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Nonlinear Stability of Solitary Waves of a Generalized Kadomtsev–Petviashvili Equation

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Abstract: We prove that the set of solitary wave solutions of a generalized Kadomtsev–Petviashvili equation in two dimensions,

$$(u_t + (u^{m+1})_x + u_{xxx})_x = u_{yy}$$

is stable for $0 < m < 4/3$.

1. Introduction

The generalized Kadomtsev–Petviashvili (GKP) equation

$$(u_t + (u^{m+1})_x + u_{xxx})_x = \sigma^2 u_{yy} \tag{1}$$

is a two dimensional analog of the generalized Korteweg–de Vries (GKdV) equation. When $m = 1$ and $\sigma^2 = 1$, (1) is known as the KPI equation while $m = 1$ and $\sigma^2 = -1$ corresponds to the KPII equation. Both are universal models for the propagation of weakly nonlinear dispersive long waves which are essentially one directional, with weak transverse effects [6]. It also describes the evolution of sound waves in antiferromagnetics [9]. It is well known that both KPI and KPII can be solved by the Inverse Scattering Transformation (IST) [1, 2].

Many local existence results for KP and GKP have recently appeared for both infinite space and periodic boundary conditions (see [19, 20, 13]). There are also some global results [22]. It is shown in [9, 20] by a virial method that GKP-I

$$(u_t + (u^{m+1})_x + u_{xxx})_x = u_{yy} \tag{2}$$

has blow-up solutions for $m \geqq 4$ while arguments in [14] indicate that blow up should occur for much lower m, namely $m > \frac{4}{3}$.

An important question is the stability and instability of solitary waves for GKP, that is, solutions of form $u(x, y, t) = \varphi(x - ct, y)$. Existence of solitary waves is shown in [14] for $1 < m < 4$ and also in [21] by a different method. For GKP-I, instability is shown in [14] for $\frac{4}{3} < m < 4$ using a method of Shatah and Strauss [3] and a completed proof is provided by de Bouard and Saut [24]. In this paper, we shall prove that the solitary waves are nonlinearly stable for $0 < m < \frac{4}{3}$. After this paper was completed, we learned that de Bouard and Saut [24] have a similar result by a different method.

The paper is organized as follows. In Sect. 2, we give the detailed proof of the existence of solitary waves for GKP-1 with $0 < m < 4$. The solutions are obtained by using the variational method and the techniques developed by Lieb [18] to solve a constrained minimization problem. In Sect. 3, we prove the set of solitary waves of KP-I is nonlinearly stable for $0 < m < \frac{4}{3}$. The proof is based on the idea of Shatah [4] and Levandosky [5]. We use the variational properties of the minimizer and a convexity lemma of Shatah to estabilish the key inequality for stability theorem.

We shall use the following notations. $|\cdot|_p$ (resp. $\|\cdot\|_s$) will stand for the norm in the Lebesque space $L^p(\mathbf{R}^2)$ (resp. the Sobolev space $H^s(\mathbf{R}^2)$) .

Because of the structure of Eq. (1), we introduce the following function space:

$$V(\Omega) = \{u | u \in L^2(\Omega), u_x \in L^2(\Omega), D_x^{-1}u_y \in L^2(\Omega)\} \tag{3}$$

equipped with a norm

$$|u|_V = \left(\int_\Omega (u^2 + |\tilde{\nabla} u|^2)\, dxdy \right)^{\frac{1}{2}} ,$$

and an inner product

$$\langle u, v \rangle = \int_\Omega (uv + \tilde{\nabla} u \cdot \tilde{\nabla} v)\, dxdy .$$

Here Ω may be $\mathbf{R}^2$ or a box $[a,b] \times [c,d]$ in $\mathbf{R}^2$, and $\tilde{\nabla} u = (u_x, D_x^{-1}u_y)^T$, $D_x^{-1} = \int_{-\infty}^{x}$ (or $\int_{-a}^{x}$).

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2. Existence of Solitary Waves

In this section, we give a more detailed proof of the existence of localized solitary waves for GKP-I given in [14]. By solitary waves, we mean traveling wave solutions of the form $u(x, y, t) = \varphi_\omega(x - \omega t, y)$. Substituting in (1), φ_ω satisfies the following equation:

$$\omega\varphi + D_x^{-2}\varphi_{yy} - \varphi_{xx} = \varphi^{m+1} . \tag{4}$$

For $m = 1$, this equation has an explicit solution (lump soliton [1])

$$v(x, y) = 8\frac{\omega - x^2/3 + y^2/(3\omega)}{(\omega + x^2/3 + y^2/(3\omega))^2} . \tag{5}$$

We shall prove the existence of decaying solutions for $0 < m < 4$ in space $V = V(\mathbf{R}^2)$ by a variational method. We first introduce the following functionals on $V(\mathbf{R}^2)$,

$$Q(u) = \frac{1}{2} \int_{\mathbf{R}^2} u^2\, dxdy , \tag{6}$$

$$E(u) = \int_{\mathbf{R}^2} \left(\frac{1}{2}((D_x^{-1}u_y)^2 + u_x^2) - \frac{1}{(m+2)} u^{m+2} \right) dxdy , \tag{7}$$

$$K(u) = \int_{\mathbf{R}^2} u^{m+2}\, dxdy\,, \tag{8}$$

$$I_\omega(u) = \int_{\mathbf{R}^2} (\omega u^2 + |\tilde{\nabla} u|^2)\, dxdy\,. \tag{9}$$

Remark 1. Here and in the following, we always assume that $m = m_1/m_2$, where m_1 is any even integer and m_2 any odd integer. This guarantees that $K(u)$ is non-negative.

