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EXISTENCE OF A POSITIVE SOLUTION TO A "SEMILINEAR" EQUATION INVOLVING PUCCI'S OPERATOR IN A CONVEX DOMAIN

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Abstract. In this article we prove existence of positive solutions for the nonlinear elliptic equation

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) - \gamma u + f(u) &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\mathcal{M}^+_{\lambda,\Lambda}$ denotes Pucci's extremal operator with parameters $0 < \lambda \leq \Lambda$ and Ω is convex smooth domain in $\mathbf{R}^N$, $N \geq 3$. The result applies to a class of nonlinear functions f, including the model cases: i) $\gamma = 1$ and $f(s) = s^p$, $1 < p \leq p^+$; and ii) $\gamma = 0$, $f(s) = \alpha s + s^p$, $1 < p \leq p^+$, and $0 \leq \alpha < \mu_1^+$. Here $p^+ = \tilde{N}^+/(\tilde{N}^+ - 2)$, $\tilde{N}^+ = \lambda(N-1)/\Lambda + 1$, and μ_1^+ is the first eigenvalue of $\mathcal{M}^+_{\lambda,\Lambda}$ in Ω. Analogous results are obtained for the operator $\mathcal{M}^-_{\lambda,\Lambda}$.

1. Introduction

In this article we consider the existence of positive solutions to a "semilinear" equation involving Pucci's extremal operators, in which neither the maximum principle nor the comparison principle hold. Pucci's extremal operators are perturbations of the usual Laplacian, sharing with it many properties like homogeneity, positivity, and comparison properties. However they are not in divergence form, thus deviating in a fundamental manner away from the Laplacian. Pucci's extremal operators represent an important prototype of fully nonlinear operators, sitting at the center of the theory of regularity; see Cabré and Caffarelli [2]. For general elements of theory of viscosity solutions see also [6].

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Our approach to studying the existence problem is based on degree theory for compact operators in positive cones. This approach has been successfully applied by many authors to a variety of problems. Of special interest to us is the work of de Figueiredo, Lions, and Nussbaum [16], on which we base our arguments. This approach requires a-priori bounds for the solutions, which are obtained via blow-up techniques as in the fundamental paper of Gidas and Spruck [11]. The success of this approach rests on Liouville-type theorems. So we begin to review some Liouville-type theorems that are known in the case of Pucci's operators.

Let us first briefly recall the definition of Pucci's extremal operators. Given two parameters $0 < \lambda \leq \Lambda$, the matrix operators $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$ are defined as follows: if M is a symmetric, $N \times N$ matrix,

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i$$

and

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i,$$

where $e_i = e_i(M), i = 1, \ldots, N$, are the eigenvalues of M. The Pucci's operators are obtained by applying $\mathcal{M}^+_{\lambda,\Lambda}$ or $\mathcal{M}^-_{\lambda,\Lambda}$ to the Hessian D^2u of the scalar function u. These two operators have many properties in common, but they are not equivalent. For more details and equivalent definitions see the monograph of Cabré and Caffarelli [2].

In a recent article Cutri and Leoni [7] studied a Liouville-type theorem; they consider the equation

$$\mathcal{M}^\pm_{\lambda,\Lambda}(D^2u) + u^p = 0 \quad \text{in} \quad \mathbf{R}^N, \tag{1.1}$$

where $p > 1$. For notational simplicity, here and in the rest of the paper, we denote by $\mathcal{M}^\pm_{\lambda,\Lambda}$ both operators $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$, in such a way that (1.1) represents actually the two corresponding equations. In [7] they prove the following Liouville-type theorem.

Theorem 1.1. *Let $N \geq 3$. Then there exist numbers $p^+ := \tilde{N}^+/(\tilde{N}^+ - 2) > 1$ and $p^- := \tilde{N}^-/(\tilde{N}^- - 2) > 1$ such that if $1 < p \leq p^+$ $(1 < p \leq p^-)$ then* (1.1) *does not have a viscosity supersolution. Here $\tilde{N}^+ = \lambda(N-1)/\Lambda + 1$ and $\tilde{N}^- = \Lambda(N-1)/\lambda + 1$.*

We notice that in the radial case a Liouville-type theorem for a large range of p can be obtained for a solution to (1.1). See Felmer and Quaas [14] also [13]. More precisely, they prove the following theorem.

