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Coherent States for Quantum Compact Groups

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Dedicated to Professor L.D. Faddeev on his 60th birthday

Abstract: Coherent states are introduced and their properties are discussed for simple quantum compact groups A_l, B_l, C_l and D_l. The multiplicative form of the canonical element for the quantum double is used to introduce the holomorphic coordinates on a general quantum dressing orbit. The coherent state is interpreted as a holomorphic function on this orbit with values in the carrier Hilbert space of an irreducible representation of the corresponding quantized enveloping algebra. Using Gauss decomposition, the commutation relations for the holomorphic coordinates on the dressing orbit are derived explicitly and given in a compact R-matrix formulation (generalizing this way the q-deformed Grassmann and flag manifolds). The antiholomorphic realization of the irreducible representations of a compact quantum group (the analogue of the Borel–Weil construction) is described using the concept of coherent state. The relation between representation theory and non-commutative differential geometry is suggested.

1. Introduction

It is difficult to overestimate the importance of the concept of coherent states in theoretical and mathematical physics. They found various applications in quantum optics, quantum field theory, quantum statistical mechanics and other branches of physics as well as in some purely mathematical problems [21, 34]. The last-named include Lie group representations, special functions, automorphic functions, reproducing kernels, etc. In the Lie group representation theory there is a remarkable relation between the geometry on the coadjoint orbits and the irreducible representations, which is reflected by the method of orbits (geometric quantization) due to Kirillov, Kostant and Souriau [53]. On the other hand the concept of coherent states leads naturally to Berezin's quantization scheme [5]. The important sources of both methods are induced representations and the Borel–Weil theory. The intrinsic relationship between the geometric and Berezin quantization has been established. There are many papers devoted to this subject (e.g. [32, 37] and many others). Recently the coherent states were used to construct examples of non-commutative manifolds [14].

The first papers [1, 44], which can be viewed as those generalizing coherent states to quantum groups appeared even before the formal birth of quantum groups [11]. A number of papers followed subsequently ([17] and many others). Nevertheless no definition seems to be completely satisfactory. The coherent states are introduced mainly for the simplest quantum groups (q-deformations of the Heisenberg–Weyl, $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ algebras) in a rather straightforward way which does not suggest a proper generalization to the more general case. Moreover, these states are assumed to be elements of the representation space for an irreducible representation of the corresponding quantized enveloping algebra and they do not reflect the whole underlying Hopf algebra structure.

Recently the representation theory for the algebras of quantum functions on compact groups was studied in great detail. It is worth emphasizing too that these results were obtained with the help of the method of orbits [49, 45, 46]. This lead finally to a quite general definition of the coherent state given by Soibelman [48] related to generalized Pontryagin duals of simple compact groups. So this is in some sense the case dual to the one we wish to consider in the present paper.

According to the general philosophy of non-commutative geometry it would be more natural to view the coherent state as a function on an appropriate q-homogeneous space of the corresponding quantum group (dual to the quantized enveloping algebra) with values in the representation space. We hope that such a more sophisticated generalization of the coherent states method to the case of quantum groups could be of interest not only for the representation theory but also for potential applications of quantum groups in physics. Many important ingredients needed for this generalization are already prepared. First of all the representation theory of quantum groups [15, 30, 40] and the method of induced representations are well developed [33]. The deformations of manifolds playing an important role in the Lie group representation theory (such as Schubert cells, flag and Grassmann manifolds) are also known [23, 25, 52, 3, 47, 50] through different approaches. Further there is the notion of the quantum dressing transformation [36] which is the substituent for the coadjoint action from the classical case. It is important already for classical groups [41], has interesting applications in physics [4] and is closely related to the notions of generalized Pontryagin dual and the Iwasawa decomposition [29]. Finally there is also a proper definition of the quantum momentum map [28]. One of the expected results of the coherent state approach for quantum groups would be to put all these ingredients together in a natural way. The second expected result would be a variant of the q-generalization of the Borel–Weil theory which follows more closely the classical Borel–Weil construction than the one described in [7] for the case of $U_q(n)$ (another approach is presented also in [3]). Finally as in the classical case it is natural to achieve a link between the representation theory and the non-commutative differential geometry on quantum groups [55]. We hope to meet these goals in the present paper.

The present paper extends some ideas from the papers [19, 50], but now the leading idea is a proper definition of the coherent state for quantum groups, using the rich structure contained in the quantum double [11, 35].

The paper is organized as follows. Section 2 contains a very brief account of the classical theory. Section 3 has a preliminary character: some basic notions of the quantum group theory are recalled. Section 4 adapts to our purposes some well known results from the representation theory of quantum groups. Section 5 which contains the definition of coherent state for the compact quantum group and discusses its basic properties is one of the more important parts of the paper. Here we would like to mention that, similarly to the classical case, we can start in the definition (5.1) of the coherent state Γ from any Hopf algebra and any state in the carrier Hilbert space $\mathscr{H}$ of some irreducible representation τ whenever (5.1) does make sense. Nevertheless the restriction to the quantized universal enveloping $\mathscr{U}_h(\mathfrak{k})$ for $\mathfrak{k}$ compact and the choice of the lowest (or equivalently the highest) weight state e_λ are the most relevant for the rest of the paper. This section also contains a definition of the (quantum) isotropy subgroup $K_0 \subset K$ (K is the spectrum of the Hopf algebra $\mathscr{A}_q(K)$ dual to the $\mathscr{U}_h(\mathfrak{k})$) of e_λ. Our coherent state can be then naturally viewed as a function on the q-homogeneous space $K_0\backslash K$ with values in the representation space $\mathscr{H}_\lambda$ for the lowest weight representation τ_λ of $\mathscr{U}_h(\mathfrak{k})$ corresponding to the lowest weight λ. Section 6 contains a detailed description of the canonical element ρ (universal R-matrix) of the quantum double (particularly inspired by [13]) which makes possible a more explicit expression for the coherent state Γ and a definition of holomorphic coordinates on a general quantum dressing orbit. Explicit commutation relations for the holomorphic coordinates in the R-matrix formulation are derived in Sect. 7. They present a compact generalization of the definition relations for the quantized flag manifold. Section 8 describes the antiholomorphic realization of the irreducible representation τ_λ which is very close to the classical Borel–Weil theory. The presentation of Sect. 8 can also be, if wished, reinterpreted as a non-commutative version of the Berezin quantization. Finally in Sect. 9 we make an attempt to relate the representation theory to the non-commutative differential geometry, which as we hope could be helpful for understanding the non-commutative version of the method of orbits.

Let us make here a few comments on some points not included in the paper.

The discussion of Sects. 8 and 9 is done using the local coordinates on an appropriate cell of the dressing orbit. There is no doubt that a globalization using the quantum Weyl elements is possible. As in the classical case it should lead to a "quantization condition" for the quantum dressing orbit and to an interpretation of the elements of $\mathscr{H}_\lambda$ as antiholomorphic sections of an appropriate quantum line bundle [8, 51].

There is also no doubt that Sect. 9 could be formulated purely in terms of the holomorphic coordinates z and their conjugates z^*. However, this requires an explicit description of the restriction of the bicovariant differential calculus on the quantum group K to the quantum homogeneous space $K_0\backslash K$. An introduction of the partial derivatives ∂_{z^*} with respect to the antiholomorphic coordinates would make it possible to interpret the formula (9.15) expressed only through coordinates z^* and partial derivatives ∂_{z^*} as a natural Fock space representation of $\mathscr{U}_h(\mathfrak{k})$.

