ASYMPTOTIC BEHAVIOR AND UNIQUENESS RESULTS FOR BOUNDARY BLOW-UP SOLUTIONS

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(Submitted by: Moshe Marcus)

Abstract. We estimate the blow-up rate and then improve some existing uniqueness results for boundary blow-up solutions to certain quasi-linear elliptic equations with a weight function. The weight function is allowed to vanish on the part of the boundary where the solution blows up. Our approach is based on the construction of certain upper and lower solutions on small annuli with partial boundary blow-up, and on a modified version of an iteration technique due to Safonov.

1. INTRODUCTION

We are interested in the asymptotic behavior and uniqueness of positive solutions to the boundary blow-up problem

$$\begin{cases}
-\Delta_p u = a(x)u^{p-1} - b(x)u^q & \text{in } \Omega, \\
u = \infty & \text{on } \Gamma_\infty, \\
u = 0 & \text{on } \partial\Omega \setminus \Gamma_\infty,
\end{cases}$$

(1.1)

where \(q > p - 1 > 0\), \(\Delta_p u = \text{div}(|Du|^{p-2}Du)\) is the so-called p-Laplacian operator, \(\Omega\) is a smooth, bounded domain in \(\mathbb{R}^N\) (\(N \geq 2\)), and \(\Gamma_\infty\) is a nonempty open and closed subset of \(\partial\Omega\) (with \(\Gamma_\infty = \partial\Omega\) allowed). We assume that \(a(x)\) and \(b(x)\) are continuous functions on \(\overline{\Omega}\) with \(b(x) > 0\) on \(\overline{\Omega} \setminus \Gamma_\infty\).

Problem (1.1) with \(p = 2\) arises in the study of degenerate logistic equations (see, e.g., [7]) and has been considered in several recent papers. While the existence of a positive solution to (1.1) is not hard to establish by making use of classical results from [9] and [14], the uniqueness problem is not...
well understood when $b(x)$ is allowed to vanish on $\Gamma_\infty$. In [7], under the restriction
\[
\lim_{d(x,\Gamma_\infty) \to 0} \frac{b(x)}{d(x,\Gamma_\infty)^\alpha} = \beta
\] (1.2)
for some constants $\alpha \geq 0$ and $\beta > 0$, it was proved that (1.1) with $p = 2$ has a unique solution $u$, and it satisfies
\[
\lim_{d(x,\Gamma_\infty) \to 0} \frac{u(x)}{d(x,\Gamma_\infty)^\gamma} = \mu, \quad \text{where } \gamma = \frac{\alpha + 2}{q - 1}, \quad \mu = \left[ \frac{\gamma(\gamma + 1)}{\beta} \right]^{1/(q-1)}.
\] (1.3)

Subsequently, more general nonlinearities were treated and better asymptotic estimates were obtained in [2, 3, 4] and [8] for the case $p = 2$.

In [5], the above-mentioned result of [7] was extended to the general case $p > 1$, and the restriction (1.2) was relaxed to the following condition:
\[
\lim_{x \to x_* \in \Gamma_\infty} \frac{b(x)}{d(x,\Gamma_\infty)^\alpha} = \beta(x_*) \quad \text{uniformly for } x_* \in \Gamma_\infty,
\] (1.4)
for some constant $\alpha \geq 0$ and positive, continuous function $\beta$ on $\Gamma_\infty$. It was shown that under condition (1.4), problem (1.1) has a unique positive solution, and it satisfies
\[
\lim_{x \to x_* \in \Gamma_\infty} \frac{u(x)}{d(x,\Gamma_\infty)^\xi} = \eta(x_*),
\] (1.5)
where
\[
\xi = \frac{\alpha + p}{q - p + 1}, \quad \eta(x_*) = \left[ \frac{(p - 1)\xi^{p-1}(\xi + 1)}{\beta(x_*)} \right]^{1/(q-p+1)}.
\]
(To be accurate, in [7] and [5] only the case that $a(x)$ is a constant was considered. But the proofs carry over to the nonconstant case straightforwardly.)

More recently, condition (1.2) was relaxed in a paper by Lopez-Gomez [11]. He proved the following results (see [11, Theorem 1.1]):

(i) If for some given point $x_* \in \Gamma_\infty$, there exist constants $\alpha = \alpha(x_*) \geq 0$ and $\beta = \beta(x_*) > 0$ such that
\[
\lim_{x \to x_*} \frac{b(x)}{d(x,\Gamma_\infty)^\alpha} = \beta(x_*),
\] (1.6)
and if $\Gamma_\infty = \partial\Omega$, then any positive solution of (1.1) with $p = 2$ satisfies
\[
\lim_{x \to x_* \in C_\sigma(x_*)} \frac{u(x)}{d(x,\Gamma_\infty)^\gamma} = \mu(x_*), \forall \sigma > 0,
\] (1.7)
where \( C_\sigma(x_*) = \{ x \in \Omega : (x - x_*) \cdot \nu_{x_*} \geq \sigma \} \), \( \nu_{x_*} \) is the unit-inward normal of \( \partial \Omega \) at \( x_* \), and
\[
\gamma(x_*) = \frac{\alpha(x_*) + 2}{q - 1}, \quad \mu(x_*) = \left[ \frac{\gamma(x_*) (\gamma(x_*) + 1)}{\beta(x_*)} \right]^{1/(q-1)}.
\]

(ii) If \( \partial \Omega \neq \Gamma_\infty \), then (1.7) should be replaced by the following weaker result:
\[
\lim_{x \to x_*} \frac{u(x)}{d(x, \Gamma_\infty)^{-\gamma(x_*)}} \leq \mu(x_*), \forall \sigma > 0.
\]

(iii) Suppose that \( \Gamma_\infty = \partial \Omega \) and that (1.6) holds uniformly for every \( x_* \in \Gamma_\infty \), with \( \alpha(x_*) \geq 0 \) and \( \beta(x_*) > 0 \) continuous on \( \Gamma_\infty \); then (1.1) with \( p = 2 \) has a unique positive solution, and it satisfies (1.7) uniformly in \( x_* \in \Gamma_\infty \).

