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HYDRODYNAMIC LIMITS OF A VLASOV-FOKKER-PLANCK EQUATION FOR GRANULAR MEDIA*

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Abstract. This paper, which is a sequel to Benedetto-Caglioti-Golse-Pulvirenti [Comput. Math. Appl. **38** (1999), 121–131], considers as a starting point a mean-field equation for the dynamics of a gas of particles interacting via dissipative binary collisions. More precisely, we are concerned with the case where these particles are immersed in a thermal bath modeled by a linear Fokker-Planck operator. Two different scalings are considered for the resulting equation. One concerns the case of a thermal bath at finite temperature and leads formally to a nonlinear diffusion equation. The other concerns the case of a thermal bath at infinite temperature and leads formally to an isentropic Navier-Stokes system. Both formal limits rest on the mathematical properties of the linearized mean-field operator which are established rigorously, and on a Hilbert or Chapman-Enskog expansion.

Key words. Granular media, Vlasov-Fokker-Planck equation, Hydrodynamic limits, Hilbert expansion, Chapman-Enskog expansion.

AMS subject classifications. 82C40, 76T25, 76M45

1. The Vlasov-Fokker-Planck Model

A simple 1D model for granular media was proposed in [13]. It consists of a gas of N like particles restricted to move on an infinite line and subject to instantaneous inelastic binary collisions. In the course of any collision, a fraction (denoted by $\varepsilon \in [0,1]$) of the relative velocity of the colliding pair of particles is dissipated. We refer for instance to [9] for a general survey on kinetic and hydrodynamic models for granular media.

A Vlasov type kinetic model was formally derived from this particle model in the limit as $N \to +\infty$, $\varepsilon \to 0$ with $N\varepsilon \to \lambda$ for some $\lambda > 0$: see [13], [8], [3],[4]. This model reads:

$$\partial_t f + v\partial_x f + \lambda \partial_v (F(f)f) = 0\,, \tag{1.1}$$

with

$$F(f) \equiv F(f)(t,x,v) = \int_{\mathbb{R}} |v'-v|(v'-v)f(t,x,v')dv'\,, \tag{1.2}$$

and where $f \equiv f(t,x,v)$ denotes the density of particles which, at time t, are in position x with velocity v. For further mathematical analysis see also [14] and [5], as well as [11] and [6].

The present paper considers a 1D inelastic particle system as above, modeled as in (1.1) but immersed in a thermal bath at a constant temperature. As in [1], [2]

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the effect of the thermal bath is modeled by adding to the Vlasov equation (1.1) a Fokker-Planck term: instead of (1.1), the phase space density f must satisfy

$$\partial_t f + v\partial_x f + \lambda\partial_v(F(f)f) = \beta\partial_v(vf) + \sigma\partial_{vv} f \,, \tag{1.3}$$

where β is the friction coefficient and σ/β the temperature of the thermal bath.

In [2], a first hydrodynamic limit of the model (1.3) was formally established in the case of a thermal bath at infinite temperature (equivalently, in the case where the friction coefficient $\beta = 0$). In this limit, the macroscopic observables

$$\rho(t,x) = \int f(t,x,v)dv \,, \quad u(t,x) = \frac{1}{\rho(t,x)} \int vf(t,x,v)dv \,, \tag{1.4}$$

respectively the macroscopic density and bulk velocity, were shown formally to satisfy the following system of conservation laws

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \,, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + k\rho^{1/3}) &= 0 \,, \end{aligned} \tag{1.5}$$

where k is a constant proportional to $\lambda^{-2/3}$ (we take this opportunity to correct the misprint in Eq. (2.13) of [2], where k was wrongly set proportional to $\lambda^{1/3}$).

The purpose of the present paper is twofold. First, we seek the Navier-Stokes correction to Eqs. (1.5), which comes in the form of a viscous perturbation of the momentum equation in (1.5). Then, in the case where the thermal bath has finite temperature (or equivalently, in the case where $\beta > 0$), the granular gas looses momentum due to friction, so that its bulk velocity u is 0 at leading order. In this situation, the macroscopic density is governed by a nonlinear diffusion equation. Both derivations are formal and based on the Hilbert or Chapman-Enskog expansions. The next section contains a description of the scalings under which the limits mentioned above are derived, while the Hilbert and Chapman-Enskog expansions are analyzed in section 4 and 5. The key argument in writing any of these expansions is that the linearized mean-field operator satisfies the Fredholm alternative. Section 3 is devoted to the mathematical analysis of this linearized mean-field operator.

2. Scalings and Hydrodynamic Limits

2.1. Hyperbolic vs. Parabolic Scalings. First we recall the dimensionless form of the Vlasov-Fokker-Planck model as discussed in [2] but in a slightly different form. The hydrodynamic limits considered here are infinite volume and long time limits of Eq. (1.3).

More precisely, one defines a macroscopic length scale L and two a priori different time scales $T \le \tau$ such that L/T is the scale of the thermal speed of particles in the granular gas while L/τ is the scale of the speed of macroscopic material motion in the same granular gas. Consider then the dimensionless variables

$$\overline{t} = \frac{t}{\tau} \,, \quad \overline{x} = \frac{x}{L} \,, \quad \overline{v} = \frac{vT}{L} \,. \tag{2.1}$$

Defining

$$\overline{f}(\overline{t},\overline{x},\overline{v}) = \frac{L^2}{T} f(t,x,v) \tag{2.2}$$

we arrive to the dimensionless form of Eq. (1.3)

$$\frac{1}{\tau}\partial_{\overline{t}}\overline{f} + \frac{1}{T}\overline{v}\partial_{\overline{x}}\overline{f} + \frac{\lambda}{T}\partial_{\overline{v}}(\overline{F}\overline{f}) = \beta\partial_{\overline{v}}(\overline{v}\overline{f}) + \frac{\sigma T^2}{L^2}\partial_{\overline{v}\overline{v}}\overline{f}\,, \tag{2.3}$$

with

$$\overline{F}(\overline{f})(\overline{t},\overline{x},\overline{v}) = \int_{\mathbb{R}} |\overline{v}' - \overline{v}|(\overline{v}' - \overline{v})\overline{f}(\overline{t},\overline{x},\overline{v}')d\overline{v}'\,. \tag{2.4}$$

The dimensionless form (2.3) of the Vlasov-Fokker-Planck equation involves three additional time scales

$$T_f = \frac{1}{\beta}\,, \quad T_F = \frac{T}{\lambda}\,, \quad T_d = \frac{L^2}{\sigma T^2}\,.$$

Their physical meaning is as follows:

- T_f is the typical length of time during which a particle of the granular gas looses half of its kinetic energy due to friction on the thermal bath;
- T_F is the typical length of time during which a particle of the granular gas looses half of its kinetic energy due to inelastic collisions;
- T_d is the typical length of time during which a particle is accelerated by the molecular agitation in the thermal bath so that its speed increases from 0 to L/T.

