LIAPUNOV OPERATORS AND STABILIZATION IN STRONGLY ORDER PRESERVING DYNAMICAL SYSTEMS

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Introduction. In this note we deal with the asymptotic behavior of (eventually) strongly monotone semigroups, \( S(t) \), on strictly ordered Banach spaces. In our considerations, the continuity in \( t \) does not enter and so our results hold for \( t \) in \( \mathbb{R}_0^+ \) or \( \mathbb{Z}_0^+ \) alike; in particular, they hold for discrete semigroups generated by a single map. Moreover, no smoothness assumptions, besides the continuity of the maps \( S(t) \) for fixed \( t \), are introduced.

The paper addresses the following basic question: under which conditions are all relatively compact orbits convergent? Our answer is very simple and geometrical. For a semigroup \( S(\cdot) \) as above, with initial conditions chosen from the order interval \([a, b]\), with \( a \) a subsolution, and \( b \) a supersolution, all precompact orbits are convergent if there is a continuous, strongly ordered arc \( r \) connecting \( a \) to \( b \), and which, in general, may consist of two pieces \( r_1 \) and \( r_2 \), with the lower one made up of subsolutions and the upper one of supersolutions.

The most interesting, dynamically, is the case where (part of) \( r \) consists of a continuum of equilibria. In that case, \( r \) can be split into \( r_1 \) and \( r_2 \) in an infinity of ways. We point out that \( r \) need not be invariant under \( S(\cdot) \), a feature that makes the result flexible in applications. We note that our structure hypotheses do not allow the existence of a non-degenerate unstable equilibrium on \( r \) except possibly at the end points. This feature, in general, is in the nature of things for stabilization of all precompact orbits to hold, for otherwise only generic results can be expected ([11], [12]).

The idea of the stabilization result is this: Given any element in the order interval \([a, b]\), there is either a maximal element of \( r_1 \) (subsolution) below it or a minimal element of \( r_2 \) (supersolution) above it. This fact, together with the strong monotonicity hypothesis, forces the \( \omega \)-limit set into a single element, an equilibrium on \( r \). In particular, \( r \) contains all the equilibria in the order interval \([a, b]\).

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Our theorem is an abstraction of the stabilization results on Fisher’s equation. Its aim is to show how far towards stabilization of all orbits one can go by making use only of the monotonicity of the system. In particular, it does not require any gradient structure, and appears to cover a large collection of stabilization results in the literature that are based on the maximum principle. In Section 2, we illustrate its use by giving a few examples. Related work in the past can be found in [1-3].

1. Liapunov operators, invariance principle and stabilization. Setting.

Let \( X \) be an ordered Banach space, with order cone \( K \). For \( x, y \in X \) we write \( x \leq y \) if \( y-x \in K \), \( x < y \) if \( y-x \in K - \{0\} \), \( x \ll y \) if \( y-x \in \text{Int}(K) \) which we assume to be nonempty. Such ordered Banach spaces are called strongly ordered. Let \( U \) be a subset of \( X \), and let \( S(\cdot) \) be a one parameter family of maps, \( \{S(t)\} \), from \( U \) into itself and satisfying the semigroup property\( S(t+s) = S(t) \cdot S(s) \) for \( t, s \in \mathbb{R}_0^+ \) or \( \mathbb{Z}_0^+ \).

We call \( S(\cdot) \) order preserving if \( S(t)x \leq S(t)y \) for all \( t \) whenever \( x \leq y \). \( S(\cdot) \) is called strongly order preserving if \( S(t)x \ll S(t)y \) for all \( t > 0 \), whenever \( x < y \). For each fixed \( t \), \( S(t) \) is assumed continuous as a map from \( U \) into itself. In this work, no continuity hypothesis on the dependence of \( S(t) \) on \( t \) is introduced.

We call an element \( x \) in \( U \) an equilibrium if \( S(t)x = x \) for all \( t \), a subsolution (supersolution) if \( S(t)x ;::; x \) (\( S(t)x ::; x \)) for all \( t \).