In this paper, we will show that (1.7) is true even if \( \partial \Omega \neq \Gamma_\infty \). In fact, we will extend the results in [11] in three directions: Firstly we drop the restriction \( \partial \Omega = \Gamma_\infty \); secondly we show the result is true for a general \( p > 1 \); thirdly we prove that a similar formula to (1.7) (for \( p > 1 \)) holds even if the value of \( u(x) \) on \( \partial \Omega \setminus \Gamma_\infty \) is not specified; in other words, as widely expected, the blow-up rate of a solution \( u(x) \) to (1.1) is not affected by its behavior away from \( \Gamma_\infty \). We will also reveal a hidden restriction of condition (1.6). Our boundary blow-up estimates will be presented in Section 2, which will be used in Section 3 to establish some uniqueness results.

We would like to mention two particular techniques which have played a crucial role in enabling us to improve various earlier results. One is the construction of a family of lower solutions on small annuli with partial boundary blow-up in the proof of Lemma 2.2, which is the key that allows us to drop the condition \( \partial \Omega = \Gamma_\infty \). This construction is related in spirit to a technique used in [6] and [13] to construct certain upper solutions with partial boundary blow-up. The other is a modified version of an iteration technique due to Safonov which we learned from [10]. It is used in the proof of our Theorem 3.2, which considerably improves the uniqueness result in [7]. Other applications of this technique can be found in [10] and [1].

We also want to make some comments on condition (1.6), which looks much less restrictive than (1.4). It is tempting to think that if \( \alpha(x) \geq 0 \) and \( \beta(x) > 0 \) are continuous functions on \( \overline{\Omega} \), then \( b(x) := \beta(x) d(x, \Gamma_\infty)^{\alpha(x)} \) satisfies (1.6) uniformly on \( \Gamma_\infty \). It is a little surprising that this is not the case unless \( \alpha(x) \) takes a constant value on each component of \( \Gamma_\infty \). In fact, we will show in Proposition 2.7 that for a general \( b(x) \), if (1.6) holds uniformly
on $\Gamma_\infty$, then $\alpha(x_\ast)$ has to be a constant on each component of $\Gamma_\infty$. Therefore, when $\Gamma_\infty$ is connected, requiring (1.6) to hold uniformly on $\Gamma_\infty$ is in fact equivalent to requiring (1.4).

We refer the interested reader to [12], [13], [10], [1], and the references therein for some related but different uniqueness results.

2. Estimates of blow-up rate

Let us recall that $a(x)$ and $b(x)$ are continuous functions on $\bar{\Omega}$ with $b(x) > 0$ on $\Omega \setminus \Gamma_\infty$, and $\Gamma_\infty$ is a nonempty open and closed subset of $\partial \Omega$, which is smooth (say $C^2$). Let $B$ be an open ball in $\mathbb{R}^N$ such that $\Gamma_\infty \cap B \neq \emptyset$. We are going to estimate the blow-up rate of positive solutions to the problem

$$-\Delta_p u = a(x)u^p - b(x)u^q \text{ in } \Omega \cap B, \ u = \infty \text{ on } \Gamma_\infty \cap B. \quad (2.1)$$

The following comparison principle will be repeatedly used.

**Proposition 2.1.** ([5, Proposition 2.2]) Suppose that $D$ is a bounded domain in $\mathbb{R}^N$, $\alpha(x)$ and $\beta(x)$ are continuous functions on $D$ with $\|\alpha\|_{L^\infty(D)} < \infty$ and $\beta(x) \geq 0$, $\beta(x) \not\equiv 0$ on $D$. Let $u_1, u_2 \in C^1(D)$ be positive in $D$ and satisfy in the sense of distribution

$$-\Delta_p u_1 - \alpha(x)u_1^{p-1} + \beta(x)g(u_1) \geq 0 \geq -\Delta_p u_2 - \alpha(x)u_2^{p-1} + \beta(x)g(u_2) \text{ in } D$$

and

$$\lim_{d(x,\partial D) \to 0} \frac{(u_2^p - u_1^p)}{d(x,\Gamma_\infty)} \leq 0,$$

where $g \in C([0, \infty))$ and $g(s)/s^{p-1}$ is increasing for

$$s \in (\inf_D \{u_1, u_2\}, \sup_D \{u_1, u_2\}).$$

Then $u_1 \geq u_2$ in $D$.

**Lemma 2.2.** Suppose $x_\ast \in \Gamma_\infty \cap B$ and there exist constants $\alpha_1 = \alpha_1(x_\ast) \geq 0$, $\beta_1 = \beta_1(x_\ast) > 0$ such that

$$\lim_{x \to x_\ast, x \in \Omega} \frac{b(x)}{d(x,\Gamma_\infty)^{\alpha_1}} \leq \beta_1. \quad (2.2)$$

Then any solution $u(x)$ of (2.1) satisfies

$$\lim_{x \to x_\ast, x \in C_\sigma(x_\ast)} \frac{u(x)}{d(x,\Gamma_\infty)^{-\gamma_1}} \geq \mu_1, \quad (2.3)$$
where
\[ \gamma_1 = \frac{\alpha_1 + p}{q - p + 1}, \quad \mu_1 = \left[ \frac{(p-1)\gamma_1^{p-1}(\gamma_1 + 1)}{\beta_1} \right]^{1/(q-p+1)}. \]

