The Nucleus of the Free Alternative Algebra

I. R. Hentzel and L. A. Peresi

CONTENTS

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We use a computer procedure to determine a basis of the elements of degree 5 in the nucleus of the free alternative algebra. In order to save computer memory, we do our calculations over the field Z_{103}. All calculations are made with multilinear identities. Our procedure is also valid for other characteristics and for determining nuclear elements of higher degree.

1. INTRODUCTION

Alternative algebras are nonassociative algebras satisfying the identities

$$(a, a, b) = 0, \qquad (1–1)$$

$$(a, b, b) = 0. \qquad (1–2)$$

(The *associator* (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$.) In an alternative algebra, the associator is an alternating function of its three arguments. The most familiar example of an alternative algebra is the octonions, which appear as distant generalizations of the reals in the chain

Reals $\subset$ Complexes $\subset$ Quaternions $\subset$ Octonions.

Cayley–Dickson algebras are eight-dimensional alternative algebras that generalize the octonions. See [Zhevlakov et al. 82, Chapter 2].

We shall assume that all algebras are over a field F of characteristic zero or of characteristic greater than the degree of the identities in question. We can therefore limit our discussion to multilinear identities (see the discussion of linearization in Chapter 1 of [Zhevlakov et al. 82]). The linearized forms of (1–1) and (1–2) are

$$(a, b, c) + (b, a, c) = 0, \qquad (1–3)$$

$$(a, b, c) + (a, c, b) = 0. \qquad (1–4)$$

Definition 1.1. The *nucleus* of a nonassociative algebra $\mathcal{A}$ is the set

$$N(\mathcal{A}) = \{p \in \mathcal{A} \mid (p, x, y) = (x, p, y) = (x, y, p) = 0, \ \forall x, y \in \mathcal{A}\}.$$

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In [Kleinfeld 53a], Kleinfeld showed that for any x and y in an alternative algebra, the element $[x,y]^4$ is in the nucleus. (The *commutator* $[x,y]$ is defined by $[x,y] = xy - yx$.) This element was used by Kleinfeld to prove that a simple alternative algebra is associative or a Cayley–Dickson algebra [Zhevlakov et al. 82, Chapter 7]. Kleinfeld's element is of degree 8. The elements of degree 5 that are known to be in the nucleus of an alternative algebra are

$$[[a,b][a,b],a] \tag{1–5}$$

and

$$[[[[b,a],c],a],a] - [[[[b,a],a],c],a] + 2\,[[[b,a],[c,a]],a]. \tag{1–6}$$

We state now our first main result.

Theorem 1.2. *The following elements of degree* 5 *are in the nucleus of an alternative algebra:*

$$([a,b][a,c])a - (a[a,b])[a,c], \tag{1–7}$$

$$[[[a,b],[a,c]],a] - 2([a,b],[a,c],a), \tag{1–8}$$

and

$$\begin{aligned} &a(b(a(ac))) + a(a(b(ca))) + b(a(c(aa))) \\ &\quad - a(a(b(ac))) - b(a(a(ca))) - a(b(c(aa))). \end{aligned} \tag{1–9}$$

There are two traditional approaches to finding elements of the nucleus. One approach is to use a Cayley–Dickson algebra for initial screening of nuclear elements. One looks for all elements of the free alternative algebra that evaluate into the nucleus of a Cayley–Dickson algebra. Then one looks among these elements to find those that are also in the nucleus of the free alternative algebra. The other approach [Zhevlakov et al. 82, Chapter 7, Lemma 3]) is to use the identity

$$(ab,c,d) + (a,b,[c,d]) = a(b,c,d) + (a,c,d)b, \tag{1–10}$$

which is a consequence of (1–2) [Kleinfeld 53b]. From this identity we see that if $(x,c,d) = 0$ for all x, then $[c,d]$ is in the nucleus. This allows one to work with expressions of lower degree, but it also limits the results to nuclear elements that are expressible as sums of commutators. It would not be expected to pick up the nuclear element (1–9).

Any alternative algebra satisfies the identity $([a,b][a,b],a,c) = 0$ [Zhevlakov et al. 82, Chapter 13, Lemma 15]. Therefore it follows from (1–10) that (1–5) is in the nucleus. From a lemma in [Filippov 75] we obtain that

$$([[[b,a],c],a] - [[[b,a],a],c] + 2\,[[b,a],[c,a]],a,d) = 0.$$

Therefore by (1–10) the element (1–6) is in the nucleus.

We want to make precise what is meant by an element of the *nucleus of the free alternative algebra*. In the free alternative algebra on two generators, any element is in the nucleus because the free alternative algebra on two generators is associative. We do not want to find the nucleus of a particular algebra, or an algebra with a prescribed number of generators. Rather, we want elements that evaluate into the nucleus in all alternative algebras over any field of characteristic 0 or sufficiently large characteristic.