It is also natural to think about limiting cases of our construction. The limit $q \longrightarrow 1$ gives of course the classical scheme recalled in Sect. 2. Nevertheless as in the classical case [43, 5] there is a second type of limit leading to the classical dressing orbits with their natural Poisson structure. This kind of limit is achieved by using the sequence of irreducible representations corresponding to the sequence of lowest weights $n\lambda$. A rescaling of $q \longrightarrow q^{1/n}$ and a subsequent limit $n \longrightarrow \infty$ gives the desired result.

2. The Classical Scheme

Let us start by recalling the classical situation [34]. Denote by G a simple and simply connected complex Lie group and by $K \subset G$ its compact form. Let $\mathscr{T}^\lambda$ be an irreducible unitary representation of K in $\mathscr{H}_\lambda$ corresponding to a lowest weight λ. $\mathscr{T}^\lambda$ extends unambiguously as a holomorphic representation of G in $\mathscr{H}_\lambda$. Let $e_\lambda \in \mathscr{H}_\lambda$ be the normalized weight vector and set

$$\Gamma : G \to \mathscr{H}_\lambda : g \mapsto \mathscr{T}^\lambda(g^{-1})\, e_\lambda\,.$$

The vector-valued function Γ is a coherent state in the sense of Perelomov. Denote further by $K_0 \subset K$ respectively $P_0 \subset G$ the isotropy subgroups of the point $\mathbb{C}e_\lambda \in \mathbb{P}(\mathscr{H}_\lambda)$. This means that there exists a character χ of P_0, unitary on $K_0 \subset P_0$, such that

$$\mathscr{T}^\lambda(k)\, e_\lambda = \chi(k)\, e_\lambda, \text{ for } \forall k \in P_0\,.$$

Thus we have the following transformation property of Γ:

$$\Gamma(kg) = \chi(k^{-1})\Gamma(g), \quad \forall k \in P_0,\ \forall g \in G\,.$$

The mapping

$$\mathscr{H}_\lambda \ni u \mapsto \langle \Gamma(\,\cdot\,), u\rangle \in C^\infty(K)$$

is injective and so one embeds in this way $\mathscr{H}_\lambda$ into the vector space of χ-equivariant functions on K. Here we adopt the physical convention according to which the inner product is linear in the second argument and conjugate linear in the first one. Furthermore, we recall that by a χ-equivariant function f on K it is meant that $f(kg) = \chi(k)\, f(g)$, $\forall k \in K_0$, $\forall g \in K$. Sending $(g,k) \in K \times K_0$ to $k^{-1}g \in K$ we get a principal K_0-bundle $K \to K_0\backslash K$ and using the 1-dimensional representation χ one associates to it a line bundle over the base space $K_0\backslash K = P_0\backslash G$. Hence χ-equivariant functions on K are identified with sections in this line bundle. Set

$$w_\lambda := \langle e_\lambda, \mathscr{T}^\lambda(\,\cdot\,)e_\lambda\rangle \in C^{\rm hol}(G)\,.$$

The function w_λ is χ-equivariant and coincides with $\langle \Gamma, e_\lambda\rangle$ on K and thus determines a trivialization of the line bundle over the cell given by $w_\lambda(g) \neq 0$. The Gauss decomposition provides a standard way to choose holomorphic coordinates $\{z_j\}$ on this cell. For a given $u \in \mathscr{H}_\lambda$, the function $\psi_u := w_\lambda^{-1}\langle \Gamma, u\rangle$ defined on the cell (which is an open subset in $K_0\backslash K$ with a complement of zero measure) can be viewed also as a P_0-invariant function defined on the pre-image of the cell under the projection $G \to P_0\backslash G$:

$$\psi_u(g) = (\langle \mathscr{T}^\lambda(g^{-1})e_\lambda, e_\lambda\rangle)^{-1}\langle \mathscr{T}^\lambda(g^{-1})e_\lambda, u\rangle\,.$$

Thus one finds that the vectors u from $\mathscr{H}_\lambda$ are represented by polynomials ψ_u in the variables $\{z_j^*\}$ and so the representation $\mathscr{T}^\lambda$ acts in the space of antiholomorphic functions living on the cell.

Finally we also recall that every operator $B \in \mathrm{Lin}(\mathscr{H}_\lambda)$ is represented by its symbol $\sigma(B) \in C^a(K_0\backslash K)$ (here the superscript "a" means real analytic) or, this is the same, by a real analytic K_0-invariant function on K,

$$\sigma(B) := \{g \mapsto \langle \Gamma(g), B\,\Gamma(g)\rangle\}\,.$$

The mapping $B \mapsto \sigma(B)$ is injective [21, 43].

The aim of the present paper is to demonstrate that this scheme applies also for quantum groups.

3. Preliminaries, Notation

Let us recall some basic notions related to the duality and the dressing transformation for quantum groups [19]. All deformed (twisted) algebras are considered over the ring $\mathbb{C}[[h]]$ of formal power series in the formal deformation parameter h. We set also $q = e^{-h}$. As far as $*$-algebras are considered we require $h^* = h$. An important role plays the duality between the quantum groups K_q and AN_q following from the Iwasawa decomposition $G = K \cdot AN$. The deformed enveloping algebra $\mathcal{U}_h(\mathfrak{k})$ is the $*$-Hopf algebra dual to $\mathcal{A}_q(K)$. $\mathcal{A}_q(AN)$ is identical to $\mathcal{U}_h(\mathfrak{k})$ as a $*$-algebra and opposite as a coalgebra. Observe that necessarily the antipode of $\mathcal{A}_q(AN)$ is the inverse of the antipode of $\mathcal{U}_h(\mathfrak{k})$. We note further that $\mathcal{A}_q(G)$ is the same Hopf algebra as $\mathcal{A}_q(K)$ but the compact form is equipped in addition with the $*$-involution. We shall also denote by $\mathcal{U}_h(\mathfrak{g})$ the Hopf algebra $\mathcal{U}_h(\mathfrak{k})$ when having forgotten about the $*$-operation. We denote by T, U and Λ the vector corepresentations for $\mathcal{A}_q(G), \mathcal{A}_q(K)$ and $\mathcal{A}_q(AN)$, respectively. This means that $X = T$ resp. U resp. Λ is an $N \times N$ matrix (N is the same in all three cases) with entries from the corresponding algebra and $\Delta X = X \dot{\otimes} X$, $\varepsilon(X) = \mathbf{I}$ and $S(X)X = XS(X) = \mathbf{I}$. Moreover, the entries of X generate the algebra according to the well known rules. We have used the adjective "vector" (corepresentation) also in the case of $\mathcal{A}_q(AN)$ as this corepresentation is closely related to the vector corepresentations of $\mathcal{A}_q(K)$ and $\mathcal{A}_q(G)$ via the quantum Iwasawa decomposition [19]. While the matrix U is unitary-like, the matrix Λ is upper triangular and the diagonal elements are self-adjoint ($\Lambda_{ii}^* = \Lambda_{ii}$). The $*$-algebras $\mathcal{A}_q(K)$ and $\mathcal{A}_q(AN)$ are defined by the relations [38]:

$$RU_1U_2 = U_2U_1R, \qquad U^* = U^{-1} ,$$

$$R\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1R, \qquad \Lambda_1^*R^{-1}\Lambda_2 = \Lambda_2R^{-1}\Lambda_1^* , \tag{3.1}$$

and for the B_l, C_l and D_l series also by

$$CU^tC^{-1} = U^{-1}, \qquad C\Lambda^tC^{-1} = \Lambda^{-1} .$$

In all expressions, $(X^*)_{ij} := (X_{ji})^*$. R is the standard R-matrix [16, 38], and C is given in [38]. The relations (3.1) (as well as the duality) can be found in [38] expressed in terms of matrices $L^{\pm}$ with the explicit transcription $\Lambda = S(L^+)$ and $\Lambda^* = L^-$. Since, as mentioned, Λ is upper triangular and $S(\Lambda) = \Lambda^{-1}$, we have $S(\Lambda_{ii})\Lambda_{ii} = \Lambda_{ii}S(\Lambda_{ii}) = 1$ (or, in terms of $L^{\pm}$, $L_{ii}^+L_{ii}^- = L_{ii}^-L_{ii}^+ = 1$). Furthermore, the pairing between $\mathcal{A}_q(AN)$ and $\mathcal{A}_q(K)$ is given by [38, 19]

$$\langle \Lambda_1; U_2 \rangle = R_{21}^{-1}, \qquad \langle \Lambda_1^*; U_2 \rangle = R_{12}^{-1} .$$

Let us introduce the canonical element

$$\rho = \sum x_s \otimes a_s \in \mathcal{A}_q(AN) \otimes \mathcal{A}_q(K) ,$$

with $\{x_s\}$ and $\{a_s\}$ being mutually dual bases. Its basic properties are (S is the antipode, Δ is the comultiplication)

$$\rho^* = \rho^{-1} = (\mathrm{id} \otimes S)\rho ,$$

$$(\Delta \otimes \mathrm{id})\rho = \rho_{23}\rho_{13}, \qquad (\mathrm{id} \otimes \Delta)\rho = \rho_{12}\rho_{13} . \tag{3.2}$$

Remark. The canonical element ρ is given by an infinite series and, strictly speaking, it is not a proper element of $\mathscr{A}_q(AN) \otimes \mathscr{A}_q(K)$. One way of treating ρ is to embed $\mathscr{A}_q(AN) \otimes \mathscr{A}_q(K)$ into $\mathrm{Lin}(\mathscr{A}_q(K))$ and then ρ corresponds to the identity (though the multiplication in $\mathscr{A}_q(AN) \otimes \mathscr{A}_q(K)$ has nothing to do with composition of linear mappings) [19]. We use ρ below to define the dressing transformation (relation (3.3)). But, in principle, one can use directly (3.5) instead of (3.3). Another occurrence of ρ in this paper is in expressions $(\tau \otimes \mathrm{id})\rho$, with τ being a finite dimensional representation of $\mathscr{A}_q(AN)$. But in this case for an explicitly constructed basis of $\mathscr{A}_q(AN) \equiv \mathscr{U}_h(\mathfrak{k})$ the infinite series truncates. The construction of this basis was given in papers devoted to the universal R-matrix to which ρ is closely related [20, 25]. In fact, making use of these results we give in Sect. 6 an explicit formula for ρ (Proposition 6.2).

Using ρ one defines the dressing transformation as a coaction

$$\mathscr{R} : \mathscr{A}_q(AN) \to \mathscr{A}_q(AN) \otimes \mathscr{A}_q(K) : u \mapsto \rho(u \otimes 1)\rho^{-1} \,. \tag{3.3}$$

We emphasize that the identification of the algebras $\mathscr{U}_h(\mathfrak{k})$ and $\mathscr{A}_q(AN)$ plays in this situation the role of the classical momentum mapping. We also note that in the literature one often identifies the dressing transformation with the quantum adjoint action,

$$\mathrm{ad}_x\, u = \sum x^{(1)} u\, S x^{(2)}, \quad \text{with } \Delta x = \sum x^{(1)} \otimes x^{(2)} \,, \tag{3.4}$$

where $x, u \in \mathscr{U}_h(\mathfrak{k})$ and the coproduct Δ and the antipode S are taken in $\mathscr{U}_h(\mathfrak{k})$ (rather than in $\mathscr{A}_q(AN)$). However these two notions are closely related since ($u \in \mathscr{A}_q(AN) \equiv \mathscr{U}_h(\mathfrak{k})$)

$$(\mathrm{id} \otimes \langle x, \cdot \rangle)\, \mathscr{R} u = \mathrm{ad}_x\, u \,. \tag{3.5}$$

The coaction $\mathscr{R}$ defined by (3.3) fulfills the usual axioms: $(\mathrm{id} \otimes \Delta) \circ \mathscr{R} = (\mathscr{R} \otimes \mathrm{id}) \circ \mathscr{R}$, $(\mathrm{id} \otimes \varepsilon) \circ \mathscr{R} = \mathrm{id}$. Because of the formula (3.4), $\mathscr{R}$ can be viewed as the Hopf algebra analogue of the adjoint representation of a Lie group on its Lie algebra. Finally we note that the dressing transformation can be calculated explicitly on the elements of the matrix $\Lambda^* \Lambda$,

$$\mathscr{R}(\Lambda^* \Lambda) = U^* \Lambda^* \Lambda U \,, \tag{3.6}$$

provided on the RHS one identifies $\mathscr{A}_q(AN)$ with $\mathscr{A}_q(AN) \otimes 1$ and similarly for $\mathscr{A}_q(K)$ (cf. [19], Proposition 4.2).

4. The "Vacuum" Functional

According to the results of Rosso and Lusztig [30, 40], to every lowest weight λ from the weight lattice there is related a unique irreducible $*$-representation τ_λ of $\mathscr{U}_h(\mathfrak{k})$ acting in $\mathscr{H}_\lambda$, $\dim \mathscr{H}_\lambda < \infty$, and correspondingly a unitary corepresentation of $\mathscr{A}_q(K)$, $\mathscr{T}^\lambda = (\tau_\lambda \otimes \mathrm{id})\rho \in \mathrm{Lin}(\mathscr{H}_\lambda) \otimes \mathscr{A}_q(K)$ (i.e., $(\mathrm{id} \otimes \Delta)\mathscr{T}^\lambda = \mathscr{T}^\lambda_{12}\mathscr{T}^\lambda_{13}$, $(\mathrm{id} \otimes \varepsilon)\mathscr{T}^\lambda = \mathbf{I}$ and $(\mathrm{id} \otimes S)\mathscr{T}^\lambda = (\mathscr{T}^\lambda)^* = (\mathscr{T}^\lambda)^{-1}$). There is some danger of formally working with ρ but we focus, as mentioned, only on finite-dimensional representations of $\mathscr{U}_h(\mathfrak{k})$ determined by the lowest weight λ; particularly, the elements q^H (H in Cartan subalgebra) are sent by the lowest weight to $\mathrm{e}^{-h\,\lambda(H)}$ (as $q = e^{-h}$). In what follows, e_λ stands again for the normalized weight vector.