An equilibrium \( p \) of the semigroup \( S(\cdot) \) is called stable if given a neighborhood \( V_p \) of \( p \) in \( U \) there exists a neighborhood \( V_s \) of \( p \) in \( U \) such that for \( x \in V_s \), \( S(t)x \) is in \( V_p \) for all \( t \). Otherwise \( p \) is called an unstable equilibrium. For a continuous map \( S \) a fixed point is called stable if it is so as an equilibrium for the discrete semigroup \( \{S^n\} \) that is generated by \( S \).

We call \( x \) a strict subsolution (strict supersolution) if \( S(t)x > x \) \( (S(t)x < x) \) for \( t \neq 0 \). In this work we will take as \( U \) an order interval \([a, b]=[x \in X : a \leq x \leq b]\), with \( a \) a subsolution and \( b \) a supersolution of \( S(\cdot) \). We will consider only \( a \ll b \).

Note that \([a, b]\) is automatically mapped into itself by \( S(t) \) for each \( t \). Let \( c \) be an equilibrium in \([a, b]\), \( a \ll c \ll b \). Let \( Q_2 = [c, b] \), \( Q_1 = [a, b] - Q_2 \). Note that \( Q_1 \neq [a, c] \) if \( c \ll b \), and that \( Q_2 \) is positively invariant under \( S \), that is, \( S(t)x \in Q_2 \) if \( x \in Q_2 \) for all \( t \). We will consider the case where a strongly ordered continuous arc \( \Gamma_1 \) exists that connects \( a \) to \( c \), and such that \( \Gamma_1 \) consists entirely of subsolutions. More precisely this arc is the range of a strongly order preserving map\( \phi_1 : [0, 1] \rightarrow [a, c], \)

with \( \phi_1(0) = a \), \( \phi_1(1) = c \), and with \( s_1 < s_2 \implies \phi_1(s_1) < \phi_1(s_2) \).

\[ (A) \text{ Definition of Liapunov operators.} \] For \( \xi \) in \( Q_1 \) define \( V(\xi) = (\xi - \partial K) \cap \Gamma_1 \).

We call \( V \) the lower Liapunov operator.

Note: This definition is motivated by Dafermos’ use of the max and min as Liapunov functionals for the equation \( u_t + (f(u))_x = 0 \) ([6]).
Proposition 1. Let the setting be as above and assume that $S(\cdot)$ is order preserving.

(i) $V(\xi)$ is a singleton for $\xi \in Q_1$.

(ii) Alternatively $V(\xi)$ can be characterized as the largest element of $\Gamma_1$ below $\xi$:
$$V(\xi) = \max\{\phi_1(s) : \phi_1(s) \leq \xi\}.$$  

(iii) $V(\xi)$ is order preserving on $Q_1$:
$$\xi_1 \leq \xi_2 \text{ implies } V(\xi_1) \leq V(\xi_2), \xi_1 \ll \xi_2 \text{ implies } V(\xi_1) \ll V(\xi_2).$$

(iv) Let $S(t)x$ such that $S(t)x \in Q_1$ for $t \geq 0$. Then $V(S(t)x)$ is increasing:
$$V(S(t_1)x) \leq V(S(t_2)x) \text{ for } t_2 > t_1.$$

(v) $V$ is continuous in $Q_1$ in the following sense:
Let $\xi_n \rightarrow \xi_0$ as $n \rightarrow \infty$, $\xi_n, \xi_0 \in Q_1$, with $\{V(\xi_n)\}$ increasing: $V(\xi_n) \leq V(\xi_{n+1})$. Then
$$V(\xi_n) \rightarrow V(\xi_0) \text{ as } n \rightarrow \infty.$$

Proof: The proof of this proposition is based on simple geometrical considerations.