**Proof.** For any given small \( \epsilon > 0 \), by (2.2), there exists \( r_0 > 0 \) such that
\[ B_{r_0}(x_*) := \{ x \in \mathbb{R}^N : |x - x_*| < r_0 \} \subset B, \]
and
\[ b(x) \leq (\beta_1 + \epsilon)d(x, \Gamma)^{\alpha_1}, \quad \forall x \in B_{r_0}(x_*) \cap \Omega. \tag{2.4} \]
If we choose \( R > 0 \) and \( \delta_0 > 0 \) small enough, and denote \( x^\delta = x_* - (R+\delta)x_* \),
the family of annuli
\[ A_{R,\delta} := \{ x \in \mathbb{R}^N : R < |x - x^\delta| < 2R, \ \delta \in [0, \delta_0] \} \]
satisfies
\[ A_{R,\delta} \cap \Omega \subset B_{r_0}(x_*) \cap \Omega, \quad \forall \delta \in [0, \delta_0]; \]
\[ \partial_1 A_{R,\delta} \cap \Omega = \emptyset \text{ if } \delta \in (0, \delta_0), \quad \partial_1 A_{R,0} \cap \Omega = \{ x_* \}, \]
where for \( j = 1, 2 \), \( \partial_j A_{R,\delta} = \{ x \in \mathbb{R}^N : |x - x^\delta| = jR \} \).

Let us denote \( r = |x - x^\delta| \). Then
\[ d(x, \Gamma_\infty) \leq r - R, \quad \forall x \in \Omega \cap A_{R,\delta}, \ \delta \in [0, \delta_0]. \]
It now follows from (2.4) that
\[ b(x) \leq (\beta_1 + \epsilon)(r - R)^{\alpha_1}, \quad \forall x \in \Omega \cap A_{R,\delta}, \ \delta \in [0, \delta_0]. \] \tag{2.5}
We are going to construct a family of lower solutions of (2.1) over \( \Omega \cap A_{R,\delta} \).
Let \( C = C(\epsilon) \) be the positive constant satisfying
\[ C^{q-p+1} = \frac{(p-1)\gamma_1^{p-1}(\gamma_1 + 1) - \epsilon}{\beta_1 + \epsilon}. \tag{2.6} \]
Then we define
\[ \psi(x) = \psi_\delta(x) = C(r - R)^{-\gamma_1} \left( 2 - \frac{r}{R} \right)^{\xi}, \quad r = |x - x^\delta|, \]
with \( \xi > 1 \) to be determined later.

Let us denote \( m := (\gamma_1 + 1)(p - 1) + 1 = \gamma_1 q - \alpha_1 \). A direct calculation gives, for \( R < r < 2R \),
\[ \Delta_p \psi = C^{p-1}(p-1)(r - R)^{-m}J_1^{p-2}J_2, \]
where
\[ J_1 = \gamma_1 \left( 2 - \frac{r}{R} \right)^{\xi} + \xi \left( \frac{r}{R} - 1 \right) \left( 2 - \frac{r}{R} \right)^{\xi-1}, \]
and
\[ J_2 = \gamma_1(\gamma_1 + 1) \left( 2 - \frac{r}{R} \right)^\xi + 2\gamma_1\xi \left( \frac{r}{R} - 1 \right) \left( 2 - \frac{r}{R} \right)^{\xi-1} \]
\[ + \xi(\xi - 1) \left( \frac{r}{R} - 1 \right)^2 \left( 2 - \frac{r}{R} \right)^{\xi-2}. \]

We may assume that \(|a(x)| \leq M\) in \(\Omega\). Then
\[ |a(x)\psi^{p-1}| \leq MC^{p-1}(r - R)^{-m}(r - R)^p \left( 2 - \frac{r}{R} \right)^{\xi(p-1)}. \quad (2.7) \]

Using (2.5), we obtain
\[ b(x)\psi^q \leq (\beta_1 + \epsilon)C^q(r - R)^{-m}(2 - \frac{r}{R})^{\xi q}. \quad (2.8) \]

For \(\eta \in (0, 1)\) to be specified in a moment, we suppose now \(R < r < (1 + \eta)R\). Then
\[ J_1 \geq \gamma_1(2 - \frac{r}{R})^\xi, \quad J_2 \geq \gamma_1(\gamma_1 + 1) \left( 2 - \frac{r}{R} \right)^\xi, \]
and hence
\[ \Delta_p \psi \geq C^{p-1}(p - 1)\gamma_1^{-1}(\gamma_1 + 1)(r - R)^{-m}(2 - \frac{r}{R})^{\xi(p-1)}. \]

Since \((2 - \frac{r}{R}) \in (0, 1)\) and \(\xi q > \xi(p-1)\), we easily see from (2.8) that
\[ b(x)\psi^q \leq (\beta_1 + \epsilon)C^q(r - R)^{-m}(2 - \frac{r}{R})^{\xi(p-1)}. \]

Let us also note from (2.7) that
\[ -a(x)\psi^{p-1} \leq MC^{p-1}(r - R)^{-m}(2 - \frac{r}{R})^{\xi(p-1)} \eta^p. \]

Therefore, we have, for \(x \in \Omega\) satisfying \(R < r < (1 + \eta)R\),
\[ -\Delta_p \psi - a(x)\psi^{p-1} + b(x)\psi^q \leq (r - R)^{-m}(2 - \frac{r}{R})^{\xi(p-1)}J_3, \]
where
\[ J_3 = -C^{p-1}(p - 1)\gamma_1^{-1}(\gamma_1 + 1) + MC^{p-1}\eta^p + (\beta_1 + \epsilon)C^q. \]

By (2.6), we see that if we fix \(\eta > 0\) small enough (depending on \(\epsilon\), but independent of \(\xi\)), then \(J_3 < 0\). We henceforth let \(\eta\) be so fixed, and obtain
\[ -\Delta_p \psi - a(x)\psi^{p-1} + b(x)\psi^q \leq 0 \text{ if } x \in \Omega, \ R < |x - x_\delta^*| < (1 + \eta)R. \quad (2.9) \]
We next choose $\xi$ so that (2.9) is satisfied in the rest of $\Omega \cap A_{R,\delta}$. So we now suppose $(1 + \eta)R \leq r < 2R$. Then

$$J_1 \geq \xi \eta \left(2 - \frac{r}{R}\right)^{\xi - 1}, \quad J_2 \geq \xi (\xi - 1) \eta^2 \left(2 - \frac{r}{R}\right)^{\xi - 2}.$$ 