Let us define the "vacuum" functional $\langle \cdot \rangle$ on $\mathscr{U}_h(\mathfrak{k})$,

$$\langle x \rangle := \langle e_\lambda, \tau_\lambda(x)\, e_\lambda \rangle \in \mathbb{C}[[h]] \,. \tag{4.1}$$

Proposition 4.1. *It holds*

$$\langle x\rangle = \langle x, w_\lambda\rangle, \quad \text{where } w_\lambda := \langle e_\lambda, \mathcal{T}^\lambda e_\lambda\rangle \in \mathcal{A}_q(K)\,. \tag{4.2}$$

This means that $\langle\,\cdot\,\rangle$ *if viewed as an element from* $\mathcal{A}_q(K)$, *the dual space to* $\mathcal{U}_h(\mathfrak{k})$, *is equal to* w_λ.

Proof. One can verify (4.2) easily using the identity

$$(\mathrm{id}\otimes\langle x,\,\cdot\,\rangle)\rho = x, \quad x\in\mathcal{U}_h(\mathfrak{k})\equiv\mathcal{A}_q(AN)\,. \qquad \square$$

Let us note at this place that, likewise in the classical case,

Proposition 4.2. *It holds*

$$w_{\lambda_1+\lambda_2} = w_{\lambda_1}w_{\lambda_2} = w_{\lambda_2}w_{\lambda_1}\,, \tag{4.3}$$

and so it is enough to determine w_λ *only for the fundamental weights* $\lambda=\omega_j$. *Furthermore* (ε *is the counit*),

$$S\,w_\lambda = w_\lambda^*, \qquad \varepsilon(w_\lambda) = 1\,. \tag{4.4}$$

Proof. To see (4.3) it suffices to observe that $\mathcal{H}_{\lambda_1+\lambda_2}$ can be identified with the cyclic submodule $\mathcal{M}$ in $\mathcal{H}_{\lambda_1}\otimes\mathcal{H}_{\lambda_2}$ corresponding to the cyclic vector $e_{\lambda_1}\otimes e_{\lambda_2}$ with respect to the representation $(\tau_{\lambda_1}\otimes\tau_{\lambda_2})\circ\Delta$ (Δ is the coproduct in $\mathcal{U}_h(\mathfrak{k})$). Since

$$\mathcal{T}^{\lambda_1+\lambda_2} = \mathcal{T}_{23}^{\lambda_2}\mathcal{T}_{13}^{\lambda_1}|\mathcal{M}\,,$$

we have

$$\begin{aligned} w_{\lambda_1+\lambda_2} &= \langle e_{\lambda_1}\otimes e_{\lambda_2}, \mathcal{T}^{\lambda_1+\lambda_2}e_{\lambda_1}\otimes e_{\lambda_2}\rangle \\ &= \langle e_{\lambda_2}, \mathcal{T}^{\lambda_2}e_{\lambda_2}\rangle\langle e_{\lambda_1}, \mathcal{T}^{\lambda_1}e_{\lambda_1}\rangle \\ &= w_{\lambda_2}w_{\lambda_1}\,. \qquad \square \end{aligned}$$

Using the identification $\mathcal{U}_h(\mathfrak{k})\equiv\mathcal{A}_q(AN)$ one can also describe the "vacuum" functional in the following way. It holds

$$\tau_\lambda(\Lambda)e_\lambda = \Lambda_\lambda e_\lambda, \quad \text{where } \Lambda_\lambda := \mathrm{diag}(\lambda(\Lambda_{ii}))\,. \tag{4.5}$$

Remark. Λ can be expressed explicitly in terms of the generators of $\mathcal{U}_h(\mathfrak{k})$ and its diagonal is then given in terms of Cartan elements; see also the relation (5.26) below). Thus Λ_λ is a diagonal matrix fulfilling the $R\Lambda_\lambda\Lambda_\lambda$-equation and possibly also $C\Lambda_\lambda^t C^{-1} = \Lambda_\lambda^{-1}$. Besides, the relation (3.1) enables one to define a normal ordering on $\mathcal{A}_q(AN)$ by requiring the elements of the matrix Λ^* to stand to the left and those of the matrix Λ to stand to the right. It doesn't matter that this ordering prescription is not quite unambiguous since the subalgebras generated by the entries of Λ^* and Λ, respectively, intersect in the Cartan elements. We have

$$\langle 1\rangle = 1, \qquad \langle\Lambda^*\rangle = \langle\Lambda\rangle = \Lambda_\lambda\,, \tag{4.6}$$

and

$$\langle x_{i_1}\cdots x_{i_k}\rangle = \langle x_{i_1}\rangle\cdots\langle x_{i_k}\rangle\,, \tag{4.7}$$

provided the product $x_{i_1}\cdots x_{i_k}$ is normally ordered. Clearly it holds also: if $x_{i_1}\cdots x_{i_k}$ is normally ordered then the same is true for $x_{i_k}^*\cdots x_{i_1}^*$. Either from this property and (4.7) or from the fact that τ_λ is a $*$-representation and the Definition (4.1) we get

$$\langle x^*\rangle = \overline{\langle x\rangle}\,. \tag{4.8}$$

5. The Quantum Coherent State

The following definition is very analogous to the classical case and is crucial for the rest of the paper.

Definition 5.1. *The quantum coherent state is the element $\Gamma \in \mathscr{H}_\lambda \otimes \mathscr{A}_q(K)$ defined by*

$$\Gamma := (\mathscr{T}^\lambda)^{-1}(e_\lambda \otimes 1) = (\tau_\lambda \otimes S)\rho \cdot (e_\lambda \otimes 1)\,. \tag{5.1}$$

Γ should be interpreted as a quantum function on K with values in $\mathscr{H}_\lambda$. Thus one can relate to every vector $u \in \mathscr{H}_\lambda$ a quantum function on K,

$$u \mapsto \langle \Gamma, u\rangle := (\langle e_\lambda, (\,\cdot\,)u\rangle \otimes \mathrm{id})\mathscr{T}^\lambda \in \mathscr{A}_q(K)\,. \tag{5.2}$$

Furthermore, the operators in $\mathscr{H}_\lambda$ can be again represented by their symbols:

$$\sigma : \mathrm{Lin}(\mathscr{H}_\lambda) \to \mathscr{A}_q(K) : B \mapsto \langle \Gamma, B\Gamma\rangle\,. \tag{5.3a}$$

More formally, we should write $(B \otimes \mathrm{id})\Gamma$ instead of $B\Gamma$. We have extended the inner product from $\mathscr{H}_\lambda$ to $\mathscr{H}_\lambda \otimes \mathscr{A}_q(K)$, with values lying in $\mathscr{A}_q(K)$, by

$$\langle x \otimes a, y \otimes b\rangle := \langle x, y\rangle a^* b\,.$$

Proposition 5.2. *The mapping σ is injective.*

Proof. The proof goes through as in the classical case [21, 43]. Let us sketch it. $\sigma(B) = 0$ means that

$$\langle e_\lambda, \mathscr{T}^\lambda B\, S(\mathscr{T}^\lambda)\, e_\lambda\rangle = 0\,.$$

Applying k-times the comultiplication to the LHS, pairing with the elements $X_{i_k}^- \otimes \cdots \otimes X_{i_1}^-$ and using the fact that e_λ is the lowest weight vector and that $(X_i^\pm)^* = X_i^\mp$, we obtain

$$\langle \tau_\lambda(X_{i_1}^+) \cdots \tau_\lambda(X_{i_k}^+)\, e_\lambda, B e_\lambda\rangle = 0\,.$$

