z}_l} \Lambda(\Gamma_k)$, and indeed $\mathrm{Det}\,x = f$ has values satisfying the congruences

$$f(\chi)^l \equiv \varPsi(f(\psi_l\chi)) \mod l\Lambda^{\mathrm{c}}(\Gamma_k)\,,$$

which define the subgroup $\mathrm{HOM}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times)$ of Hom^* (see [RW3,§2]).
Above, $\varPsi$ is the $\mathbb{Z}_l{}^{\mathrm{c}}$-algebra endomorphism of $\Lambda^{\mathrm{c}}(\Gamma_k)$ induced by $\gamma \mapsto \gamma^l$ on Γ_k, and ψ_l is the l-th Adams operation on $R_l(G_\infty)$.
However, the values $L_{K_\infty/k}(\chi)$ are not in $\Lambda^{\mathrm{c}}(\Gamma_k)^\times$ but in $\Lambda^{\mathrm{c}}_\bullet(\Gamma_k)^\times$, where $\Lambda^{\mathrm{c}}_\bullet(\Gamma_k) = \mathbb{Z}_l{}^{\mathrm{c}} \otimes_{\mathbb{Z}_l} \Lambda(\Gamma_k)_\bullet$ with $\Lambda(\Gamma_k)_\bullet$ the localization of $\Lambda(\Gamma_k)$ at l. We work with the completion $\Lambda(\Gamma_k)_\wedge$ of $\Lambda(\Gamma_k)_\bullet$ at l because logarithmic methods apply to $K_1(\Lambda(G_\infty)_\wedge)$ (see [RW3, beginning of §5]). We arrive at

$$\mathrm{Det}\,: K_1(\Lambda(G_\infty)_\wedge) \to \mathrm{HOM}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}_\wedge(\Gamma_k)^\times)\,,$$

with $\Lambda^{\mathrm{c}}_\wedge(\Gamma_k) = \mathbb{Z}_l{}^{\mathrm{c}} \otimes_{\mathbb{Z}_l} \Lambda(\Gamma_k)_\wedge$, and now $L_{K_\infty/k} \in \mathrm{HOM}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}_\wedge(\Gamma_k)^\times)\,.$

The induction techniques that we are going to apply will also involve $\Lambda^{\mathfrak{O}}(G) = \mathfrak{O} \otimes_{\mathbb{Z}_l} \Lambda(G)$ and $\Lambda^{\mathfrak{O}}(G)_\wedge$, where $\mathfrak{O}$ is the ring of integers of a finite unramified extension $N/\mathbb{Q}_l$. All that has been said so far remains true except that the $G_{\mathbb{Q}_l{}^c/\mathbb{Q}_l}$-invariance on Hom^* gets replaced by $G_{\mathbb{Q}_l{}^c/N}$-invariance to define Hom^N and that the Frobenius automorphism Fr of $N/\mathbb{Q}_l$ appears (see [RW3, Proposition 4]).

2. $\mathbb{Q}_l$ - l - elementary groups G_∞

In this section the Galois group $G_\infty = G_{K_\infty/k}$ is assumed to be $\mathbb{Q}_l$ - l - *elementary*, i.e., a semidirect product $G_\infty = \langle s\rangle \rtimes U$ of a finite cyclic group $\langle s\rangle$ of order prime to l and an open l-subgroup U whose action on $\langle s\rangle$ induces a homomorphism $U \to G_{\mathbb{Q}_l(\zeta)/\mathbb{Q}_l}$, where ζ is a root of unity of order $|\langle s\rangle|$.

We fix a set $\{\beta_i\}$ of representatives of $G_{\mathbb{Q}_l{}^c/\mathbb{Q}_l}$-orbits of the $\mathbb{Q}_l{}^c$-irreducible characters of $\langle s\rangle$ and denote the stabilizer group of β_i by $U_i = \{u \in U : \beta_i^u = \beta_i\}$. Note that $U_i \lhd U$ and set $A_i = U/U_i \leqslant G_{N_i/\mathbb{Q}_l}$, with N_i the field of character values of β_i

Theorem 1. *1. There are natural maps r, r' so that*

$$\begin{array}{ccc} K_1(\Lambda(G_\infty)) & \xrightarrow{r} & \prod_i K_1(\Lambda^{\mathfrak{O}_i}(U_i)) \\ \mathrm{Det}\downarrow & & \mathrm{Det}\downarrow \\ \mathrm{Hom}^*(R_l(G_\infty), \Lambda^c(\Gamma_k)^\times) & \overset{r'}{\rightarrowtail} & \prod_i \mathrm{Hom}^{N_i}(R_l(U_i), \Lambda^c(\Gamma_{k_i})^\times) \end{array}$$

commutes and r' is injective. Here $k_i = K_\infty{}^{U_i}$ and $\mathfrak{O}_i$ is the ring of integers of N_i. Moreover, r induces an isomorphism

$$\mathrm{Det}\, K_1(\Lambda(G_\infty)) \to \prod_i (\mathrm{Det}\, K_1(\Lambda^{\mathfrak{O}_i}(U_i)))^{A_i}\,.$$

2. The same holds in the completed situation, i.e., with Λ replaced by $\Lambda_\wedge$.

Proof. (Compare [Ty, p.67-71] or [Fr, p.89-96].) In order to use subscripts we abbreviate G_∞ by G.

Set $G_i = \langle s\rangle \rtimes U_i$, $e_i = \frac{1}{|\langle s\rangle|}\sum_{j \bmod |\langle s\rangle|} \mathrm{tr}_{N_i/\mathbb{Q}_l}(\beta_i(s^{-j}))s^j \in \mathbb{Z}_l\langle s\rangle$ and let $R_l^{(e_i)}(G) \subset R_l(G)$ be the span of the irreducible $\chi \in R_l(G)$ with $\chi(e_i) \neq 0$. Observe that e_i is a central idempotent of $\Lambda(G_\infty)$.

We first glue the following squares together

$$\begin{array}{ccc} K_1(\Lambda(G)) & \xrightarrow{\mathrm{res}_G^{G_i}} & K_1(\Lambda(G_i)) \\ \mathrm{Det}\downarrow & & \mathrm{Det}\downarrow \\ \mathrm{Hom}^*(R_l(G), \Lambda^c(\Gamma_k)^\times) & \xrightarrow{\mathrm{res}_G^{G_i}} & \mathrm{Hom}^*(R_l(G_i), \Lambda^c(\Gamma_{k_i})^\times) \end{array}$$

$$\begin{array}{ccc} K_1(\Lambda(G_i)) & \to & K_1(e_i\Lambda(G_i)) \\ \mathrm{Det}\downarrow & & \mathrm{Det}\downarrow \\ \mathrm{Hom}^*(R_l(G_i), \Lambda^c(\Gamma_{k_i})^\times) & \to & \mathrm{Hom}^*(R_l^{(e_i)}(G_i), \Lambda^c(\Gamma_{k_i})^\times) \end{array} \quad .$$

Actually, both diagrams should have the field $k_i' = K_\infty{}^{G_i}$ in place of k_i; however, $\Gamma_{k_i'}$ and Γ_{k_i} get identified as subgroups of Γ_k since $[k_i : k_i'] = |\langle s\rangle|$ is not divisible by l.

The upper diagram commutes by [RW2, Lemma 9], and $\Lambda(G_i) = e_i\Lambda(G_i) \times (1 - e_i)\Lambda(G_i)$ implies the commutativity of the bottom one. Note that there is no ambiguity in writing $\mathrm{Hom}^*(R_l^{(e_i)}(G_i), \Lambda^c(\Gamma_{k_i})^\times)$ because $\chi(e_i) = (\chi\rho)(e_i)$ for characters ρ of G_i of type W.

There are natural actions of $A_i = G/G_i$ on $K_1(\Lambda(G_i))$ and on

$$\mathrm{Hom}^*(R_l(G_i), \Lambda^c(\Gamma_{k_i})^\times);$$

moreover,

$$\mathrm{res}\,_G^{G_i}(K_1(\Lambda(G))) \subset K_1(\Lambda(G_i))^{A_i},$$
$$\mathrm{res}\,_G^{G_i}(\mathrm{Hom}^*(R_l(G), \Lambda^c(\Gamma_k)^\times)) \subset (\mathrm{Hom}^*(R_l(G_i), \Lambda^c(\Gamma_{k_i})^\times))^{A_i}.$$

The maps in the bottom diagram are all A_i-equivariant. For this we only need to check the A_i-equivariance of $\mathrm{Det} : K_1(\mathcal{Q}(G_i)) \to \mathrm{Hom}^*(R_l(G_i), \mathcal{Q}^c(\Gamma_{k_i})^\times)$:
Set $H_i = \ker(G_i \to \Gamma_{k_i})$. Further, let $[P, \alpha]$ represent an element of $K_1(\mathcal{Q}(G_i))$, with α an automorphism of the projective module P. If $a \in A_i$ has preimage $g \in G$, then $[P, \alpha]^a = [P^{[g]}, \alpha^{[g]}]$ where $P^{[g]} = \{[p] : p \in P\}$ with $y[p] = [y^{g^{-1}}p]$ for $y \in G_i$ and $\alpha^{[g]}([p]) = [\alpha(p)]$. Taking $V = V_{\chi^{g^{-1}}}$, so $V^{[g]} = V_\chi$, it suffices to show that

$$\mathrm{Hom}_{\mathbb{Q}_l{}^c[H_i]}(V, \mathbb{Q}_l{}^c \otimes_{\mathbb{Q}_l} P) \to \mathrm{Hom}_{\mathbb{Q}_l{}^c[H_i]}(V^{[g]}, \mathbb{Q}_l{}^c \otimes_{\mathbb{Q}_l} P^{[g]}),$$
$$\varphi \mapsto [\varphi] \text{ with } [\varphi]([v]) = [\varphi(v)]$$

is a $\mathcal{Q}^c(\Gamma_{k_i})$-vector space isomorphism which is natural for the respective actions of α. Now,

$$\begin{aligned}&(y[\varphi])([v]) = y([\varphi](y^{-1}[v])) = y([\varphi]([y^{-g^{-1}}v]))\\ &= y[\varphi(y^{-g^{-1}}v)] = [y^{g^{-1}}(\varphi(y^{-g^{-1}}v))] = [(y^{g^{-1}}\varphi)(v)],\end{aligned}$$

and taking $y \in H_i$ implies that $[\varphi] \in \mathrm{Hom}_{\mathbb{Q}_l{}^c[H_i]}(V^{[g]}, \mathbb{Q}_l{}^c \otimes_{\mathbb{Q}_l} P^{[g]})$. Reading the above for $y \in \Gamma_{k_i}$ we see the map is $\mathcal{Q}^c(\Gamma_{k_i})$-linear.