Therefore,

$$\Delta_p \psi \geq C^{p-1}(p-1)\eta^p \xi^{p-1}(\xi - 1)(r - R)^{-m} \left(2 - \frac{r}{R}\right)^{\xi(p-1) - p} \geq C^{p-1}(p-1)\eta^p \xi^{p-1}(\xi - 1)(r - R)^{-m} \left(2 - \frac{r}{R}\right)^{\xi(p-1)},$$

$$-a(x)\psi^{p-1} \leq MC^{p-1} R^p (r - R)^{-m} \left(2 - \frac{r}{R}\right)^{\xi(p-1)},$$

$$b(x)\psi^q \leq (\beta_1 + \epsilon) C^q (r - R)^{-m} \left(2 - \frac{r}{R}\right)^{\xi q} \leq (\beta_1 + \epsilon) C^q (r - R)^{-m} \left(2 - \frac{r}{R}\right)^{\xi (p-1)}.$$ 

It follows that

$$-\Delta_p \psi - a(x)\psi^{p-1} + b(x)\psi^q \leq (r - R)^{-m} \left(2 - \frac{r}{R}\right)^{\xi(p-1)} J_4,$$

where

$$J_4 = -C^{p-1}(p-1)\eta^p \xi^{p-1}(\xi - 1) + MC^{p-1} R^p + (\beta_1 + \epsilon) C^q.$$ 

Clearly, we can now choose $\xi > 1$ large enough (depending on $\eta$ and $\epsilon$) so that $J_4 < 0$. We let $\xi$ be such fixed and find that (2.9) now also holds for $x \in \Omega$ satisfying $(1 + \eta)R \leq |x - x_\delta^R| < 2R$. Thus it holds on $\Omega \cap A_{R,\delta}$ for every $\delta \in [0, \delta_0]$.

We now apply Proposition 2.1 to compare a positive solution $u(x)$ of (2.1) with $\psi_\delta$ over $A_{R,\delta} \cap \Omega$, $\delta > 0$ small. We have

$$-\Delta_p u - a(x)u^{p-1} + b(x)u^q = 0 \geq -\Delta_p \psi_\delta - a(x)\psi_\delta^{p-1} + b(x)\psi_\delta^q$$

in $A_{R,\delta} \cap \Omega$. On $\partial \Omega \cap A_{R,\delta}$, we have $u = \infty$ and $\psi_\delta$ is finite. On $\partial A_{R,\delta} \cap \Omega \subset \partial_2 A_{R,\delta}$, $u > 0$ and $\psi_\delta = 0$. Therefore we always have

$$\lim_{x \to \partial(A_{R,\delta} \cap \Omega)} (\psi_\delta^p - u^p) \leq 0.$$ 

By Proposition 2.1, we conclude that $u \geq \psi_\delta$ in $\Omega \cap A_{R,\delta}$ for all small $\delta > 0$. Letting $\delta \to 0$, we deduce $u \geq \psi_0$ in $\Omega \cap A_{R,0}$. Therefore

$$\lim_{x \to x_*, x \in C_\delta(x_*)} \frac{u(x)}{\overline{d}(x, \Gamma_\infty)^{-\gamma}} \geq \lim_{x \to x_*, x \in C_\delta(x_*)} \frac{\psi_0(x)}{\overline{d}(x, \Gamma_\infty)^{-\gamma}} = C(\epsilon).$$
Letting $\epsilon \to 0$, we obtain, due to $C(\epsilon) \to \mu_1$,
\[
\lim_{x \to x^*, x \in C_\delta(x^*)} \frac{u(x)}{d(x, \Gamma_\infty)^{-\gamma_1}} \geq \mu_1.
\]

Parallel to Lemma 2.2, we have the following result.

**Lemma 2.3.** Suppose $x^* \in \Gamma_\infty \cap B$ and there exist constants $\alpha_2 = \alpha_2(x^*) \geq 0$, $\beta_2 = \beta_2(x^*) > 0$ such that
\[
\lim_{x \to x^*, x \in \Omega} \frac{b(x)}{d(x, \Gamma_\infty)^{\alpha_2}} \geq \beta_2.
\]

Then any solution $u(x)$ of (2.1) satisfies
\[
\lim_{x \to x^*, x \in C_\delta(x^*)} \frac{u(x)}{d(x, \Gamma_\infty)^{-\gamma_2}} \leq \mu_2,
\]
where
\[
\gamma_2 = \frac{\alpha_2 + p}{q - p + 1}, \quad \mu_2 = \left[\frac{(p - 1)\gamma_2^{-1}(\gamma_2 + 1)}{\beta_2}\right]^{1/(q-p+1)}.
\]

**Proof.** For any given small $\epsilon > 0$, by (2.10), there exists $r_0 > 0$ such that $B_{r_0}(x^*) \subset B$ and
\[
b(x) \geq (\beta_2 - \epsilon)d(x, \Gamma)^{\alpha_2}, \quad \forall x \in B_{r_0}(x^*) \cap \Omega.
\]

We can choose $R > 0$ and $\delta_0 > 0$ small enough so that, with $\tilde{x}^\delta_\nu = x^* + (2R + \delta)\nu_{x^*}$, the family of annuli
\[
\tilde{A}_{R,\delta} := \{x \in \mathbb{R}^N : R < |x - \tilde{x}^\delta_\nu| < 2R\}, \quad \delta \in [0, \delta_0]
\]
satisfies
\[
\tilde{A}_{R,\delta} \subset B_{r_0}(x^*) \cap \Omega, \quad \forall \delta \in (0, \delta_0);\]
\[
\partial_2 \tilde{A}_{R,\delta} \cap \partial \Omega = \emptyset \quad \text{if } \delta \in (0, \delta_0], \quad \partial_2 \tilde{A}_{R,0} \cap \overline{\Omega} = \{x^*\},
\]
where for $j = 1, 2$, $\partial_j \tilde{A}_{R,\delta} = \{x \in \mathbb{R}^N : |x - x^\delta_\nu| = jR\}$.