Let p be an element of the free nonassociative algebra $F[X]$ in generators $X = \{x_1, x_2, \ldots, x_n\}$. We say that p is an element of the nucleus of the free alternative algebra in generators X if in the free alternative algebra on generators $X \cup \{x_{n+1}, x_{n+2}\}$ one has that $(p, x_{n+1}, x_{n+2}) = 0$.

We show that the minimal degree of the nonzero elements in the nucleus of the free alternative algebra is 5. We find a basis of the nuclear elements of degree 5 of the free alternative algebra. Our calculations are done in Z_{103} (see Section 6). It does not guarantee that there are not elements of degree less than 5 for all finite characteristics.

The nuclear elements that we found were checked by the computer program ALBERT [Jacobs et al. 96] for various finite characteristics, but we do not have a dependency relation over the integers that would show that the elements are in the nucleus for all but a finite number of characteristics.

We state more precisely our second main result.

Theorem 1.3. *In the free alternative algebra over* Z_{103} *on generators* $\{a,b,c,\,d,e\}$ *we have the following:*

(i) *There are no nonzero nuclear elements of degree less than* 5.

(ii) *All the nuclear elements of degree* 5 *are consequences of the alternative identities of degree* 5 *(i.e., the identities of degree* 5 *implied by the alternative identities of degree* 3*) and* $([a,b][a,c])a - (a[a,b])[a,c]$.

In the last section, we use our computer procedure to verify that the first four elements, in a series of nuclear elements in the free alternative algebra, are nonzero. This series of nuclear elements is defined in [Shestakov and Zhukavets 06a]. All the nuclear elements in this series are proved to be nonzero in [Shestakov and Zhukavets 06b].

2. REPRESENTATION OF ALGEBRAIC EXPRESSIONS

Let $X = \{x_1, \dots, x_n\}$ be a set of variables. We construct the set $M[X]$ of (noncommutative and nonassociative) *monomials* inductively as follows: $X \subset M[X]$; if $x_i,\ x_j \in X$ then $x_i x_j \in M[X]$; if $u, v \in M[X] - X$ then $x_i(u), (u)x_i, (u)(v) \in M[X]$.

An *association type* of degree n is a way to put parentheses in a product of degree n. The number of association types of degree n is given by the *Catalan number*

$$\mathrm{cat}[n] = \frac{1}{n}\binom{2n-2}{n-1}.$$

Here are some Catalan numbers:

n	1	2	3	4	5	6	7	8	9	10
cat[n]	1	1	2	5	14	42	132	429	1430	4862

A *term* is a scalar multiple of a monomial. To each monomial or term corresponds a unique association type. The ordering on the association types is given by the following rules. For terms of different degree,

$$A < B \text{ if and only if } \deg(A) > \deg(B).$$

For terms of the same degree we proceed lexicographically on the factors. Thus

$$AB < CD \text{ if } A < C \text{ or if } A = C \text{ and } B < D.$$

Let

$$F[X] = \Big\{ \sum_{i=1}^{n} \alpha_i\ u_i \mid n \in \mathbb{N},\ \alpha_i \in F,\ u_i \in M[X] \Big\}$$

be the vector space over F spanned by $M[X]$. The elements of $F[X]$ are called (nonassociative) *polynomials* in the variables x_i. We define in $F[X]$ a multiplication by the following rules:

$$x_i.x_j = x_i x_j, \quad x_i.u = x_i(u),$$
$$u.x_i = (u)x_i, \quad u.v = (u)(v),$$

and

$$\Big(\sum_{i=1}^{n} \alpha_i u_i\Big).\Big(\sum_{j=1}^{m} \beta_j v_j\Big) = \sum_{i,j=1}^{n,m} \alpha_i \beta_j u_i.v_j,$$

where $x_i, x_j \in X,\ u, u_i, v_j \in M[X] - X$. We obtain then an algebra called the *free nonassociative algebra generated by* X. We also denote this algebra by $F[X]$ (or $F[x_1, \dots, x_n]$). An element $\sum_{i=1}^{n} \alpha_i u_i \in F[X]$ is called *multilinear of degree* k if the u_i's are monomials in the set of variables $\{x_{t_1}, \dots, x_{t_k}\} \subset X$ and x_{t_j} $(j = 1, \dots, k)$ appears exactly once in u_i $(i = 1, \dots, n)$.