By composing the above two squares we arrive at

$$\text{(D1)} \qquad \begin{array}{ccc} K_1(\Lambda(G)) & \to & \prod_i K_1(e_i\Lambda(G_i))^{A_i} \\ \mathrm{Det} \downarrow & & \mathrm{Det} \downarrow \\ \mathrm{Hom}^*(R_l(G), \Lambda^c(\Gamma_k)^\times) & \to & \prod_i \mathrm{Hom}^*(R_l^{(e_i)}(G_i), \Lambda^c(\Gamma_{k_i})^\times)^{A_i}. \end{array}$$

We claim that the lower horizontal map in (D1) is injective. To see this we first observe that it is also the composite

$$\begin{aligned}\mathrm{Hom}^*(R_l(G), \Lambda^c(\Gamma_k)^\times) &\to \prod_i \mathrm{Hom}^*(R_l^{(e_i)}(G), \Lambda^c(\Gamma_k)^\times)\\ &\to \prod_i \mathrm{Hom}^*(R_l^{(e_i)}(G_i), \Lambda^c(\Gamma_{k_i})^\times)\end{aligned}$$

and that $R_l(G) = \bigoplus_i R_l^{(e_i)}(G)$. Hence, as induction on characters is restriction on Hom^*, we are done once we know $\mathrm{ind}\,_{G_i}^{G_\infty}(R_l^{(e_i)}(G_i)) = R_l^{(e_i)}(G)$. However, if $\chi \in R_l(G)$ is irreducible, then Clifford theory [CR I, 11.8, p.265] implies $\chi = \mathrm{ind}\,_{G_i}^{G}(\tilde{\beta}_i^\sigma \xi)$

for some irreducible $\xi \in R_l(U_i)$ and the i and $\sigma \in G_{N_i/\mathbb{Q}_l}$ so that β_i^σ appears in $\mathrm{res}_G^{\langle s\rangle}(\chi)$; here $\tilde\beta_i \in R_l(G_i)$ is defined by $\tilde\beta_i(s^j u) = \beta_i(s^j)$.

Note that $e_i\Lambda(G_i) = e_i\mathbb{Z}_l\langle s\rangle \otimes_{\mathbb{Z}_l} \Lambda(U_i)$ is, via β_i, isomorphic to $\mathfrak{O}_i \otimes_{\mathbb{Z}_l} \Lambda(U_i) = \Lambda^{\mathfrak{O}_i}(U_i)$. We next show that the square

$$\text{(D2)}\qquad \begin{array}{ccc} K_1(e_i\Lambda(G_i)) & \xrightarrow{\beta_i} & K_1(\Lambda^{\mathfrak{O}_i}(U_i)) \\ \mathrm{Det}\downarrow & & \mathrm{Det}\downarrow \\ \mathrm{Hom}^*(R_l^{(e_i)}(G_i), \Lambda^{\mathrm{c}}(\Gamma_{k_i})^\times) & \overset{\beta_i^*}{\rightarrowtail} & \mathrm{Hom}^{N_i}(R_l(U_i), \Lambda^{\mathrm{c}}(\Gamma_{k_i})^\times) \end{array}$$

commutes, with the top horizontal map induced by β_i and β_i^* defined by $f \mapsto f'$, $f'(\xi) = f(\tilde\beta_i\xi)$. The map β_i^* is injective because $R_l^{(e_i)}(G_i)$ is spanned by the $\tilde\beta_i^\sigma\xi$.

Turning to the commutativity of (D2), it suffices to show that $(\mathrm{Det}\,(\alpha))' = \mathrm{Det}\,(\beta_i(\alpha))$ for units $\alpha \in e_i\Lambda(G_i)$, by [CR II, p.76]. Now, with V_ξ denoting a ${\mathbb{Q}_l}^{\mathrm{c}}$-realization of $\xi \in R_l(G_i)$,

$$\mathrm{Det}\,(\beta_i(\alpha))(\xi) = \det_{\mathcal{Q}^{\mathrm{c}}(\Gamma_{k_i})}(\beta_i(\alpha) \mid \mathrm{Hom}_{{\mathbb{Q}_l}^{\mathrm{c}}[H_i']}(V_\xi, {\mathbb{Q}_l}^{\mathrm{c}} \otimes_{N_i} \mathcal{Q}^{N_i}(U_i))) \quad \text{and}$$
$$\mathrm{Det}\,(\alpha)(\tilde\beta_i\xi) = \det_{\mathcal{Q}^{\mathrm{c}}(\Gamma_{k_i})}(\alpha \mid \mathrm{Hom}_{{\mathbb{Q}_l}^{\mathrm{c}}[H_i]}(V_{\tilde\beta_i\xi}, {\mathbb{Q}_l}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} (e_i\mathbb{Q}_l\langle s\rangle \otimes_{\mathbb{Q}_l} \mathcal{Q}(G_i))))$$

where H_i, as before, equals $\ker(G_i \to \Gamma_{k_i})$ and $H_i' = H_i/\langle s\rangle$; see [RW2, §3]. Hence it suffices to exhibit a $\mathcal{Q}^{\mathrm{c}}(\Gamma_{k_i})$-isomorphism

$$\mathrm{Hom}_{{\mathbb{Q}_l}^{\mathrm{c}}[H_i']}(V_\xi, \mathcal{Q}^{\mathrm{c}}(U_i)) \longrightarrow \mathrm{Hom}_{{\mathbb{Q}_l}^{\mathrm{c}}[H_i]}(V_{\tilde\beta_i\xi}, ({\mathbb{Q}_l}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} e_i\mathbb{Q}_l\langle s\rangle) \otimes_{{\mathbb{Q}_l}^{\mathrm{c}}} \mathcal{Q}^{\mathrm{c}}(U_i))$$

which is natural for the respective actions of α. Such a map is given by multiplying $\varphi' \in \mathrm{Hom}_{{\mathbb{Q}_l}^{\mathrm{c}}[H_i']}$ by the idempotent $\varepsilon_i = \frac{1}{|\langle s\rangle|}\sum_{j \bmod |\langle s\rangle|} \beta_i(s^{-j}) \otimes e_i s^j$ of ${\mathbb{Q}_l}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} e_i\mathbb{Q}_l\langle s\rangle$. This map is surjective since ε_i acts as the identity on $V_{\tilde\beta_i\xi}$, hence every $\varphi \in \mathrm{Hom}_{{\mathbb{Q}_l}^{\mathrm{c}}[H_i]}$ has image in $\varepsilon_i({\mathbb{Q}_l}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} e_i\mathbb{Q}_l\langle s\rangle) \otimes_{{\mathbb{Q}_l}^{\mathrm{c}}} \mathcal{Q}^{\mathrm{c}}(U_i) = \varepsilon_i \otimes_{{\mathbb{Q}_l}^{\mathrm{c}}} \mathcal{Q}^{\mathrm{c}}(U_i)$.

Combining (D1) and (D2) gives the commutative square in 1. of the theorem. To complete the proof we are left with showing

$$\mathrm{Det}\,K_1(\Lambda(G)) \simeq \prod_i (\mathrm{Det}\,K_1(\Lambda^{\mathfrak{O}_i}(U_i)))^{A_i}.$$

We first check that the maps in (D2) are all A_i-equivariant. The left Det has already been dealt with. The right Det will follow since β_i is an isomorphism.