(i) First we show that $V(\xi) \neq \emptyset$. If $a \in \xi - \partial K$ or $c \in \xi - \partial K$ then $V(\xi) \neq \emptyset$. Hence let $a \in \xi - \text{Int}(K)$, $c \not\in \xi - \partial K$. For $\xi \in Q_1$ clearly $\xi \gg c$, that is, $c \not\in \xi - \text{Int}(K)$. Hence in our case $c \not\in \xi - K$. Since $\Gamma_1$ is connected, it follows that $(\xi - \partial K) \cap \Gamma_1 \neq \emptyset$, and so $V(\xi) \neq \emptyset$.

Next we show that $V(\xi)$ is a singleton. For assume $v, w \in V(\xi)$, $v \neq w$. Since $\Gamma_1$ is strongly ordered, without loss of generality we may assume that $v \gg w$. Now
$$w = (w - v) + v = (w - v) + (\xi - \partial k) \quad (\partial k \in \partial K)$$
and so $w \in \xi - \text{Int}(K)$, contradicting the definition of $V(\xi)$.

(ii) There exists $s_0 \in [0, 1]$ such that $V(\xi) = \phi_1(s_0)$; from the definition $V(\xi) \leq \xi$. Assume $\phi_1(s_0)$ is not the largest element in $\Gamma_1$ below $\xi$. Since $\Gamma_1$ is strongly ordered, there exists $s_0' > s_0$ with $\phi_1(s_0') \ll \phi_1(s_0) \leq \xi$. Hence $\phi_1(s_0) \in \xi - \text{Int}(K)$, contradicting the definition of $V(\xi)$.

(iii) That $\xi_1 \leq \xi_2$ implies $V(\xi_1) \leq V(\xi_2)$ is clear from part (ii). To see that $\xi_1 \ll \xi_2$ implies $V(\xi_1) \ll V(\xi_2)$, it suffices to show that $V(\xi_1) < V(\xi_2)$. Assume for the sake of contradiction that
$$V(\xi_1) = V(\xi_2),$$
which means that
$$\xi_1 - \partial k_1 = \xi_2 - \partial k_2$$
for some elements $\partial k_1, \partial k_2 \in \partial K$. Then
$$0 \ll \xi_2 - \xi_1 = \partial k_2 - \partial k_1 \leq \partial k_2,$$
from which it follows that $\partial k_2 \in \text{Int}(K)$, a contradiction.
(iv) We give the details of the proof for discrete semigroups generated by a single continuous map \( S \). \( S^n(x) \geq V(S^n(x)) \) by definition. Since \( S \) is order preserving,

\[
S^m(S^n(x)) \geq S^m(V(S^n(x))), \quad \text{for } m \in \mathbb{N}.
\]

Since \( V(S^n(x)) \) is in \( \Gamma_1 \), it is a subsolution, and therefore

\[
S^m(V(S^n(x))) \geq V(S^n(x)).
\]

By (iii),

\[
V(S^{m+n}(x)) \geq V(S^n(x)) = V(S^n(x)),
\]

and so the monotonicity of \( V(S^n(x)) \) is established.

(v) \( V(\xi_n) \in \Gamma_1 \) for all \( n, \Gamma_1 \) is compact and \( V(\xi_n) \) is increasing, hence

\[
V(\xi_n) \to q, \quad q \in \Gamma_1, \quad \text{as } n \to \infty.
\]

First we show \( q \leq V(\xi_0) \). Indeed, \( \xi_n \geq V(\xi_n) \) for all \( n \) and so \( \xi_0 \geq q \). Therefore, by (iii), \( V(\xi_0) \geq V(q) = q \).

Next assume for the sake of contradiction that \( q < V(\xi_0) \). Since both \( V(\xi_0) \) and \( q \) are elements of \( \Gamma_1 \), it follows by the strong ordering of \( \Gamma_1 \) that \( V(\xi_0) \gg q \).