Since the vectors $\tau_\lambda(X_{i_1}^+) \cdots \tau_\lambda(X_{i_1}^+)\, e_\lambda$ span $\mathscr{H}_\lambda$, it follows that $B e_\lambda = 0$. Applying instead the comultiplication $(k+1)$-times one finds that the same argument is valid also provided B is replaced in $\mathscr{T}^\lambda B\, S(\mathscr{T}^\lambda) \in \mathrm{Lin}(\mathscr{H}_\lambda) \otimes \mathscr{A}_q(K)$ and so $B S(\mathscr{T}^\lambda)\, e_\lambda = 0$. The same reasoning as above gives $B = 0$. □

The symbol σ can be extended naturally as a mapping from $\mathscr{U}_h(\mathfrak{k}) \equiv \mathscr{A}_q(AN)$ to $\mathscr{A}_q(K)$ by writing $\sigma(u)$ instead of $\sigma(\tau_\lambda(u))$, i.e.

$$\sigma(u) := \langle \Gamma, \tau_\lambda(u)\Gamma\rangle, \text{ or equivalently, } \sigma = (\langle\,\cdot\,\rangle \otimes \mathrm{id}) \circ \mathscr{R}\,. \tag{5.3b}$$

Lemma 5.3. *It holds true that*

$$\Delta \circ \sigma = (\sigma \otimes \mathrm{id}) \circ \mathscr{R}\,, \tag{5.4}$$

$$\varepsilon \circ \sigma = \langle\,\cdot\,\rangle, \tag{5.5}$$

$$\sigma(x^*) = \sigma(x)^*, \quad \forall x \in \mathscr{U}_h(\mathfrak{k}) \equiv \mathscr{A}_q(AN)\,. \tag{5.6}$$

Proof. The relation (5.4) follows immediately from (5.3b) and from the property $(\mathrm{id}\otimes\Delta)\mathscr{R}=(\mathscr{R}\otimes\mathrm{id})\mathscr{R}$. Concerning (5.5), we have

$$\varepsilon\circ(\langle\cdot\rangle\otimes\mathrm{id})\circ\mathscr{R}=\langle\cdot\rangle\circ(\mathrm{id}\otimes\varepsilon)\circ\mathscr{R}=\langle\cdot\rangle\,.$$

The equality (5.6) follows from (5.3b) and (4.8). □

Now we proceed to the definition of the isotropy subgroup as a $*$-Hopf algebra $\mathscr{A}_q(K_0)$ with the vector representation U_0 and the projection ("restriction morphism") $p_0:\mathscr{A}_q(K)\to\mathscr{A}_q(K_0)$, $p_0(U)=U_0$. We require

$$((\langle\cdot\rangle\otimes p_0)\circ\mathscr{R}=\langle\cdot\rangle\,1 \tag{5.7}$$

as a mapping from $\mathscr{A}_q(AN)$ to $\mathbb{C}[[h]]\otimes\mathscr{A}_q(K_0)\equiv\mathscr{A}_q(K_0)$, i.e.

$$p_0(\sigma(Y))=\langle Y\rangle\,1,\quad\text{for }\forall Y\in\mathscr{A}_q(AN)\equiv\mathscr{U}_h(\mathfrak{k})\,. \tag{5.8}$$

Let $\langle\langle\mathscr{J}\rangle\rangle$ be the two-sided ideal in $\mathscr{A}_q(K)$ generated by the elements of the set

$$\mathscr{J}:=\{\sigma(Y)-\langle Y\rangle 1;\ Y\in\mathscr{U}_h(\mathfrak{k})\}\,. \tag{5.9}$$

Lemma 5.4. *$\langle\langle\mathscr{J}\rangle\rangle$ is a two-sided coideal in $\mathscr{A}_q(K)$, $\langle\langle\mathscr{J}\rangle\rangle^*=\langle\langle\mathscr{J}\rangle\rangle$, ε vanishes on $\langle\langle\mathscr{J}\rangle\rangle$ and $1\notin\langle\langle\mathscr{J}\rangle\rangle$.*

Proof. By Definition (5.3b) and by virtue of (5.4) we have

$$\begin{aligned}\Delta(\sigma(Y)-\langle Y\rangle 1)&=(\sigma\otimes\mathrm{id})\mathscr{R}(Y)-\langle Y\rangle\,1\otimes 1\\&=((\sigma-\langle\cdot\rangle 1)\otimes\mathrm{id})\mathscr{R}(Y)+1\otimes(\sigma(Y)-\langle Y\rangle 1)\\&\in(\mathscr{J}\otimes\mathscr{A}_q(K))+(\mathscr{A}_q(K)\otimes\mathscr{J})\,.\end{aligned}$$

Hence $\mathscr{J}$ itself is already a two-sided coideal. $\langle\langle\mathscr{J}\rangle\rangle^*=\langle\langle\mathscr{J}\rangle\rangle$ is true owing to (5.6) and (4.8). The counit vanishes on $\langle\langle\mathscr{J}\rangle\rangle$ because of (5.5). Once this is known, $1\notin\langle\langle\mathscr{J}\rangle\rangle$ since $\varepsilon(1)=1$. □

Definition 5.5. *We define the $*$-Hopf algebra $\mathscr{A}_q(K_0)$ (the quantum stabilizer) by*

$$\mathscr{A}_q(K_0):=\mathscr{A}_q(K)/\langle\langle\mathscr{J}\rangle\rangle\,. \tag{5.10}$$

Denote by $p_0:\mathscr{A}_q(K)\to\mathscr{A}_q(K_0)$ the corresponding $*$-Hopf algebra morphism and set

$$U_0:=p_0(U)\,. \tag{5.11}$$

The $*$-Hopf algebra $\mathscr{U}_h(\mathfrak{k}_0)$ is defined as the subalgebra in $\mathscr{U}_h(\mathfrak{k})$ annihilating $\langle\langle\mathscr{J}\rangle\rangle$ (Ann $\equiv$ annihilator),

$$\mathscr{U}_h(\mathfrak{k}_0):=\mathrm{Ann}(\langle\langle\mathscr{J}\rangle\rangle)=\{u\in\mathscr{U}_h(\mathfrak{k});\ \langle u,a\rangle=0\ \textit{for all}\ a\in\langle\langle\mathscr{J}\rangle\rangle\}\,. \tag{5.12}$$

Clearly, U_0 fulfills

$$RU_{01}U_{02} = U_{02}U_{01}R, \qquad U_0^* = U_0^{-1}, \tag{5.13}$$

and for B_l, C_l, D_l also

$$CU_0^t C^{-1} = U_0^{-1}. \tag{5.14}$$

Notice that according to (3.6) and (4.6),

$$\sigma(\Lambda^*\Lambda) = U^{-1}A_\lambda^2 U. \tag{5.15}$$

By the requirement (5.7), and since $\langle \Lambda^*\Lambda\rangle = A_\lambda^2$, U_0 should fulfill, too,

$$U_0^{-1}A_\lambda^2 U_0 = A_\lambda^2. \tag{5.16}$$

The condition (5.16) is formally the same as in the classical case. In fact, it amounts in annulation of some entries of the matrix U when taking the projection $p_0(U) = U_0$. Observe also that one can replace, in (5.16), A_λ^2 by A_λ since this matrix is diagonal and $(A_{\lambda,ii})^2 \neq (A_{\lambda,jj})^2$ iff $A_{\lambda,ii} \neq A_{\lambda,jj}$.