A polynomial $f = f(x_1, x_2, \dots, x_n) \in F[X]$ is called an *identity* of an algebra $\mathcal{A}$ if $f(a_1, a_2, \dots, a_n) = 0$ for all $a_1, a_2, \dots, a_n \in \mathcal{A}$. When f is an identity of $\mathcal{A}$ we say also that $\mathcal{A}$ *satisfies* $f = 0$.

We say that an ideal I of $F[X]$ is a *T-ideal* if $\psi(I) \subset I$ for all homomorphisms $\psi : F[X] \to F[X]$.

Let $\mathrm{Alt}[X]$ denote the ideal of $F[X]$ generated by the elements (f_1, f_1, f_2), (f_2, f_1, f_1) $(f_1, f_2 \in F[X])$. This ideal is a T-ideal called the *T-ideal of* $F[X]$ *generated by (1–1) and (1–2).* We denote by $\mathrm{Alt}_n[X]$ the subspace of multilinear elements of degree n in $\mathrm{Alt}[X]$.

Definition 2.1. The *free alternative algebra generated by* X is the quotient algebra

$$\mathrm{ALT}[X] = F[X]/\,\mathrm{Alt}[X].$$

Let $\mathcal{A}$ be an alternative algebra over F. Let $\phi : X \to \mathcal{A}$ be a mapping. Then there is a unique homomorphism $\overline{\phi} : \mathrm{ALT}[X] \to \mathcal{A}$ such that $\overline{\phi}(x) = \phi(x)$ for all $x \in X$. If $p(x_1, \dots, x_n) \in N(\mathrm{ALT}[X])$ and $a_1, \dots, a_n \in \mathcal{A}$, then $p(a_1, \dots, a_n) \in N(A)$.

A major decision was to represent the association type by an integer. The association types are listed in order for degrees 1 through 5. The "x" is a placeholder. Here, we are interested only in how the terms are associated, not the particular generators from which the product is made:

Degree 1: T_1 x Degree 2: T_1 xx Degree 3: T_1 $(xx)x$, T_2 $x(xx)$

Degree 4:

T_1	T_2	T_3	T_4	T_5
$((xx)x)x$	$(x(xx))x$	$(xx)(xx)$	$x((xx)x)$	$x(x(xx))$

Degree 5:

T_1	T_2	T_3	T_4
$(((xx)x)x)x$	$((x(xx))x)x$	$((xx)(xx))x$	$(x((xx)x))x$
T_5	T_6	T_7	T_8
$(x(x(xx)))x$	$((xx)x)(xx)$	$(x(xx))(xx)$	$(xx)((xx)x)$
T_9	T_{10}	T_{11}	T_{12}
$(xx)(x(xx))$	$x(((xx)x)x)$	$x((x(xx))x)$	$x((xx)(xx))$
T_{13}	T_{14}		
$x(x((xx)x))$	$x(x(x(xx)))$		

Having given a theoretical discussion on terms, we now discuss data structures in the computational implementation. The representation of a term A has four parts: $A.c$ is the coefficient, $A.d$ is the degree, $A.t$ is the type, and $A.x$ is the string of generators that are multiplied together to get A. For example, the term $3((ca)(bd))e$ is represented by A, where $A.c = 3, A.d = 5, A.t = 3$, and $A.x = cabde$.

The computer procedure to multiply terms concatenates the terms to be multiplied. It then computes how many association types of the same degree come at or before the association type of the resulting term in the listing of all association types.

The computer procedure to factor a term first establishes the degree of the separate factors, and then establishes the actual association types of the factors. Each term has a unique factorization. For example, the factors of the term $((ab)c)d$ are $(ab)c$ and d.

We study the free nonassociative algebra $F[X]$ on generators X. We need to represent elements of $F[X]$ and to recognize all elements in the T-ideal $\mathrm{Alt}[X]$. The basic unit of the elements of $F[X]$ is the term. We have ways to represent terms, ways to multiply terms, and ways to factor terms. These basic processes can then be combined to create the T-ideal $\mathrm{Alt}[X]$.

3. CREATING THE T-IDEAL

Computing the multilinear alternative identities for a particular degree n (i.e., the consequences in that degree of the alternative identities in degree 3) is done in two steps. We first create what we shall call the *type identities* of degree n. The subspace $\mathrm{Alt}_n[X]$ of multilinear elements of degree n of the T-ideal $\mathrm{Alt}[X]$ is spanned by the set of type identities after that set has been augmented to include the additional (but equivalent) identities obtained by permuting their arguments in all possible ways.

The type identities of degree n are created in the following manner: For each association type of degree n we find the $n-2$ possible ways that the term could be reassociated. This gives $n-2$ *buried* associators. For example, $((ab)c)d$ can be reassociated as $(a(bc))d$ and $(ab)(cd)$. This gives the buried associators $(a, b, c)d$ and (ab, c, d). Using (1–1) and (1–2), each associator creates two identities. We attempt to avoid duplicating the same identity as much as possible. It does no harm to have a particular identity more than once, but when done to excess, it can make the process too big to run.