Now for every \( n \)

\[
\xi_k \to_{k \to \infty} \xi_0 \geq V(\xi_0) \gg q \geq V(\xi_n),
\]

thus \( \xi_n \gg V(\xi_n) \) for large \( n \), contradicting the definition of \( V(\xi_n) \) by (ii). Therefore we have that \( q = V(\xi_0) \).

**Remark.** If we assume there is a strongly ordered arc \( \Gamma_2 \) in \( Q_2 \) joining \( c \) with \( b \), consisting of supersolutions, we may define the *upper Liapunov operator* \( \bar{V} \) on \( Q_2 \) by

\[
\bar{V}(\xi) = (\xi + \partial K) \cap \Gamma_2.
\]

Results analogous to Proposition 1 are then true for \( \bar{V} \), with obvious modifications.

(B) **The invariance principle** (cf. [13]). Let \( S(\cdot) \) be order preserving and assume the same setting as in Proposition 1. Consider \( x \in Q_1 \) such that \( S(t)x \in Q_1 \) for all \( t \geq 0 \) (\( t \in \mathbb{R}_+^+ \) or \( \mathbb{Z}_+^+ \)). Assume moreover that the semi-orbit

\[
\gamma^+(x) = \{S(t)x : t \geq 0\}
\]

is relatively compact. Then it is well-known that the \( \omega \)-limit set \( \omega(x) \) is nonempty.

**Proposition 2.**

\[
\left. V \right|_{\omega(x)} = q.
\]

**Proof:** We give the details for discrete semigroups. The sequence \( \{V(S^n(x))\} \) is increasing by Proposition 1(iv), and relatively compact since \( V \) takes values in \( \Gamma_1 \). Hence

\[
V(S^n(x)) \to q \quad \text{as } n \to \infty.
\]
Let $y \in \omega(x)$; it follows that there exists a subsequence $\{n_k\} \not\to \infty$ such that

$$S^{n_k}(x) \to y.$$ 

Hence

$$V(S^{n_k}(x)) \to q,$$

and by Proposition 1(v),

$$V(S^{n_k}(x)) \to V(y).$$

Therefore $V(y) = q$, for all $y \in \omega(x)$.

(C) **Stabilization.** Let $x \in Q_1$ be such that $S(t)x \in Q_1$ for $t \geq 0$ ($t \in \mathbb{R}_0^+$ or $\mathbb{Z}_0^+$), and assume that the semi-orbit $\{S(t)x : t \geq 0\}$ is relatively compact. Moreover, assume that $S(\cdot)$ is *eventually strongly order preserving* in $Q_1 : S(\cdot)$ is order preserving in $[a, b]$, and to any pair of elements $q, \xi \in Q_1$ with $q < \xi$ there exists $t_0 > 0$ such that

$$S(t_0)q \ll S(t_0)\xi.$$ 

(Note: it would suffice to assume that at least one of $q, \xi$ belongs to $\Gamma_1$.)

**Theorem 3.**

$$\lim_{t \to \infty} S(t)x = q, \quad \text{where } q \text{ is an equilibrium, } q \in \Gamma_1.$$ 

**Proof:** By Proposition 2, $V_{\omega(x)} = q$. Let $\xi \in \omega(x)$; we claim that $\xi = q$ and so $\omega(x) = \{q\}$. We argue by contradiction. So assume that $\xi \neq q$. Since $q = V(\xi) \leq \xi$, it follows that $q < \xi$; $q$ is in $\Gamma_1$ and so by the hypothesis on $S$ there exists $t_0 > 0$ such that $S(t_0)q \ll S(t_0)\xi$. Since $q$ is subsolution, $q \leq S(t_0)q$, and so $S(t_0)\xi \gg q$. By Proposition 1(iii), since $S(t_0)\xi \in \omega(x)$, we have $q = V(S(t_0)\xi) \gg V(q) = q$, a contradiction. Therefore $\xi = q$, and the proof is complete.