1. The natural embedding $a \mapsto \sigma_a : A_i \to G_{N_i/\mathbb{Q}_l}$ is determined by $\beta_i(s^a) = \beta_i(s)^{\sigma_a}$ and we transport the conjugation action of G on $e_i\mathbb{Z}_l\langle s\rangle \otimes_{\mathbb{Z}_l} \Lambda(U_i)$ to $\Lambda^{\mathfrak{O}_i}(U_i)$ by β_i, hence $\beta_i : K_1(e_i\mathbb{Z}_l\langle s\rangle \otimes_{\mathbb{Z}_l} \Lambda(U_i)) \to K_1(\Lambda^{\mathfrak{O}_i}(U_i))$ is A_i-equivariant.
2. We show that β_i^* is A_i-equivariant, with the action of A_i on $\varphi \in \mathrm{Hom}^{N_i}(R_l(U_i), \Lambda^{\mathrm{c}}(\Gamma_{k_i})^\times)$ defined by $\varphi^a(\xi) = \varphi(\xi^{a^{-1}})^{\sigma_a}$, where $\sigma_a \in G_{N_i/\mathbb{Q}_l}$ is extended to ${\mathbb{Q}_l}^{\mathrm{c}}$ so that it is the identity on l-power roots of unity; this is possible since $N_i/\mathbb{Q}_l$ is unramified. Note that φ^a is well-defined since changing σ_a to $\sigma\sigma_a$, with $\sigma \in G_{{\mathbb{Q}_l}^{\mathrm{c}}/\mathbb{Q}_l}$ the identity on $N_i(\zeta_{l^\infty})$, gives $\varphi(\xi^{a^{-1}})^{\sigma\sigma_a} = \varphi(\xi^{a^{-1}\sigma})^{\sigma_a} =$

$\varphi(\xi^{a^{-1}})^{\sigma_a}$ as $\xi^{a^{-1}}$ is a character of the l-group U_i. Moreover, $\varphi^a \in \mathrm{Hom}^{N_i}$: If $\sigma \in G_{\mathbb{Q}_l{}^c/N_i}$, then $\varphi^a(\xi^\sigma) = \varphi(\xi^{\sigma a^{-1}})^{\sigma_a} = \varphi(\xi^{a^{-1}\sigma})^{\sigma_a} = \varphi(\xi^{a^{-1}})^{\sigma\sigma_a} = (\varphi(\xi^{a^{-1}})^{\sigma\sigma_a\sigma^{-1}})^\sigma = \varphi^a(\xi)^\sigma$, because $\sigma\sigma_a\sigma^{-1}$ is also an admissible extension of σ_a.

The A_i-equivariance of the map β_i^* now follows from $\beta_i^a = \beta_i^{\sigma_a^{-1}}$ (which is a reformulation of $\beta_i(s^{a^{-1}}) = \beta_i(s)^{\sigma_a^{-1}}$). Namely, let $f' \in \mathrm{Hom}^{N_i}$ be the image of $f \in \mathrm{Hom}^*$ and let $f'' \in \mathrm{Hom}^{N_i}$ be that of f^a. Then $f''(\xi) = f^a(\tilde\beta_i\xi) = f(\tilde\beta_i^{a^{-1}}\xi^{a^{-1}}) = f((\tilde\beta_i\xi^{a^{-1}})^{\sigma_a}) = f(\tilde\beta_i\xi^{a^{-1}})^{\sigma_a} = f'(\xi^{a^{-1}})^{\sigma_a} = (f')^a(\xi)$.

For 1. of Theorem 1 it now remains to show that r' induces an epimorphism $\mathrm{Det}\, K_1(\Lambda(G)) \twoheadrightarrow \prod_i (\mathrm{Det}\, K_1(\Lambda^{\mathfrak{O}_i}(U_i)))^{A_i}$. From

$$\begin{array}{ccccc}
K_1(\Lambda(G)) & \xrightarrow{\mathrm{res}_G^{G_i}} & K_1(\Lambda(G_i)) & & \\
\downarrow & & \downarrow & & \\
K_1(e_i\Lambda(G)) & \xrightarrow{\mathrm{res}_G^{G_i}} & K_1(e_i\Lambda(G_i)) & \xrightarrow{\beta_i,\simeq} & K_1(\Lambda^{\mathfrak{O}_i}(U_i))
\end{array}$$

and the surjectivity of the left vertical arrow we deduce

$$\mathrm{im}\,(r) \supset \prod_i \beta_i \mathrm{res}_G^{G_i}(K_1(e_i\Lambda(G))) \supset \prod_i \beta_i \mathrm{res}_G^{G_i}\mathrm{ind}_{G_i}^G(K_1(e_i\Lambda(G_i)))\,.$$

Hence, by [RW2, Lemma 9] and [RW3, Lemma 1],

$$\begin{array}{l}
r'(\mathrm{Det}\, K_1(\Lambda(G))) \supset \prod_i \beta_i^* \mathrm{res}_G^{G_i}\mathrm{ind}_{G_i}^G(\mathrm{Det}\, K_1(e_i\Lambda(G_i))) \\
\mathrel{\mathring{=}} \prod_i \beta_i^* \mathrm{N}_{A_i}(\mathrm{Det}\, K_1(e_i\Lambda(G_i))) = \prod_i \mathrm{N}_{A_i}(\mathrm{Det}\, K_1(\Lambda^{\mathfrak{O}_i}(U_i)))
\end{array}$$

where $\mathrel{\mathring{=}}$ is due to Mackey's subgroup theorem and $G/G_i = A_i$:

$$\mathrm{res}_G^{G_i}\mathrm{ind}_{G_i}^G(f_i)(\tilde\beta_i^\sigma\xi) = f_i(\mathrm{res}_G^{G_i}\mathrm{ind}_{G_i}^G(\tilde\beta_i^\sigma\xi)) = (\prod_{a\in A_i} f_i^a)(\tilde\beta_i^\sigma\xi) = (\mathrm{N}_{A_i}f)(\tilde\beta_i^\sigma\xi)\,.$$

All arguments above apply to 2. of Theorem 1 without changes.

The proposition below now finishes the proof of Theorem 1. □

Proposition 2. $\mathrm{N}_{A_i}(\mathrm{Det}\, K_1(\Lambda^{\mathfrak{O}_i}(U_i))) = (\mathrm{Det}\, K_1(\Lambda^{\mathfrak{O}_i}(U_i)))^{A_i}$ *and the same with* Λ *replaced by* $\Lambda_\wedge$.

Since the U in $G_\infty = \langle s\rangle \rtimes U$ will not occur in the proof of the proposition, we drop the index i throughout, so U $(= U_i)$ is now a pro-l group and we need to consider the A-module $\mathrm{Det}\, K_1(\Lambda^{\mathfrak{O}}(U))$. Recall that A acts on U by group automorphisms and on $\mathfrak{O}$ by $A \rightarrowtail G_{N/\mathbb{Q}_l}$.

Let $\mathfrak{a}$ denote the kernel of $\Lambda(U) \to \Lambda(U^{\mathrm{ab}})$ and set $\mathfrak{A} = \mathfrak{O} \otimes_{\mathbb{Z}_l} \mathfrak{a}$.

By surjectivity of $(\Lambda^{\mathfrak{O}}(U))^\times \to K_1(\Lambda^{\mathfrak{O}}(U))$ (see [CR II, p.76]) we have $\mathrm{Det}\,(\Lambda^{\mathfrak{O}}(U)^\times) = \mathrm{Det}\, K_1(\Lambda^{\mathfrak{O}}(U))$.

We start out the proof of the proposition from the diagram

$$\begin{array}{ccccc}
1+\mathfrak{A} & \rightarrowtail & \Lambda^{\mathfrak{O}}(U)^\times & \twoheadrightarrow & \Lambda^{\mathfrak{O}}(U^{\mathrm{ab}})^\times \\
\mathrm{Det}\downarrow & & \mathrm{Det}\downarrow & & \mathrm{Det}\downarrow \\
\mathrm{Det}\,(1+\mathfrak{A}) & \rightarrowtail & \mathrm{Det}\,(\Lambda^{\mathfrak{O}}(U)^\times) & \twoheadrightarrow & \mathrm{Det}\,(\Lambda^{\mathfrak{O}}(U^{\mathrm{ab}}))^\times)
\end{array}$$

with the top row exact because $\mathfrak{a}$ is contained in the radical of $\Lambda(U)$. The right square of the diagram commutes [RW2, Lemma 9] and the right Det is an isomorphism (see [CR II, 45.12, p.142]). Therefore the whole diagram commutes and its bottom sequence is exact.

We claim that $\mathrm{Det}\,(1+\mathfrak{A}) \simeq \tau(\mathfrak{A})$ with $\tau(\mathfrak{A})$ the image of $\mathfrak{A} \subset \Lambda^{\mathfrak{O}}(G_\infty)$ in $T(\Lambda^{\mathfrak{O}}(G_\infty)) = \Lambda^{\mathfrak{O}}(G_\infty)/[\Lambda^{\mathfrak{O}}(G_\infty), \Lambda^{\mathfrak{O}}(G_\infty)]$ (see [RW3,§3]) . Since $\mathbf{L} : \mathrm{Det}\,(1+\mathfrak{A}) \to \mathrm{Tr}(\tau(\mathfrak{A}))$ is an isomorphism by the Corollary to Theorem $\mathrm{B}_{\wedge}$ in [RW3], it remains to see that $\mathbf{L}$ and Tr are A-equivariant. For $\mathbf{L}$ this follows as Ψ is induced by $\gamma \mapsto \gamma^l$ for $\gamma \in \Gamma_k$. For Tr it follows from Lemma 6 and Proposition 3 of [RW3] : Let $a \in A$, $\omega \in \mathfrak{O}$, and $u \in U$. Then

$$\begin{aligned} &\mathrm{Tr}(\omega u)^a(\chi) = \mathrm{Tr}(\omega u)(\chi^{a^{-1}})^{\sigma_a} = \mathrm{trace}(\omega u \mid \mathfrak{V}_{\chi^{a^{-1}}})^{\sigma_a} = (\omega \chi^{a^{-1}}(u)\overline{u})^{\sigma_a} \\ &= \omega^{\sigma_a}\chi(u^a)\overline{u} = \mathrm{trace}(\omega^{\sigma_a}u^a \mid \mathfrak{V}_\chi) = \mathrm{Tr}(\omega^{\sigma_a}u^a)(\chi). \end{aligned}$$

Collecting everything so far, the starting diagram gives the exact A-module sequence

$$\tau(\mathfrak{A}) \rightarrowtail \mathrm{Det}\,(\Lambda^{\mathfrak{O}}(U)^\times) \twoheadrightarrow \Lambda^{\mathfrak{O}}(U^{\mathrm{ab}})^\times .$$

So the proof of the proposition will be finished once we have shown that

$$\tau(\mathfrak{A}) \text{ and } \Lambda^{\mathfrak{O}}(U^{\mathrm{ab}})^\times \text{ are } A\text{-cohomologically trivial}.$$

For $\tau(\mathfrak{A})$ this holds because $\tau(\mathfrak{A}) = \mathfrak{O} \otimes_{\mathbb{Z}_l} \tau(\mathfrak{a})$ has diagonal A-action and $\mathfrak{O}$ is $\mathbb{Z}_l[A]$-cohomologically trivial, as $\mathfrak{O}/\mathbb{Z}_l$ is unramified. By [Se1, Theorem 9, p.152] then the tensor product is cohomologically trivial as well.