The computer procedure to create the type identities of degree n is recursive. We run through each of the $\mathrm{cat}[n]$ association types of degree n. Let the term A represent an association type. If $\deg(A)$ is 1, there are no type identities. If $\deg(A)$ is greater than 1, then factor $A = BC$. Now the type identities from A contain those of B multiplied on the right by C as well as those of C multiplied on the left by B. Furthermore, if $B = B_1B_2$, we add the identities

$$(B_1, B_2, C) + (B_2, B_1, C) \quad (\text{if } B_1 \le B_2),$$
$$(B_1, B_2, C) + (B_1, C, B_2) \quad (\text{if } B_2 \le C).$$

This gives us the type identities with a minimum of duplication. Any instance of the alternative identities expands to four terms. We capture this particular type identity when we create the identities for the term (or perhaps terms) of smallest type of the four.

As an example we compute the type identities of degree 4. Let $A = ((ab)c)d$ be a term with association type 1. We factor A as $A = BC$, where $B = ((ab)c)$ and $C = d$. Now $B = B_1B_2$ with $B_1 = (ab)$ and $B_2 = c$. Association type 1 gives the following type identities:

$$(a, b, c)d + (b, a, c)d = 0, \quad (a, b, c)d + (a, c, b)d = 0,$$
$$(ab, c, d) + (c, ab, d) = 0, \quad (ab, c, d) + (ab, d, c) = 0.$$

The association type 2 gives

$$(a, bc, d) + (a, d, bc) = 0.$$

The association type 3 gives

$$(a, b, cd) + (b, a, cd) = 0.$$

The association type 4 gives

$$a(b, c, d) + a(c, b, d) = 0, \quad a(b, c, d) + a(b, d, c) = 0.$$

The association type 5 gives no type identities.

4. GROUP REPRESENTATION APPROACH

No effective basis is known for the free alternative algebra $\mathrm{ALT}[X]$. This leads to complicated calculations. One way to reduce the size of the calculations is to apply the theory of superalgebras. The calculations are done in the free alternative algebra in one generator. For an application of this technique see [Shestakov and Zhukavets 06b]. Another technique is the group representation approach that we describe in this section.

With the base field any field of characteristic 0 or greater than n, the symmetric group algebra on n letters FS_n is isomorphic to a direct sum of complete matrix algebras [James and Kerber 81].

We use this isomorphism to replace computations involving permutations with computations involving matrices. Since the two systems are isomorphic, any computations done in one system can be done in the other system. The character of the computations is usually quite different. What might be a trivial computation in one system would involve a tremendous amount of computation in the other. If we show that something is true in the matrix system, the corresponding proof in the group algebra notation may involve linear combinations of huge numbers of terms, which would not be possible even to display, much less publish. The traditional way to prove something is to calculate the rank of the subspace of multilinear elements of degree k in a T-ideal. Then one adds the hypothetical identity of degree k to this subspace and checks the rank again. If the rank remains the same, the hypothetical identity is true. If the rank increases, the hypothetical identity is false.

The origins of this method of using the representations of the symmetric group can be found in the papers by Malcev [Malcev 50] and Specht [Specht 50]. Starting in the 1970s, the method was further developed by Regev [Regev 88]. As a computer technique to study polynomial identities in nonassociative algebras, the method was introduced by Hentzel [Hentzel 77]. The method to calculate the representation matrices was simplified by Clifton [Clifton 81]. This makes the computation of the matrices easy to program in a computer.

There are three significant reasons for introducing the representation technique. The first is that the size of the problem is reduced considerably. In the group-algebra approach, the elements are expressed in terms of $n!\,\mathrm{cat}[n]$ different terms. In the matrix approach, the $n!$ is replaced by the size of the representation. The maximal representation is of size at most $\sqrt{n!}$ [McKay 76]. These differences are significant. In degree 6 the maximal representation size is 16. If we work in the group algebra, we have to work with a matrix of identities having 30240 columns ($30240 = 42 \times 6!$, where $42 = \mathrm{cat}[6]$). In the matrix approach, the largest matrix we have to deal with has 672 columns, where $672 = 42 \times 16$. Of course, the matrix technique has to be done one time for each representation, but this is acceptable.

The second advantage is that the basic unit in the matrix approach is the identity, rather than all permutations of the arguments into the identity. When there is one identity, you find one identity, rather than the whole set of equivalent identities. In other words, we work with a set of module generators for the S_n-module of identities, rather than a set of basis vectors for the vector space of identities.