Under the hypothesis that there exists a strongly ordered arc $\Gamma = \Gamma_1 \cup \Gamma_2$ consisting of sub- and supersolutions, respectively, and that $S(\cdot)$ is eventually strongly order preserving in $[a, b]$, by employing analogous considerations in $Q_2$ we can prove

**Theorem 3'.** For $x$ in the order interval $[a, b]$, and under the hypothesis that the semi-orbit $\{S(t)x : t \geq 0\}$ is relatively compact we have

$$\lim_{t \to \infty} S(t)x = q, \quad \text{where } q \text{ is an equilibrium, } q \in \Gamma.$$ 

2. **Applications.** In this section we give various applications of our stabilization result.

**Example 1: Sublinear maps.** Consider a continuous map $S$ defined on the order interval $[0, b]$ in a strongly ordered Banach space $X$ with cone $K$. Assume that

(i) $S(0) = 0$, $S(b) \leq b$.

(ii) $\{S^n\}$ is eventually strongly order preserving in $[0, b]$.

(iii) $S$ is sublinear:

$$\alpha S(x) \leq S(\alpha x)$$

for all $x$ in $[a, b]$ and any number $\alpha$ in $[0, 1]$. 

Takáč [14] proved that (at least if $S$ is strongly order preserving) relatively compact orbits of the discrete semigroup generated by $S$ are convergent.

We now give a proof of this result by employing Theorem 3.

It follows that $S^n(b)$ converges, as $n \to \infty$, to an equilibrium $c : \lim_{n \to \infty} S^n(b) = c$. If $c = 0$ then clearly by comparison $S^n(x) \to 0$ as $n \to \infty$, for all $x$ in $[0, b]$. If $c > 0$, by property (ii) above $c > 0$. Set $\Gamma_1 = \{ \alpha c : 0 \leq \alpha \leq 1 \}$. By property (iii), $S(\alpha c) \geq \alpha S(c) = \alpha c$, and so $\Gamma_1$ consists of subsolutions. From $c > 0$ we conclude that $\Gamma_1$ is a continuous strongly ordered arc. Let $Q_1 = [0, b] - [c, b]$, and consider first an orbit that lies entirely in $Q_1$. Theorem 3 applies and stabilization follows.

If on the other hand the orbit enters $Q_2 = [c, b]$ then it clearly stays there for all subsequent ‘times’ and stabilization to $c$ follows by comparison.

Example 2: Nonexpansive maps. Let $X$ be a reflexive strongly ordered Banach space and let $\{ S(t) \}$ be a semi-group of nonexpansive maps on $X$, $t$ taking values either in $\mathbb{R}_0^+$ or in $\mathbb{Z}_0^+$ (no continuity assumptions in $t$). Let $u, \tilde{u}$ be a pair of sub- and supersolutions with $u < \tilde{u}$, and assume that $S(\cdot)$ is eventually strongly order preserving in $[u, \tilde{u}]$. Then relatively compact orbits $\{ S(t)u : t \geq 0 \}$ in $[u, \tilde{u}]$ are convergent; i.e., \( \lim_{t \to \infty} S(t)u = q \), where $q$ is an equilibrium.

This result represents an improvement over Theorem 1 in [2] since there is no continuity requirement on $t$. We present below a proof by verifying the hypotheses of Theorem 3.

Since $u, \tilde{u}$ are sub- and supersolutions,

$$ S(t)u \not\nearrow a, \quad S(t)\tilde{u} \not\searrow b, \quad \text{as} \quad t \to \infty, $$

where $a, b$ are equilibria, $a \leq b$.