The proof of the cohomological triviality of $\Lambda^{\mathfrak{O}}(U^{\mathrm{ab}})^\times$ uses the following fact: If $(X_n, f_n : X_n \to X_{n-1})$ is a projective system of A-modules with surjective maps f_n, then $X = \varprojlim X_n$ is cohomologically trivial if all the X_n are. This holds because of the exact sequence $X \rightarrowtail \prod_n X_n \twoheadrightarrow \prod X_n$ in which $(\cdots, x_n, \cdots) \mapsto (\cdots, f_{n+1}(x_{n+1}) - x_n, \cdots)$ is the second map. Note that the X_n are cohomologically trivial, if X_1 and all $\ker(X_{n+1} \to X_n)$ are so.

Set $\mathfrak{g} = \ker(\Lambda(U^{\mathrm{ab}}) \to \Lambda(\Gamma_k))$ and $\mathfrak{G} = \mathfrak{O} \otimes_{\mathbb{Z}_l} \mathfrak{g}$. Since some power of $\mathfrak{g}$ is contained in $l\Lambda(U^{\mathrm{ab}})$ (compare the beginning of the proof of [RW3, Theorem 8]), $\Lambda(U^{\mathrm{ab}})$ is complete with respect to its $\mathfrak{g}$-adic topology. Also, $1 + \mathfrak{g} \subset \Lambda(U^{\mathrm{ab}})^\times$, and thus the short exact sequence $1 + \mathfrak{G} \rightarrowtail \Lambda^{\mathfrak{O}}(U^{\mathrm{ab}})^\times \twoheadrightarrow \Lambda^{\mathfrak{O}}(\Gamma_k)^\times$ implies the cohomological triviality of $\Lambda^{\mathfrak{O}}(U^{\mathrm{ab}})^\times$, if $1 + \mathfrak{G}$ and $\Lambda^{\mathfrak{O}}(\Gamma_k)^\times$ are A-cohomologically trivial.

Setting $X_n = \frac{1+\mathfrak{G}}{1+\mathfrak{G}^n}$, $\ker(X_{n+1} \to X_n) \simeq \mathfrak{O} \otimes_{\mathbb{Z}_l} \frac{\mathfrak{g}^n}{\mathfrak{g}^{n+1}}$, which is cohomologically trivial by [Se1, loc.cit.].

For the right term of the above short exact sequence we identify $\Lambda^{\mathfrak{O}}(\Gamma_k)$ and $\mathfrak{O}[[T]]$, as usual, and set $X_n = \frac{\mathfrak{O}[[T]]^\times}{1+T^n\mathfrak{O}[[T]]}$; so $X_1 = \mathfrak{O}^\times$ and $\ker(X_{n+1} \to X_n) = \mathfrak{O}$, which both are cohomologically trivial.

Adding $\Lambda_{\wedge}$ at the appropriate places, Proposition 2 is established.

Corollary (to Theorem 1) . *Let G_∞ be $\mathbb{Q}_l$ - l - elementary. Then*

$$\mathrm{Det}\, K_1(\Lambda(G_\infty)_{\wedge}) \cap \mathrm{Hom}^*(R_l(G_\infty), \Lambda^{c}(\Gamma_k)^\times) \subset \mathrm{Det}\, K_1(\Lambda(G_\infty)) .$$

Namely, by Theorem 1,

$$\begin{aligned}
&\operatorname{Det} K_1(\Lambda(G_\infty)_{\wedge}) \cap \operatorname{Hom}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times) \\
&\subset \textstyle\prod_i (\operatorname{Det} K_1(\Lambda^{\mathfrak{O}_i}(U_i)_{\wedge})^{A_i} \cap \prod_i \operatorname{Hom}^{N_i}(R_l(U_i), \Lambda^{\mathrm{c}}(\Gamma_{k_i})^\times)^{A_i} \\
&\subset \textstyle\prod_i \left(\operatorname{Det} K_1(\Lambda^{\mathfrak{O}_i}(U_i)_{\wedge}) \cap \operatorname{Hom}^{N_i}(R_l(U_i), \Lambda^{\mathrm{c}}(\Gamma_{k_i})^\times)\right)^{A_i} \\
&\mathrel{\dot{\subset}} \textstyle\prod_i (\operatorname{Det} K_1(\Lambda^{\mathfrak{O}_i}(U_i)))^{A_i} \subset \operatorname{Det} K_1(\Lambda(G_\infty))
\end{aligned}$$

with $\dot{\subset}$ by [RW3, Theorem $\mathrm{B}_{\wedge}$].

Proposition 3. *Let G_∞ be $\mathbb{Q}_l$ - l - elementary. Then $L_{K_\infty/k} \in \operatorname{Det} K_1(\Lambda(G_\infty)_{\wedge})$ if, and only if, $L_{K'/k'} \in \operatorname{Det} K_1(\Lambda(G_{K'/k'})_{\wedge})$ whenever $G_{K'/k'}$ is an l-elementary section of G_∞.*

If $L_{K_\infty/k} \in \operatorname{Det} K_1(\Lambda(G_\infty)_{\wedge})$ and if $G_{K'/k'} = G_{K_\infty/k'}\Big/G_{K_\infty/K'}$ is an l-elementary section of G_∞ with $k \subset k' \subset K' \subset K_\infty$, then $\operatorname{defl}^{G_{K'/k'}}_{G_{K_\infty/k'}} \operatorname{res}^{G_{K_\infty/k'}}_{G_\infty} L_{K_\infty/k} = L_{K'/k'}$ (see [RW2,§4]). And by [RW2, Lemma 9], $L_{K'/k'} \in \operatorname{Det} K_1(\Lambda(G_{K'/k'})_{\wedge})$.

For the converse it may help to review the notation of that part of the proof of Theorem 1 where (D2) appears. The point is that $\overline{G}_i \overset{\text{def}}{=} G_i/\ker\beta_i = \langle \overline{s}_i\rangle \times U_i$, with $\langle \overline{s}_i\rangle = \langle s\rangle/\ker\beta_i$, is an l-elementary section. And as $G_i = \langle s\rangle \rtimes U_i$,

$$\operatorname{Hom}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}_{\wedge}(\Gamma_k)^\times) \xrightarrow{\text{res}} \prod_i \operatorname{Hom}^*(R_l(G_i), \Lambda^{\mathrm{c}}_{\wedge}(\Gamma_{k_i})^\times)^{A_i} \xrightarrow{\text{defl}} \prod_i \operatorname{Hom}^*(R_l(\overline{G}_i), \Lambda^{\mathrm{c}}_{\wedge}(\Gamma_{k_i})^\times)^{A_i}$$

takes $L_{K_\infty/k}$ to $\prod_i L_{K_i'/k_i'}$ where $k_i' = K_\infty{}^{G_i}$ and $K_i' = K_\infty{}^{\ker\beta_i}$. Note here that the ith deflation map is A_i-equivariant since $\langle s\rangle \to \langle \overline{s}_i\rangle$ is so.

By assumption, $L_{K_i'/k_i'} = \operatorname{Det} y_i$ where $y_i \in K_1(\Lambda(\overline{G}_i)_{\wedge})$ and so $\operatorname{Det} y_i \in (\operatorname{Det} K_1(\Lambda(\overline{G}_i)_{\wedge}))^{A_i}$. Projecting to $e_i(\Lambda(\overline{G}_i))_{\wedge}$, $L_{K_i'/k_i'}$ induces a function in $\operatorname{Hom}^*(R_l^{(e_i)}(\overline{G}_i), \Lambda^{\mathrm{c}}_{\wedge}(\Gamma_{k_i})^\times)^{A_i}$. But $e_i(\Lambda(\overline{G}_i))_{\wedge} = \overline{e}_i(\Lambda(\overline{G}_i))_{\wedge} = e_i\mathbb{Z}_l\langle s\rangle \otimes_{\mathbb{Z}_l} \Lambda(U_i)_{\wedge}$, so $\overline{e}_i y_i \in K_1(e_i\mathbb{Z}_l\langle s\rangle \otimes_{\mathbb{Z}_l} \Lambda(U_i)_{\wedge})$ and $\operatorname{Det}(\overline{e}_i y_i) \in (\operatorname{Det} K_1(e_i\mathbb{Z}_l\langle s\rangle \otimes_{\mathbb{Z}_l} \Lambda(U_i)_{\wedge}))^{A_i}$. Now $\prod_i (\operatorname{Det} K_1(e_i\mathbb{Z}_l\langle s\rangle \otimes_{\mathbb{Z}_l} \Lambda(U_i)_{\wedge}))^{A_i} = \operatorname{Det} K_1(\Lambda(G_\infty)_{\wedge})$, by Theorem 1, and the proof is finished.