Theorem 1.2. *Let $N \geq 3$. Then there exist numbers $p_*^+ > 1$ and $p_*^- > 1$ such that if $1 < p < p_*^+$ $(1 < p < p_*^-)$ then* (1.1) *does not have a radially symmetric C^2 solution.*

The numbers p_*^+ and p_*^- are called critical exponents for the operators $M_{\lambda,\Lambda}^+$ and $M_{\lambda,\Lambda}^-$, respectively. These exponents are optimal in the sense that for the critical exponents there exists a solution for (1.1). When the parameters λ and Λ are equal, then $p_*^+ = p_*^- = p_N = (N+2)/(N-2)$, the usual Sobolev critical exponent; see Pohozaev [18] and Caffarelli, Gidas, and Spruck [5]. Notice that in the case $\lambda < \Lambda$, we have $p_*^+ > \max\{p_N, p^+\}$ and $p^- < p_*^- < p_N$, so there is a gap between the exponent of Theorem 1.2 and Theorem 1.1. A general Liouville-type theorem for (1.1) is still an open problem in the range $p^\pm < p < p_*^\pm$.

Another interesting open problem is the Liouville-type theorem for the same equation of (1.1) but in the half space, which is unknown for any range of p. In order to overcome this difficulty we will assume that the domain is convex. In this case we can control the values of the solution near the boundary using the moving-planes method.

For other related results concerning Pucci's operator see the papers of Labutin [8] and [9].

It is the main purpose of this article to prove existence theorems for positive solutions for the equation

$$\begin{aligned} \mathcal{M}_{\lambda,\Lambda}^{\pm}(D^2u) - \gamma u + f(u) &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.2}$$

where Ω is a convex domain in $\mathbf{R}^N$ with boundary $\partial\Omega$ of class $C^{2,\alpha}$ and f is an appropriate nonlinearity.

As for equation (1.1), equation (1.2) represents the two equations corresponding to $\mathcal{M}_{\lambda,\Lambda}^+$ and $\mathcal{M}_{\lambda,\Lambda}^-$. We observe that when $\lambda = \Lambda = 1$ then $\mathcal{M}_{\lambda,\Lambda}^{\pm}$ simply reduce to the Laplace operator, so (1.2) becomes

$$\begin{aligned} \Delta u - \gamma u + f(u) &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.3}$$

This equation has been studied by many authors, not only in a convex domain, but on general domains. We refer the reader to the review paper by P.L. Lions [10] and the references therein.

Continuing with the description of our results, let us introduce the precise assumptions on our nonlinearity f:

(f0) $f \in C([0, +\infty))$ and is locally Lipschitz.

(f1) $f(s) \geq 0$ and there is $1 < p \leq p^{\pm}$ and a constant $C^* > 0$ such that

$$\lim_{s \to +\infty} \frac{f(s)}{s^p} = C^*.$$

(f2) There is a constant $c^* \geq 0$ such that $c^* - \gamma < \mu_1^{\pm}$ and

$$\lim_{s \to 0} \frac{f(s)}{s} = c^*,$$

where μ_1^+ (μ_1^-) is the first eigenvalue for $\mathcal{M}_{\lambda,\Lambda}^+$ ($\mathcal{M}_{\lambda,\Lambda}^-$) in Ω. See Remark 1.2 below.

The first model problem is $\gamma = 1$ and $f(s) = s^p$, $1 < p \leq p^{\pm}$. The second model problem is $\gamma = 0$ and $f(s) = \alpha s + s^p$, $1 < p < p^{\pm}$ and $0 \leq \alpha < \mu_1^{\pm}$.

Now we are in a position to state our main theorem.

Theorem 1.3. *Assume $N \geq 3$ and f satisfies the hypotheses (f0), (f1), and (f2). Then there exists a positive $C^2(\Omega)$ solution of* (1.2).

Remark 1.1. The missing part to cover the range $p^{\pm} < p < p_*^{\pm}$ in (f1) is the general Liouville-type theorem in $\mathbf{R}^N$, which is still open. In the case in which Ω is a ball, all the range of p is covered. This fact follows from Theorem 1.2; for the proof see Felmer and Quaas [15].

Remark 1.2. In Section 3 we will prove the existence of the first eigenvalue and eigenfunction for the nonlinear operators $\mathcal{M}_{\lambda,\Lambda}^+$ ($\mathcal{M}_{\lambda,\Lambda}^-$). This fact seems not to have been proved, as far as we know. With the first eigenvalue we can allow more general nonlinearities f, so we can obtain a more complete existence theorem for equation (1.2) in analogy to the Laplacian case.

In order to prove our main theorem we use degree theory on positive cones as presented in [16]. A priori bounds for solutions are obtained by the blow-up method introduced by Gidas and Spruck [11] in combination with the Liouville-type Theorem 1.1.