The first situation studied here, which will be referred to as "the hyperbolic scaling", corresponds to

$$\tau = T\,, \quad T_F << T << \sqrt{T_F T_f}\,, \quad T_d = Const\, T_F\,. \tag{2.5}$$

A variant of (2.5) is the hyperbolic scaling with relaxation

$$\tau = T\,, \quad T_f = Const\, T\,, \quad T_F << T\,, \quad T_d = Const\, T_F\,. \tag{2.6}$$

The second situation considered in this work is the so-called "parabolic scaling"

$$\tau >> T\,, \quad \frac{T_F}{T} = \frac{T}{\tau}\,, \quad T_f = Const\, T_F\,, \quad T_d = Const\, T_F\,. \tag{2.7}$$

In the hyperbolic scaling (2.5), we set

$$\varepsilon = \frac{T_F}{T}\,, \quad \overline{\sigma} = \frac{T_F}{T_d}\,.$$

In the relaxation scaling (2.6), we set

$$\varepsilon = \frac{T_F}{T}\,, \quad \overline{\sigma} = \frac{T_F}{T_d}\,, \quad \overline{\beta} = \frac{T}{T_f}\,.$$

After dropping bars, the dimensionless Vlasov-Fokker-Planck model (2.3) becomes

$$\partial_t f_\varepsilon + v\partial_x f_\varepsilon + \frac{1}{\varepsilon}\partial_v\left[F(f_\varepsilon)f_\varepsilon - \sigma\partial_v f_\varepsilon\right] = 0 \tag{2.8}$$

in the hyperbolic scaling (2.5), and

$$\partial_t f_\varepsilon + v\partial_x f_\varepsilon - \beta\partial_v(v f_\varepsilon) + \frac{1}{\varepsilon}\partial_v\left[F(f_\varepsilon)f_\varepsilon - \sigma\partial_v f_\varepsilon\right] = 0 \tag{2.9}$$

in the relaxation scaling (2.6).

In the parabolic scaling (2.7), we set

$$\varepsilon = \frac{T_F}{T} = \frac{T}{\tau}\,, \quad \overline{\sigma} = \frac{T_F}{T_d}\,, \quad \overline{\beta} = \frac{T_F}{T_f}\,.$$

Again after dropping bars, the dimensionless Vlasov-Fokker-Planck model (2.3) in the parabolic scaling becomes

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} v \partial_x f_\varepsilon + \frac{1}{\varepsilon^2}\partial_v \left[F(f_\varepsilon) f_\varepsilon - \beta v f_\varepsilon - \sigma \partial_v f_\varepsilon \right] = 0\,. \tag{2.10}$$

2.2. The Free Energy and the Stationary States. In [1], it was recognized that the space homogeneous dynamics of (1.3) can be written as follows:

$$-\partial_v \left[F(f) f - \beta v f - \sigma \partial_v f \right] = \partial_v \left(f \partial_v D\eta_{\beta,\sigma}(f) \right)\,. \tag{2.11}$$

Here, the free energy $\eta_{\beta,\sigma}$ is defined on the class of all measurable, a.e. nonnegative functions of the variable v by the formula

$$\eta_{\beta,\sigma}(f) = \sigma \int f \ln f dv + \tfrac{1}{2}\beta \int v^2 f dv + \tfrac{1}{6} \iint |v - v'|^3 f(v) f(v') dv dv' \tag{2.12}$$

and

$$D\eta_{\beta,\sigma} = \frac{\delta}{\delta f}\eta_{\beta,\sigma} = \sigma(1 + \ln f) + \beta \frac{v^2}{2} + \tfrac{1}{3} \iint |v - v'|^3 f(v') dv'.$$

In all the scalings discussed above, the limiting form of the Vlasov-Fokker-Planck equation (2.8) to (2.10) is, according to (2.11) above,

$$\partial_v \left(f \partial_v D\eta_{\beta,\sigma}(f) \right) = 0\,, \tag{2.13}$$

with $\beta = 0$ in the case of the hyperbolic or relaxation scalings, while $\beta > 0$ in the case of the parabolic scaling. Thus, a special role in all hydrodynamic limits of (1.3) is played by stationary states, in other words critical points of $\eta_{\beta,\sigma}$, i.e. densities f_0 such that

$$D\eta_{\beta,\sigma}(f_0) = 0\,, \tag{2.14}$$

which are particular solutions of the limiting form (2.13) of the Vlasov-Fokker-Planck equation. Notice that stationary states may depend on the variables t and x which are just parameters in (2.13); such stationary states are referred to as “local”, while those which are constant in t and x are referred to as “uniform” stationary states.

A natural question is that of the existence and uniqueness of stationary states with respect to the free energy $\eta_{\beta,\sigma}$ above; these states minimize the free energy under the constraint of given total mass.

LEMMA 1. *For all $\beta > 0$ and $\sigma > 0$, $\eta_{\beta,\sigma}$ is strictly convex with values in $[0, +\infty]$ on the set of probability densities on $\mathbb{R}$ with finite second moment, and reaches its minimum there at a single point denoted by $G_{\beta,\sigma}$. This function $G_{\beta,\sigma}$ is even and belongs to $C^\infty(\mathbb{R})$. Moreover, the local steady state with spatial density ρ is*

$$f_0(v) = \rho G_{\beta/\rho, \sigma/\rho}(v). \tag{2.15}$$

This is precisely Theorem 2.1 of [1] in the case $\beta > 0$ — see also [12] for a more general result. In the case $\beta = 0$, uniqueness is lost due to translation invariance (in v)

of $\eta_{0,\sigma}$. Uniqueness is recovered by also imposing the constraint of given momentum, as follows. Define, for all $\rho \geq 0$ and $u \in \mathbb{R}$

$$\mathbf{K}_{\rho,u} = \{\phi \in L^1(\mathbb{R},(1+v^2)dv) \mid \phi \geq 0 \text{ a.e. } , \int_{\mathbb{R}} \phi dv = \rho\,, \int_{\mathbb{R}} v\phi dv = \rho u\}\,. \tag{2.16}$$

Corollary 1. *For all $\sigma > 0$ and all $\rho > 0$, $u \in \mathbb{R}$,*

$$\inf_{\mathbf{K}_{\rho,u}} \eta_{0,\sigma} \text{ is attained by the unique function } v \mapsto \rho G_{0,\sigma/\rho}(v-u)\,.$$