Let's also define the following minimization problem:

$$M(\omega) = \inf_{u \in V(\mathbf{R}^2)} \frac{I_\omega(u)}{(K(u))^{\frac{2}{m+2}}}\,.$$

It is easy to see that

$$\frac{I_\omega(\lambda u)}{(K(\lambda u))^{\frac{2}{m+2}}} = \frac{I_\omega(u)}{(K(u))^{\frac{2}{m+2}}} \quad \text{for } \lambda \neq 0\,.$$

Similarly, we have

$$M(\omega) = \inf_{u \in V} \{I_\omega(u) \,|\, K(u) = 1\}\,.$$

By change of variable $u(x,y) = w^{\frac{1}{m}} V(\sqrt{w}x, wy)$, one easily obtains that

$$M(w) = w^{\frac{4-m}{2(m+2)}} M(1) \quad w > 0\,. \tag{10}$$

Note that Eq. (4) is the Euler–Lagrange equation of the functional

$$L_\omega(u) = \int \left(\frac{\omega}{2} u^2 + \frac{1}{2} |\tilde{\nabla} u|^2 - \frac{1}{(m+1)(m+2)} u^{m+2} \right) dxdy\,. \tag{11}$$

Therefore, if there is a function $\varphi \in V(\mathbf{R}^2)$, such that

$$I_\omega(\varphi) = M(\omega) \quad \text{with } K(\varphi) = 1\,,$$

then φ is a solution of

$$\omega\varphi - \varphi_{xx} + D_x^{-2}\varphi_{yy} = \lambda\varphi^{m+1}\,,$$

where λ is the Lagrange multiplier. Hence $\psi = \lambda^{\frac{1}{m}}\varphi$ is the solution of (4). We call such a solution a ground state.

Theorem 1 (*Existence of solitary waves*). *Let* $0 < m < 4$, $\omega > 0$ *and* $m = m_1/m_2$, *where* m_1 *it is any even integer and* m_2 *any odd integer. Then there exists a minimizer* $\varphi_w \in N$ *such that*

$$I_\omega(\varphi_\omega) = \inf_{u \in N} I_\omega(u) = M(\omega)\,,$$

where $N = \{u | u \in V(\mathbf{R}^2),\ K(u) = 1\}$ *and* $K(u)$ *given by* (8).

To prove this theorem, we use the techniques used in [15, 18, 23 and 17]. The following lemmas are needed for the proof of Theorem 1. We shall prove the lemmas later.

Lemma 1. *Let $V(\Omega)$ be the space defined in* (3) *and Ω may be $\mathbf{R}^2$ or a box in $\mathbf{R}^2$. Then for any $2 < n < 6$, there exists a constant C, such that for any $u \in V(\Omega)$,*

$$\left(\int_\Omega |u|^n\right)^{\frac{1}{n}} \leqq C\left(\int_\Omega |\tilde{\nabla} u|^2 + u^2\right)^{\frac{1}{2}} . \tag{12}$$

Lemma 2. *Let $u \in V(B)$, where $B \subset \mathbf{R}^2$ is an arbitary box. Then $\forall \varepsilon > 0$, there exist integers $N_\varepsilon, M_\varepsilon$, s.t.*

$$\int_B u^2 \leqq \sum_{n_1=1}^{M_\varepsilon} \sum_{n_2=1}^{N_\varepsilon} \left(\int_B u w_{n_1,n_2}\right) + \varepsilon \int_B |\tilde{\nabla} u|^2 ,$$

where w_{n_1,n_2} are orthonomal basis functions in $V(B)$.

Lemma 3. *Let $\{u^j\}$ denote a minimizing sequence of $I_1 = I_{\omega=1}$. That is, $\lim_{j\to\infty} I_1(u^j) = \inf_{u\in N} I_1(u)$. Then there exist $\varepsilon, \delta > 0$ such that for all j,*

$$\mu([|u^j| > \varepsilon]) \geqq \delta,$$

where $\mu(\cdot)$ denotes the Lebesgue measure.

Lemma 4. *Let u be a function such that $|u|_V \leqq C$ and $\mu([|u| > \varepsilon]) \geqq \delta > 0$. Then there exists a shift $T_{s,t}u(x,y) = u(x+s, y+t)$, such that for some constant $\alpha = \alpha(C,\delta,\varepsilon) > 0$,*

$$\mu(B \cap [|T_{s,t}u| > \varepsilon/2]) > \alpha ,$$

where B is the unit box (i.e. box centered at origin and has length 1×1) in $\mathbf{R}^2$.

Lemma 2 is used in [11]. Lemmas 3 and 4 are similar to the results in [15 and 18].

Proof of Theorem 1. By (10), it suffices to show that there exists a minimizer $u_0 \in N$ such that $I_1(u_0) = M(1)$. We denote $I(u) = I_1(u)$ and $I^0 = \inf_{u\in N} I(u)$. Let $\{u_j\}$ be a minimizing sequence, i.e $I(u_j) \to I^0$ with $I(u_j) \leqq C$, $|u_j|_{m+2} = 1$. We then have $u_j \rightharpoonup u_0$ weakly in V, $u_j \rightharpoonup u_0$ weakly in L^{m+2}. It follows from Lemma 2, $u_j \to u_0$ strongly in L^2 on any bounded domain. And $u_j \to u_0$ a.e. in $\mathbf{R}^2$.