Let us denote $r = |x - \tilde{x}^\delta_\nu|$. Then
\[
d(x, \Gamma_\infty) \geq 2R - r, \quad \forall x \in \tilde{A}_{R,\delta}, \quad \delta \in [0, \delta_0].
\]

It now follows from (2.12) that
\[
b(x) \geq (\beta_2 - \epsilon)(2R - r)^{\alpha_2}, \quad \forall x \in \tilde{A}_{R,\delta}, \quad \delta \in [0, \delta_0].
\]
We are going to construct a family of upper solutions to (2.1) over $\tilde{A}_{R, \delta}$. To this end, we let $A = A(\epsilon)$ be the positive constant determined by
\begin{equation}
A^q - p + 1 = \frac{(p - 1)\gamma_2^{p-1}(\gamma_2 + 1) + \epsilon}{\beta_2 - \epsilon}.
\end{equation}
Then define
\[ \phi(x) = \phi_\delta(x) = A(2R - r)^{-\gamma_2} + \zeta, \quad r = |x - \tilde{x}_\delta|, \]
with $\zeta > 0$ to be determined later. Using the notation
\[ m := (\gamma_2 + 1)(p - 1) + 1 = \gamma_2 q - \alpha_2, \]
we obtain from a simple calculation
\begin{equation}
\Delta_p \phi = A^p - \gamma_2^{p-1}(\gamma_2 + 1)(2R - r)^{-m}.
\end{equation}
We may assume that $|a(x)| \leq M$ on $\Omega$. Then making use of the inequality
\[ ab \leq a^\xi/\xi + b^\eta/\eta \quad \text{for} \ a, b, \xi, \eta > 0, \xi^{-1} + \eta^{-1} = 1, \]
we obtain
\begin{align*}
-a(x)\phi^{p-1} & \geq -M(2R - r)^{-m}(2R - r)^p \left[ A + \zeta(2R - r)^{\gamma_2} \right]^{p-1} \\
& \geq -M(2R - r)^{-m}(2R - r)^p \left( \frac{A + \zeta(2R - r)^{\gamma_2}}{q/(p-1)} + \frac{1}{q/(q-p+1)} \right). 
\end{align*}
By (2.13), we have
\[ b(x)\phi^q \geq (\beta_2 - \epsilon)A^q(2R - r)^{-m} \left[ A + \zeta(2R - r)^{\gamma_2} \right]^q. \]
Therefore,
\[ -\Delta_p \phi - a(x)\phi^{p-1} + b(x)\phi^q \geq (2R - r)^{-m} \left( I_1 + I_2 \left[ A + \zeta(2R - r)^{\gamma_2} \right]^q \right), \]
where
\[ I_1 = -A^p - \gamma_2^{p-1}(\gamma_2 + 1)(p - 1) - M^{q-p+1/q}(2R - r)^p, \]
and
\[ I_2 = (\beta_2 - \epsilon)A^q - M^{p-1/q}(2R - r)^p. \]
Due to (2.14), there exists $\eta > 0$ small enough (depending on $\epsilon$ but independent of $\zeta$) such that when $0 \leq (2R - r) \leq \eta$, it holds that $I_2 > 0$ and $I_1 + I_2 A^q > 0$. It follows that, for such $r$,
\[ (2R - r)^{-m} \left( I_1 + I_2 \left[ A + \zeta(2R - r)^{\gamma_2} \right]^q \right) \geq (2R - r)^{-m} (I_1 + I_2 A^q) \geq 0, \]
and hence
\[ -\Delta_p \phi - a(x)\phi^{p-1} + b(x)\phi^q \geq 0 \quad \text{if} \quad 0 \leq (2R - r) \leq \eta. \tag{2.15} \]
We next choose \( \zeta \) so that (2.15) holds in the rest of \( \bar{A}_{R,\delta} \). So suppose from now on \( \eta \leq 2R - r \leq 2R \). Then
\[
- a(x)\phi^{p-1} \geq -MA^{p-1}(2R - r)^{-m}(2R - r)^p[A + \zeta(2R - r)^{\gamma_2}]^{p-1},
\]
\[
b(x)\phi^q \geq (\beta_2 - \epsilon)A^q(2R - r)^{-m}[A + \zeta(2R - r)^{\gamma_2}]^q.
\]
We thus obtain
\[
-\Delta_p \phi - a(x)\phi^{p-1} + b(x)\phi^q \geq (2R - r)^{-m} I_3,
\]
where
\[
I_3 = -A^{p-1}\gamma_2^{p-1}(\gamma_2 + 1)(p - 1) + I_4[A + \zeta(2R - r)^{\gamma_2}]^{p-1},
\]
and
\[
I_4 = (\beta_2 - \epsilon)A^q[A + \zeta(2R - r)^{\gamma_2}]^{q-p+1} - MA^{p-1}(2R)^p.
\]
It is now easy to see that if \( \zeta_0 > 0 \) is large enough (depending on \( \eta \) and \( \epsilon \)), then \( I_4 > 0 \) and \( I_3 > 0 \), \( \forall \zeta \geq \zeta_0 \). Hence, for \( \zeta \geq \zeta_0 \), (2.15) also holds when \( \eta \leq 2R - r \leq 2R \). In conclusion, we find that (2.15) holds on \( \bar{A}_{R,\delta} \) for every \( \delta \in [0, \delta_0] \) if we choose \( \zeta \geq \zeta_0 \) in the definition of \( \phi_\delta \).