Let us now explain, quite informally, what characterization of $\mathscr{U}_h(\mathfrak{k}_0)$ we wish to derive. There should exist a subset Π_0 of the set of simple roots Π so that $\mathscr{U}_h(\mathfrak{k}_0)$ is generated by all Cartan elements H_i and only by those elements $X_i^\pm$ for which $\alpha_i \in \Pi_0$. On the dual level we have an injection $\mathscr{U}_h(\mathfrak{k}_0) \hookrightarrow \mathscr{U}_h(\mathfrak{k})$. An element X from $\mathscr{U}_h(\mathfrak{k})$ belongs to $\mathscr{U}_h(\mathfrak{k}_0)$ if and only if

$$\langle X, f\,\sigma(Y)\,g\rangle = \langle X, fg\rangle\langle Y\rangle$$

holds for every $Y \in \mathscr{U}_h(\mathfrak{k})$ and $f, g \in \mathscr{A}_q(K)$. Letting $f = g = 1$ we have (cf. (3.5))

$$\langle \mathrm{ad}_X\, Y\rangle = \langle X, \sigma(Y)\rangle = \varepsilon(X)\,\langle Y\rangle. \tag{5.17}$$

Let us substitute the elements H_i and X_i^+ for X in (5.17). Using $\tau_\lambda(H_i)\,e_\lambda = \lambda(H_i)\,e_\lambda$ and $\tau_\lambda(X_i^-)\,e_\lambda = 0$ we find that (5.17) is true for all Cartan elements H_i and only for those elements X_i^+ which fulfill

$$\langle Y X_i^+\rangle = \langle \tau_\lambda(Y^*)\,e_\lambda, \tau_\lambda(X_i^+)\,e_\lambda\rangle = 0, \quad \forall Y \in \mathscr{U}_h(\mathfrak{k}).$$

Putting $Y^* = X_{i_1}^+ \cdots X_{i_k}^+$ we conclude that the expected condition on Π_0 reads:

$$\alpha_i \in \Pi_0 \quad \text{iff } \tau_\lambda(X_i^+)\,e_\lambda = 0. \tag{5.18}$$

Accept (5.18) for the definition of the subset $\Pi_0 \subset \Pi$. Furthermore, let the symbol $\mathscr{U}_{h0}$ stand for the $*$-Hopf subalgebra in $\mathscr{U}_h(\mathfrak{k})$ generated by all Cartan elements H_i and by those elements $X_i^\pm$ for which $\alpha_i \in \Pi_0$. Take into account the fact that $\mathscr{J}$ is a coideal as well as the well-known rules for comultiplication in $\mathscr{U}_h(\mathfrak{k})$, and observe that according to the above discussion and the Definition (5.12) it is true that

$$\mathscr{U}_{h0} \subset \mathscr{U}_h(\mathfrak{k}_0). \tag{5.19}$$

Denote by $\langle\langle \tilde{\mathscr{J}}\rangle\rangle$ the two-sided ideal in $\mathscr{A}_q(K)$ generated by the elements of the set

$$\tilde{\mathscr{J}} := \{(A_\lambda U - UA_\lambda)_{ij};\ 1 \leqq i,j \leqq \text{dimension of } U\}. \tag{5.20}$$

The equality (5.16) means that $p_0(\tilde{\mathscr{J}}) = \{0\}$ and hence

$$\langle\langle \tilde{\mathscr{J}}\rangle\rangle \subset \langle\langle \mathscr{J}\rangle\rangle \quad \text{and} \quad \mathrm{Ann}(\langle\langle \tilde{\mathscr{J}}\rangle\rangle) \supset \mathrm{Ann}(\langle\langle \mathscr{J}\rangle\rangle). \tag{5.21}$$

Lemma 5.6. *$\langle\langle\tilde{\mathscr{J}}\rangle\rangle$ is a two-sided coideal in $\mathscr{A}_q(K)$, $\langle\langle\tilde{\mathscr{J}}\rangle\rangle^* = \langle\langle\tilde{\mathscr{J}}\rangle\rangle$, ε vanishes on $\langle\langle\tilde{\mathscr{J}}\rangle\rangle$ and $1 \notin \langle\langle\tilde{\mathscr{J}}\rangle\rangle$.*

Proof. We still use the notation: $(E \dot{\otimes} F)_{ij} := \sum_k E_{ik} \otimes F_{kj}$. Clearly, if D is a diagonal matrix with entries from $\mathbb{C}[[h]]$, then $ED \dot{\otimes} F = E \dot{\otimes} DF$. Since $\Delta U = U \dot{\otimes} U$, we have

$$\Delta(A_\lambda U - UA_\lambda) = (A_\lambda U - UA_\lambda) \dot{\otimes} U + U \dot{\otimes} (A_\lambda U - UA_\lambda) .$$

The rest follows from the facts that $U^* = U^{-1}$ and $\varepsilon(U) = \mathbf{I}$. □

Moreover, since the pairing of $\mathscr{U}_h(\mathfrak{k})$ with U gives the vector representation and from the well-known rules for comultiplication in $\mathscr{U}_h(\mathfrak{k})$ one easily finds that

$$\mathscr{U}_{h0} \subset \operatorname{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle) . \tag{5.22}$$

Proposition 5.7. *It holds true that*

$$Ann(\langle\langle\tilde{\mathscr{J}}\rangle\rangle) = \mathscr{U}_{h0} . \tag{5.23}$$

Corollary 5.8. *It holds true that*

$$Ann(\langle\langle\mathscr{J}\rangle\rangle) = \mathscr{U}_{h0} , \tag{5.24}$$

i.e. the $*$-Hopf algebra $\mathscr{U}_h(\mathfrak{k}_0)$ is generated by all Cartan elements H_i and by those elements $X_i^\pm$ for which $\alpha_i \in \Pi_0$. On the other hand, the $*$-Hopf algebra $\mathscr{A}_q(K_0)$ is generated by the entries of the matrix U_0 and is determined by the relations (5.13), (5.14) and by $\Delta U_0 := U_0 \dot{\otimes} U_0$.

Proof. The proof follows from (5.21), (5.19) and (5.23). □

To prove Proposition 5.7 we shall need the quantum Poincaré–Birkhoff–Witt (PBW) theorem. It was proved by Jimbo for $\mathfrak{sl}(2)$ [15], by Rosso for $\mathfrak{sl}(n)$ [39] and by Lusztig in the general case [31] (but see also [24, 26, 6]). An exhaustive survey is presented in [10]. Set $l :=$ rank $\mathfrak{g}$ and let $d := |\Delta^+|$ be the number of positive roots. The Chevalley generators of $\mathscr{U}_h(\mathfrak{k})$ are chosen as usual: $H_1, \ldots, H_l, X_1^\pm, \ldots, X_l^\pm$. There exists a braid group action on $\mathscr{U}_h(\mathfrak{k})$ which we do not recall explicitly but just note that it is generated by l automorphisms T_i related to the simple roots $\alpha_i \in \Pi$ (in addition to the above papers the reader can consult also [30, 20, 25]). Let $w_0 = s_{i_1} s_{i_2} \cdots s_{i_d}$ be a reduced decomposition of the longest element in the Weyl group ($s_i =$ the reflection corresponding to $\alpha_i \in \Pi_i$). It is known that every positive root occurs exactly once in the following set:

$$\beta_1 := \alpha_{i_1}, \qquad \beta_2 := s_{i_2}(\alpha_{i_2}), \ldots, \qquad \beta_d := s_{i_1} s_{i_2} \cdots s_{i_{d-1}}(\alpha_{i_d}) .$$

One defines

$$X_{\beta_\nu}^\pm := T_{i_1} T_{i_2} \cdots T_{i_{\nu-1}}(X_{i_\nu}^\pm) .$$

We do not need the explicit formulas for $X_\beta^\pm$, but the following facts are important: if $\beta = \sum_i k_i \alpha_i$ is a root then, classically, $X_\beta^\pm$ is expressed in $\mathscr{U}(\mathfrak{k})$ as a multiple commutator of those "simple root vectors" $X_i^\pm$ for which $k_i > 0$; in $\mathscr{U}_h(\mathfrak{k})$ similar formulas are valid but with deformed commutators (of the type $[x, y]_q = xy - q\, yx$).