We next show $u_j \to u_0$ strongly in L^{m+2}. To do that it suffices to prove $|u_0|_{L^{m+2}} = 1$; i.e. that $u_0 \in N$. Since $u_j \rightharpoonup u_0$ weakly in L^{m+2}, this implies $|u_0|_{L^{m+2}} \leqq 1$. Next, we want to show that $|u_0|_{L^{m+2}} \neq 0$, i.e. $u_0 \not\equiv 0$. From Lemmas 3 and 4, there exists α, such that

$$\mu(B \cap [|T_{s_j,t_j}u_j| > \varepsilon]) > \alpha .$$

Let $T_{s_j,t_j}u_j$ be the new sequence denoted also by u_j, we have

$$\mu(B \cap [|u_j| > \varepsilon]) > \alpha .$$

Since $u_j \to u_0$ a.e., it follows that $\mu(B \cap [|u_0| > \frac{\varepsilon}{2}]) > \alpha$, therefore $u_0 \not\equiv 0$ and $\|u_0\|_{L^{m+2}} \neq 0$.

Next, we show that if $0 < \lambda = \int u_0^{m+2} < 1$, we have a contradiction. Denote $v_j = u_j - u_0$, so that $v_j \rightharpoonup 0$ weakly in L^{m+2}. Observe that due to a theorem of Brezis and Lieb (a refined Fatou lemma [16])

$$\int |v_j|^{m+2} \to 1 - \int u_0^{m+2} = 1 - \lambda . \tag{13}$$

Note that

$$I(u_j) = I(v_j + u_0) = I(v_j) + I(u_0) + 2\int v_{jx}u_{0_x} + 2\int (D_x^{-1}v_{jy})(D_x^{-1}u_{0y}) + 2\int v_j u_0 .$$

The last three terms converge to zero by weak convergence of $v_j \rightharpoonup 0$ in $V(\mathbf{R}^2)$.

Hence

$$I^0 = \lim_{j\to\infty} I(v_j) + I(u_0)\,. \tag{14}$$

Let $\tilde{u}_0 = \frac{u_0}{\lambda^{\frac{1}{m+2}}}$, then we have $\int |\tilde{u}_0|^{m+2} = 1$. By homogeneity,

$$I(u_0) = \lambda^{\frac{2}{m+2}} I\left(\frac{u_0}{\lambda^{\frac{1}{m+2}}}\right) = \lambda^{\frac{2}{m+2}} I(\tilde{u}_0) \geqq \lambda^{\frac{2}{m+2}} I^0\,. \tag{15}$$

Let $\tilde{v}_j = \frac{v_j}{(1-\lambda)^{\frac{1}{m+2}}}$. From (13), we have $\int |\tilde{v}_j|^{m+2} = \frac{\int |v_j|^{m+2}}{1-\lambda} \to 1$. Similarly,

$$I(v_j) = (1-\lambda)^{\frac{2}{m+2}} I\left(\frac{v_j}{(1-\lambda)^{\frac{1}{m+2}}}\right) = (1-\lambda)^{\frac{2}{m+2}} I(\tilde{v}_j)\,,$$

$$\lim_{j\to\infty} I(v_j) = (1-\lambda)^{\frac{2}{m+2}} \lim_{j\to\infty} I(\tilde{v}_j) \geqq (1-\lambda)^{\frac{2}{m+2}} I^0\,.$$

Therefore

$$I^0 - I(u_0) \geqq (1-\lambda)^{\frac{2}{m+2}} I^0\,. \tag{16}$$

Equations (15) and (16) give

$$I^0 \geqq (\lambda^{\frac{2}{m+2}} + (1-\lambda)^{\frac{2}{m+2}}) I^0 > I^0\,,$$

which is a contradiction. Therefore

$$\lambda = \int |u_0|^{m+2} = 1\,,$$

i.e. $u_j \to u_0$ strongly in L^{m+2}.

Moreover, because $u_j \rightharpoonup u_0$ weakly in V, $I(u_0) \leqq \inf I(u_j)$ and u_0 minimizes I in N. Hence $I(u_j) \to I(u_0)$. Therefore $u_j \to u_0$ strongly in $V(\mathbf{R}^2)$, and this establishes Theorem 1.

We now prove the lemmas.

Proof of Lemma 1. We shall prove (12) for $\Omega = \mathbf{R}^2$ only. The case with Ω being a box can be proved similarly. Consider the Fourier transform representations of u, u_x and $D_x^{-1}u_y$,

$$u = \int \hat{u}(p,q) e^{ipx+iqy}\, dpdq\,, \qquad u_x = \int \widehat{u_x}(p,q) e^{ipx+iqy}\, dpdq\,,$$

$$D_x^{-1}u_y = \int \widehat{D_x^{-1}u_y}(p,q) e^{ipx+iqy}\, dp\, dq\,.$$

Then, we have

$$\int |u|^2\, dxdy = \int |\hat{u}|^2 dpdq\,, \qquad \int |\tilde{\nabla} u|^2\, dxdy = \int |\widehat{\tilde{\nabla} u}|^2 dp\, dq\,.$$

For some $0 \leqq \alpha \leqq 1$ and $p, q \neq 0$, we have

$$\begin{aligned} \hat{u}(p,q) &= \int e^{-ipx-iqy} u(x,y)\, dxdy \\ &= \alpha \int e^{-ipx-iqy} u(x,y)\, dxdy + (1-\alpha) \int e^{-ipx-iqy} u(x,y)\, dxdy \\ &= \frac{\alpha}{-ip} \int e^{-ipx-iqy} u_x(x,y)\, dxdy + \frac{(1-\alpha)p}{q} \int e^{-ipx-iqy} D_x^{-1} u_y\, dxdy\,. \end{aligned}$$