Remark. In Proposition 3, the Iwasawa L-function $L_{K_\infty/k}$ may be replaced by any function $f \in \operatorname{Hom}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}_{\wedge}(\Gamma_k)^\times)$ on setting $f_{K'/k'} = \operatorname{defl}^{G_{K'/k'}}_{G_{K_\infty/k'}} \operatorname{res}^{G_{K_\infty/k'}}_{G_\infty} f$ for all l-elementary sections $G_{K'/k'}$ of G_∞.

3. $\mathbb{Q}_l$ - q - elementary groups G_∞

In this section q is a prime number $\neq l$.

We say that the Galois group $G_\infty = G_{K_\infty/k}$ is a $\mathbb{Q}_l$ - q - *elementary* group, if $G_\infty = H \times \Gamma$ for some central open $\Gamma \leqslant G_\infty$ and a finite $\mathbb{Q}_l$ - q - elementary group H. Recall that a finite group H is called $\mathbb{Q}_l$ - q - elementary if it is a semidirect product $\langle s\rangle \rtimes H_q$ of a cyclic normal subgroup $\langle s\rangle$ of order prime to q and a q-group H_q whose action on $\langle s\rangle$ induces a homomorphism $H_q \to G_{\mathbb{Q}_l(\zeta)/\mathbb{Q}_l}$, where ζ is a root of unity of order $|\langle s\rangle|$.

Lemma 4.

1. *If Γ is a central open subgroup of G_∞ so that (the finite group) G_∞/Γ is a $\mathbb{Q}_l$ - q - elementary group, then G_∞ is $\mathbb{Q}_l$ - q - elementary.*
2. *Let G_∞ be $\mathbb{Q}_l$ - q - elementary, $G_\infty = H \times \Gamma$, $H = \langle s\rangle \rtimes H_q$. Then each irreducible character $\chi \in R_l(G_\infty)$ can be written as $\chi = \rho \cdot \mathrm{ind}_{G'}^{G_\infty}(\xi)$ with an abelian character ρ of G_∞ of type W and an abelian character ξ of a subgroup $G' \supset \langle s\rangle \times \Gamma$ of G_∞ so that $\xi = 1$ on Γ.*

In order to see 1. we pick a Sylow-l subgroup U of G_∞ containing the central open Γ. Then U/Γ is an l-subgroup of the finite $\mathbb{Q}_l$ - q - elementary group G_∞/Γ, hence cyclic and normal in G_∞/Γ. We conclude that U is an abelian normal subgroup of G_∞, and, moreover, that $G_\infty = U \rtimes H'$ with a finite $\mathbb{Q}_l$ - q - elementary group H' of order prime to l. Writing the abelian U as $U = H_l \times \Gamma_1$ with H_l finite (cyclic) and $\Gamma_1 \simeq \mathbb{Z}_l$, so $H_l \lhd G_\infty$, the usual Maschke argument provides a $\mathbb{Z}_l[H']$-decomposition $U = H_l \times \Gamma_2$ with $\Gamma_2 \simeq \mathbb{Z}_l$, by $|H'| \in {\mathbb{Z}_l}^\times$. We infer from $\Gamma^{l^n} \subset \Gamma_2$ for some n that H' acts trivially on Γ_2. Thus $G_\infty = H \times \Gamma_2$ with $H = H_l \rtimes H'$ a finite $\mathbb{Q}_l$ - q - elementary group and Γ_2 central open in G_∞.

For 2. we first restrict χ to Γ and obtain $\mathrm{res}^{\Gamma}_{G_\infty}\chi = \chi(1) \cdot \rho_1$ for some abelian character ρ_1 of Γ. Via $G_\infty/H = \Gamma_k$, ρ_1 is the restriction of a type W character ρ of G_∞. Since $\chi\rho^{-1}$ is trivial on Γ, we may henceforth assume that χ is trivial on Γ, whence is inflated from an irreducible ${\mathbb{Q}_l}^{\mathrm{c}}$-character of H. By Clifford theory [CR I, p.265] the ${\mathbb{Q}_l}^{\mathrm{c}}$-irreducible characters of H are of the form $\mathrm{ind}_{\tilde{H}}^{H}(\tilde{\xi} \cdot \omega)$ with an abelian character $\tilde{\xi}$ of some subgroup $\tilde{H} \geqslant \langle s\rangle$ and an irreducible character ω of $\tilde{H}/\langle s\rangle$ (inflated to $\tilde{H}$). The group $\tilde{H}/\langle s\rangle$ is a q-group, so monomial, from which we deduce an equality $\mathrm{ind}_{\tilde{H}}^{H}(\tilde{\xi} \cdot \omega) = \mathrm{ind}_{H'}^{H}(\xi)$ with $\langle s\rangle \leqslant H' \leqslant \tilde{H}$ and an abelian character ξ of H'. Setting $G' = H' \times \Gamma$ finishes the proof of 2. and of the lemma.

Lemma 5. *Assume that $G_\infty = H \times \Gamma$ with H of order prime to l. Then $\mathcal{Q}(G_\infty)$ is the group algebra of the finite group H over the field $\mathcal{Q}(\Gamma)$ and each $f \in \mathrm{Hom}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times)$ is a $\mathrm{Det}\, z$ for some $z \in \Lambda(G_\infty)^\times$.*

This is straightforward: $\mathcal{Q}(G_\infty) = \mathcal{Q}(\Gamma)[H] = \mathcal{Q}(\Gamma) \otimes_{\mathbb{Q}_l} \mathbb{Q}_l[H]$ is isomorphic to a product of matrix rings over the character fields $\mathcal{Q}(\Gamma)(\chi)$ (see [CR II, 74.11, p.740]), where χ runs through the ${\mathbb{Q}_l}^{\mathrm{c}}$-irreducible characters of H modulo $G_{{\mathbb{Q}_l}^{\mathrm{c}}/\mathbb{Q}_l}$-action. By $l \nmid |H|$, $\Lambda(\Gamma)[H] = \Lambda(\Gamma) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[H]$ is a maximal order in $\mathcal{Q}(\Gamma)[H]$, hence a product of matrix rings over the integral closures of $\Lambda(\Gamma)$ in the centre fields $\mathcal{Q}(\Gamma)(\chi)$.

Proposition 6. *Assume that G_∞ is $\mathbb{Q}_l$ - q - elementary. Let $f \in \mathrm{Hom}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times)$ satisfy $(\mathrm{res}^{G'}_{G_\infty} f)(\chi')^l \equiv \Psi((\mathrm{res}^{G'}_{G_\infty} f)(\psi_l \chi')) \mod l\Lambda^{\mathrm{c}}(\Gamma_{k'})$ for all open subgroups G' of G_∞ (with $k' = {K_\infty}^{G'}$) and all $\chi' \in R_l(G')$. Then there exists a $z \in \mathrm{Det}\, K_1(\Lambda(G_\infty))$ such that $((\mathrm{Det}\, z)^{-1} f)^{l^m} \in \mathrm{Hom}^*(R_l(G_\infty), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k))$ for some power l^m. The same holds with Λ replaced by $\Lambda_{\wedge}$.*

For the proof (compare also [Ty, p.94/95]) we set $\overline{G} = G_\infty/H_l = \overline{H} \times \Gamma$ with $\overline{H}$ finite of order prime to l. In particular, $\Lambda(\overline{G}) = \Lambda(\Gamma)[\overline{H}]$. We proceed from the commutative square (see [RW2, Lemma 9])

$$\begin{array}{ccc}
K_1(\Lambda(G_\infty)) & \xrightarrow{\mathrm{defl}} & K_1(\Lambda(\overline{G})) \\
\mathrm{Det} \downarrow & & \mathrm{Det} \downarrow \\
\mathrm{HOM}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times) & \xrightarrow{\mathrm{defl}} & \mathrm{HOM}^*(R_l(\overline{G}), \Lambda^{\mathrm{c}}(\Gamma_k)^\times)
\end{array}$$

and consider $\mathrm{defl} f$. By Lemma 5, $\mathrm{defl} f = \mathrm{Det}\, \overline{z}$ is solvable for some $\overline{z} \in \Lambda(\overline{G})^\times$. Lift $\overline{z}$ to a unit $z \in \Lambda(G_\infty)^\times$, which is possible as $H_l = \ker(G_\infty \to \overline{G})$ is an l-group, and read this z in $K_1(\Lambda(G_\infty))$ (via $\Lambda(G_\infty)^\times \twoheadrightarrow K_1(\Lambda(G_\infty))$). Then $f' \overset{\mathrm{def}}{=} (\mathrm{Det}\, z)^{-1} f \in \mathrm{Hom}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}(\Gamma_k)^\times)$ and $\mathrm{defl}(f') = 1$.

Next, pick an irreducible $\chi \in R_l(G_\infty)$ which is trivial on Γ. So $\chi = \mathrm{ind}^{G_\infty}_{G'}(\xi)$, with a ${\mathbb{Q}_l}^{\mathrm{c}}$-irreducible character ξ of G' which is trivial on Γ, by 2. of Lemma 4. We define $\overline{\chi} = \mathrm{ind}^{G_\infty}_{G'}(\overline{\xi})$ where $\xi = \xi_l \cdot \overline{\xi}$ has been decomposed into its l-singular and l-regular components $\xi_l, \overline{\xi}$, respectively. As $\overline{\xi}$ is trivial on H_l, $\overline{\chi}$ is inflated from $\overline{G}$.