Proof. Let $\phi \in \mathbf{K}_{\rho,u}$; then $\psi : v \mapsto \frac{1}{\rho}\phi(v+u)$ belongs to $\mathbf{K}_{1,0}$ and a trivial change of variables shows that

$$\eta_{0,\sigma}(\phi) = \rho^2 \eta_{0,\sigma/\rho}(\psi) + \sigma\rho\log\rho\,. \tag{2.17}$$

One then concludes by a direct application of Lemma 1. ☐

As explained in [1], $\eta_{0,\sigma}$ is a Lyapunov function for the spatially homogeneous version of (1.3) and the functions $G_{0,\sigma}$ above are the spatially homogeneous steady states for Eq. (1.3). They also satisfy the self consistent equation

$$G_{0,\sigma} = \frac{e^{-\frac{1}{3\sigma}\int |v'-v|^3 G_{0,\sigma}(v')dv'}}{\int e^{-\frac{1}{3\sigma}\int |v'-v|^3 G_{0,\sigma}(v')dv'} dv}\,, \tag{2.18}$$

which is a consequence of the stationary condition

$$\partial_v(Ff) - \sigma\partial_{vv} f = 0$$

for the homogeneous equation associated to (1.3). Also, it follows from Eq. (2.18) that

$$G_{0,\sigma}(v) = O(e^{-|v|^3}) \text{ as } |v| \to +\infty\,. \tag{2.19}$$

Corollary 2. *For all $\sigma > 0$ and all $v \in \mathbb{R}$,*

$$G_{0,\sigma}(v) = \sigma^{-1/3} G_{0,1}(\sigma^{-1/3}v)\,.$$

This is just another formulation of Lemma 3 in our previous work [2], and the reason why the pressure law in Eqs. (1.5) is of the form proposed in [2] and recalled in the statement of (1.5), that is

$$P(\rho) = \int v^2 \rho G_{0,\sigma/\rho}(v)dv = k\rho^{1/3} \text{ with } k = \sigma^{2/3}\int v^2 G_{0,1}(v)dv\,. \tag{2.20}$$

2.3. The Hydrodynamic Models. The first such hydrodynamic model is the system (1.5) and corresponds to the zeroth-order (in ε) approximation of the Vlasov-Fokker-Planck equation (2.8) in the hyperbolic scaling. The same zeroth-order approximation on the Vlasov-Fokker-Planck equation (2.9) in the relaxation scaling leads to

$$\begin{aligned} &\partial_t\rho + \partial_x(\rho u) = 0\,,\\ &\partial_t(\rho u) + \partial_x(\rho u^2 + k\rho^{1/3}) = -\beta\rho u\,. \end{aligned} \tag{2.21}$$

In both cases, one expects to reconstruct the leading order of the solutions of the Vlasov-Fokker-Planck equation in either the hyperbolic (2.8) or the relaxation (2.9) scaling in terms of solutions (ρ, u) of (1.5) or (2.21) respectively by the formula

$$f_\varepsilon(t,x,v) = \rho(t,x) G_{0,\sigma/\rho(t,x)}(v - u(t,x)) + o(1)\,.$$

Notice that

$$\rho(t,x)G_{0,\sigma/\rho(t,x)}(v-u(t,x)) = \frac{\rho(t,x)^{4/3}}{\sigma^{1/3}}G\left(\frac{\rho(t,x)^{1/3}}{\sigma^{1/3}}(v-u(t,x))\right) \tag{2.22}$$

in view of Corollary 2.

The next model considered here is the first order (in ε) approximation of the Vlasov-Fokker-Planck in the relaxation scaling, a viscous correction Eqs. (2.21) analogous to compressible Navier-Stokes models as described for example in [10]. It reads

$$\begin{aligned} &\partial_t\rho + \partial_x(\rho u) = 0\,, \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + k\rho^{1/3}) = -\beta\rho u + \varepsilon\partial_x(\mu\rho^{-\frac{1}{3}}\partial_x u) + \frac{3}{4}\varepsilon\beta\mu\partial_x\rho^{-\frac{1}{3}}\,, \end{aligned} \tag{2.23}$$

where $\mu > 0$ is a (positive) viscosity coefficient which admits the following remarkable expression

$$\mu = -\tfrac{4}{3}\partial_\beta \int v^2 G_{\beta,\sigma}(v)dv_{\,|\beta=0}\,. \tag{2.24}$$

How to reconstruct the first order (in ε) approximation of the microscopic density relies on somewhat less explicit computations, by taking the Hilbert or the Chapman-Enskog expansions analyzed in the next sections.

Consider finally the Vlasov-Fokker-Planck equation in the parabolic scaling, i.e. Eq. (2.10); its 0-th order (in ε) approximation is governed by the nonlinear diffusion equation

$$\partial_t\rho - \partial_{xx}D_{\sigma,\beta}(\rho) = 0\,, \tag{2.25}$$

where the nonlinear diffusion flux is given by the formula

$$D_{\sigma,\beta}(\rho) = \frac{\rho}{\beta}\int v^2 G_{\beta/\rho,\sigma/\rho}(v)dv\,. \tag{2.26}$$

The vague resemblance between formulas (2.24) for the kinematic viscosity and (2.26) leading to a diffusion coefficient $\partial_\rho D_{\sigma,\beta}(\rho)$ might suggest that in some appropriate scaling limit, the diffusion equation (2.25) can be derived from the compressible Navier-Stokes model (2.23). This seems however totally misleading. Rather, the nonlinear diffusion equation (2.25) can be loosely related to the large β limit of (2.21) as follows. Following the general method outlined in [7] to obtain diffusion limits of systems of conservation laws, we see that the large β limit of (2.21) is governed by the nonlinear diffusion equation

$$\partial_t\rho - \partial_x\left(\frac{1}{\beta}\partial_x(k\rho^{1/3})\right) = 0\,. \tag{2.27}$$

Because of (2.20), one can recast the nonlinear diffusion flux in (2.27) as

$$\overline{D}_{\beta,\sigma}(\rho) = \frac{1}{\beta}\int v^2 G_{0,\sigma/\rho}(v)dv\,.$$

Thus

$$\beta\overline{D}_{\beta,\sigma}(\rho) = \lim_{\beta\to 0}\frac{\beta}{\rho}D_{\beta,\sigma}(\rho)\,.$$

In other words, the nonlinear diffusion limit (2.25) involves the relaxation of the speed of the granular gas due to collisions with the thermal bath, and not the viscosity of the granular gas, solely due to the inelastic collision process in this gas, fed by the Brownian kicks due to collisions with the thermal bath.

3. The Linearized Mean-Field Operator

In this section we analyze the linearized mean-field operator. We prove that it satisfies the Fredholm alternative. This allows us to perform, in the next sections, the Hilbert and Chapman-Enskog expansions.