In order to set up our abstract scheme we give an existence and regularity result of solutions for the equation

$$\begin{aligned} \mathcal{M}_{\lambda,\Lambda}^{\pm}(D^2 u) - \gamma u &= g(x) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.4}$$

for a given $g \in C^{\alpha}$. In Section 3 we consider the eigenvalue problem for Pucci's operator, basing our arguments on the Krein-Rutman theorem that is proved by Rabinowitz in [19]. In Section 4 we describe the abstract setting in [16] and we prove the necessary a-priori bounds that allow us to use the abstract theory. We prove here Theorem 1.3.

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2. Preliminary

Is this section we review some basic existence and regularity theorems for the equation (1.4), as consequences of the theory of viscosity solution. We will see also the maximum principle and comparison results for Pucci's operators.

To start let us recall some basic properties of the matrix operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$. See Lemma 2.10 in [2] for the proof.

Lemma 2.1. *Let M and N be two symmetric matrices. Then*
i) $\mathcal{M}^{+}_{\lambda,\Lambda}(M) + \mathcal{M}^{-}_{\lambda,\Lambda}(N) \leq \mathcal{M}^{+}_{\lambda,\Lambda}(M+N) \leq \mathcal{M}^{+}_{\lambda,\Lambda}(M) + \mathcal{M}^{-}_{\lambda,\Lambda}(N)$
ii) $\mathcal{M}^{-}_{\lambda,\Lambda}(M) + \mathcal{M}^{-}_{\lambda,\Lambda}(N) \leq \mathcal{M}^{-}_{\lambda,\Lambda}(M+N) \leq \mathcal{M}^{-}_{\lambda,\Lambda}(M) + \mathcal{M}^{+}_{\lambda,\Lambda}(N)$.

Now we present the maximum and comparison principles:

Proposition 2.1. *Let Ω be a bounded domain in $\mathbf{R}^N$.*

1) If u is continuous in $\bar{\Omega}$ and u is a $C^2(\Omega)$ solution of $\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) \leq 0$ in Ω, with $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω.

2) Let u and v be continuous functions in $\bar{\Omega}$. If u and v are $C^2(\Omega)$ and

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) - \gamma u \leq g(x), \quad \textit{and} \quad \mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2v) - \gamma v \geq g(x)$$

in Ω, with $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.

Proof. 1) Let us consider the function $v_\varepsilon = u + \varphi_\varepsilon$, where $\varphi_\varepsilon(x) = \varepsilon(d^2 - |x|^2)$, with d such that $v_\varepsilon \geq 0$ on $\partial\Omega$. Since $D^2v_\varepsilon = D^2u - 2\varepsilon I_N$, Lemma 2.1 implies that

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2v_\varepsilon(x)) < 0, \quad \text{for} \quad x \in \Omega.$$

But then v_ε cannot have a minimum. Thus we conclude that $v_\varepsilon \geq 0$ in Ω, for all $\varepsilon > 0$. Hence $u \geq 0$ in Ω.

2) Consider $w_\varepsilon = u + \varphi_\varepsilon - v$. Then, using Lemma 2.1, we find that

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2w_\varepsilon) - \gamma w_\varepsilon < 0.$$

Then w_ε cannot have a negative minimum. □

Let us now recall a very well-known fact about Pucci's equation; that is, the Alexandroff, Bakelman, Pucci estimate holds; for the proof see [2], Theorem 3.6; see also [3] for a refined form.

Theorem 2.1. **ABP** *Let Ω be a bounded domain in $\mathbf{R}^N$ and f a continuous and bounded function in Ω. Suppose u is continuous in $\overline{\Omega}$ and satisfies $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \le f(x)$ in Ω and $u \le 0$ on $\partial\Omega$. Then*

$$\sup u^- \le C diam(\Omega) \|f^+\|_{L^N(\Omega)}.$$

Here C is a universal constant.

The next corollary is a maximum principle for small domain and is first noted by Bakelman and used extensively in [1]. We will use it to prove that the maximum of any solution of (1.2) is away from the boundary by using the moving-planes method.

Corollary 2.1. *Let Ω be a bounded domain in $\mathbf{R}^N$. Suppose u is continuous in $\overline{\Omega}$ and satisfies $\mathcal{M}^-_{\lambda,\Lambda}(D^2u)+c(x)u(x) \le 0$ in Ω, $u \ge 0$ on $\partial\Omega$ and $c(x) \le b$. If the measure of Ω is small enough, then $u \ge 0$ in Ω.*

The proof is an adaptation of the proof in [1].