Let $\xi$ in $\omega(u)$. Then for an appropriate sequence $t_n \to \infty$, $S(t_n)u \to \xi$, and from

$$ S(t_n)u \leq S(t_n)u \leq S(t_n)\tilde{u} $$

it follows that $\omega(u) \subset [a, b]$. Therefore the only case of interest is $a < b$. A posteriori $a \ll b$. Consider now the order interval $[a, b]$. Since the set of fixed points of nonexpansive maps on reflexive spaces is convex [5], the strongly ordered segment $\Gamma = \{ \lambda a + (1 - \lambda) b : 0 \leq \lambda \leq 1 \}$ consists of equilibria of $S(\cdot)$. By applying Theorem 3 on $[a, b]$ with $c = b$ and $\Gamma_1 = \Gamma$, taking an element $x \in \omega(u)$ we conclude that $\omega(u)$ contains an equilibrium from the set $\Gamma$. By nonexpansiveness it follows that $\omega(u)$ is a singleton and the proof is complete.

An immediate consequence of the result above is the following.

**Corollary.** Let $\{ S(t) \}$, $t \in \mathbb{R}_0^+$ or $t \in \mathbb{Z}_0^+$, be a nonexpansive, eventually strongly order preserving semigroup on a strongly ordered reflexive Banach space $X$. Let $p, q$ be two order related equilibria of $S(\cdot)$. Then the set of equilibria of $S(\cdot)$ in the order interval $[p, q]$ coincides with the segment $\{ \lambda p + (1 - \lambda) q : 0 \leq \lambda \leq 1 \}$.

Example 3: Stable maps. For discrete semigroups that are generated by a single continuous map $S$ and with a certain extra compactness property, the previous two examples can be subsumed into a more general result concerning semigroups with only stable equilibria in an arbitrary strongly ordered Banach space $X$. More precisely, let $a < b$ be a pair of sub/supersolutions, and let $S : [a, b] \subset X \to [a, b]$ be continuous, with $S([a, b])$ relatively compact. Also assume that $\{ S^n \}$ is eventually strongly order preserving on $[a, b]$. Finally assume that all equilibria of $\{ S^n \}$ in $[a, b]$ are stable.
Theorem 4. All semi-orbits in \([a, b]\) converge.

For strongly order preserving semigroups, the first version of this result is due to Alikakos, Hess, Matano [16], under the assumption that each semi-orbit \(\gamma^+(u)\) is stable; for any \(\epsilon > 0\) there exists \(\delta > 0\) such that \(S^n(v) \in B(S^n(u), \epsilon)\), for all \(v \in B_{[a,b]}(u, \delta) = \{w \in [a, b] : \|w - u\| < \delta\}\), for all \(n \geq 0\). Subsequently Takáč [15] obtained the same result, assuming only the stability of all equilibria. Both \[16\] and \[15\] require a lattice structure on the underlying space. Dancer and Hess [7], manage to relax this structure hypotheses on the space. The theorem above is taken from there. We present here a different proof by employing Theorem 3. We need first

Lemma 5. Let \(S\) be order preserving and \(x < y\) stable equilibria of the discrete semigroup \(\{S^n\}\). Then there exists another equilibrium between \(x\) and \(y\).

Proof: (cf. [7]). We argue by contradiction. So assume there is no other equilibrium in the order interval \(V := [x, y]\). Let

\[
S_t(u) := tS(u) + (1 - t)y, \quad u \in V, \ 0 \leq t \leq 1.
\]

\[
\tilde{S}_t(u) := tS(u) + (1 - t)x,
\]

If \(S_t(u) = u\) for some \(t \in [0, 1]\) and \(u \in V \setminus \{x, y\}\) then \(u\) is a strict supersolution of \(S(\cdot)\). Indeed from

\[
tS(u) + (1 - t)y = tu + (1 - t)u
\]

it follows that

\[
t[S(u) - u] = (1 - t)[u - y].
\]

The possibility of \(t = 1\) is excluded by the contradiction hypothesis and so we have \(S(u) < u\). Similarly, if \(\tilde{S}_t(v) = v\) for some \(t \in [0, 1]\), \(v \in V \setminus \{x, y\}\), then \(v\) is a strict subsolution.