Now, $f'(\chi - \overline{\chi}) = f'(\mathrm{ind}^{G_\infty}_{G'}(\xi - \overline{\xi})) = (\mathrm{res}^{G'}_{G_\infty} f')(\xi - \overline{\xi})$.

The assumption on f and the above Remark imply that

$$f'(\chi - \overline{\chi})^{l^m} \equiv 1 \mod l\Lambda^{\mathrm{c}}(\Gamma_{k'})$$

if m is big enough so that $\psi_l^m(\xi) = \psi_l^m(\overline{\xi})$:

$$f'(\chi - \overline{\chi})^{l^m} = (\mathrm{res}^{G'}_{G_\infty} f')(\xi - \overline{\xi})^{l^m} \equiv \Psi^m((\mathrm{res}^{G'}_{G_\infty} f')(\psi_l^m \xi - \psi_l^m \overline{\xi})) \mod l\Lambda^{\mathrm{c}}(\Gamma_{k'}) .$$

And since $\mathrm{defl}(f') = 1$ and $\overline{\chi}$ is inflated from $\overline{G}$, $f'(\overline{\chi}) = 1$, we arrive at $(f')^{l^m}(\chi) \equiv 1 \mod l\Lambda^{\mathrm{c}}(\Gamma_{k'})$.

By 2. of Lemma 4 every irreducible character of G_∞ is of the form $\chi\rho$ with a χ as above (i.e., χ is trivial on Γ) and ρ of type W. Hence $(f')^{l^m}(\chi\rho) = \rho^\sharp((f')^{l^m}(\chi)) \equiv 1 \mod l\Lambda^{\mathrm{c}}(\Gamma_{k'})$ (see [RW2, Definition in §2]).

Remark. Observe that the above hypothesis is satisfied by $f = L_{K_\infty/k}$ (see [RW3, 2. of Corollary to Theorem 9; RW2, 2. of Proposition 12]) and by every $f \in \mathrm{Det}\, K_1(\Lambda(G_\infty))$ (see [RW2, Lemma 9; RW3, Proposition 4, 1. of Proposition 11]).

4. Proofs of Theorem B and C

In this section we prove Theorems B and C in full generality. This is done by using character actions on K_1 and Hom^* (as well as the Corollary to Theorem 1 and Proposition 3).

For an open subgroup U of G_∞, we denote by $R_{\mathbb{Q}_l}(U)$ the ring of all characters of finite dimensional $\mathbb{Q}_l$-representations of U with open kernel. We view $R_{\mathbb{Q}_l}$ as a Frobenius functor of the open subgroups of G_∞ in the sense of [CR II, 38.1].

We make $\mathrm{Hom}^*(R_l(U), \Lambda^c(\Gamma_{k_U})^\times)$, with $k_U = {K_\infty}^U$, into an $R_{\mathbb{Q}_l}(U)$-module by

$$(\kappa f)(\chi) = f(\check{\kappa}\chi) \quad \text{for} \quad f \in \mathrm{Hom}^*\,,\ \kappa \in R_{\mathbb{Q}_l}(U)\,,\ \chi \in R_l(U)\,,$$

with $\check{\kappa}$ the contragredient of κ.

We make $K_1(\Lambda(U))$ into an $R_{\mathbb{Q}_l}(U)$-module as follows. If κ is a character in $R_{\mathbb{Q}_l}(U)$, and if $[P, \alpha]$ represents an element in $K_1(\Lambda(U))$, then choosing $U' \subset \ker \kappa$, an open subgroup of U, and a $\mathbb{Z}_l[U/U']$-lattice with character κ, we define

$$(*) \qquad \kappa \cdot [P, \alpha] = [M \otimes_{\mathbb{Z}_l} P, \mathrm{id}_M \otimes_{\mathbb{Z}_l} \alpha]$$

(compare [CR II, p.175]).

Lemma 7. $\mathrm{Det} : K_1(\Lambda(-)) \to \mathrm{Hom}^*(R_l(-), \Lambda^c(\Gamma_{k_-})^\times)$ *is a morphism of Frobenius modules over the Frobenius functor* $U \mapsto R_{\mathbb{Q}_l}(U)$.

The lemma is shown in the same way as its analogue in the case of group rings of finite groups. We only need to observe that the $\Lambda(U)$-module structure of $M \otimes_{\mathbb{Z}_l} P$ is derived from the diagonal action of U on $M \otimes_{\mathbb{Z}_l} P$:
First, the $\Lambda(U')$-module structure on P gives $M \otimes_{\mathbb{Z}_l} P$ a $\Lambda(U')$-structure. The pushout diagram then determines a unique $\Lambda(U)$-module structure.

$$\begin{array}{ccc} \mathbb{Z}_l[U'] & \rightarrowtail & \mathbb{Z}_l[U] \\ \downarrow & & \downarrow \\ \Lambda(U') & \rightarrowtail & \Lambda(U) \end{array}$$

In order to check $\Lambda(U)$-projectivity of $M \otimes_{\mathbb{Z}_l} P$, it suffices to take $P = \Lambda(U)$ and then Frobenius reciprocity $M \otimes_{\mathbb{Z}_l} \mathrm{ind}_{U'}^{U}(\Lambda(U')) = \mathrm{ind}_{U'}^{U}(\mathrm{res}_{U}^{U'}(M) \otimes_{\mathbb{Z}_l} \Lambda(U'))$ takes care of this, since M is $\mathbb{Z}_l$-free.

We next recall Swan's theorem (see [CR II, 39.10, p.47]) which implies the independence of $(*)$ from the choice of the lattice M. Indeed, given κ and $U' \subset \ker \kappa$ as above, then two $\mathbb{Z}_l[U/U']$-lattices M_1, M_2 with character κ induce the same element in the Grothendieck group $G_0^{\mathbb{Z}_l}(\mathbb{Z}_l[U/U'])$ of finitely generated $\mathbb{Z}_l[U/U']$-lattices (see [CR I,§16B]). Moreover, it is readily checked from [CR II, 38.20, 38.24, p.14,16] that $[M_1 \otimes_{\mathbb{Z}_l} P, \mathrm{id}_{M_1} \otimes_{\mathbb{Z}_l} \alpha] = [M_2 \otimes_{\mathbb{Z}_l} P, \mathrm{id}_{M_2} \otimes_{\mathbb{Z}_l} \alpha]$ in $K_1(\Lambda(U))$.

It remains to show that Det is a Frobenius module homomorphism. Let $\chi \in R_l(G_\infty)$ and let $[P, \alpha] \in K_1(\Lambda(G_\infty))$, $[M] \in G_0^{\mathbb{Z}_l}(\mathbb{Z}_l[U/U'])$ as in $(*)$; set ${\mathbb{Q}_l}^c \otimes_{\mathbb{Z}_l} M = V_\kappa$. Then

$$\begin{aligned}
&(\mathrm{Det}\,[M \otimes_{\mathbb{Z}_l} P, 1 \otimes_{\mathbb{Z}_l} \alpha])(\chi) \\
&= \det_{\mathcal{Q}^c(\Gamma_k)}(1 \otimes_{\mathbb{Z}_l} \alpha \mid \mathrm{Hom}_{{\mathbb{Q}_l}^c[H]}(V_\chi, {\mathbb{Q}_l}^c \otimes_{\mathbb{Z}_l} (M \otimes_{\mathbb{Z}_l} P))) \\
&= \det_{\mathcal{Q}^c(\Gamma_k)}(1 \otimes_{\mathbb{Z}_l} \alpha \mid \mathrm{Hom}_{{\mathbb{Q}_l}^c[H]}(V_\chi, (V_\kappa \otimes_{{\mathbb{Q}_l}^c} ({\mathbb{Q}_l}^c \otimes_{\mathbb{Z}_l} P)))) \\
&\stackrel{1}{=} \det_{\mathcal{Q}^c(\Gamma_k)}(\alpha \mid \mathrm{Hom}_{{\mathbb{Q}_l}^c[H]}(V_\chi, \mathrm{Hom}_{{\mathbb{Q}_l}^c}(V_{\check{\kappa}}, {\mathbb{Q}_l}^c \otimes_{\mathbb{Z}_l} P))) \\
&\stackrel{2}{=} \det_{\mathcal{Q}^c(\Gamma_k)}(\alpha \mid \mathrm{Hom}_{{\mathbb{Q}_l}^c[H]}(V_{\check{\kappa}} \otimes_{{\mathbb{Q}_l}^c} V_\chi, {\mathbb{Q}_l}^c \otimes_{\mathbb{Z}_l} P)) \\
&= (\mathrm{Det}\,[P, \alpha])(\check{\kappa}\chi) = (\kappa\,\mathrm{Det}\,[P, \alpha])(\chi)\,,
\end{aligned}$$

with $\stackrel{1}{=}$ and $\stackrel{2}{=}$ due to the naturality on H-fixed points of the isomorphisms [CR I, 10.30, 2.19], respectively.