Proof. Suppose for the sake of contradiction that there exists a nonempty, open, connected component A of $\Omega^- := \{x \in \Omega : u(x) < 0\}$. Let us write $c = c^+ - c^-$; we have

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - c^-u \le c^+u^-.$$

So

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \le c^+u^- \quad \text{in} \quad A.$$

Notice that $u = 0$ on ∂A. If we apply the ABP estimate in A we infer

$$\sup u^- \le C diam(A) b |A|^{\frac{1}{N}} \sup u^-.$$

If the measure of Ω is small enough we have $C diam(A) b |A|^{\frac{1}{N}} < 1$, getting a contradiction. □

We continue by noting that Pucci's operators are convex or concave operators in the Hessian, so the regularity result of [2], Chapter 8, for example applies. So for any $u \in C^\alpha(\Omega)$ a viscosity solution of $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = g(x)$ with a $g \in C^\alpha(\Omega)$ it follows that $u \in C^{2,\alpha}(\Omega)$.

Let us continue with a compactness result for a viscosity solution.

Proposition 2.2. *Let u_n be a sequence of continuous viscosity solutions to*

$$\begin{aligned} \mathcal{M}^\pm_{\lambda,\Lambda}(D^2u_n) - \gamma u_n &= g_n(x) \quad \text{in} \quad \Omega, \\ u_n &= \phi_n \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.1}$$

with $\gamma \geq 0$. Suppose that there exists $K > 0$ such that $\|g_n\|_{L^\infty(\Omega)} \leq K$ and $\|\phi_n\|_{L^\infty(\partial\Omega)} \leq K$. There exists a subsequence such that $u_n \to u$ uniformly in $\overline{\Omega}$. Moreover, if $g_n \to g$ uniformly in a compact set of Ω, then u is a viscosity solution to $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) - \gamma\, u = g(x)$.

Proof. First we will suppose that u_n is bounded; then u_n satisfies

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) = h_n(x)$$

with $\|h_n\|_\infty \leq K$ and $u_n = \phi_n$ on $\partial\Omega$. By Proposition 4.14 in [2] it follows that u_n is an equicontinuous sequence in $\overline{\Omega}$, so by the Arzela-Ascoli theorem, the first part follows. The other part is a direct consequence of the definition of a viscosity solution; see Proposition 2.9 in [2]. Suppose now that u_n is unbounded; define $v_n = u_n/\|u_n\|_{L^\infty(\Omega)}$. Then $\|v_n\|_{L^\infty(\Omega)} = 1$ and satisfies (2.1) with the right side $g_n/\|u_n\|_{L^\infty(\Omega)}$ and $v_n = \phi_n/\|u_n\|_{L^\infty(\Omega)}$ on $\partial\Omega$. Using the result just proved we conclude that $v_n \to v$ uniformly in $\overline{\Omega}$ and satisfies (1.4) with $g \equiv 0$. This implies by Proposition 2.1 and the regularity that $v \equiv 0$, contradicting $\|v\|_{L^\infty(\Omega)} = 1$. Thus u_n is bounded. □

Let us finally give the main results of this section.

Theorem 2.2. *Let $g \in C^\alpha(\Omega)$ be a nonpositive function and $\gamma \geq 0$. Then there exists a unique $C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$, a positive solution to (1.4).*

Proof. If $g \in C^2(\Omega)$ the existence follows from a result in [4] (see the remark on page 243) or [12], Theorem 17.8. If $g \in C^\alpha(\Omega)$ we take a sequence g_n in $C^2(\Omega)$ such that $g_n \to g$ uniformly in a compact set of Ω and $\|g_n\|_\infty$ is bounded. Let u_n be the respective solution to (1.4) with the left side g_n. Using the Proposition 2.2 we conclude that $u_n \to u$ uniformly, and u is the desired solution, since we have the regularity result. □

3. Eigenvalue problem for $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$.

Is this section we study the eigenvalue problem for Pucci’s operators. In the Introduction we described our hypotheses (f2) on the nonlinearity f in terms of the first eigenvalue of the operator. Here we prove that such an eigenvalue is positive and that the first eigenfunction is positive also. The proof applies to a general bounded domain with a smooth boundary, not necessary convex.

Theorem 3.1. *Let Ω be a convex domain. Then the eigenvalue problem*

$$\begin{aligned} &-\mathcal{M}^\pm_{\lambda,\Lambda}(D^2u) = \mu u \qquad \text{in } \Omega \\ &u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

has a solution $(\mu_1^\pm, u_1^\pm)$*, with* $\mu_1^\pm$ *and* $u_1^\pm$ *positive. Moreover, all positive solutions to* (3.1) *are of the form* $(\mu_1^\pm, \alpha u_1^\pm)$*, with* $\alpha > 0$.