It follows that

$$|\hat{u}(p,q)| \leqq \frac{\alpha}{|p|}|\widehat{u_x}| + \frac{(1-\alpha)|p|}{|q|}|\widehat{D_x^{-1}u_y}|$$
$$\leqq \sqrt{\left(\frac{\alpha}{p}\right)^2 + \left(\frac{(1-\alpha)p}{q}\right)^2} \cdot \sqrt{|\widehat{u_x}|^2 + |\widehat{D_x^{-1}u_y}|^2} .$$

Let $\alpha = \frac{p^2}{|q|+p^2}$. Then

$$|\hat{u}| \leqq \frac{\sqrt{2}|p|}{|q|+p^2}\sqrt{|\widehat{u_x}|^2 + |\widehat{D_x^{-1}u_y}|^2} ,$$
$$|\hat{u}|^m \leqq \left(\frac{\sqrt{2}|p|}{|q|+p^2}\right)^m (|\widehat{u_x}|^2 + |\widehat{D_x^{-1}u_y}|^2)^{\frac{m}{2}} .$$

From Hölder's inequality, we have

$$\int_{p^2+q^2\geqq 1} |\hat{u}|^m dpdq \leqq \int_{p^2+q^2\geqq 1} \left(\frac{|p|}{|q|+p^2}\right)^m \{|\widehat{u_x}|^2 + |\widehat{D_x^{-1}u_y}|^2\}^{\frac{m}{2}} dpdq$$
$$\leqq \left(\int_{p^2+q^2\geqq 1} \left(\frac{|p|}{|q|+p^2}\right)^{\frac{2m}{2-m}} dpdq\right)^{\frac{2-m}{2}}$$
$$\times \left\{\int_{p^2+q^2\geqq 1} |\widehat{u_x}|^2 + |\widehat{D_x^{-1}u_y}|^2 dpdq\right\}^{\frac{m}{2}} .$$

It is easy to see that $\int_{p^2+q^2\geqq 1}(\frac{|p|}{|q|+p^2})^{\frac{2m}{2-m}} dpdq$ is convergent if $\frac{6}{5} < m < 2$. Hence

$$\int_{p^2+q^2\geqq 1} |\hat{u}|^m dpdq \leqq C_1 \left\{\int_{p^2+q^2\geqq 1} |\widehat{u_x}|^2 + |\widehat{D_x^{-1}u_y}|^2 dpdq\right\}^{\frac{m}{2}}$$
$$\leqq C_1 \left\{\int_{\mathbf{R}^2} |\widehat{u_x}|^2 + |\widehat{D_x^{-1}u_y}|^2 dpdq\right\}^{\frac{m}{2}} ,$$

where $C_1 = \int_{p^2+q^2\geqq 1}(\frac{|p|}{|q|+p^2})^{\frac{2m}{2-m}} dpdq$. On the other hand, we have

$$\int_{p^2+q^2<1} |\hat{u}|^m dpdq \leqq C_2 \left(\int_{p^2+q^2<1} |\hat{u}|^2 dpdq\right)^{\frac{m}{2}} \leqq C_2 \left(\int |\hat{u}|^2 dpdq\right)^{\frac{m}{2}} .$$

Again, $\int$ denotes the integral over $\mathbf{R}^2$. It follows then,

$$\int |\hat{u}|^m dpdq \leqq C_1 \left(\int |\widehat{u_x}|^2 + |\widehat{D_x^{-1}u_y}|^2 dpdq\right)^{\frac{m}{2}} + C_2 \left(\int |\hat{u}|^2 dpdq\right)^{\frac{m}{2}}$$
$$\leqq C_0 \left(\int |\widehat{u_x}|^2 + |\widehat{D_x^{-1}u_y}|^2 + |\hat{u}|^2 dpdq\right)^{\frac{m}{2}} .$$

By the theorem of Hausdorff–Young (p. 72 in [7]), we have

$$\left(\int |u|^n dxdy\right)^{\frac{1}{n}} \leqq \left(\int |\hat{u}|^m\right)^{\frac{1}{m}} ,$$

where $\frac{1}{n} + \frac{1}{m} = 1$. Therefore, if $\frac{6}{5} < m < 2$, we have $2 < n < 6$, and

$$\left(\int_{\mathbf{R}^2} |u|^n dxdy\right)^{\frac{1}{n}} \leqq C_0 \left(\int_{\mathbf{R}^2} u_x^2 + (D_x^{-1}u_y)^2 + u^2 dxdy\right)^{\frac{1}{2}} . \quad \square$$

Proof of Lemma 2. Let us assume that the box is of length $2\pi \times 2\pi$. The general case can be obtained by a scaling.