The mean-field operator is

$$-\partial_v[F(f)f - \beta v f - \sigma \partial_v f] = \partial_v(f \partial_v D\eta_{\beta,\sigma}(f)),$$

and the local stationary state f_0 is given by Eq. (2.15), and satisfies

$$-\partial_v[F(f_0)f_0 - \beta v f_0 - \sigma \partial_v f_0] = 0. \tag{3.1}$$

The linearized mean-field operator L is defined by

$$L(f) = \partial_v[\beta v f + \sigma \partial_v f] + Q(f, f_0),$$

where

$$Q(f,g) = -\partial_v[F(f)g] - \partial_v[F(g)f]. \tag{3.2}$$

We have to exploit the invertibility of L. Before doing this we observe that, as follows from Eq. (3.1)

$$\begin{aligned} L(f) =& \partial_v \left(\sigma(\partial_v f - f\partial_v \log(f_0)) - F(f)f_0\right) = \\ =& \partial_v \left(f_0 \, \partial_v \left(\sigma \frac{f}{f_0} + \frac{1}{3} \int |v - \overline{v}|^3 f(\overline{v}) d\overline{v} \right) \right). \end{aligned} \tag{3.3}$$

L can be also expressed in terms of the free energy functional for the linearized case: let

$$\begin{aligned} \theta_{\beta,\sigma}(f,g) = \frac{1}{2} D^2 \eta_{\beta,\sigma} \Big|_{f_0} (f,g) = \\ \frac{\sigma}{2} \int \frac{fg}{f_0} dv + \frac{1}{6} \int |v - \overline{v}|^3 f(v) g(\overline{v}) dv \, d\overline{v} \,, \end{aligned} \tag{3.4}$$

then

$$L(f) = \partial_v (f_0 \partial_v D\theta_{\beta,\sigma}(f,f)). \tag{3.5}$$

LEMMA 2. *For all $f \in C^2(\mathbb{R})$ s.t. $(1+v^2)f(v) \in L_1(\mathbb{R})$*
i) $\int L(f)(v) \, dv = 0$.
ii) $\int vL(f)(v) \, dv = -\beta \int v f(v) dv$.
iii) If f is even (odd), $L(f)$ is even (odd).

Proof. Statement i) is obvious; ii) follows by the identity

$$\int [fF(g) + gF(f)] dv = 0 \, .$$

Finally, iii) follows from direct inspection. □

Inserting Eq. (2.15) for f_0 in Eq. (3.1) and deriving with respect to ρ we get

$$L(\alpha) = 0 \tag{3.6}$$

where

$$\alpha = \partial_\rho f_0 = G_{\beta/\rho,\sigma/\rho} + \rho\partial_\rho G_{\beta/\rho,\sigma/\rho}. \tag{3.7}$$

Differentiating with respect to v:

$$L(\partial_v f_0) = -\beta\partial_v f_0. \tag{3.8}$$

Obviously α is even and $\int \alpha\, dv = 1$, while $\partial_v f_0$ is odd and $\int v\partial_v f_0(v)\, dv = -\rho$. Now we can analyze the equation

$$L(f) = g, \tag{3.9}$$

for g such that $\int g dv = 0$.

Setting

$$h(v) = \int_{-\infty}^{v} g(w) dw\,, \tag{3.10}$$

we have by Eq. (3.3)

$$\sigma(\partial_v f - f\partial_v \log f_0) = f_0 F(f) + h, \tag{3.11}$$

$$\sigma\partial_v\left(\frac{f}{f_0}\right) = F(f) + \frac{h}{f_0}, \tag{3.12}$$

$$\frac{f}{f_0}(v) = \frac{f}{f_0}(0) + \frac{1}{\sigma}\int_0^v F(f)\, dv + \frac{1}{\sigma}\int_0^v \frac{h(w)}{f_0(w)} dw. \tag{3.13}$$

We claim that there exists a unique solution of Eq. (3.13) such that

$$\int \frac{f^2(w)}{f_0(w)} dw < +\infty \text{ and } \int f dv = 0\,.$$

We first remark that

$$\begin{aligned}
&\int_0^v dw\, F(f)(w) = \int_0^v dw \int (\overline{v} - w)|\overline{v} - w| f(\overline{v}) d\overline{v} = \\
&= -\int_0^v dw\, \partial_w \int \frac{|\overline{v} - w|^3}{3} f(\overline{v}) d\overline{v} = -\frac{1}{3}\int |\overline{v} - v|^3 f(\overline{v}) d\overline{v} + \frac{1}{3}\int |\overline{v}|^3 f(\overline{v}) d\overline{v}\,.
\end{aligned} \tag{3.14}$$

Thus Eqs. (3.13)-(3.14) imply

$$\begin{aligned}
f(v) &= f_0(v)\frac{f}{f_0}(0) \\
&\quad - \frac{1}{3\sigma} f_0(v) \int (|\overline{v} - v|^3 - |\overline{v}|^3) f(\overline{v}) d\overline{v} + \frac{1}{\sigma} f_0(v) \int_0^v \frac{h(w)}{f_0(w)} dw.
\end{aligned} \tag{3.15}$$

Integrating with respect to v, the condition $\int f dv = 0$ implies

$$\rho\frac{f(0)}{f_0(0)} - \frac{1}{3\sigma}\int (|v - \overline{v}|^3 - |\overline{v}|^3) f(\overline{v}) f_0(v) dv\, d\overline{v} + \frac{1}{\sigma}\int f_0(v) \int_0^v \frac{h}{f_0}(w) dw dv = 0. \tag{3.16}$$

Therefore we can recover $f(0)$ as a functional of f:

$$
\begin{aligned}
f(v) = &- f_0(v)\frac{1}{3\sigma}\int |v-\overline{v}|^3 f(\overline{v})d\overline{v} + f_0(v)\frac{1}{3\sigma\rho}\int |w-\overline{v}|^3 f(\overline{v}) f_0(w)dwd\overline{v} \\
&+ \frac{1}{\sigma} f_0(v)H(g)(v) - \frac{1}{\sigma\rho} f_0(v)\int f_0(w)H(g)(w)dw\,,
\end{aligned}
\tag{3.17}
$$

where

$$
H(g)(v) = \int_0^v \frac{h}{f_0}(w)dw = \int_0^v \frac{dw}{f_0(w)}\int_{-\infty}^v g(\overline{v})d\overline{v}\,. \tag{3.18}
$$

Setting:

$$
f(v) = \tilde{f}(v) f_0^{\frac{1}{2}}(v), \tag{3.19}
$$

(so that $\int \tilde{f} f_0^{\frac{1}{2}} dv = 0$), the equation above becomes

$$
\begin{aligned}
\tilde{f}(v) = &- f_0^{\frac{1}{2}}(v)\frac{1}{3\sigma}\int |v-\overline{v}|^3 f_0^{\frac{1}{2}}(\overline{v})\tilde{f}(\overline{v})d\overline{v} + \\
&+ f_0^{\frac{1}{2}}(v)\frac{1}{3\sigma\rho}\int d\overline{v}\,|w-\overline{v}|^3 f_0^{\frac{1}{2}}(\overline{v})\tilde{f}(\overline{v}) f_0(w)dw + S(v),
\end{aligned}
\tag{3.20}
$$

where $S(v)$ is known, and, as follows from the $\exp(-|v|^3)$ decay of f_0 (see Eq. (2.19)), it is in $L^2(\mathbb{R})$.