Corollary. *$SK_1(\mathcal{Q}(G_\infty)) = 0$ if $SK_1(\mathcal{Q}(G') = 0$ for all open $\mathbb{Q}_l$-elementary subgroups G' of G_∞.*

This follows because $SK_1(\mathcal{Q}(-))$ is a Frobenius module over $R_{\mathbb{Q}_l}(-)$, by Lemma 7 with Λ replaced by $\mathcal{Q}$. Now apply the Witt-Berman induction theorem (see [CR I, 21.6, p.459]) to the finite group G_∞/Γ where Γ is a central open subgroup: There exist $\mathbb{Q}_l$-elementary subgroups $\overline{G}_i \leqslant G_\infty/\Gamma$ and (virtual) ${\mathbb{Q}_l}^{\mathrm{c}}$-characters $\overline{\xi}_i$ of $\overline{G}_i$ such that $1_{G_\infty} = \sum_i \mathrm{ind}\,_{G_i}^{G_\infty}(\xi_i)$, with G_i the full preimage of $\overline{G}_i$ in G_∞ and $\xi_i = \mathrm{infl}\,_{\overline{G}_i}^{G_i}(\overline{\xi}_i)$. By Lemma 4 the groups G_i are $\mathbb{Q}_l$-elementary (this is trivial for the prime number l). Now let $z \in SK_1(\mathcal{Q}(G_\infty))$ and apply the above character relation to get from $\mathrm{res}\,_{G_\infty}^{G_i} z = 0$

$$z = 1_{G_\infty} \cdot z = \sum_i \mathrm{ind}\,_{G_i}^{G_\infty}(\xi_i) \cdot z = \sum_i \mathrm{ind}\,_{G_i}^{G_\infty}(\xi_i \cdot \mathrm{res}\,_{G_\infty}^{G_i} z) = 0\,.$$

Lemma 8. $\mathrm{Det}\,K_1(\Lambda(G_\infty)) \cap \mathrm{Hom}^*(R_l(G_\infty), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k))$ *is a $\mathbb{Z}_l$-module, and the same with Λ replaced by $\Lambda_\wedge$.*

It suffices to show $(\mathrm{Hom}^*(R_l(G_\infty), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k)))^m \subset \mathrm{Det}\,K_1(\Lambda(G_\infty))$ for some nonzero integer m, as this implies that $\mathrm{Det}\,K_1(\Lambda(G_\infty)) \cap \mathrm{Hom}^*(R_l(G_\infty), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k))$ is a $\mathbb{Z}_l$-submodule of the $\mathbb{Z}_l$-module $\mathrm{Hom}^*(R_l(G_\infty), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k))$:
For if $f \in \mathrm{Det}\,K_1(\Lambda(G_\infty)) \cap \mathrm{Hom}^*(R_l(G_\infty), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k))$ and $c \in \mathbb{Z}_l$, then, writing $c = a + mb$ with $a \in \mathbb{Z}$, $b \in \mathbb{Z}_l$, $f^c = f^a(f^b)^m$, and $f^a \in \mathrm{Det}\,K_1(\Lambda(G_\infty)) \cap \mathrm{Hom}^*(R_l(G_\infty), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k))$, $f^b \in \mathrm{Hom}^*(R_l(G_\infty), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k))$, so $(f^b)^m \in \mathrm{Det}\,K_1(\Lambda(G_\infty)) \cap \mathrm{Hom}^*(R_l(G_\infty), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k))$.

We next prove the containment claimed above when $G_\infty = H \times \Gamma$ is abelian. Let $f \in \mathrm{Hom}^*(R_l(G_\infty), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k))$, whence $f^{|H|} \in \mathrm{Hom}^*(R_l(G_\infty), 1 + |H|\Lambda^{\mathrm{c}}(\Gamma_k))$. Moreover, by $(*)$ in the proof of [RW2, Theorem 8] and [CR II, 45.12, p.142],

$$f^{|H|} = \mathrm{Det}\,q \quad \text{with} \quad q = \sum_{h \in H} q_h h \quad \text{in} \quad \mathcal{Q}(G_\infty) = \mathcal{Q}(\Gamma)[H]\,.$$

Hence, by [RW3, Proposition 3], $f^{|H|}(\chi) = \sum_{h \in H} \overline{q}_h \chi(h)$ for every irreducible character $\chi \in R_l(G_\infty)$ which is trivial on Γ, where $^-$ is the isomorphism $\Gamma \to \Gamma_k$. It follows that

$$|H|\overline{q}_h = \sum_\chi f^{|H|}(\chi)\chi(h^{-1}) \equiv \sum_\chi \chi(h^{-1}) \equiv 0 \mod |H|\Lambda^{\mathrm{c}}(\Gamma_k)\,,$$

i.e., $q_h \in \Lambda^{\mathrm{c}}(\Gamma) \cap \mathcal{Q}(\Gamma) = \Lambda(\Gamma)$. By [RW3, Lemma 10], $q \in \Lambda(G_\infty)^\times$.

For the general case we apply Artin induction: If Γ is central open of index n in G_∞, then there exist subgroups $\Gamma \subset A_i \subset G_\infty$ with A_i/Γ cyclic so that $n \cdot 1_{G_\infty} = \sum_i \mathrm{ind}\,_{A_i}^{G_\infty}(1_{A_i})$. It follows that the A_i are abelian, and whence, with $k_i = {K_\infty}^{A_i}$, $\mathrm{Hom}^*(R_l(A_i), 1 + l\Lambda(\Gamma_{k_i}))^{m_i} \subset \mathrm{Det}\,K_1(\Lambda(A_i))$ for suitable integers m_i. Setting $m = \prod_i m_i$, we get $\mathrm{Hom}^*(R_l(A_i), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k))^m \subset \mathrm{Det}\,K_1(\Lambda(A_i))$. Thus, if $f^m \in \mathrm{Hom}^*(R_l(G_\infty), 1 + l\Lambda^{\mathrm{c}}(\Gamma_k))^m$, then the above character relation yields

$$f^{mn} = \prod_i \operatorname{ind}_{A_i}^{G_\infty}(1_{A_i}) f^m = \prod_i \operatorname{ind}_{A_i}^{G_\infty}((\operatorname{res}_{G_\infty}^{A_i} f)^m) \subset \operatorname{Det} K_1(\Lambda(G_\infty)),$$

by Lemma 7 and [RW3, Lemma 1].
This proves the lemma.

Proof of Theorem B.

Choose a central open subgroup Γ and apply the Witt-Berman induction theorem to G_∞/Γ. By [Se2, Theorem 28, p.98] there are $\mathbb{Q}_l$ - l - elementary open subgroups $U_i \leqslant G_\infty$ containing Γ together with characters $\xi_i \in R_{\mathbb{Q}_l}(U_i)$ so that we have

$$n \cdot 1_{G_\infty} = \textstyle\sum_i \operatorname{ind}_{U_i}^{G_\infty}(\xi_i) \tag{1}$$

for an integer $n \mid [G_\infty : \Gamma]$ prime to l. Now, let $d \in \operatorname{Det} K_1(\Lambda(G_\infty)_\wedge) \cap \operatorname{Hom}^*(R_l(G_\infty), \Lambda(\Gamma_k)^\times)$ and apply this character relation to it:

$$d^n = \prod_i \operatorname{ind}_{U_i}^{G_\infty}(\xi_i) d = \prod_i \operatorname{ind}_{U_i}^{G_\infty}(\xi_i \operatorname{res}_{G_\infty}^{U_i} d)\ .$$ [2]

But $\operatorname{res}_{G_\infty}^{U_i} d \in \operatorname{Det} K_1(\Lambda(U_i)_\wedge) \cap \operatorname{Hom}^*(R_l(U_i), \Lambda^c(\Gamma_{k_i})^\times)$, with $k_i = {K_\infty}^{U_i}$, and so, by the Corollary to Theorem 1, $\operatorname{res}_{G_\infty}^{U_i} d \in \operatorname{Det} K_1(\Lambda(U_i))$. It follows first that $\xi_i \operatorname{res}_{G_\infty}^{U_i} d \in \operatorname{Det} K_1(\Lambda(U_i))$ and then, from [RW3, Lemma 1], that

$$d^n \in \operatorname{Det} K_1(\Lambda(G_\infty))\,. \tag{2}$$

On the other hand, by 1. of Lemma 4 we find, for each prime number q dividing n, $\mathbb{Q}_l$ - q - elementary subgroups U_j' of G_∞ containing Γ, characters $\xi_j' \in R_{\mathbb{Q}_l}(U_j')$ and an integer $n' \mid [G_\infty : \Gamma]$ prime to q such that

$$n' \cdot 1_{G_\infty} = \textstyle\sum_j \operatorname{ind}_{U_j'}^{G_\infty}(\xi_j')\,. \tag{3}$$

And, setting $f_j = \operatorname{res}_{G_\infty}^{U_j'} d \in \operatorname{Det} K_1(\Lambda(U_j')_\wedge) \cap \operatorname{Hom}^*(R_l(U_j'), \Lambda^c(\Gamma_{k_j'})^\times)$, with $k_j' = {K_\infty}^{U_j'}$, then f_j is a function f as in Proposition 6 (compare the Remark following the proposition) and so there exist $z_j \in K_1(\Lambda(U_j'))$ such that

$$((\operatorname{Det} z_j)^{-1} f_j)^{l^{m_j'}} \in \operatorname{Hom}^*(R_l(U_j'), 1 + l\Lambda^c(\Gamma_{k_j'}))\,,$$

for some power $l^{m_j'}$. Combining this with (2), and setting $m' = \max_j\{m_j'\}$, we obtain

$$((\operatorname{Det} z_j)^{-1} f_j)^{nl^{m'}} \in \operatorname{Det} K_1(\Lambda(U_j')) \cap \operatorname{Hom}^*(R_l(U_j'), 1 + l\Lambda^c(\Gamma_{k_j'}))\,.$$

By Lemma 8 the group on the right is a $\mathbb{Z}_l$-module, hence, as $l \nmid n$,

$$((\operatorname{Det} z_j)^{-1} f_j)^{l^{m'}} \in \operatorname{Det} K_1(\Lambda(U_j'))$$

[2] The notation is an additive-multiplicative compromise.

and consequently $f_j^{l^{m'}} = (\operatorname{res}^{U'_j}_{G_\infty} d)^{l^{m'}} \in \operatorname{Det} K_1(\Lambda(U'_j))$. Now (3) yields $d^{n'l^{m'}} \in \operatorname{Det} K_1(\Lambda(G_\infty))$ and then, by (2), $d^{n'} \in \operatorname{Det} K_1(\Lambda(G_\infty))$. Letting q vary we obtain Theorem B.