Let \( u \) be a positive solution to (2.1). Set
\[
\zeta_R := \max\{u(x) : x \in \cup_{\delta \in [0, \delta_0]} \partial_1 \bar{A}_{R,\delta}\}.
\]
Then fix \( \zeta \) in the definition of \( \phi_\delta \) such that \( \zeta > \max\{\zeta_0, \zeta_R\} \). We now apply Proposition 2.1 to compare \( u \) and \( \phi_\delta \) over \( \bar{A}_{R,\delta} \) for \( \delta \in (0, \delta_0] \).

By our choice of \( \zeta \), we have \( \phi_\delta > u \) on \( \partial_1 \bar{A}_{R,\delta} \) for all \( \delta \in [0, \delta_0] \). On \( \partial_2 \bar{A}_{R,\delta} \) with \( \delta \in (0, \delta_0] \), we have \( \phi_\delta = \infty \) but \( u < \infty \). Therefore we have
\[
\lim_{x \to \partial_2 \bar{A}_{R,\delta}} (u^p - \phi_\delta^p) \leq 0, \forall \delta \in (0, \delta_0].
\]
As \( \phi_\delta \) is an upper solution to (2.1) over \( \bar{A}_{R,\delta} \), by Proposition 2.1, we conclude that \( u \leq \phi_\delta \) in \( \bar{A}_{R,\delta} \). Letting \( \delta \to 0 \), we deduce \( u \leq \phi_0 \) in \( \bar{A}_{R,0} \). It follows that
\[
\lim_{x \to x^+, x \in C_\delta(x^+)} \frac{u(x)}{d(x, \Gamma_\infty)^{-\gamma_2}} \leq \lim_{x \to x^+, x \in C_\delta(x^+)} \frac{\phi_0(x)}{d(x, \Gamma_\infty)^{-\gamma_2}} = A(\epsilon).
\]
Letting $\epsilon \to 0$, we obtain, due to $A(\epsilon) \to \mu_2$,
\[
\lim_{x \to x_*, x \in C_{\delta}(x_*)} \frac{u(x)}{d(x, \Gamma_{\infty})^{-\gamma_2}} \leq \mu_2. \quad \square
\]

**Remark 2.4.** (i) We could have used Theorem 4.4 of [5] and a less explicit family of lower solutions to prove Lemma 2.2. More precisely, we can replace $\psi_{\delta}$ in the proof of Lemma 2.2 by the unique positive solution of the problem
\[
-\Delta_p \psi = -\|a\|_{L^\infty(\Omega)} \psi^{p-1} - (\beta_1 + \epsilon)(r - R)^{\alpha_1} \psi^q \quad \text{in } A_{R,\delta},
\]
\[
\psi|_{\partial A_{R,\delta}} = 0, \quad \psi|_{\partial \tilde{A}_{R,\delta}} = \infty,
\]
where $r = |x - x_{\delta}^*|$.

(ii) Similarly, in the proof of Lemma 2.3, we can replace $\phi_{\delta}$ by the unique positive solution of
\[
-\Delta_p \phi = \|a\|_{L^\infty(\Omega)} \phi^{p-1} - (\beta_2 - \epsilon)(r - R)^{\alpha_2} \phi^q \quad \text{in } \tilde{A}_{R,\delta}, \quad \phi|_{\partial \tilde{A}_{R,\delta}} = \infty,
\]
where $r = |x - \tilde{x}_{\delta}^*|$.

Combining Lemmas 2.2 and 2.3, we immediately obtain the following result.

**Theorem 2.5.** Suppose $x_* \in \Gamma_{\infty} \cap B$ and there exist constants $\alpha = \alpha(x_*) \geq 0$ and $\beta = \beta(x_*) > 0$ such that
\[
\lim_{x \to x_*, x \in \Omega} \frac{b(x)}{d(x, \Gamma_{\infty})^{\alpha(x_*)}} = \beta(x_*). \quad (2.16)
\]
Then any solution $u(x)$ of (2.1) satisfies
\[
\lim_{x \to x_*, x \in C_{\sigma}(x_*)} \frac{u(x)}{d(x, \Gamma_{\infty})^{-\xi(x_*)}} = \eta(x_*), \quad (2.17)
\]
where
\[
\xi(x_*) = \frac{\alpha(x_*)}{q} + \frac{p}{q - p + 1}, \quad \eta(x_*) = \left[ \frac{(p-1)\xi(x_*)^{p-1} + 1}{\beta(x_*)} \right]^{1/(q-p+1)}.
\]

**Remark 2.6.** From the proofs of Lemmas 2.2 and 2.3, we see that if $\alpha(x_*) \geq 0$ and $\beta(x_*) > 0$ are continuous functions on $\Gamma_{\infty}$, then the argument in the last part of the proof of Theorem 1.1 in [11] can be easily adapted to show that when (2.16) holds uniformly for every $x_* \in \Gamma_{\infty}$, then (2.17) holds uniformly in $x_* \in \Gamma_{\infty}$ for any positive solution $u$ of (2.1), where we now should replace $B$ by a neighborhood of $\Gamma_{\infty}$.

We now examine condition (2.16) more closely.
Proposition 2.7. Suppose $B$ is a ball such that $B \cap \Gamma_\infty$ is nonempty and connected. If (2.16) holds uniformly on $B \cap \Gamma_\infty$, then $\alpha|_{B \cap \Gamma_\infty}$ is a constant function.

Proof. Let $x_\infty^0$ be an arbitrary point on $B \cap \Gamma_\infty$. By (2.16), there exists $\delta > 0$ such that $B_\delta(x_\infty^0) \subset B$ and

$$\frac{\beta(x_\infty)}{d(x, \Gamma_\infty)^{\alpha(x_\infty)}} - \beta(x_\infty^0) \leq \beta(x_\infty^0)/2, \forall x \in \Omega \cap B_\delta(x_\infty^0). \quad (2.18)$$

We show that $\alpha(x_\infty) \equiv \alpha(x_\infty^0)$ on $\Gamma_\infty \cap B_\delta(x_\infty^0)$. Clearly this implies the conclusion of the proposition.