Here equation (3.1) represents the two eigenvalue problems, for $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$. The eigenpairs (μ_1^+, u_1^+) and (μ_1^-, u_1^-) correspond to the operators $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$, respectively. For the proof of this theorem we rely on ideas from Rabinowitz [19]. The starting point is the Krein-Rutman theorem, which can be proved using a general result on existence of a one-parameter family of fixed points; see [19].

Theorem 3.2. *Let* $(E, \|\cdot\|)$ *be a Banach space and* K *be a closed cone in* E *with a vertex at* 0. *Let* $T : \mathbf{R}^+ \times K \to K$ *be a compact operator such that* $T(0,u) = 0$ *for all* $u \in E$*; then there exists an unbounded, connected component* $\mathcal{C}$ *of* $\mathbf{R}^+ \times K$ *of solutions of* $u = T(\mu, u)$ *and starting from* $(0,0)$.

Remark 3.1. Here we denote by $\mathbf{R}^+$ the interval $[0, +\infty)$.

Let us consider the cone of nonnegative, continuous functions

$$C_\# = \{w \in C^\alpha(\Omega) \cap C(\overline{\Omega}) : w \geq 0 \text{ in } \Omega,\ w = 0 \text{ on } \partial\Omega\},$$

and define $\mathcal{L}^\pm : C_\# \to C_\#$ as the inverse of $-\mathcal{M}^\pm_{\lambda,\Lambda}$. The operator $\mathcal{L}^\pm$ is well defined after Theorem 2.2, and it is compact by Proposition 2.2.

Remark 3.2. If $g \in C^\alpha(\Omega)$, then by the regularity result $L(g) \in C^{2,\alpha}(\Omega)$.

In the next three lemmas we describe the main properties of the operator $\mathcal{L}^\pm$.

Lemma 3.1. *The operator* $\mathcal{L}^\pm$ *is monotone; that is, if* $g_1, g_2 \in C_\#$ *such that* $g_1 \leq g_2$*, then* $\mathcal{L}^\pm(g_1) \leq \mathcal{L}^\pm(g_2)$.

Proof. Direct from Proposition 2.1. □

Lemma 3.2. *Let* $g_1, g_2 \in C_\#$*; then*

a) $\mathcal{L}^-(g_1 + g_2) \geq \mathcal{L}^-(g_1) + \mathcal{L}^-(g_2)$

b) $\mathcal{L}^+(g_1 + g_2) \leq \mathcal{L}^+(g_1) + \mathcal{L}^+(g_2)$.

Proof. Let $u_i = \mathcal{L}^-(g_i)$, $i = 1, 2$. Then, using Lemma 2.1, we obtain

$$-\mathcal{M}^-_{\lambda,\Lambda}(D^2u_1 + D^2u_2) \leq -\mathcal{M}^-_{\lambda,\Lambda}(D^2u_1) - \mathcal{M}^-_{\lambda,\Lambda}(D^2u_2) = g_1 + g_2,$$

from which the inequality follows, taking $\mathcal{L}^-$ on both sides. The case $\mathcal{L}^+$ is analogous. □

Lemma 3.3. *Let* $g \in C_\#$ *and* $u = \mathcal{L}^\pm(g)$*. If* $g \neq 0$ *then* $u(x) > 0$ *for all* $x \in \Omega$*. Moreover* $\frac{\partial u}{\partial \nu}(x_0) > 0$*, where* ν *is the outer normal and* $x_0 \in \partial\Omega$.

Remark 3.3. The outer-normal derivative always exists; this is due to the utilization of Theorem 1 of Safonov in [20] for this case, but we don't really need this regularity result, we need only that condition (3.11) in [12] holds (see comments after Lemma 3.4 there).

Proof. The same proof as in [12] holds (see Lemma 3.4). □

Now we are in a position to prove the existence of first eigenvalues.

Proof of Theorem 3.1. Take $u_0 \in C_\# \setminus \{0\}$; we claim that there exists $M > 0$ such that $M\mathcal{L}^\pm u_0 \geq u_0$. Suppose that $\mathcal{L}^\pm u_0 - u_0/M \notin C_\#$ for all $M > 0$. Taking the limit as $M \to +\infty$ we have $\mathcal{L}^\pm u_0 \notin Int(C_\#)$, getting a contradiction with Lemma 3.3.

Define now $T_\varepsilon : \mathbf{R}^+ \times C_\# \to C_\#$ as $T_\varepsilon(\mu, u) = \mu\mathcal{L}^\pm(u) + \mu\varepsilon\mathcal{L}^\pm(u_0)$, for $\varepsilon > 0$. From Theorem 3.1, there exists a connected component $\mathcal{C}_\varepsilon$ of the solution to $T_\varepsilon(\mu, u) = u$.