Proof of Theorem C.

We only check the nontrivial implication and proceed as above. We start with $L_{K_\infty/k} \in \operatorname{HOM}^*(R_l(G_\infty), \Lambda^{\mathrm{c}}_{\wedge}(\Gamma_k)^\times)$ and first use (1). Because $\operatorname{res}^{U_i}_{G_\infty} L_{K_\infty/k} = L_{K_\infty/k_i}$, it follows from the hypothesis and Proposition 3 that $L^n_{K_\infty/k} \in \operatorname{Det} K_1(\Lambda(G_\infty)_\wedge)$. For each $q|n$ we next turn to (3) and use that $L_{K_\infty/k'_j} \in \operatorname{Hom}^*(R_l(U'_j), \Lambda^{\mathrm{c}}_\wedge(\Gamma_{k'_j})^\times)$ is a function f as in Proposition 6. Thus there is a $z_j \in K_1(\Lambda(U'_j)_\wedge)$ with $((\operatorname{Det} z_j)^{-1} L_{K_\infty/k'_j})^{l^{m'_j}} \in \operatorname{Hom}^*(R_l(U'_j), 1 + l\Lambda^{\mathrm{c}}_\wedge(\Gamma_{k'_j}))$. Combining as before, we see that $((\operatorname{Det} z_j)^{-1} L_{K_\infty/k'_j})^{nl^{m'}} \in \operatorname{Det} K_1(\Lambda(U'_j)_\wedge)$, whence already $L^{l^{m'}}_{K_\infty/k'_j} \in \operatorname{Det} K_1(\Lambda(U'_j)_\wedge)$, by $l \nmid n$. Now apply (3) and get first $L^{n'l^{m'}}_{K_\infty/k} \in \operatorname{Det} K_1(\Lambda(G_\infty)_\wedge)$ and then, from (2), $L^{n'}_{K_\infty/k} \in \operatorname{Det} K_1(\Lambda(G_\infty)_\wedge)$. Varying q, this finishes the proof of Theorem C.

Remark 1. The proof shows that the definition of a section of G_∞ could be strengthened to require K_∞/K' to be finite cyclic of order prime to l.

Remark 2. As before we may generalize Theorem C by replacing the Iwasawa L-functions $L_{K'/k'}$ by the functions $f_{K'/k'}$ of the Remark after Proposition 3.

5. Complements

We begin this section by presenting some examples:

Example 1. *If the Sylow-l subgroups of G_∞ are abelian, then* $L_{K_\infty/k} \in \operatorname{Det} K_1(\Lambda(G_\infty)_\wedge)$.

Indeed, Theorem C requires us to check whether $L_{K_\infty{}^C/K_\infty{}^U} \in \operatorname{Det} K_1(\Lambda(E)_\wedge)$ whenever $E = G_{K_\infty{}^C/K_\infty{}^U}$ is an l-elementary section of G_∞. But the assumption on the Sylow-l subgroups of G_∞ implies that the Sylow-l subgroup of E is abelian, whence E itself. Now apply 1. of the Corollary to Theorem 9 in [RW3].

Concerning the full "main conjecture" we have

Example 2. *If $G_\infty = H \rtimes \Gamma$ satisfies $l \nmid |H|$, then $SK_1(\mathcal{Q}(G_\infty)) = 1$. In particular, the "main conjecture" is true for these groups.*

The second assertion holds as the Sylow-l subgroup Γ of G_∞ is abelian; moreover, the first assertion now guaranties uniqueness of $\tilde{\Theta}_S$ (see [RW2,§3, especially Remark E]).

For the proof of this first assertion, $SK_1(\mathcal{Q}(G_\infty)) = 1$, we may assume that G_∞ is $\mathbb{Q}_l$-elementary, by the Corollary to Lemma 7.

If G_∞ is $\mathbb{Q}_l$ - q - elementary with $q \neq l$, then $G_\infty = H \times \Gamma$ with H a finite $\mathbb{Q}_l$ - q - elementary group. Since $l \nmid |H|$, Lemma 5 implies that $\mathcal{Q}(G_\infty)$ is totally split.

Next, let G_∞ be $\mathbb{Q}_l$ - l - elementary, so $G_\infty = \langle s \rangle \rtimes \Gamma$ by $l \nmid |H|$, whence $U = \Gamma$ in the notation of Theorem 1 which we continue to use (in particular, β_i is a ${\mathbb{Q}_l}^{\mathrm{c}}$-irreducible character of $\langle s \rangle$ with stabilizer subgroup $\Gamma_i = U_i \leqslant \Gamma$, $G_i = \langle s \rangle \rtimes \Gamma_i$, and e_i is the idempotent associated to the $G_{\mathbb{Q}_l{}^c/\mathbb{Q}_l}$-orbit of β_i).
Because $SK_1(\mathcal{Q}(G_\infty)) = \prod_i SK_1(e_i\mathcal{Q}(G_\infty))$, it suffices to show that each $e_i\mathcal{Q}(G_\infty)$ is a (full) ring of matrices over a (commutative) field. Recall first that $e_i\Lambda(G_i) = \Lambda^{\mathfrak{O}_i}(\Gamma_i)$. Therefore

$$e_i\Lambda(G_\infty) = \Lambda^{\mathfrak{O}_i}(\Gamma_i) \circ [\Gamma/\Gamma_i]$$

is the crossed product order of the cyclic group Γ/Γ_i over the ring $\Lambda^{\mathfrak{O}_i}(\Gamma_i)$, with the Galois action on $\mathfrak{O}_i$ resulting from $\Gamma/\Gamma_i \xrightarrow{\simeq} G_{N_i/N_i'} \leqslant G_{N_i/\mathbb{Q}_l}$. If γ_i is a generator of Γ_i, then by [Re, p.259/260] the algebra $\mathcal{Q}^{N_i}(\Gamma_i) \circ [\Gamma/\Gamma_i]$ splits if, and only if, γ_i is a norm in $\mathcal{Q}^{N_i}(\Gamma_i)/\mathcal{Q}^{N_i'}(\Gamma_i)$. But γ_i is already a norm in $\Lambda^{\mathfrak{O}_i}(\Gamma_i)/\Lambda^{\mathfrak{O}_i'}(\Gamma_i)$ by Proposition 2.
Finally we give a bound on the order of $L_{K_\infty/k}$ mod $\mathrm{Det}\, K_1(\Lambda(G_\infty)_{\wedge})$.

Proposition 9. *Set $l^a = [G' : Z(G')]$, where G' is a Sylow-l subgroup of G_∞ and $Z(G')$ is its centre. Then $L_{K_\infty/k}^{l^a} \in \mathrm{Det}\, K_1(\Lambda(G_\infty)_{\wedge})$.*

We first note that obviously $a = a(G_\infty)$ is an invariant of G_∞ and that $a(G_\infty) \geqslant a(G_{K'/k'})$ for all sections K'/k' of K_∞/k. Hence, if we can show that $L_{K'/k'}^{l^{a'}} \in \mathrm{Det}\, K_1(\Lambda(G_{K'/k'})_{\wedge})$ for all l-elementary sections K'/k' of K_∞/k, with $a' = a(G_{K'/k'})$, then, by Remark 2 following the proof of Theorem C, we have also verified Proposition 9. Hence, from now on, G_∞ is l-elementary.

In this case $l^a = [G_\infty : Z(G_\infty)]$ and we proceed by induction on a. If $a = 0$, then G_∞ is abelian and 1. of Corollary to Theorem 9 in [RW3] gives what we want. If $a > 0$, then G_∞ is nonabelian and consequently $G_\infty/Z(G_\infty)$ noncyclic. We infer the existence of a normal subgroup G' of G_∞ containing $Z(G_\infty)$ so that $\overline{G} \stackrel{\text{def}}{=} G_\infty/G'$ is noncyclic of order l^2. From it we obtain the character relation $l\cdot 1_{\overline{G}} = \sum_{\overline{M}} \mathrm{ind}_{\overline{M}}^{\overline{G}}(1_{\overline{M}}) - \mathrm{ind}_{\overline{1}}^{\overline{G}}(1_{\overline{1}})$ with $\overline{M}$ running through the maximal subgroups of $\overline{G}$. Inflation yields $l \cdot 1_{G_\infty} = \sum_j n_j \mathrm{ind}_{M_j}^{G_\infty}(1_{M_j})$ with proper open subgroups $M_j \leqslant G_\infty$ containing $Z(G_\infty)$ and with integers n_j. Because $a(M_j) < a$, induction implies that $L_{K_\infty/k_j}^{l^{a-1}} \in \mathrm{Det}\, K_1(\Lambda(G_{K_\infty/k_j})_{\wedge})$ for all j (with $k_j = {K_\infty}^{M_j}$), and then the last character relation gives $L_{K_\infty/k}^{l^a} \in \mathrm{Det}\, K_1(\Lambda(G_\infty)_{\wedge})$.

Proposition 9 is established.

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