Now if for every \(\epsilon > 0\) there are \(t_{\epsilon} \in [0, 1]\) and \(u_{\epsilon} \in \partial BV(y, \epsilon)\) with

\[
S_{t_{\epsilon}}(u_{\epsilon}) = u_{\epsilon},
\]

then there are strict supersolutions as close to \(y\) as we wish. Otherwise there is an \(\epsilon_0 > 0\) with \(S_t(u) \neq u\) for all \(t \in [0, 1]\) and all \(u \in \partial BV(y, \epsilon_0)\). We now employ an index argument [4] in the order interval \(V\). By the homotopy invariance of the index,

\[
i(S, V, y) = i(S_1, V, y) = i(S_0, V, y) = 1.
\]

On the other hand, from the normalization property of the index, \(i(S, V, V) = 1\). Therefore \(i(S, V, x) = 0\). Consequently there is no homotopy of \(S = \tilde{S}_1\) to \(\tilde{S}_0\) on \(BV(x, \epsilon)\) for any \(\epsilon > 0\) small. It follows that for every such \(\epsilon > 0\) there is \(t_{\epsilon}\) in \([0, 1]\) and \(v_{\epsilon}\) in \(\partial BV(x, \epsilon)\) with

\[
\tilde{S}_{t_{\epsilon}}(v_{\epsilon}) = v_{\epsilon}.
\]

Therefore there are strict subsolutions as close to \(x\) as we wish. In either case, iteration yields instability (of \(x\) from above or of \(y\) from below).
Proof of Theorem 4: Let $a' = \lim S^n(a)$, $b' = \lim S^n(b)$. Clearly $a \leq a' \leq b' \leq b$, and $a'$, $b'$ are equilibria. If $a' = b'$ then the statement of the theorem is obvious. Hence assume that $a' < b'$ and so a posteriori $a' \ll b'$ since $\{S^n\}$ is eventually strongly order preserving. Clearly it is sufficient to establish stabilization for orbits in the order interval $[a', b']$. For this purpose we need to produce the arc $\Gamma$. In fact we will show that the set of equilibria of $S(\cdot)$ in $[a', b']$ is order-homeomorphic to a closed interval of real numbers. To see this we proceed as follows. Let

$$\mathcal{E} = \{\text{equilibria of } S(\cdot) \text{ in } [a', b']\}.$$ 

Clearly $\mathcal{E}$ is compact. Next consider the collection

$$\mathcal{C} = \{\text{chains from } \mathcal{E}\}$$

and order it by inclusion. Zorn’s lemma applies and provides a maximal element $\Gamma$ of $\mathcal{C}$. Clearly by maximality, $\Gamma$ is closed and so compact. Of course $a', b' \in \Gamma$. Choose now a positive functional $z^* \in X^*$; i.e., $z^*$ with $\langle z^*, u \rangle \geq 0$ for $u \in K$ and $\langle z^*, u_0 \rangle > 0$ for some $u_0 \in K$; note that a priori $u_1 < u_2$ in $\Gamma$ but a posteriori $u_1 \ll u_2$ since $\{S^n\}$ is eventually strongly order preserving. The desired homeomorphism is provided by $z^*$.

By Lemma 5 the image of $\Gamma$ under $z^*$ is clearly an interval. Theorem 3 (or equally well Theorem 3’) now applies and finishes the proof; in particular $\Gamma = \mathcal{E}$.

Remark. It is surprising, perhaps, that as a consequence of Theorem 4 we obtain that the stability of all equilibria implies the stability of all semi-orbits.

Let us take $u \in [a, b]$ and show the stability of its semi-orbit $\gamma^+(u)$. By Theorem 4, $q = \lim_{n \to \infty} S^n(u)$ exists and is an equilibrium. Let $\epsilon > 0$ be given. By stability of $q$, there exists $\sigma > 0$ (without loss of generality) such that

$$S^n(p) \in B(q, \frac{\epsilon}{2}), \quad \text{for all } p \in B_{[a,b]}(q, \sigma), \quad \text{all } n \geq 0.$$ 