We show next that $\mathcal{C}_\varepsilon \subset [0, M] \times C_\#$. In fact, let $(\mu, u) \in \mathcal{C}_\varepsilon$. Then

$$u = \mu\mathcal{L}^\pm u + \mu\varepsilon\mathcal{L}^\pm u_0;$$

hence, $u \geq \mu\varepsilon\mathcal{L}^\pm u_0 \geq \frac{\mu}{M}\varepsilon u_0$. If we apply $\mathcal{L}^\pm$ we get

$$\mathcal{L}^\pm u \geq \frac{\mu}{M}\varepsilon\mathcal{L}^\pm u_0 \geq \frac{\mu}{M^2}\varepsilon u_0.$$

But $u \geq \mu\mathcal{L}^\pm u$; then $u \geq (\frac{\mu}{M})^2\varepsilon u_0$. By recurrence we get

$$u \geq (\frac{\mu}{M})^n\varepsilon u_0 \quad \text{for all} \quad n \geq 2.$$

This implies that $\mu \leq M$, and thus $\mathcal{C}_\varepsilon \subset [0, M] \times C_\#$.

Now we conclude. Since $\mathcal{C}_\varepsilon$ is unbounded there exist μ_ε and u_ε so that $(\mu_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon$ and $\|u_\varepsilon\|_\infty = 1$. Then, by the compactness of $\mathcal{L}^\pm$ we find $\mu_1 \in [0, M]$ and u_1 with $\|u_1\|_\infty = 1$ such that $u_1 = \mu_1\mathcal{L}^\pm u_1$. From here we also deduce that $\mu_1 > 0$.

To complete the proof of Theorem 2.1 we need the following lemma, whose proof is simple and can be seen in [19].

Lemma 3.4. *Let K be a closed cone with nonempty interior and $y_0 \in int(K)$. Then for all $y \notin K$ there exists a unique number $\delta_{y_0}(y)$ such that*

i) if $\mu \in [0, \delta_{y_0}(y)]$, then $y_0 + \mu y \in K$,

ii) if $\mu \geq \delta_{y_0}(y)$, then $y_0 + \mu y \notin K$.

Moreover, if $y_0 + \mu y \in int(K)$, then $\mu < \delta_{y_0}(y)$.

Continuing with the proof of Theorem 2.1, let us consider $u \in C_\#, u \neq 0$, and $\mu > 0$ such that $u = \mu\mathcal{L}^\pm u$. Here we split the proof.

Case $\mathcal{L}^-$: Define $\gamma_1 = \delta_{u_1}(-u)$ and $\gamma_2 = \delta_u(-u_1)$ as given by the previous lemma, with $K = C_\#$. Using Lemma 3.2 a) we have that

$$\mathcal{L}^-(u_1 - \gamma_1 u) \leq \mathcal{L}^-(u_1) - \gamma_1 \mathcal{L}^-(u) = \frac{1}{\mu_1}(u_1 - \gamma_1 \frac{\mu_1}{\mu} u),$$

and similarly

$$\mathcal{L}^-(u - \gamma_2 u_1) \leq \frac{1}{\mu}(u - \gamma_2 \frac{\mu}{\mu_1} u_1).$$

If $u - \gamma_2 u_1 \neq 0$, then $\mathcal{L}^-(u - \gamma_2 u_1) \in int(C_\#)$, so $\mu/\mu_1 < 1$. Since $\mathcal{L}^-(u_1 - \gamma_1 u) \in C_\#$, then $\mu_1/\mu \leq 1$, which is a contradiction. Thus $u = \gamma_2 u_1$.

Case $\mathcal{L}^+$: Using the previous lemma with $K = -C_\#$, we define $\gamma_1 = \delta_{-u_1}(u)$ and $\gamma_2 = \delta_{-u}(u_1)$. From Lemma 3.2 b) we have then

$$\mathcal{L}^+(\gamma_1 u - u_1) \geq \frac{1}{\mu_1}(\gamma_1 \frac{\mu_1}{\mu} u - u_1), \quad \mathcal{L}^+(\gamma_2 u_1 - u) \geq \frac{1}{\mu}(\gamma_2 \frac{\mu}{\mu_1} u_1 - u).$$

If $-u + \gamma_2 u_1 \neq 0$, then $\mathcal{L}^+(-u + \gamma_2 u_1) \in int(C_\#)$ so $\mu/\mu_1 < 1$. Since $\mathcal{L}^+(-u_1 + \gamma_2 u) \in C_\#$, then $\mu_1/\mu \leq 1$, which is a contradiction. Thus $u = \gamma_2 u_1$. □

4. A-priori bounds and proof of Theorem 1.3

Our existence theorem will be proved by using the approach of de Figueiredo, Lions, and Nussbaum [16]. It consists in using degree theory for compact operators in cones. This abstract tool is combined with appropriate a-priori bounds and computation of degree.