The third advantage is that the problem is converted into a standard matrix problem in which the computational techniques are already well established. The identities implied by a system of identities correspond to the row space of a matrix. Two sets of identities are equivalent if they have the same row space. One set of identities implies another if the row space of one contains the row space of the other.

We refer to [Hentzel 77] and [Hentzel and Peresi 97] for more details on the theory behind these computational methods.

4.1 The Identification of Multilinear Elements in $F[X]$ with Elements in FS_n

Let p be a multilinear element of $F[X]$ of degree n. We sort the terms of p by association type:

		T_1		T_2		$\dots$		$T_{\mathrm{cat}[n]}$
p	$=$	p_1	$+$	p_2	$+$	$\dots$	$+$	$p_{\mathrm{cat}[n]}$

Within each association type, the terms differ only by their coefficients and the permutation of their (distinct) arguments. Suppose that

$$p_i = \sum_{\pi \in S_n} c_{i\pi}\ (x_1 x_2 \dots x_n)_\pi .$$

All the terms are associated according to association type T_i, and π represents the permutation of the elements that arranges them as they appear in the term of p_i. The permutation π applies to the position, not the subscript. Thus

$$(x_1 x_2 x_3)_{(123)} = x_3 x_1 x_2 .$$

This is not $x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} = x_2 x_3 x_1$. The representation of p_i as an element of the group algebra is

$$p_i = \sum_{\pi \in S_n} c_{i\pi} \pi .$$

For example, the identity $(ab, c, d) + (c, ab, d) = 0$ becomes

T_1	T_3	T_2	T_4
$((ab)c)d$	$-(ab)(cd)$	$+(c(ab))d$	$-c((ab)d)$

T_1	T_3	T_2	T_4
I	$-I$	$+(123)$	$-(123)$

Representation	Partition	Rank
1	3	1
2	21	2
3	111	0

TABLE 1. Alternative laws.

The identity $(ab, cd, e) + (cd, ab, e) = 0$ becomes

T_3	T_8
$((ab)(cd))e$	$-(ab)((cd)e)$
$((cd)(ab))e$	$-(cd)((ab)e)$

T_3	T_8
$I + (13)(24)$	$-I - (13)(24)$

Representing the alternative identities (1–3) and (1–4) by matrices, and reducing to row canonical form, we obtain the ranks given in Table 1.

In Section 3 we obtained the 8 type identities of degree 4. The ranks given by these identities are displayed in Table 2.

Representation	Partition	Rank
1	4	4
2	31	12
3	22	8
4	211	10
5	1111	2

TABLE 2. Type identities of degree 4.

5. NUCLEAR ELEMENTS OF DEGREE n

To show that an element p of $F[x_1, x_2, \ldots, x_n]$ is in the nucleus, we have to show that (p, x_{n+1}, x_{n+2}) is in the T-ideal $\mathrm{Alt}[x_1, x_2, \ldots, x_{n+2}]$ generated by (1–1) and (1–2) on the generators $\{x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}\}$.

The process of finding the nuclear elements of degree n requires setting up a matrix with

$$\mathrm{repsize}\left(\mathrm{cat}[n+2]\binom{n+2}{2} + \mathrm{cat}[n]\right)$$

columns, where repsize is the size of the representation. This matrix is obtained from the group algebra expressions of Table 3 by replacing each element by its representation matrix.

$T_1 \ldots T_{\mathrm{cat}[n+2]\binom{n+2}{2}}$	$(T'_1, x_{n+1}, x_{n+2}) \ldots (T'_{\mathrm{cat}[n]}, x_{n+1}, x_{n+2})$
Augmented type identities of degree $n+2$	Zero matrix
Expansion of associators	Identity matrix

TABLE 3. Nuclear elements.

We will now explain the various steps used to create this matrix.

1. *We create all the type identities of degree* $n+2$. These type identities and the identities obtained by applying all possible permutations of their entries form a spanning set of the subspace $\mathrm{Alt}_{n+2}[x_1, x_2, \ldots, x_{n+2}]$ of the T-ideal $\mathrm{Alt}[x_1, x_2, \ldots, x_{n+2}]$. We work the problem using the group algebra FS_n on n symbols, not $n+2$ symbols. Since the group algebra FS_n does not include permutations involving the symbols x_{n+1} and x_{n+2}, we have to include those permutations separately.

2. *In each type identity, we interchange* x_{n+1} *and* x_{n+2} *with the elements in all possible ways.* Because the associator (p, x_{n+1}, x_{n+2}) is skew-symmetric in the letters x_{n+1} and x_{n+2}, we can assume that all expressions we work with are also skew-symmetric in x_{n+1} and x_{n+2}. When we specify the positions of x_{n+1} and x_{n+2} in a term, there will automatically be a second term that has the positions of x_{n+1} and x_{n+2} reversed and the sign changed. This is done to keep the space required for the computations manageable.