Consider the Fourier series representations of u, u_x and $D_x^{-1}u_y$

$$u = \sum a_{m,n} e^{imx+iny} , \qquad u_x = \sum b_{m,n} e^{imx+iny} ,$$

$$D_x^{-1}u_y = \sum c_{m,n} e^{imx+iny} .$$

Then,

$$\int |u|^2 dxdy = \sum_{m,n} a_{m,n}^2 , \qquad \int |\tilde{\nabla} u|^2 dxdy = \sum_{m,n} b_{m,n}^2 + c_{m,n}^2 .$$

For some $0 \leqq \alpha \leqq 1$ and $p, q \neq 0$, we have

$$\begin{aligned} a_{m,n} &= \int e^{-imx-iny} u(x,y)\, dxdy \\ &= \alpha \int e^{-imx-iny} u(x,y)\, dxdy + (1-\alpha) \int e^{-imx-iny} u(x,y)\, dxdy \\ &= \frac{\alpha}{-im} \int e^{-ipx-iqy} u_x(x,y)\, dxdy + \frac{(1-\alpha)m}{n} \int e^{-imx-iny} D_x^{-1} u_y\, dxdy . \end{aligned}$$

It follows that

$$\begin{aligned} |a_{m,n}| &\leqq \frac{\alpha}{|m|} |b_{m,n}| + \frac{(1-\alpha)|m|}{|n|} |c_{m,n}| \\ &\leqq \sqrt{\left(\frac{\alpha}{m}\right)^2 + \left(\frac{(1-\alpha)m}{n}\right)^2} \cdot \sqrt{|b_{m,n}|^2 + |c_{m,n}|^2} . \end{aligned}$$

Let $\alpha = \frac{m^2}{|n|+m^2}$. Then

$$|a_{m,n}| \leqq \frac{|m|}{|n| + m^2} \sqrt{|b_{m,n}|^2 + |c_{m,n}|^2} ,$$

$$|a_{m,n}|^2 \leqq \left(\frac{|m|}{|n| + m^2}\right)^2 (|b_{m,n}|^2 + |c_{m,n}|^2) .$$

$\forall \varepsilon > 0$, $\exists N_\varepsilon$, M_ε, *s.t.* for $n > N_\varepsilon$, $m > M_\varepsilon$, we have $(\frac{m}{n+m^2})^2 < \varepsilon$. Hence

$$\begin{aligned} \int u^2 &= \sum_{m,n}^{\infty} a_{m,n}^2 \leqq \sum_{m,n}^{N_\varepsilon, M_\varepsilon} a_{m,n} + \varepsilon \left(\sum_{m,n} b_{m,n}^2 + c_{m,n}^2\right) \\ &= \sum_{m=1}^{M_\varepsilon} \sum_{n=1}^{N_\varepsilon} \left(\int_\Omega u w_{m,n}\right) + \varepsilon \left(\int_\Omega u_x^2 + (D_x^{-1}u_y)^2\right) . \quad \square \end{aligned}$$

Proof of Lemma 3. Since $\{u_j\}$ is a minimizing sequence, we have

$$1 = \int |u_j|^{m+2} = \int_{[|u_j| \geqq \frac{1}{\varepsilon}]} |u_j|^{m+2} + \int_{[|u_j| \leqq \varepsilon]} |u_j|^{m+2} + \int_{[\varepsilon < |u_j| < \frac{1}{\varepsilon}]} |u_j|^{m+2}$$

$$\leqq \int_{[|u_j| \geqq \frac{1}{\varepsilon}]} \frac{|u_j|^{m+2+\gamma}}{|u_j|^{\gamma}} + \varepsilon^m \int_{[|u_j| \leqq \varepsilon]} |u_j|^2 + \left(\frac{1}{\varepsilon}\right)^{m+2} \mu([|u_j| > \varepsilon])$$

$$\leqq \varepsilon^{\gamma} \int_{[|u_j| \geqq \frac{1}{\varepsilon}]} |u_j|^{m+2+\gamma} + \varepsilon^m \int_{[|u_j| \leqq \varepsilon]} |u_j|^2 + C_\varepsilon \mu([|u_j| > \varepsilon]),$$

where $0 < \gamma < 4 - m$. From Lemma 1, we have

$$\int_{[|u_j| \geqq \frac{1}{\varepsilon}]} |u_j|^{m+2+\gamma} \leqq \int_{\mathbf{R}^2} |u_j|^{m+2+\gamma} \leqq C_1 ,$$

and

$$\int_{[|u_j| \geqq \varepsilon]} |u_j|^2 \leqq \int_{\mathbf{R}^2} |u_j|^2 \leqq C_2 .$$

By choosing ε small enough, we have

$$\mu([|u_j| > \varepsilon]) \geqq \frac{1 - \varepsilon^{\gamma} C_1 - \varepsilon^m C_2}{C_\varepsilon} = \delta . \qquad \square$$

Proof of Lemma 4. For simplicity, we assume $|u|_V \leqq 1$. Consider any function v, such that $|v|_V \leqq 1$ and $|v|_2 \neq 0$. Let $k = 1 + \frac{1}{|v|_2^2}$. We first prove that there is some x, such that

$$\int_{B_x} (v^2 + |\tilde{\nabla} v|^2)\, dxdy < mk \int_{B_x} v^2 dxdy , \tag{17}$$

where B_x is the unit box centered at x and m is a certain integer.