Eq. (3.20) is of the form

$$
\tilde{f} - K\tilde{f} = S. \tag{3.21}
$$

We study this equation in the Hilbert space $\tilde{\mathbf{V}}_0 = \{\tilde{f} \in L^2(\mathbb{R})\,|\,\tilde{f} \perp f_0^{\frac{1}{2}}\}$. Note that K is self-adjoint in $\tilde{\mathbf{V}}_0$, and is the sum of a compact and a finite rank operator. Moreover $S \in \tilde{\mathbf{V}}_0$.

By the Fredholm alternative Eq. (3.21) has solution if $(I-K)\tilde{f} = 0$ is equivalent to $\tilde{f} = 0$ (K is self-adjoint).

Indeed

$$
\begin{aligned}
\int \tilde{f}(I-K)\tilde{f} = &\int \frac{f^2}{f_0} + \frac{1}{3\sigma}\int |v-\overline{v}|^3 f(v) f(\overline{v})dv\,d\overline{v} + \\
&+ \frac{1}{3\sigma\rho}\left(\int f(v)dv\right)\int |w-\overline{v}|^3 f(w) f_0(\overline{v})dw\,d\overline{v}\,,
\end{aligned}
\tag{3.22}
$$

and hence:

$$
\int \tilde{f}(I-K)\tilde{f} = \int \frac{f^2}{f_0} + \frac{1}{3\sigma}\int f(v)f(\overline{v})|v-\overline{v}|^3 dv\,d\overline{v} = \frac{2}{\sigma}\theta_{\beta,\sigma}(f,f), \tag{3.23}
$$

where $\theta_{\beta,\sigma}$ is the functional defined in (3.4). The term

$$
\int f(v)f(\overline{v})|v-\overline{v}|^3 dv\,d\overline{v}
$$

is positive on the space of the functions f with

$$
\int f(v)dv = \int v f(v)dv = 0\,,
$$

as can be seen by passing to Fourier transforms. Therefore $\theta_{\beta,\sigma}$ restricted to that space is a scalar product. Moreover if $\beta > 0$, $\theta_{\beta,\sigma}$ is also a scalar product in the space of the functions f with $\int f dv = 0$.

Namely, indicating with $c = \int v f(v) dv$, a direct computation leads to

$$\theta_{\beta,\sigma}(f,f) = \theta_{\beta,\sigma}\left((f - \frac{c}{\rho}\partial_v f_0), (f - \frac{c}{\rho}\partial_v f_0)\right) + \frac{\beta c^2}{2\rho}. \tag{3.24}$$

The function $f - \frac{c}{\rho}\partial_v f_0$ has zero mean and zero first momentum, then $\theta_{\beta,\sigma}(f,f)$ is 0, on the zero-mean functions, only if $f = 0$.

The above analysis proves the main part of the following proposition.

Proposition 1. *Let* $\mathbf{V} = \{f | \int \frac{f^2}{f_0} dv < +\infty\}$, $\mathbf{V}_0 = \{f \in \mathbf{V} | \int f = 0\}$, $\mathbf{V}_{00} = \{f \in \mathbf{V_0} | \int v f(v) dv = 0\}$, *endowed with the scalar product*

$$(f,g) \mapsto \int \frac{fg}{f_0} dv\,.$$

If $\beta > 0$ *the equation*

$$L(f) = g \tag{3.25}$$

is uniquely solvable in $\mathbf{V}_0$, *and*

$$Ker(L) = \{c\alpha | \, c \in \mathbb{R}\}, \tag{3.26}$$

where α *is as in Eq. (3.7) If* $\beta = 0$, *Eq. (3.25) is uniquely solvable in* $\mathbf{V}_{00}$, *and*

$$Ker(L) = \{c_1\alpha + c_2\partial_v f_0 | \, c_1, c_2 \in \mathbb{R}\}. \tag{3.27}$$

The assertion on the kernel of L follows from the fact that if $\beta > 0$, L maps a dense subset of $\mathbf{V}_0$ in itself (see i) of Lemma 2) and that if $\beta = 0$, L maps a dense subset of $\mathbf{V}_{00}$ in itself (see ii) of Lemma 2).

We conclude this section describing some other useful formal properties of the functional L.

Proposition 2. *For* $\beta > 0$

i) *the free energy functional* $\theta_{\beta,\sigma}(f,g)$ *is a scalar product in* $\mathbf{V}_0$ *equivalent to* $(f,g) \mapsto \int \frac{fg}{f_0} dv$;

ii) *for all regular functions* f, $g \in \mathbf{V}_0$,

$$\theta_{\beta,\sigma}(L(f), g) = \theta_{\beta,\sigma}(f, L(g)),$$

i.e. L *is formally self-adjoint in the norm defined from* $\theta_{\beta,\sigma}$;

iii) *for any* $g \in \mathbf{V}_0$

$$\int v g(v) dv = -\frac{2}{\beta}\theta_{\beta,\sigma}(\partial_v f_0, g);$$

iv) *for any* $g \in \mathbf{V}_0$

$$\int v L^{-1}(g)(v) dv = -\frac{1}{\beta}\int v g(v) dv\,.$$

Proof. Statement i) is an easy consequence of the Cauchy-Schwartz inequality and of Eq. (3.24). Points ii) and iii) follow by direct computation, using the expression (3.3) and Eq. (2.13) respectively. Point iv) follows from iii), i) and from Eq. (3.8). ◻

4. The Hilbert Expansion

4.1. Hydrodynamic scaling. The Hilbert expansion

$$f(t,x,v) = f_0(t,x,v) + \varepsilon f_1(t,x,v) + \varepsilon^2 f_2(t,x,v)\dots \tag{4.1}$$

for the kinetic Fokker-Plank equation under the hydrodynamic scaling (2.8), yields:

$$O(\varepsilon^{-1}): \quad f_0(t,x,v) = \rho(t,x)^{\frac{4}{3}} G_{0,\sigma}(\rho(t,x)(v-u(t,x))), \tag{4.2}$$

as follows from Corollary 2, where ρ and u are unknown, and

$$\partial_t f_k + v\partial_x f_k = L(f_{k+1}) + \sum_{h=1}^{k} Q(f_{k+1-h}, f_h), \tag{4.3}$$

for $k \geq 0$. Integrating Eq. (4.3) in dv and $v\,dv$

$$\partial_t \int f_k(v)dv + \partial_x \left(u \int f_k(v)dv + \int (v-u) f_k(v)dv \right) = 0\,, \tag{4.4}$$

$$\begin{aligned} &\partial_t \left(u \int f_k(v)dv + \int (v-u) f_k(v)dv \right) + \\ &+ \partial_x \left(u^2 \int f_k(v)dv + 2u \int (v-u) f_k(v)dv + \int (v-u)^2 f_k(v)dv \right) = 0\,. \end{aligned} \tag{4.5}$$

For $k=0$, Eqs. (4.4), (4.5) are the closed hydrodynamic system described in [2], see Eq. (1.5).