Because of the convergence there exists $\bar{n}$ such that $S^n(u) \in B(q, \frac{\sigma}{2})$ for all $n \geq \bar{n}$. By continuity of $S$, there exists further $\delta > 0$ such that

$$S^n(v) \in B(S^n(u), \frac{\sigma}{2}), \quad \text{for all } v \in B_{[a,b]}(u, \delta), \quad \text{all } n = 0, \ldots, \bar{n}.$$ 

Now

$$\|S^n(v) - q\| \leq \|S^n(v) - S^n(u)\| + \|S^n(u) - q\| < \sigma$$

for all $v \in B_{[a,b]}(u, \delta)$. Therefore

$$\|S^n(v) - S^n(u)\| \leq \|S^{\bar{n} - \bar{n}}S^{\bar{n}}(v) - q\| + \|S^{\bar{n}}(u) - q\| < \frac{\epsilon}{2} + \frac{\sigma}{2} \leq \epsilon,$$

for all $v \in B_{[a,b]}(u, \delta)$, all $n \geq \bar{n}$. The stability of $\gamma^+(u)$ follows.

Example 4: Fisher’s equation ([8]). Consider the initial value problem

$$\begin{cases} u_t = m_1(t)\Delta u + m_2(t)g(u, \nabla u) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial n} |_{\partial \Omega} = 0 & \text{on } \partial \Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases} \quad \text{(FE)}$$
where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $m_i(\cdot)$ are Hölder continuous $T$-periodic functions, and $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a $C^1$ function satisfying $g(c,0) = g(d,0) = 0$, $g(u,0) > 0$ for $u$ in $(c,d)$, and $u_0(\cdot)$ is a measurable function satisfying $c \leq u_0(x) \leq d$, almost everywhere in $\Omega$. We assume $m_1$ to be positive on $\mathbb{R}$, and consider two different sets of hypotheses on $m_2$:

(H1) $\int_0^T m_2(t) \, dt = 0$

(H2) $\int_0^T m_2(t) \, dt \neq 0$.

**Proposition 6.** 1. Under (H1) any solution to (FE) converges (uniformly), as $t \to \infty$, to a periodic solution of the ordinary differential equation $\dot{p}(t) = m_2(t)g(p,0)$, in the sense that

$$\|u(\cdot, t) - p(t)\|_{C(\Omega)} \to 0 \quad \text{as} \quad t \to \infty.$$

2. Under (H2) any nontrivial solution to (FE) converges (uniformly), as $t \to \infty$, to the constant $c$ or $d$ according to whether

$$\int_0^T m_2(t) \, dt < 0 \quad \text{or} \quad \int_0^T m_2(t) \, dt > 0.$$

The proposition above partly extends a result of Hess and Weinberger [10] by allowing gradient dependence. We give a geometric proof by applying the stabilization theorem.

**Proof:** 1. Consider the discrete semigroup generated by the period map $S$, $S(u_0) = u(x,T,u_0)$. We begin by considering $S(\cdot)$ on the order interval $[c,d] \subset C(\Omega)$ (which, by the maximum principle, is positively invariant under $S(\cdot)$). We denote by $\Gamma$ the set of constant functions in the order interval $[c,d]$. From (H1) it follows that constants are equilibria of $S(\cdot)$, and the strong maximum principle implies that $S(\cdot)$ is strongly order preserving. Standard regularity theory [9] implies that orbits in $[c,d]$ are relatively compact in $C(\Omega)$. Therefore Theorem 3' applies and we obtain stabilization,

$$\lim S^n(u_0) = q, \quad q \text{ equilibrium of } S(\cdot).$$

The result, as stated, follows by a standard continuous dependence argument [9]: the case of measurable initial conditions is reduced immediately to the case just treated by utilizing well-known regularizing properties of parabolic partial differential equations.

2. The proof is essentially as before. The only different point now is that constants between $c$ and $d$ are either (strict) subsolutions or (strict) supersolutions for the period map $S(\cdot)$.

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REFERENCES