Note that the symbols $X_\beta^\pm$ now make sense in both the classical and the deformed cases. Denote by τ_v the "vector" representation of $\mathcal{U}_h(\mathfrak{k})$ given by pairing with U,

$$\tau_v(x) := \langle x, U \rangle ,$$

and let τ_v^{class} be its classical counterpart. Then an entry of $\tau_v(X_\beta^\pm)$ vanishes if and only if the same entry vanishes for $\tau_v^{\text{class}}(X_\beta^\pm)$ and

$$\text{all entries of } \tau_v(X_\beta^\pm) - \tau_v^{\text{class}}(X_\beta^\pm) \quad \text{belong to } h\mathbb{C}[[h]] . \tag{5.25}$$

Furthermore, using the canonical isomorphism $\mathfrak{h}^* \simeq \mathfrak{h} \subset \mathcal{U}_h(\mathfrak{k})$ for the dual of the Cartan algebra, one can show the equality

$$A_\lambda = \exp(h\, \tau_v^{\text{class}}(\lambda)) . \tag{5.26}$$

Consequently,

$$[\tau_v(X_\beta^\pm), A_\lambda] = 0 \quad \text{iff } [\tau_v^{\text{class}}(X_\beta^\pm), \tau_v^{\text{class}}(\lambda)] = 0 \quad \text{iff } \langle \lambda, \beta \rangle = 0 \quad \text{iff } X_\beta^\pm \in \mathcal{U}_{h0} . \tag{5.27}$$

Returning back to the PBW theorem we recall its content: the following elements form the PBW basis in $\mathcal{U}_h(\mathfrak{k})$ over $\mathbb{C}[[h]]$:

$$(X_{\beta_1}^-)^{r_1} \cdots (X_{\beta_d}^-)^{r_d} H_1^{s_1} \cdots H_l^{s_l} (X_{\beta_d}^+)^{t_d} \cdots (X_{\beta_1}^+)^{t_1} \equiv (X^-)^r H^s (X^+)^t ,$$

with $(r, s, t) \in \mathbb{Z}_+^{d+l+d}$. Hence every element $f \in \mathcal{U}_h(\mathfrak{k})$ can be expressed unambiguously as a sum

$$f = \sum_{r,s,t} c_{r,s,t}(f)\, (X^-)^r H^s (X^+)^t , \tag{5.28}$$

with $c_{r,s,t}(f) \in \mathbb{C}[[h]]$ depending linearly on f. The sum (5.28) is, in principle, infinite, but one requires that for each $n \in \mathbb{N}$, only finitely many coefficients $c_{r,s,t}(f)$ have a non-zero factor image in $\mathbb{C}[[h]]/h^n\mathbb{C}[[h]]$. Let $\{\varepsilon_1, \ldots, \varepsilon_d\}$ be the standard basis in $\mathbb{Z}_+^d$. We define the functionals

$$\psi_i^+ : \mathcal{U}_h(\mathfrak{k}) \to \mathbb{C}[[h]]; \;\; \psi_i^+(f) := c_{\bar{0}, \bar{0}, \varepsilon_i}(f), \quad i = 1, \ldots, d . \tag{5.29}$$

Set

$$D_i^+ := (\psi_i^+ \otimes \mathrm{id}) \circ \Delta : \mathcal{U}_h(\mathfrak{k}) \to \mathbb{C}[[h]] \otimes \mathcal{U}_h(\mathfrak{k}) \equiv \mathcal{U}_h(\mathfrak{k}) . \tag{5.30}$$

The PBW basis makes it possible to define also the derivatives $\partial_i^+ \equiv \partial_{X_{\beta_i}^+}$ by

$$\partial_i^+ (X^-)^r H^s (X^+)^t := t_i\, (X^-)^r H^s (X^+)^{t-\varepsilon_i} . \tag{5.31}$$

Clearly,

$$(D_i^+ - \partial_i^+) \mathcal{U}_h(\mathfrak{k}) \subset h\, \mathcal{U}_h(\mathfrak{k}) . \tag{5.32}$$

One can proceed analogously to define D_j^-, ∂_j^- and D_k^0, ∂_k^0 for the elements X_j^- and H_k, respectively.

Proof of Proposition 5.7. We have to show that $\mathrm{Ann}(\langle\langle \tilde{\mathcal{J}} \rangle\rangle)/\mathcal{U}_{h0} = 0$. It is sufficient to show that

$$\mathrm{Ann}(\langle\langle \tilde{\mathcal{J}} \rangle\rangle)/\mathcal{U}_{h0} \subset h(\mathrm{Ann}(\langle\langle \tilde{\mathcal{J}} \rangle\rangle)/\mathcal{U}_{h0}) , \tag{5.33}$$

since then, by induction, $\mathrm{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle)/\mathscr{U}_{h0} \subset h^n(\mathrm{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle)/\mathscr{U}_{h0})$, for all $n \in \mathbb{N}$. To this end, we shall assume that, contrary to (5.33), there exists an element $f' \in \mathrm{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle)/\mathscr{U}_{h0}$ which does not belong to $h(\mathrm{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle)/\mathscr{U}_{h0})$. The element f' is then the factor image of an element $f \in \mathrm{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle)$ of the form (5.28) such that at least one coefficient $c_{r,s,t}(f)$ is invertible in $\mathbb{C}[[h]]$ (but, on the other hand, the number of these coefficients is finite). Among all these coefficients choose one for which $|r|+|s|+|t|$ is maximal. We can assume that $c_{r,s,t}(f)=1$. To get a contradiction it will be enough to show that whenever $t_i > 0$, then $X^+_{\beta_i} \in \mathscr{U}_{h0}$, and similarly for the indices s_i.

At this point we need the operators D^+_j. Since $\mathrm{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle)$ is a coalgebra we have $D^\sigma_j(\mathrm{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle)) \subset \mathrm{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle)$, $\sigma = +,-,0$. Set

$$g := (D^-)^r(D^0)^s(D^+)^{t-\varepsilon_i} f \in \mathrm{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle)$$

(here $(D^-)^r(D^0)^s(D^+)^t$ stands for the composition of linear mappings). Owing to (5.32) and maximality of $|r|+|s|+|t|$, we have

$$g = r!s!t!\,X^+_{\beta_i} + \sum_{j\neq i} c'_j X^+_{\beta_j} + \sum_j c''_j X^-_{\beta_j} + \sum_k c'''_k H_k + g'\,,$$

where $c'_j, c''_j, c'''_k \in \mathbb{C}[[h]]$ and $g' \in h\,\mathscr{U}_h(\mathfrak{k})$.