If (17) is not true, then we can cover $\mathbf{R}^2$ with unit boxes $\{B_{x_i}\}$ so that each point x is covered by at most m unit boxes and we have

$$m \int_{\mathbf{R}^2} (v^2 + |\tilde{\nabla} v|^2)\, dxdy \geqq \sum_i \int_{B_{x_i}} (v^2 + |\tilde{\nabla} v|^2)\, dxdy \geqq mk \sum_i \int_{B_{x_i}} v^2 dxdy$$

$$\geqq mk \int_{\mathbf{R}^2} v^2 dxdy = m \left(1 + \int_{\mathbf{R}^2} v^2 \right) > m .$$

Therefore

$$|v|_V^2 = \int_{\mathbf{R}^2} (v^2 + |\tilde{\nabla} v|^2)\, dxdy > 1 ,$$

which is a contradiction. By Lemma 1, we have, for x satisfies (17) and some $p > 2$,

$$\left(\int_{B_x} |v|^p dxdy \right)^{\frac{2}{p}} \leqq C_1 \int_{B_x} (|v|^2 + |\tilde{\nabla} v|^2)\, dxdy < C_1 mk \int_{B_x} |v|^2 dxdy$$

$$\leqq C_1 mk \left(\int_{B_x} |v|^p dxdy \right)^{\frac{2}{p}} (\mu(B_x \cap \operatorname{supp}(v)))^{\frac{p-2}{p}} \tag{18}$$

and

$$\mu(B_x \cap \operatorname{supp}(v)) > \left(\frac{1}{C_1 mk}\right)^{\frac{p-2}{p}} . \tag{19}$$

Equation (19) is true for any v satisfying $|v|_V \leqq 1$ and $|v|_2 \neq 0$. Now let's take $v = \max(|u| - \frac{\varepsilon}{2}, 0)$, we have $|v|_V \leqq 1$ and $\int_{\mathbf{R}^2} v^2 dxdy \geqq (\frac{\varepsilon}{2})^2 \delta$ and we have $k \leqq 1 + \frac{1}{(\frac{\varepsilon}{2})^2 \delta}$. From (19), there exists x, s.t.

$$\mu(B_x \cap \operatorname{supp}(v)) > \alpha, \quad \alpha = \left(\frac{1}{C_1 mk}\right)^{\frac{p-2}{p}} ,$$

i.e.

$$\mu(B_x \cap [|u| > \varepsilon/2]) > \alpha .$$

The conclusion of the lemma follows with a shift. □

3. Stability of the Solitary Wave

In this section, we show that the solitary wave of GKP-I is nonlinearly stable if $0 < m < \frac{4}{3}$. In order to study the stability of solitary wave of GKP-I, we need to consider the local existence for GKP-I. There are many results on local existence for GKP-I (see [19, 20, 13]). For our purpose, we state here the local existence results by Saut [20]. Let

$$\dot{H}_x^{-2}(\mathbf{R}^2) = \left\{ f \in S'(\mathbf{R}^2),\ \frac{1}{\xi_1^2} \hat{f}(\xi_1, \xi_2) \in L^2(\mathbf{R}^2) \right\}$$

equipped with the norm

$$\|f\|_{-2,x} = \left| \frac{1}{\xi_1^2} \hat{f} \right|_2 ,$$

and

$$X_s = \left\{ f \in H^s(\mathbf{R}^2), \mathscr{F}^{-1}\left(\frac{\hat{f}}{\xi_1}\right) \in H^s(\mathbf{R}^2) \right\},$$

with

$$\|f\|_{X_s} = \|f\|_s + \left\| \mathscr{F}^{-1}\left(\frac{\hat{f}}{\xi_1}\right) \right\|_s .$$

Theorem 2. *Let $\phi \in X_s$, $s \geqq 3$, such that $\phi_{yy} \in \dot{H}_x^{-2}$. There exists $T > 0$ such that* (1) *has a unique solution u with $u(0) = \phi$ satisfying*

$$u \in C([-T, T]; H^s(\mathbf{R}^2)) \cup C^1([-T, T]; H^{s-3}(\mathbf{R}^2)) ,$$

$$D_x^{-1} u_y \in C([-T, T]; H^{s-1}(\mathbf{R}^2)) .$$

Moreover, $Q(u)$ and $E(u)$ are well defined and independent of t.

We next give our definition of stability of the solitary waves.

Definition 1. *A set $S \subset X$ is X-stable with respect to GKP-I if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for any $u_0 \in X \cap X_s$ and $\partial_y^2 u_0 \in \dot{H}_x^{-2}$, $s \geqq 3$ with*

$$\inf_{v \in S} \|u_0 - v\|_X < \delta\,, \tag{20}$$

the solution $u(t)$ of (1) *with $u(0) = v$ can be extended to a global solution in $C([0,\infty); X \cap X_s)$, $s \geqq 3$ and*

$$\sup_{0 \leqq t < \infty} \inf_{v \in S} \|u(t) - v\|_X < \varepsilon\,. \tag{21}$$

Otherwise S is called X-unstable.

Now define the set of all ground state with speed $\omega > 0$ as

$$S_\omega = \{\varphi \in V(\mathbf{R}^2);\ K(\varphi) = I_\omega(\varphi) = (M(\omega))^{\frac{m+2}{m}}\}\,.$$

Let φ_ω be a ground state of GKP-I. For simplicity, we denote φ_ω by φ. Then

$$\omega\varphi + D_x^{-2}\varphi_{yy} - \varphi_{xx} - \varphi^{m+1} = 0\,.$$

It is well known in [8] that the stability of the solitary waves depends on the behavior of the following functional:

$$d(\omega) = E(\varphi_\omega) + \omega Q(\varphi_\omega) \quad \varphi_\omega \in S_\omega\,. \tag{22}$$

It follows that

$$\begin{aligned} d(\omega) &= \frac{1}{2} I_\omega(\varphi_\omega) - \frac{1}{m+2} K(\varphi_\omega) \\ &= \frac{m}{2(m+2)} I_\omega(\varphi_\omega) = \frac{m}{2(m+2)} K(\varphi_\omega)\,. \end{aligned} \tag{23}$$

Theorem 3 (*Nonlinear stability*). *Let $0 < m < \frac{4}{3}$ with $m = m_1/m_2$, where m_1 is any even integer and m_2 any odd integer and $w > 0$. Then S_ω is V-stable, where V is defined in* (3).