We will show that we can solve Eq. (4.3) and Eqs. (4.4), (4.5), for each k. We start with f_1. From Eq. (4.3)

$$f_1 = L^{-1}(\partial_t f_0 + v\partial_x f_0) + \mu_1 \alpha(t,x,v) + \frac{\nu_1}{\rho}\partial_v f_0, \tag{4.6}$$

where $\mu_1 = \mu_1(t,x)$ and $\nu_1 = \nu_1(t,x)$ are unknown coefficients of the two independent elements of the kernel of L (see Eq. (3.27)). Taking into account that

$$\begin{aligned} &\int \alpha(v)dv = 1 && \frac{1}{\rho}\int \partial_v f_0(v)dv = 0 \\ &\int (v-u)\alpha(v)dv = 0 && \frac{1}{\rho}\int (v-u)\partial_v f_0(v)dv = -1 \\ &\int (v-u)^2 \alpha(v)dv = -\tfrac{1}{3}k\rho^{-\frac{2}{3}} && \frac{1}{\rho}\int (v-u)^2 \partial_v f_0(v)dv = 0, \end{aligned} \tag{4.7}$$

(as follows from Eqs. (4.2) and from the definition of $\alpha = \partial_\rho f_0$), we obtain, from Eqs. (4.4)-(4.5), for $k=1$, the following linear hyperbolic system for μ_1 and ν_1:

$$\begin{aligned} &\partial_t \mu_1 + \partial_x (u\mu_1 - \nu_1) = 0 \\ &\partial_t (u\mu_1 - \nu_1) + \partial_x (u(u\mu_1 - 2\nu_1) + k\rho^{\frac{1}{3}}\mu_1) = s_1, \end{aligned} \tag{4.8}$$

where s_1 is known:

$$s_1 = -\partial_x \int (v-u)^2 L^{-1}(\partial_t f_0 + v\partial_x f_0)dv\,. \tag{4.9}$$

We can iterate this procedure, to find f_k. First, from Eq. (4.3), we find the component with zero mean and zero momentum of f_k and then, from Eqs. (4.4) and (4.5) we recover the zero mean and the zero momentum components. We obtain the following linear hyperbolic non-homogeneous systems:

$$\begin{aligned} &\partial_t \mu_k + \partial_x(u\mu_k - \nu_k) = 0 \\ &\partial_t(u\mu_k - \nu_k) + \partial_x(u(u\mu_k - 2\nu_k) + k\rho^{\frac{1}{3}}\mu_k) = s_k, \end{aligned} \tag{4.10}$$

where

$$s_k = -\partial_x \int (v-u)^2 L^{-1}\left(\partial_t f_{k-1} + v\partial_x f_{k-1} - \sum_{h=1}^{k-1} Q(f_{k-h}, f_h)\right) dv\,,$$

is a known source term.

When $\beta > 0$ (see Eq. (2.9)) the analysis can be carried out along the same lines.

4.2. Diffusive scaling. The Hilbert expansion

$$f(t,x,v) = f_0(t,x,v) + \varepsilon f_1(t,x,v) + \varepsilon^2 f_2(t,x,v)\ldots \tag{4.11}$$

for the model in the diffusive scaling (2.10), yields:

$$O(\varepsilon^{-2}): \quad f_0(t,x,v) = \rho(t,x) G_{\beta/\rho(t,x),\sigma/\rho(t,x)}(v), \tag{4.12}$$

where ρ is unknown;

$$O(\varepsilon^{-1}): \quad v\partial_x f_0(t,x,v) = L(f_1)(t,x,v); \tag{4.13}$$

$$O(\varepsilon^k): \quad \partial_t f_k + v\partial_x f_{k+1} = L(f_{k+2}) + \sum_{h=1}^{k+1} Q(f_{k+2-h}, f_h), \tag{4.14}$$

for $k \geq 0$.

Integrating Eq. (4.14) with respect to dv and $v\,dv$

$$\partial_t \int f_k dv + \partial_x \int v f_{k+1} dv = 0, \tag{4.15}$$

$$\partial_t \int v f_k dv + \partial_x \int v^2 f_{k+1} dv = -\beta \int v f_{k+2} dv\,. \tag{4.16}$$

Integrating with respect to $v\,dv$ Eq. (4.13)

$$\partial_x \int v^2 f_0 dv = -\beta \int v f_1 dv\,. \tag{4.17}$$

Eqs. (4.15) for $k = 0$, (4.17) and (4.12) give the closed non-linear diffusion equation described in (2.25):

$$\partial_t \rho(t,x) = \frac{1}{\beta}\partial^2_{xx} \int v^2 f_0(t,x,v) dv \, = \frac{1}{\beta}\partial^2_{xx}(\Phi(\rho(t,x))\rho(t,x)), \tag{4.18}$$

where $\Phi(\rho) = \int v^2 G_{\beta/\rho,\sigma/\rho}(v) dv$. Since Eq. (4.18) has a unique smooth solution, we next determine each term of the Hilbert expansion. The method consists in solving repeatedly (4.14) for f_k in terms of $f_0, \ldots f_{k-1}$.

If there exists a solution f_1 to Eq. (4.13), then $\int v\partial_x f_0 dv = 0$. Pick f_0 to be even in v. Then the above condition holds and

$$f_1 = L^{-1}(v\partial_x f_0) + \mu_1\alpha, \tag{4.19}$$

where μ_1 is unknown. For $k > 1$, if the necessary condition of zero mean expressed in Eq. (4.15) is satisfied

$$f_k = L^{-1}(r_k) + \mu_k\alpha, \tag{4.20}$$

where

$$r_k = \partial_t f_{k-2} + v\partial_x f_{k-1} - \sum_{h=1}^{k-1} Q(f_{k-h}, f_h), \tag{4.21}$$

and $\mu_k = \mu_k(t,x)$ is unknown.