Arguing indirectly we suppose that there exists $x_1^\infty \in \Gamma_\infty \cap B_\delta(x_\infty^0)$ satisfying $\alpha(x_1^\infty) \neq \alpha(x_\infty^0)$. Let $\{x_n\} \subset \Omega \cap B_\delta(x_\infty^0)$ satisfy $x_n \to x_1^\infty$. By (2.16), we have

$$\lim_{n \to \infty} \frac{b(x_n)}{d(x_n, \Gamma_\infty)^{\alpha(x_\infty^0)}} = \beta(x_1^\infty).$$

It follows that

$$\lim_{n \to \infty} \frac{b(x_n)}{d(x_n, \Gamma_\infty)^{\alpha(x_\infty^0)}} = \beta(x_1^\infty) \lim_{n \to \infty} d(x_n, \Gamma_\infty)^{\alpha(x_\infty^0) - \alpha(x_\infty^0)} = \left\{ \begin{array}{ll} 0 & \text{if } \alpha(x_1^\infty) - \alpha(x_\infty^0) > 0, \\ \infty & \text{if } \alpha(x_1^\infty) - \alpha(x_\infty^0) < 0. \end{array} \right.$$ 

Therefore, if we take $x = x_n$ in (2.18) and let $n \to \infty$, we deduce

$$\beta(x_\infty^0) \leq \beta(x_\infty^0)/2, \quad \text{or} \quad \infty \leq \beta(x_\infty^0)/2.$$ 

This contradiction finishes our proof. \hfill \Box

3. Uniqueness results

In this section, we make use of our estimates in Section 2 to improve some existing uniqueness results for (1.1). Throughout this section, we make the same assumptions on $a(x), b(x), \Gamma_\infty,$ and $\Omega$ as in Section 2. Our first result follows easily from Remark 2.6, and (due to Proposition 2.7) is a slight improvement of Theorem 4.4 in [5].

Theorem 3.1. Suppose that (2.16) holds uniformly in $x_s \in \Gamma_\infty$, where $\alpha(x_s) \geq 0$ and $\beta(x_s) > 0$ are continuous functions on $\Gamma_\infty$. Then (1.1) has a unique positive solution $u(x)$. Moreover, (2.17) holds uniformly in $x_s \in \Gamma_\infty$. 
Proof. That (1.1) has a positive solution can be proved in the same way as in [5, Theorem 4.3]. Using Remark 2.6, we find that the uniqueness result follows from a standard consideration. We briefly recall this. Suppose that \( u_1 \) and \( u_2 \) are positive solutions of (1.1). Then for any small \( \epsilon > 0 \), \( w_j := (1 + \epsilon)u_j, \ j = 1, 2 \), satisfies
\[
-\Delta_p w_j \geq a(x)w_j^{p-1} - b(x)w_j^q \quad \text{in } \Omega.
\]
Due to Remark 2.6, we have
\[
\lim_{d(x, \Gamma_\infty) \to 0} \frac{u_1(x)}{u_2(x)} = 1.
\]
This allows us to apply Proposition 2.1 to compare \( u_1 \) with \( w_2 \) over \( \Omega \) to conclude that \( u_1 \leq w_2 \) in \( \Omega \). Similarly, \( u_2 \leq w_1 \) in \( \Omega \). Letting \( \epsilon \to 0 \), we hence obtain \( u_1 \leq u_2 \) and \( u_2 \leq u_1 \) in \( \Omega \). This completes the proof. \( \square \)

Next we use Lemmas 2.2 and 2.3 and an iteration technique of Safonov to obtain a uniqueness result for (1.1) with \( p = 2 \), which greatly improves the uniqueness result of [7].

Theorem 3.2. Suppose that there exist constants \( \delta > 0, \alpha \geq 0, \) and \( \beta_2 > \beta_1 > 0 \) such that
\[
\beta_1 d(x, \Gamma_\infty)^\alpha \leq b(x) \leq \beta_2 d(x, \Gamma_\infty)^\alpha \quad \text{for } x \in \Omega, \ d(x, \Gamma_\infty) \leq \delta.
\] (3.1)
Then (1.1) with \( p = 2 \) has a unique positive solution. \[ \square \]

Proof. For simplicity of notation, we write
\[
d(x) = d(x, \Gamma_\infty), \ \gamma = (\alpha + 2)/(q - 1).
\]
By (3.1) and Lemmas 2.2 and 2.3, there exist constants \( \mu_2 > \mu_1 > 0 \) and \( \delta_0 \in (0, \delta) \) such that any positive solution \( u(x) \) of (1.1) with \( p = 2 \) satisfies
\[
\mu_1 d(x)^{-\gamma} \leq u(x) \leq \mu_2 d(x)^{-\gamma} \quad \text{for } x \in \Omega, \ d(x) \leq \delta_0.
\] (3.2)
Suppose for contradiction that (1.1) with \( p = 2 \) has two distinct solutions \( u_1 \) and \( u_2 \). We necessarily have
\[
\lim_{d(x) \to 0} u_2/u_1 > 1 \quad \text{or} \quad \lim_{d(x) \to 0} u_2/u_1 < 1,
\]
for otherwise the standard argument as recalled in the proof of Theorem 3.1 above would imply \( u_1 \equiv u_2 \). Without loss of generality we assume that
\[
\lim_{d(x) \to 0} u_2/u_1 > 1.
\]
Then there exist a sequence \( \{x_n\} \subset \Omega \) and a constant \( k > 1 \) such that
\[
d(x_n) \to 0 \text{ as } n \to \infty, \ u_2(x_n) > ku_1(x_n) \ \forall n \geq 1.
\] (3.3)
Let $\delta_0 \in (0, \delta_0]$ be chosen small enough so that
\[
\lambda := (2/3)^{\gamma+2} \left[ \beta_1 \mu_1^q (k^q - 1) - M (1 + k^{-1}) \mu_2 (3/2)^2 \delta_*^2 \right] > 0,
\]
where $M = \|a\|_{L^\infty(\Omega)}$.