Remark 2. It is easy to calculate that

$$d''(\omega) = \left(\frac{4-m}{2m}\right)\left(\frac{4-3m}{2m}\right)\omega^{\frac{4-m}{2m}-2}A\,,$$

where $A = \frac{1}{3}\int\int(\varphi_x^2 + (D_x^{-1}\varphi_y)^2)\,dxdy$. Hence

$$d''(\omega) > 0 \Leftrightarrow 0 < m < \frac{4}{3}\,.$$

In order to prove Theorem 3, we need several lemmas.

Lemma 5. *$d(\omega)$ is differentiable and strictly increasing for $\omega > 0$, $0 < m < 4$ with $m = m_1/m_2$, where m_1 is any even integer and m_2 any odd integer.*

Proof. In fact, from (5), we have

$$d(\omega) = \frac{m}{2(m+2)}\omega^{\frac{4-m}{2m}}(M(1))^{\frac{m+2}{m}}$$

and

$$d'(w) = \frac{4-m}{4(m+2)}w^{\frac{4-3m}{2m}}(M(1))^{\frac{m+2}{m}} > 0$$

for $0 < m < 4$.

Lemma 6. *Let $d''(\omega) > 0$ with $\omega > 0$. Then $\exists \varepsilon > 0$, such that for $\omega_1 > 0$ with $|\omega_1 - \omega| < \varepsilon$ we have*

$$d(\omega_1) \geqq d(\omega) + d'(\omega)(\omega_1 - \omega) + \frac{1}{4}d''(\omega)|\omega - \omega_1|^2 . \tag{24}$$

Proof. This follows by Taylor's expansion at $\omega_1 = \omega$. □

Define

$$U_{\omega,\varepsilon} = \left\{ u \in V(\mathbf{R}^2);\ \inf_{\varphi \in S_\omega} \|u - \varphi\|_V < \varepsilon \right\}.$$

Since $d(\omega)$ is differentiable and strictly increasing for $\omega > 0$, it follows that for u near φ and $\varphi \in S_\omega$,

$$\omega(u) = d^{-1}\left(\frac{m}{2(m+2)}K(u)\right) \tag{25}$$

is a C^1 map:

$$\omega(\cdot):\ U_{\omega,\varepsilon} \to \mathbf{R}^+ \quad \text{for small } \varepsilon > 0 ,$$

and $\omega(\varphi_\omega) = \omega$ for any $\varphi_\omega \in S_\omega$.

The next lemma uses the variational characterization of ground states to establish the key inequality in the proof of stability.

Lemma 7. *Suppose $d''(\omega) > 0$ for $\omega > 0$. Then there exists $\varepsilon > 0$ such that for all $u \in U_{\omega,\varepsilon}$ and $\varphi_\omega \in S_\omega$,*

$$E(u) - E(\varphi_\omega) + \omega(u)(Q(u) - Q(\varphi_\omega)) \geqq \frac{1}{4}d''(\omega)\,|\omega(u) - \omega|^2 , \tag{26}$$

where $\omega(u)$ is defined by

$$\omega(u) = d^{-1}\left(\frac{m}{2(m+2)}K(u)\right) \quad \textit{for } u \in U_{\omega,\varepsilon} .$$

Proof. First of all, we have

$$E(u) + \omega(u)Q(u) = \frac{1}{2}I_{\omega(u)}(u) - \frac{1}{m+2}K(u) . \tag{27}$$

Since

$$\frac{2(m+2)}{m}d(\omega(u)) = K(u) ,$$

and

$$\frac{2(m+2)}{m}d(\omega(u)) = K(\varphi_{\omega(u)}), \quad \varphi_{\omega(u)} \in S_{\omega(u)} ,$$

then

$$K(u) = K(\varphi_{\omega(u)}) .$$

This implies that

$$I_{\omega(u)}(u) \geqq I_{\omega(u)}(\varphi_{\omega(u)}) . \tag{28}$$

Since $\varphi_{\omega(u)}$ is a minimizer of $I_{\omega(u)}$ subject to the constraint $K(u) = K(\varphi_{\omega(u)})$ and $\omega(u) \in C^1$, then by (27) and Lemma 6 we have

$$\begin{aligned} E(u) + \omega(u)Q(u) &\geqq \frac{1}{2} I_{\omega(u)}(\varphi_{\omega(u)}) - \frac{1}{m+2} K(\varphi_{\omega(u)}) = d(\omega(u)) \\ &\geqq d(\omega) + d'(\omega)(\omega(u) - \omega) + \frac{1}{4} d''(\omega)|\omega(u) - \omega|^2 \\ &= E(\varphi_\omega) + \omega(u)Q(\varphi_\omega) + \frac{1}{4} d''(\omega)|\omega(u) - \omega|^2 , \end{aligned} \tag{29}$$

where we use the fact

$$d'(\omega) = Q(\varphi_\omega) . \tag{30}$$

Now we can prove Theorem 3.