Using Eq. (4.15) and Eq. (4.16) for $k-1$, and Eq. (4.17) we have

$$\partial_t \int f_1(t,x,v)dv = \frac{1}{\beta}\partial^2_{xx} \int v^2 f_1(t,x,v)dv\,, \tag{4.22}$$

for $k = 1$, and

$$\partial_t \int f_k(t,x,v)dv = \frac{1}{\beta}\partial^2_{xx} \int v^2 f_k(t,x,v)dv + \frac{1}{\beta}\partial^2_{tx} \int v f_{k-1}(t,x,v)dv\,, \tag{4.23}$$

for $k > 1$.

Inserting (4.19) in (4.22) we obtain the equation for μ_1:

$$\partial_t\mu_1(t,x) = \frac{1}{\beta}\partial^2_{xx}(\Psi(\rho(t,x))\mu_1(t,x)), \tag{4.24}$$

where

$$\Psi(\rho) = \Phi(\rho) + \rho\partial_\rho\Phi(\rho), \tag{4.25}$$

is a function depending only on $\rho(t,x)$.

Inserting (4.20) in (4.23), we obtain:

$$\partial_t\mu_k(t,x) = \frac{1}{\beta}\partial^2_{xx}(\Psi(\rho(t,x))\mu_k(t,x)) + s_k(t,x), \tag{4.26}$$

where

$$s_k = \frac{1}{\beta}\partial^2_{xx} \int v^2 L^{-1}(r_k)(v)dv \,+ \frac{1}{\beta}\partial^2_{xt} \int vL^{-1}(r_{k-1})(v)dv\,. \tag{4.27}$$

At this point we have only to prove that the zero mean condition (4.15) (which is not in principle equivalent to Eq. (4.26)) is satisfied. This fact is consequence of the point iv) of Proposition 3:

$$\begin{aligned}&\partial_x \int v f_{k+1} dv = \partial_x \int vL^{-1}(r_{k+1})dv = -\frac{1}{\beta}\partial_x \int v r_{k+1} dv = \\ &= -\frac{1}{\beta}\partial_x \int v(\partial_t f_{k-1} + v\partial_x f_k)dv = -\frac{1}{\beta}\partial^2_{xx}\int v^2 f_k dv - \frac{1}{\beta}\partial^2_{tx}\int v f_{k-1} dv\,.\end{aligned} \tag{4.28}$$

Then Eqs. (4.15) and Eqs. (4.26) are equivalent.

5. The Chapman-Enskog expansion

In this section we calculate the first order correction to the hydrodynamic system (2.21) (or, in the same way to the system (1.5)). According to the procedure for the Chapman-Enskog expansion, we set

$$f_0 = \rho^{\frac{4}{3}} G_{0,\sigma}(\rho^{\frac{1}{3}}(v-u)), \tag{5.1}$$

where $\rho = \rho(t,x) = \int f(t,x,v)dv$, $\rho(t,x)u(t,x) = \int vf(t,x,v)dv$, and f solves (2.9). Expanding as above $f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 \ldots$, at the first order in ε, the relaxed hydrodynamic system is

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0 \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + k\rho^{\frac{1}{3}}) + \varepsilon\partial_x \int (v-u)^2 f_1 dv = -\beta\rho u. \end{aligned} \tag{5.2}$$

The equation for f_1 is

$$L(f_1) = \partial_t f_0 + v\partial_x f_0 - \beta\partial_v(vf_0). \tag{5.3}$$

All the functions f_k have vanishing mean and momentum, thus we can find f_1 as $L^{-1}(\partial_t f_0 + v\partial_x f_0 - \beta\partial_v(vf_0))$, because from Eq. (5.2) the r.h.s. of Eq. (5.3) also has zero mean and momentum at order ε^0.

We rewrite the r.h.s. of Eq. (5.3) in terms of the hydrodynamic variables

$$\begin{aligned} &\partial_t f_0 + v\partial_x f_0 - \beta\partial_v(vf_0) = \\ &= \partial_\rho f_0(\partial_t + v\partial_x)\rho + \partial_u f_0(\partial_t + v\partial_x)u - \beta(f_0 + v\partial_v f_0) = \\ &= \partial_\rho f_0\left((v-u)\partial_x\rho - \rho\partial_x u\right) + \\ &+ \partial_u f_0\left((v-u)\partial_x u - \frac{k}{\rho}\partial_x\rho^{\frac{1}{3}} - \beta u\right) + \\ &- \beta(f_0 + v\partial_v f_0) + O(\varepsilon) \end{aligned} \tag{5.4}$$

(we have used the system (5.2)). In order to compute $\int (v-u)^2 f_1 dv$, we are interested only in contributions to f_1 that are even in the variable $v-u$. The even part in the previous formula is

$$\partial_x u(-\rho\partial_\rho f_0 + (v-u)\partial_u f_0)) - \beta(f_0 + (v-u)\partial_v f_0). \tag{5.5}$$

It is useful to rewrite all the relevant quantities in terms of $G_{0,\sigma}$ and of the variable $\xi = \rho^{\frac{1}{3}}(v-u)$.

$$\begin{aligned} &\partial_\rho f_0(v) = \rho^{\frac{1}{3}}\left(\frac{4}{3}G_{0,\sigma}(\xi) + \frac{1}{3}\xi G'_{0,\sigma}(\xi)\right), \\ &\partial_u f_0(v) = -\rho^{\frac{5}{3}} G'_{0,\sigma}(\xi), \\ &\partial_v f_0(v) = -\partial_u f_0(v) = \rho^{\frac{5}{3}} G'_{0,\sigma}(\xi), \\ &L(f)(v) = \rho^{\frac{2}{3}} \overline{L}(\overline{f})(\xi), \end{aligned} \tag{5.6}$$

where $f(v) = \overline{f}(\xi)$ and

$$\overline{L}(\overline{f}) = \partial_\xi\left(G_{0,\sigma}\partial_\xi\left(\sigma\frac{\overline{f}}{G_{0,\sigma}} + \frac{1}{3}\int |\xi-\xi'|^3\overline{f}(\xi')d\xi'\right)\right). \tag{5.7}$$

The terms in Eq. (5.5) become

$$\begin{aligned}
&\partial_x u\left(-\rho^{\frac{4}{3}}\left(\frac{4}{3}G_{0,\sigma}+\frac{1}{3}\xi G'_{0,\sigma}\right)+\rho^{-\frac{1}{3}}\xi\left(-\rho^{\frac{5}{3}}G'_{0,\sigma}\right)\right)+\\
&-\beta\left(\rho^{\frac{4}{3}}G_{0,\sigma}+\rho^{-\frac{1}{3}}\xi\left(\rho^{\frac{5}{3}}G'_{0,\sigma}\right)\right)= \qquad (5.8)\\
&=-\frac{4}{3}\partial_x u\rho^{\frac{4}{3}}(\xi G_{0,\sigma})'-\beta\rho^{\frac{4}{3}}(\xi G_{0,\sigma})'=-\left(\frac{4}{3}\partial_x u+\beta\right)\rho^{\frac{4}{3}}(\xi G_{0,\sigma})'.
\end{aligned}$$