   This means that each identity generates $\binom{n+2}{2}$ (equivalent) identities. After our type identities have been augmented to include these permutations involving x_{n+1} and x_{n+2}, all possible permutations are obtained using only permutations among the elements $x_1, \ldots, x_n$. We call this augmentation of the set of type identities the *augmented type identities.*

   The positions of x_{n+1} and x_{n+2} give each association type of degree $n+2$ a system of $\binom{n+2}{2}$ variations. We assign a single number to identify the association type and the positions of x_{n+1} and x_{n+2}. There are now $\mathrm{cat}[n+2]\binom{n+2}{2}$ types, and each term of the augmented type identities can be expressed in terms of these new types.

3. *We assign a new type number that identifies the degree-*$(n+2)$ *association type as well as the position of the elements* x_{n+1} *and* x_{n+2}. We then drop the letters x_{n+1} and x_{n+2} from the listing of factors, giving us n elements in our list of factors.

   These augmented type identities along with all the permutations of the arguments $x_1, \ldots, x_n$ give the elements of the subspace $\mathrm{Alt}_{n+2}[x_1, x_2, \ldots, x_{n+2}]$ of the T-ideal $\mathrm{Alt}[x_1, x_2, \ldots, x_{n+2}]$. The next task is to see whether any of the elements in the row space of this matrix can be written in the form (p, x_{n+1}, x_{n+2}), where p is in $F[x_1, x_2, \ldots, x_n]$. Such an element p is in the nucleus. We will now show how to pick such elements, and we will later show how to determine which are not zero in the free alternative algebra $\mathrm{ALT}[x_1, x_2, \ldots, x_n]$. To locate p in $F[x_1, x_2, \ldots, x_n]$ such that (p, x_{n+1}, x_{n+2}) is zero, we add $\mathrm{cat}[n]$ new types on the right side of the matrix. These types represent each of the $\mathrm{cat}[n]$ nonassociative types of degree n (denoted by T_i') in $\{x_1, x_2, \ldots, x_n\}$ inside an associator. Finally, each associator (T_i', x_{n+1}, x_{n+2}) is expanded as $(T_i' x_{n+1})x_{n+2} - T_i'(x_{n+1}x_{n+2})$ and encoded in terms of the $\mathrm{cat}[n+2]\binom{n+2}{2}$ types of the left-hand portion.

4. *We expand the* $\mathrm{cat}[n]$ *expressions* (T_i', x_{n+1}, x_{n+2}) *in terms of the types on the left-hand side.* Now when we reduce the entire system to row canonical form, any stairstep one (i.e., a leading 1 of a row of the row canonical form of the matrix) that appears on the right-hand side of the matrix indicates an element p such that $(p, x_{n+1}, x_{n+2}) = 0$, i.e., an element in the nucleus.

5. *We reduce the matrix to row canonical form and print out those rows in which the leading stairstep one appears under the types on the right-hand side.* Naturally, any identity of $\mathrm{ALT}[x_1, x_2, \ldots, x_n]$ appears on the right-hand side, since such elements are zero and certainly are in the nucleus. But we can locate those that are not zero by simply computing $\mathrm{Alt}_n[x_1, x_2, \ldots, x_n]$ in $F[x_1, x_2, \ldots, x_n]$ and looking for stairstep ones on the right-hand side that are not identities for $\mathrm{ALT}[x_1, x_2, \ldots, x_n]$. These are the nonzero nuclear elements.

6. *We create the row canonical form for the type identities of degree n and record any stairstep ones for the right-hand side that are not consequences of the type identities.* The next chore is to find out what the nuclear elements look like. We have their matrix form, but we would like to have them expressed in notation involving associators, commutators, and nonassociative products. This is most easily done by taking the elements from the literature and seeing whether in total they explain all the nuclear elements found. This is done by converting these elements to matrix form and adding them to the identities of $\mathrm{ALT}[x_1, x_2, \ldots, x_n]$ and reducing it to row canonical form. If the new rows that appear are the exact new rows that appeared on the right-hand side, then the element used is the nuclear element we are looking for.

7. *We test whether the nuclear elements are already known by adding the known elements to the identities of* $\mathrm{ALT}[x_1, x_2, \ldots, x_n]$ *and checking the row canonical form to see whether the additional rows they produce are linear combinations of the nonzero nuclear elements of the right-hand side.*

6. THE ACTUAL CALCULATIONS

All of our calculations are done modulo the prime 103. We chose 103 because it is only one byte in length, so the matrix storage is small, and also we avoid integer overflow. It is also larger than the degree of any identities we are likely to be working with. Therefore the group algebra is semisimple and the identities are equivalent to their linearized forms.