We start by recalling the abstract setting in [16]. Let K be a closed cone with nonempty interior in the Banach space $(E, \|\cdot\|)$. Let $\Phi : K \to K$ and $F : E \times [0, \infty) \to K$ be compact operators such that $\Phi(0) = 0$ and $F(x, 0) = \Phi(x)$ for all $x \in E$. Then the following theorem is proved in [17]. See also [16], Proposition 2.1 and Remark 2.1.

Theorem 4.1. *Assume there exist numbers* $0 < R_1 < R_2$ *and* $T > 0$ *such that*

i) $x \neq \beta\Phi(x)$ *for all* $0 \leq \beta \leq 1$ *and* $\|x\| = R_1$,

ii) $F(x,t) \neq x$ *for all* $\|x\| = R_2$, $\quad t \in [0, +\infty)$ *and*

iii) $F(x,t) = x$ *has no solution* $x \in \bar{B}_{R_2}$ *for* $t = T$.

Then Φ *has a fixed point in* $\mathcal{U}$, *where* $\mathcal{U} = \{x \in K : R_1 < \|x\| < R_2\}$.

We note that solving (1.2) is equivalent to finding a fixed point of $\Phi : C_{\#} \to C_{\#}$ defined as

$$\Phi(u)(x) \stackrel{\text{def}}{=} \mathcal{L}(f(u(x))), \quad x \in \Omega,$$

where $\mathcal{L}$ is the inverse of $-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2\cdot) + \gamma\cdot$.

By Theorem 2.2, Proposition 2.1, and regularity, $\mathcal{L}$ is well defined and compact. We define next the operator F as $F(u,t)(r) = \mathcal{L}(f(u(r)+t))$.

We complete the proof of Theorem 1.3 by proving conditions i), ii), and iii) in Theorem 4.1.

To start we give a lemma which guarantees that the maximum of a $C^2(\Omega)$ solution of the equation

$$\begin{array}{ll} -M^{\pm}_{\lambda,\Lambda}(D^2u) + \gamma u = f(u+t) & \text{in } \Omega, \\ u > 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u = 0 & \text{on } \partial\Omega, \end{array} \tag{4.1}$$

is away from the boundary of Ω.

Lemma 4.1. *Let u be a $C^2(\Omega)$ solution to (4.1); then there exists $\varepsilon > 0$ depending only on the geometry of Ω such that for all $x \in \Omega^\varepsilon := \{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}$ there exists $y \in \Omega \backslash \Omega^\varepsilon$ such that $u(y) \geq u(x)$.*

For the proof we will use a moving-plane method as in [1].

Proof. Given $\beta \in \mathbf{R}$ and $\gamma \in S^1$ we define

$$\Sigma_\beta = \{x \in \Omega : x \cdot \gamma > \beta\}, \qquad T_\beta = \{x \in \Omega : x \cdot \gamma = \beta\}.$$

Let $x_\beta = 2(\beta - x\cdot\gamma)\gamma + x$ be the reflection of a point $x \in \Sigma_\beta$ with respect to the plane T_β. Define $w_\beta(x) = u(x_\beta) - u(x)$ for $x \in \Sigma_\beta$. Using Lemma 2.1 and the fact that f is locally Lipschitz, we have that w_β satisfies

$$\mathcal{M}^{-}_{\lambda,\Lambda} D^2(w_\beta) + c(x) w_\beta \leq 0 \quad \text{in} \quad \Sigma_\beta$$

and $w_\beta \geq 0$ on $\partial\Sigma_\beta$. Hence, applying Corollary 2.1 we get that $w_\beta \geq 0$ in Σ_β if the measure of Σ_β is small.