Finally, we can compute the order ε^0 part of $\int(v-u)^2 f_1 dv$:

$$\begin{aligned}
&\int(v-u)^2L^{-1}(\partial_x u(-\rho\partial_\rho f_0+(v-u)\partial_u f_0)-\beta(f_0+(v-u)\partial_v f_0))dv=\\
&=-\left(\frac{4}{3}\partial_x u+\beta\right)\rho^{-1}\rho^{\frac{4}{3}}\rho^{-\frac{2}{3}}\int\xi^2\overline{L}^{-1}\left((\xi G_{0,\sigma})'\right)(\xi)d\xi\ = \qquad (5.9)\\
&=-\mu\left(\partial_x u+\frac{3}{4}\beta\right)\rho^{-\frac{1}{3}},
\end{aligned}$$

where $\mu=\frac{4}{3}\int\xi^2\overline{L}^{-1}\left(\xi G_{0,\sigma}\right)'(\xi)d\xi$ is a scalar.

Inserting (5.9) in Eq. (5.2), we obtain

$$\begin{cases}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t(\rho u)+\partial_x(\rho u^2+k\rho^{\frac{1}{3}})=-\beta\rho u+\varepsilon\partial_x(\mu\rho^{-\frac{1}{3}}\partial_x u)+\dfrac{3}{4}\varepsilon\beta\mu\partial_x\rho^{-\frac{1}{3}}.\end{cases} \tag{5.10}$$

In order to obtain the expression (2.24) for the viscosity coefficient μ, we rewrite the equilibrium equation (3.1) solved by $G_{\beta,\sigma}$, in the variable ξ:

$$-\partial_\xi\left(G_{\beta,\sigma}(\xi)\int|\xi'-\xi|(\xi'-\xi)G_{\beta,\sigma}-\beta\xi G_{\beta,\sigma}(\xi)-\sigma\partial_\xi G_{\beta,\sigma}\right)d\xi'=0. \tag{5.11}$$

By deriving the above identity with respect to β, we get, for $\beta=0$:

$$\overline{L}(\partial_\beta G_{0,\sigma})+(\xi G_{0,\sigma})'(\xi)=0, \tag{5.12}$$

then

$$\mu=-\frac{4}{3}\partial_\beta\bigg|_{\beta=0}\int\xi^2G_{\beta,\sigma}(\xi)d\xi\,. \tag{5.13}$$

We conclude by showing that μ is positive. Let $\gamma\in\mathbf{V}_{00}$ be such that

$$\overline{L}(\gamma)=(\xi G)'. \tag{5.14}$$

Using the expression of $\overline{L}$ in (5.7), integrating in ξ, we obtain

$$G_{0,\sigma}\partial_\xi\left(\sigma\frac{\gamma}{G_{0,\sigma}}+\frac{1}{3}\int|\xi-\xi'|^3\gamma(\xi')d\xi'\right)=\xi G_{0,\sigma} \tag{5.15}$$

(the constant of integration is 0 since γ is even). Dividing by $G_{0,\sigma}$ and integrating again, we find that γ satisfies

$$\sigma\frac{\gamma}{G}+\frac{1}{3}\int|\xi-\overline{\xi}|^3\gamma(\overline{\xi})d\overline{\xi}=\frac{\xi^2}{2}+c, \tag{5.16}$$

for some constant c. Multiplying for γ and integrating in ξ, and using the definition (3.4) of the scalar product $\theta_{0,\sigma}$ we have

$$\frac{1}{2}\int \xi^2\gamma(\xi)d\xi = 2\theta_{0,\sigma}(\gamma,\gamma). \tag{5.17}$$

Then

$$\mu = \frac{4}{3}\int \xi^2\overline{L}^{-1}\left(\xi G_{0,\sigma}\right)'d\xi = \frac{4}{3}\int \xi^2\gamma(\xi)d\xi = \frac{16}{3}\theta_{0,\sigma}(\gamma,\gamma) > 0. \tag{5.18}$$

REFERENCES

[1] D. Benedetto, E. Caglioti, J. Carrillo, and M. Pulvirenti, *A non-maxwellian steady distribution for one-dimensional granular media*, J. of Stat. Phys., 91:979–990, 1999.

[2] D. Benedetto, E. Caglioti, F. Golse, and M. Pulvirenti, *A hydrodynamic model arising in the context of granular media*, Comput. Math. Appl., 38:121–131, 1999.

[3] D. Benedetto, E. Caglioti, and M. Pulvirenti, *A kinetic equation for granular media*, RAIRO Modél. Math. Anal. Numér., 31:615–641, 1997.

[4] D. Benedetto, E. Caglioti, and M. Pulvirenti, *Erratum: A kinetic equation for granular media*, Math. Mod. Num. Anal., 33:439–441, 1999.

[5] E. Caglioti and C. Villani, *Homogeneous cooling states are not always good approximations to granular flows*, Arch. Ration. Mech. Anal., 163:329–343, 2002.

[6] J. Carrillo, R.J. McCann, and C. Villani, *Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates*, Revista Mat. Iberoamericana, 19:971–1018, 2003.

[7] G.Q. Chen, C.D. Levermore, and T.P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math., 47:787–830, 1994.

[8] Y. Du, H. Li, and L.P. Kadanoff, *Breakdown of hydrodynamics in one-dimensional system of inelastic particles*, Phys. Rev. Lett., 74:1268–1271, 1995.

[9] J.W. Dufty, *Kinetic theory and hydrodynamics for rapid granular flows - A perspective*, preprint, 2001.

[10] P.L. Lions, *Mathematical Topics in Fluid Mechanics Vol. 2: Compressible Models*, Clarendon Press, Oxford, 1998.

[11] H. Li, and G. Toscani, *Long time asymptotics of kinetic models of granular flows*, Arch. Ration. Mech. Anal., to appear.

[12] R.J. McCann, *A convexity principle for interacting gases*, Adv. Math., 128:153–179., 1997.

[13] S. McNamara, W.R. Young, *Kinetic of a 1-dimensional medium in the quasi-elastic limit*, Phys. of Fluids A, 7:4–45, 1995.

[14] G. Toscani, *One-dimensional kinetic models of granular flows*, Math. Mod. Numer. Anal., 34:1277–1291, 2000.