The explanatory Section 5 on the method is valid for characteristic zero, or a large-enough prime. But in this section we work only with the prime 103. We considered doing the calculations for several characteristics, but we decided against it. No matter how many characteristics we checked, there would still be the open question about all those we did not check. Our goal is to find the nuclear elements of smallest degree. This is degree 5 over Z_{103}. We find all degree-5 nuclear elements modulo 103. Our work does not show that the minimal degree is 5 in all characteristics.

We do not have an integral dependency relation that displays how an associator containing our elements is a linear combination of elements in the T-ideal generated by (1–1) and (1–2). With such a dependency relation, we would know that the element is in the nucleus, except possibly for prime factors of the coefficient of the associator.

The first instance for a nuclear element occurs in degree 5. There are 61 type identities from (1–1) and 61 type identities from (1–2). Altogether, there are 122 type identities.

The number of augmented type identities is $122 \times \binom{7}{2} = 2562$. The number of extended types is $132 \times \binom{7}{2} = 2772$. There are 14 types on the right-hand side of the matrix in Table 3. The dimensions of the matrix that we have to reduce for each representation depends on the size of the representation. The dimensions of the irreducible representations of S_5 are $1, 4, 5, 6, 5, 4, 1$. The largest matrix we have to reduce is

Representation	Partition	Type identities	Nuclear elements
1	5	13	13
2	41	52	52
3	32	65	66
4	311	75	76
5	221	63	63
6	2111	46	46
7	11111	10	10

TABLE 4. Rank of the matrices.

$$\left\{\left(122 \times \binom{7}{2} + 14\right) \times 6\right\} \times \left\{\left(132 \times \binom{7}{2} + 14\right) \times 6\right\}$$
$$= 15456 \times 16716.$$

The matrix is sparse, with no more than four nonzero entries in each row. We use sparse-matrix techniques. That is, we use a data structure in which only the nonzero entries are stored.

The portion of the row canonical form with the leading ones in the right-hand portion of the matrix is identical to the row canonical form of the degree-5 type identities except for two of the representations. In each of these two representations there is exactly one more leading one. See Table 4.

The rows with the new leading ones occur in representations 3 and 4 and are located under type 14 in both representations. Type 14 is $x(x(x(xx)))$. This means that there is a nuclear element for which all the terms are associated as in type 14. See Table 5.

Each of these rows represents a nonzero nuclear element. One could find an element of the group algebra FS_5 that maps exactly to them, using the isomorphism of FS_5 to a direct sum of complete matrix algebras. This would most likely be an element involving a nonzero coefficient for each possible permutation of the letters x_1, x_2, x_3, x_4, x_5 associated as type 14. One usually tries to write the element with a minimal number of terms.

There are many equivalent ways to express an element. One needs only an element that when added to the type identities gives the additional rows in the row canonical form for representations 3 and 4, and no additional

Representation	Partition	Type 14	Nuclear element
3	32	$x(x(x(xx)))$	10111
4	311	$x(x(x(xx)))$	1 0 –1 –3 –1 2

TABLE 5. Nonzero nuclear elements of degree 5.

rows in representations $1, 2, 5, 6, 7$. An element with that property is (1–9). It has the fewest terms of any element in the nucleus found so far. Using alternative identities, one can rewrite this element (1–9) as element (1–7). This last element actually has eight terms, but it is compactly written with the commutators.

Element (1–5) is the nuclear element in representation 3, and (1–6) is the nuclear element in representation 4.

6.1 Additional Checking Using ALBERT

We checked that elements (1–5) through (1–9) are in the nucleus using the computer program ALBERT [Jacobs et al. 96]. Furthermore, we checked that:

(i) Element (1–7) is equivalent to (1–9).

(ii) Element (1–6) is equivalent to (1–8).

(iii) Element (1–7) implies (1–5), (1–6), and (1–8).

(iv) Element (1–7) is equivalent to the set of elements $\{(1\text{–}5), (1\text{–}6)\}$.

Here two elements are equivalent if we can verify, using consequences of the alternative identities, that each implies the other.

ALBERT is the code of the algorithm described in detail in [Hentzel and Jacobs 91]. As examples we explain how ALBERT verifies that (1–7) is in the nucleus and how (1–6) implies (1–8).