Proof. Assume that S_ω is V-unstable. Then by the definition of stability, $\exists \delta > 0$ and initial data $u_k(0) \in U_{\omega, \frac{1}{k}}$ such that

$$\sup_{t>0} \inf_{\varphi \in S_\omega} \|u_k(t) - \varphi\|_V \geqq \delta , \tag{31}$$

where $u_k(t)$ is the solution of GKP-I with initial data $u_k(0)$. By continuity in t, we can pick the first time t_k so that

$$\inf_{\varphi \in S_\omega} \|u_k(t_k) - \varphi\|_V = \delta . \tag{32}$$

Since $E(u)$ and $Q(u)$ are conserved at t and continuous for u, we can find $\varphi_k \in S_\omega$ such that

$$|E(u_k(t_k)) - E(\varphi_k)| = |E(u_k(0)) - E(\varphi_k)| \to 0 \tag{33}$$

as $k \to \infty$ and

$$|Q(u_k(t_k)) - Q(\varphi_k)| = |Q(u_k(0)) - Q(\varphi_k)| \to 0 \tag{34}$$

as $k \to \infty$. Choose δ small enough so that Lemma 7 applies,

$$E(u_k(t_k)) - E(\varphi_k) + \omega(u_k(t_k))(Q(u_k(t_k)) - Q(\varphi_k)) \geqq \frac{1}{4} d''(\omega)|\omega(u_k(t_k)) - \omega|^2 . \tag{35}$$

By (32), there exists $\psi_k \in S_\omega$ such that

$$\begin{aligned} \|u_k(t_k)\|_V &\leqq \|\varphi_k\|_V + 2\delta \quad (36) \\ &\leqq \left(1 + \frac{1}{\omega}\right) I_\omega(\varphi_k) + 2\delta \\ &\leqq c(\omega) M(\omega)^{\frac{m+2}{m}} + 2\delta < +\infty . \quad (37) \end{aligned}$$

Since $\omega(u)$ is a continuous map, $\omega(u_k(t_k))$ is uniformly bounded for k. By (35), letting $k \to \infty$, we have

$$\omega(u_k(t_k)) \to \omega\,. \tag{38}$$

Hence

$$\lim_{k\to\infty} K(u_k(t_k)) = \lim_{k\to\infty} \frac{2(m+2)}{m} d(\omega(u_k(t_k)) = \frac{2(m+2)}{m} d(\omega)\,. \tag{39}$$

On the other hand,

$$\begin{aligned} I_\omega(u_k(t_k)) &= 2(E(u_k(t_k)) + \omega Q(u_k(t_k)) + \frac{2}{m+2} K(u_k(t_k)) \\ &= 2d(\omega(u_k(t_k)) + 2(\omega - \omega(u_k(t_k))Q(u_k(t_k)) + \frac{2}{m+2} K(u_k(t_k))\,. \end{aligned} \tag{40}$$

Since

$$Q(u_k(t_k)) = Q(u_k(0)) \leqq \|u_k(t_k)\|_V \leqq C(\omega) < +\infty\,,$$

then by (39)

$$I_\omega(u_k(t_k)) \to 2d(\omega) + \frac{2}{m+2}\frac{2(m+2)}{m} d(\omega) = \frac{2(m+2)}{m} d(\omega) \quad \text{as } k \to \infty. \tag{41}$$

That is

$$I_\omega(u_k(t_k)) \to I(\varphi_\omega) = (M(\omega))^{\frac{m+2}{m}}\,. \tag{42}$$

Let

$$v_k(t_k) = (K(u_k(t_k)))^{-\frac{1}{m+2}} u_k(t_k)\,.$$

Then $K(v_k(t_k)) = 1$ and

$$\begin{aligned} I_\omega(v_k(t_k)) &= (K(u_k(t_k)))^{-\frac{2}{m+2}} I_\omega(u_k(t_k)) \\ &\to \frac{(M(\omega))^{\frac{m+2}{m}}}{(\frac{2(m+2)}{m} d(\omega))^{\frac{2}{m+2}}} = (M(\omega))^{\frac{m+2}{m}} (M(\omega))^{-\frac{2}{m}} = M(\omega)\,. \end{aligned} \tag{43}$$

Hence, $v_k(t_k)$ is a minimizing sequence. Therefore, $\exists \varphi_k \in S_\omega$ such that

$$\lim_{k\to\infty} \|v_k(t_k) - (M(\omega))^{-\frac{1}{m}} \varphi_k\|_V = 0\,, \tag{44}$$

where $K((M(\omega))^{-\frac{1}{m}} \varphi_k) = 1$. This implies that

$$\begin{aligned} \lim_{k\to\infty} \|u_k(t_k) - \varphi_k\|_V &= \lim_{k\to\infty} [(K(u_k(t_k)))^{\frac{1}{m+2}} \cdot \|(K(u_k(t_k)))^{-\frac{1}{m+2}} (u_k(t_k) - \varphi_k)\|_V] \\ &\leqq M^{\frac{1}{m}}(\omega) \left[\lim_{k\to\infty} \|v_k(t_k) - M^{-\frac{1}{m}}(\omega)\varphi_k\|_V \right] \\ &\quad + \lim_{k\to\infty} |M^{\frac{-1}{m}}(\omega) - (K(u_k(t_k)))^{-\frac{1}{m+2}}|\, \|\varphi_k\|_V = 0 \end{aligned} \tag{45}$$

since $\|\varphi_k\|_V^2 \leqq (1 + \frac{1}{\omega}) I_\omega(\varphi_k) \to (1 + \frac{1}{\omega})(M(\omega))^{\frac{m+2}{m}} < +\infty$.

Hence (45) contradicts with (32). □

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