Let us fix now $\varepsilon > 0$ small depending only on the geometry of Ω such that for any $x \in \Omega^\varepsilon$ there exists $\gamma \in S^1$ such that $x \in \Sigma_\beta$, Σ_β has small measure, and $x_\beta \in \Omega\backslash\Omega^\varepsilon$. Setting $y = x_\beta$ we conclude. □

Now, the first a-priori bound:

Proposition 4.1. *Let u be a $C^2(\Omega)$ solution of the equation (4.1) with $t \geq 0$. Then there exists a constant C, independent of u, such that*

$$\|u\|_\infty \leq C.$$

Proof. We argue by contradiction. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of positive solutions to (4.1) such that $\|u_n\|_\infty \to +\infty$ as $n \to +\infty$. Suppose that $u_n(x_n) = \|u_n\|_\infty$; from Lemma 4 we can choose $x_n \in \Omega^\varepsilon$ for some positive ε. Let us define

$$v_n(x) = \frac{1}{M_n} u_n(x_n + xM_n^{\frac{1-p}{2}}), \tag{4.2}$$

with $u_n(x_n) = M_n$; then v_n satisfies

$$-M^{\pm}_{\lambda,\Lambda}(D^2 v_n) + \gamma \frac{u_n}{M_n^p} = \frac{f(u_n + t)}{M_n^p} \quad \text{in} \quad M_n^{\frac{p-1}{2}}(\Omega - x_n),$$

and $\|v_n\|_\infty = 1$. By Proposition 2.2 we have that, up to a subsequence, $v_n \to v$ in compact sets of $\mathbf{R}^N$. Since $(f1)$ holds, we have that $f(u_n + t)/M_n^p \to C^* v^p$, and $u_n/M_n^p \to 0$; the v is a viscosity solution to

$$M^{\pm}_{\lambda,\Lambda}(D^2 v) + C^* v^p = 0 \text{ in } \mathbf{R}^N, \text{ with } p \le p^{\pm}. \tag{4.3}$$

But this contradicts Theorem 1.1. □

Our next proposition implies condition i) in Theorem 4.1.

Proposition 4.2. *There is $R_1 > 0$ so that the equation*

$$\begin{aligned} -M^{\pm}_{\lambda,\Lambda}(D^2 u) + \gamma u = \beta f(u) \quad & \text{in } \Omega \\ u > 0 \text{ in } \Omega, \quad u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{4.4}$$

$\beta \in [0, 1]$*, has no solution u with $0 < \|u\|_\infty < R_1$.*

Proof. We argue by contradiction. Let $\{(u_n, \beta_n)\}_{n\in\mathbb{N}}$ be a sequence of positive solutions to (4.4) such that $\|v_n\|_\infty \to 0$ as $n \to +\infty$. Define $v_n = u_n/\|u_n\|_\infty$; then we have that v_n satisfies

$$M^{\pm}(D^2 v_n) - \gamma v_n + \beta_n \frac{f(u_n)}{u_n} v_n = 0, \text{ in } \Omega,$$

and $\|v_n\|_\infty = 1$. Using Proposition 2.2, up to a subsequence, we find $v_n \to v$ uniformly in $\overline{\Omega}$. Moreover, $\|v\|_\infty = 1$. By hypotheses (f2) we then obtain that v satisfies

$$\begin{aligned} M^{\pm}(D^2 v) + (\beta c^* - \gamma) v = 0 \quad & \text{in } \Omega \\ v > 0 \text{ in } \Omega, \quad v = 0 \quad & \text{on } \partial\Omega. \end{aligned}$$

If $\beta c^* - \gamma \le 0$, we get a contradiction with the maximum principle, Proposition 2.1. If $0 < \beta c^* - \gamma$, then by (f2) $\beta c^* - \gamma < \lambda_1^{\pm}$, and we get a contradiction with Theorem 3.2. □

In order to prove condition iii) in Theorem 4.1 we need

Proposition 4.3. *There exists a constant $T > 0$ so that if* (4.1) *possesses a solution u, then $0 \leq t \leq T$.*

Proof. We argue by contradiction. Suppose there is a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to +\infty$ as $n \to \infty$, such that for each t_n there exists a solution u_n to (4.1). Define $v_n(x) = u_n(x) + t_n$. Let us define

$$w_n(x) = \frac{1}{M_n} v_n(x_n + x M_n^{\frac{1-p}{2}}), \tag{4.5}$$

with $v_n(x_n) = M_n := \sup_\Omega v_n$; then w_n satisfies

$$-M^{\pm}_{\lambda,\Lambda}(D^2 w_n) + \gamma \frac{v_n}{M_n^p} = \frac{f(v_n)}{M_n^p} \quad \text{in} \quad M_n^{\frac{p-1}{2}}(\Omega - x_n),$$

and $\|w_n\|_\infty = 1$. Since Proposition 2.2 holds for w_n we can argue as in the proof of Proposition 4.1 to reach a contradiction. □

Proof of Theorem 1.3. Propositions 4.1, 4.2, and 4.3 give the conditions i)–iii) in Theorem 4.1, from which the result follows. □

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