To verify that (1–7) is in the nucleus we need to verify that

$$(([a,b][a,c])a - (a[a,b])[a,c], d, e) = 0 \qquad (6\text{–}1)$$

is an identity in the free alternative algebra $\mathrm{ALT}[a,b,c,d,e]$. ALBERT constructs a finite dimension algebra that is a homomorphic image of $\mathrm{ALT}[a,b,c,d,e]$. Then ALBERT verifies that (6–1) is an identity in this finite dimension algebra. As proved in [Hentzel and Jacobs 91], this implies that (6–1) is an identity in $\mathrm{ALT}[a,b,c,d,e]$.

To verify that (1–6) implies (1–8) we have to verify that (1–8) is an identity in the free algebra $F[a,b,c]/I$, where I is the T-ideal generated by (1–1), (1–2), and (1–6). ALBERT constructs a finite dimension algebra that is a homomorphic image of $F[a,b,c]/I$. Then ALBERT verifies that (1–8) is an identity in this finite dimension algebra.

7. A SERIES OF ELEMENTS IN THE NUCLEUS

Shestakov and Zhukavets [Shestakov and Zhukavets 06a, Corollary 5.2] proved that the elements

$$a_n = \sum_{\pi} \mathrm{sgn}(\pi)([[\ldots[[[x_1,x_2],x_3],x_4], \ldots, x_n],[x_{n+1},x_{n+2}]])_\pi$$

are in the center (hence in the nucleus) of the free alternative algebra, and are nonzero in the free Malcev algebra, for $n = 4k$ $(k > 1)$ and $n = 4k+1$ $(k > 0)$, over a field of characteristic zero. They asked the question whether the a_n are nonzero in the free alternative algebra. In [Hentzel and Peresi 03] we already proved that a_5 is nonzero. Shestakov and Zhukavets [Shestakov and Zhukavets 06b, Corollary 4.6] proved that the a_n are nonzero over a field of characteristic different from 2 and 3.

Using our computer procedure we prove that the elements a_n, for $n = 4k, k = 2,3$, and $n = 4k+1, k = 1,2$, are nonzero in the free alternative algebra $\mathrm{ALT}[X]$. It follows that they are nonzero elements in the center of $\mathrm{ALT}[X]$. That is, $a_n \neq 0$ and

$$(a_n, \mathrm{ALT}[X], \mathrm{ALT}[X]) = [a_n, \mathrm{ALT}[X]] = 0.$$

The question is whether a_n is in the T-ideal generated by (1–1) and (1–2) in $F[X]$, where $X = \{x_1, x_2, \ldots, x_{n+2}\}$. To decide this we compute the rank of the type identities of degree $n+2$ in $F[X]$. Then we add a_n to the type identities and compute the rank again. If the rank stays the same, then $a_n = 0$ in $\mathrm{ALT}[X]$. If the rank increases, then $a_n \neq 0$ in $\mathrm{ALT}[X]$. Our calculations are done over the field Z_{103}.

Because a_n is skew-symmetric, its representation is easy to compute. It has zero representation in all of the representations except the last one. In the last representation, if we know the representation of

$$p = [[\ldots[[[x_1,x_2],x_3],x_4],\ldots,x_n],[x_{n+1},x_{n+2}]],$$

then the representation of a_n is $(n+2)!$ times the representation of p. Therefore it is sufficient to calculate just the last representation and just the representation of p. Because the last representation has size 1, it is possible to carry the calculations up to degree 14.

In Table 6, type I occurs when $n = 4k$ $(k > 1)$, and type II occurs when $n = 4k+1$ $(k > 0)$. The column labeled "alternative" gives the rank of the subspace $\mathrm{Alt}_{n+2}[x_1, x_2, \ldots, x_{n+2}]$ generated by type identities of degree $n+2$. The column a_n gives the rank after the T-ideal is augmented by the element a_n. Notice that the rank increases when $n = 1, 4, 5, 8, 9, 12$.

n	$n+2$	k	Type	Alternative	a_n	a_n is
1	3			0	1	nonzero
2	4			2	2	zero
3	5			10	10	zero
4	6			36	37	nonzero
5	7	1	II	123	124	nonzero
6	8			418	418	zero
7	9			1418	1418	zero
8	10	2	I	4848	4849	nonzero
9	11	2	II	16779	16780	nonzero
10	12			58767	58767	zero
11	13			207992	207992	zero
12	14	3	I	742878	742879	nonzero

TABLE 6. Elements a_n.

Therefore a_5, a_8, a_9, a_{12} are nonzero over Z_{103}.

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I. R. Hentzel, Department of Mathematics, Iowa State University, Ames, Iowa 50011-2064, U.S.A. (hentzel@iastate.edu)

L. A. Peresi, Departamento de Matemática, Universidade de São Paulo, Caixa Postal 66281, São Paulo-SP, Brazil 05311-970 (peresi@ime.usp.br)

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