We now use a modified version of Safonov’s technique (see [10]) to prove the following claim (it is here that $p = 2$ is needed).

**Claim.** If $x^* \in \Omega$ satisfies $d(x^*) \leq \delta_*$ and $u_2(x^*) > k_* u_1(x^*)$ for some $k_* \geq k$, then there exists $y^* \in \Omega$ such that
\[
|y^* - x^*| = d(x^*)/2, \quad u_2(y^*) > (1 + c_0) k_* u_1(y^*),
\]
where $c_0 := \lambda (2N \mu_2)^{-1}$.

To prove this claim, we denote $r = d(x^*)/2$ and define
\[
\Omega_0 := \{ x \in \Omega : u_2(x) > k_* u_1(x) \} \cap B_r(x^*).
\]

We note that
\[
r \leq d(x) \leq (3/2)r < \delta_*, \quad \forall x \in \Omega_0.
\]

By (3.2), we obtain
\[
\mu_1 d(x)^{-\gamma} \leq u_i(x) \leq \mu_2 d(x)^{-\gamma}, \quad \forall x \in \Omega_0, \quad i = 1, 2.
\]

Using (3.1), (3.4), and (3.5), we have, for $x \in \Omega_0$,
\[
\Delta (u_2 - k_* u_1) = -a(x) (u_2 - k_* u_1) + b(x) (u_2^q - k_* u_1^q)
\geq -M (1 + k_*) \mu_2 d(x)^{-\gamma} + \beta_1 d(x) \alpha (k_*^q - k_*) u_1^q
\geq k_* d(x)^{-\gamma-2} \left[ \beta_1 \mu_1^q (k_*^q - 1) - M (1 + k_*^{-1}) \mu_2 d(x)^2 \right]
\geq \lambda k_* r^{-\gamma-2}.
\]

Note that to obtain the last inequality we have used $k_* \geq k$. Let
\[
w(x) = (2N)^{-1} \lambda k_* r^{-\gamma-2} (r^2 - |x - x^*|^2).
\]

Then $\Delta w = -\lambda k_* r^{-\gamma-2}$, and hence
\[
\Delta (u_2 - k_* u_1 + w) \geq 0 \text{ in } \Omega_0.
\]

By the maximum principle, we obtain
\[
u_2(x^*) - k_* u_1(x^*) + w(x^*) \leq \max_{\partial \Omega_0} (u_2 - k_* u_1 + w).
\]

We observe that this maximum over $\partial \Omega_0$ has to be achieved by some $y^* \in \partial B_r(x^*)$, since any $y \in \partial \Omega_0 \setminus \partial B_r(x^*)$ necessarily satisfies $u_2(y) = k_* u_1(y)$ and hence
\[
u_2(y) - k_* u_1(y) + w(y) = w(y) \leq w(x^*) < u_2(x^*) - k_* u_1(x^*) + w(x^*).
So we can find \( y^* \in \partial \Omega_0 \subset \Omega \) satisfying \( |y^* - x^*| = r \) (and hence \( w(y^*) = 0 \)) such that
\[
u_2(y^*) - k\nu u_1(y^*) = u_2(y^*) - k\nu u_1(y^*) + w(y^*) \geq u_2(x^*) - k\nu u_1(x^*) + w(x^*).
\]
In particular,
\[
u_2(y^*) - k\nu u_1(y^*) > w(x^*). \tag{3.6}
\]
Making use of (3.4) and (3.5), we have
\[w(x^*) = (2N)^{-1}\lambda k r^{-\gamma} \geq c_0 k_1 u_1(y^*).
\]
Hence, by (3.6), \( u_2(y^*) > (1 + c_0)k\nu u_1(y^*) \). This proves our claim.

From (3.2), we deduce
\[
\frac{\nu_2(x)}{\nu_1(x)} \leq \frac{\mu_2}{\mu_1} \text{ for } x \in \Omega, \; d(x) \leq \delta_s. \tag{3.7}
\]
Let \( j \) be a large positive integer so that \((1 + c_0)^j k > \mu_2 / \mu_1\). Since \( d(x_n) \to 0 \), we can find \( n \) large enough such that \((3/2)^j d(x_n) < \delta_s\). The inequality in (3.3) allows us to apply the above claim with \( x^* = x_n \) and \( k_s = k \) to obtain \( y^* = x_1^* \) satisfying
\[|x_1^* - x_n| = d(x_n)/2, \; u_2(x_1^*) > (1 + c_0)k u_1(x_1^*).
\]
Since \( d(x_1^*) \leq (3/2)^j d(x_n) < \delta_s \), we can apply the above claim (with \( x^* = x_1^* \) and \( k_s = (1 + c_0)k \) this time) to find \( x_2^* \) satisfying
\[|x_2^* - x_1^*| = d(x_1^*)/2, \; u_2(x_2^*) > (1 + c_0)^2 k u_1(x_2^*).
\]
Repeating this process we find \( x_i^* \) for \( i = 1, \ldots, j \) satisfying
\[d(x_i^*) \leq (3/2)^i d(x_n), \; u_2(x_i^*) > (1 + c_0)^i k u_1(x_i^*).
\]
In particular, \( d(x_j^*) < \delta_s \) and
\[u_2(x_j^*) > (1 + c_0)^j k u_1(x_j^*) > (\mu_2 / \mu_1) u_1(x_j^*),
\]
a contradiction to (3.7). This finishes our proof of the theorem. \( \Box \)

References