Now we can use the fact that all Cartan generators $H_1,\ldots,H_l$ belong to $\mathrm{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle)$ and apply to the element g the adjoint action $x \mapsto q^H x q^{-H}$, where $H = \sum_k \xi_k H_k$ is a general Cartan element. Since then

$$q^H X^\pm_{\beta_j} q^{-H} = q^{\pm\sum_k \xi_k \langle\alpha_k,\beta_j\rangle} X^\pm_{\beta_j}\,,$$

we deduce that

$$X^+_{\beta_i} \in \mathrm{Ann}(\langle\langle\tilde{\mathscr{J}}\rangle\rangle) + h\,\mathscr{U}_h(\mathfrak{k})\,.$$

This means that $[\tau_v(X^+_{\beta_i}), A_\lambda] = h[g'', A_\lambda]$, for some $g'' \in \mathscr{U}_h(\mathfrak{k})$, and hence, by virtue of (5.25) and (5.26), all entries of the matrix $[\tau_v^{\mathrm{class}}(X^+_{\beta_i}), \tau_v^{\mathrm{class}}(\lambda)]$ belong to $h\mathbb{C}[[h]]$. But as the last matrix is complex it must be zero and consequently $X^+_{\beta_i} \in \mathscr{U}_{h0}$, as required. $\square$

From the equality (5.18) it follows immediately that there exists a character χ on $\mathscr{U}_h(\mathfrak{k}_0)$ such that

$$\tau_\lambda(X)\,e_\lambda = \chi(X)\,e_\lambda, \quad \text{for } X \in \mathscr{U}_h(\mathfrak{k}_0)\,. \tag{5.34}$$

Pairing both sides with e_λ one finds that $\chi(\,\cdot\,)$ is the restriction of the "vacuum" functional $\langle\,\cdot\,\rangle$. Considering χ as an element from $\mathscr{A}_q(K_0)$ we deduce that

$$\chi = p_0(w_\lambda) \quad \text{and} \quad \Delta\chi = \chi\otimes\chi\,. \tag{5.35}$$

Moreover, using (5.35), (4.4) and the relation $m\circ(S\otimes\mathrm{id})\circ\Delta = \varepsilon$ we have

$$S\chi = \chi^* = \chi^{-1}\,. \tag{5.36}$$

Let us further introduce the Hopf algebra $\mathscr{U}_h(\mathfrak{p}_0)$ as the Hopf subalgebra of $\mathscr{U}_h(\mathfrak{g})$ generated by all H_i, X^-_i and by those X^+_i for which $\alpha_i \in \Pi_0$. Let $\mathscr{A}_q(P_0)$ be the Hopf algebra generated by entries of the block-lower-triangular matrix T_0 (of the same dimension as T) whose structure of blocks is determined by the set Π_0. It is the same as for the classical subgroup $P_0 \subset G$ formed by block-lower-triangular matrices. The matrix T_0 is required to satisfy the RT_0T_0-equation, in case $B-C-D$ the orthogonality condition and $\det_q T_0 = 1$. Again, there is an algebra morphism

$$\tilde{p}_0 : \mathscr{A}_q(G) \to \mathscr{A}_q(P_0), \qquad \tilde{p}_0(T) := T_0 .$$

There is a natural non-degenerate pairing between the Hopf algebras $\mathscr{U}_h(\mathfrak{p}_0)$ and $\mathscr{A}_q(P_0)$ given by

$$\langle Y, T_0 \rangle := \langle Y, T \rangle, \quad \forall Y \in \mathscr{U}_h(\mathfrak{p}_0) \subset \mathscr{U}_h(\mathfrak{g}) .$$

Observe that all statements (5.34)–(5.36), apart from $\chi^* = \chi^{-1}$, can be made also for P_0 instead of K_0.

Let us now check the equivariance property. The relations (5.4), (5.8) and (5.3b) imply

Proposition 5.9. *It holds true that*

$$(p_0 \otimes \mathrm{id})\, \Delta\, \sigma(Y) = 1 \otimes \sigma(Y), \quad \forall Y \in \mathscr{U}_h(\mathfrak{k}) . \tag{5.37}$$

This means that every symbol $\sigma(Y) \in \mathscr{A}_q(K)$ is left K_0-invariant, i.e. $\sigma(Y) \in \mathscr{A}_q(K_0 \backslash K)$.

Before considering the equivariance of the coherent state itself, let us state a lemma necessary for performing the localization of $\mathscr{A}_q(K_0\backslash K)$ by allowing w_λ to be invertible.

Lemma 5.10. *The element w_λ is neither a right nor left divisor of zero in $\mathscr{A}_q(K)$, i.e.*

$$f w_\lambda = 0 \quad (\text{resp. } w_\lambda f = 0) \Rightarrow f = 0, \ \forall f \in \mathscr{A}_q(K) .$$

Proof. Since it holds that $\langle x, fg \rangle = \langle \Delta x, g \otimes f \rangle$, $\forall x \in \mathscr{A}_q(AN)$, $\forall f, g \in \mathscr{A}_q(K)$ (as the comultiplication in $\mathscr{A}_q(AN)$ is opposite with respect to $\mathscr{U}_h(\mathfrak{k})$) and $\Delta\Lambda = \Lambda \dot{\otimes} \Lambda$, $\Delta\Lambda^* = P(\Lambda^* \dot{\otimes} \Lambda^*)$, with P=the permutation operator, we have

$$\begin{aligned} \langle \Lambda_1^* \cdots \Lambda_j^* \Lambda_{j+1} \cdots \Lambda_{j+k}, fg \rangle^{t_1 \cdots t_j} = {} & \langle \Lambda_1^* \cdots \Lambda_j^* \Lambda_{j+1} \cdots \Lambda_{j+k}, g \rangle^{t_1 \cdots t_j} \\ & \times \langle \Lambda_1^* \cdots \Lambda_j^* \Lambda_{j+1} \cdots \Lambda_{j+k}, f \rangle^{t_1 \cdots t_j} , \end{aligned}$$

where the superscript t_i stands for transposition in the i^{th} factor of the tensor product. According to Proposition 4.1 and using (4.7), (4.6) we find that

$$\langle \Lambda_1^* \cdots \Lambda_j^* \Lambda_{j+1} \cdots \Lambda_{j+k}, w_\lambda \rangle = \langle \Lambda_1^* \cdots \Lambda_j^* \Lambda_{j+1} \cdots \Lambda_{j+k} \rangle = A_{\lambda 1} \cdots A_{\lambda j} A_{\lambda(j+1)} \cdots A_{\lambda(j+k)}$$

is invertible. Thus the relation $f w_\lambda = 0$ (resp. $w_\lambda f = 0$) implies that $\langle \Lambda_1^* \cdots \Lambda_j^* \Lambda_{j+1} \cdots \Lambda_{j+k}, f \rangle = 0$, $\forall j, k = 0, 1, 2, \ldots$ (insert 1 into the first argument if $j = k = 0$). Since the entries of the matrices $\Lambda_1^* \cdots \Lambda_j^* \Lambda_{j+1} \cdots \Lambda_{j+k}$, $j, k = 0, 1, 2, \ldots$, span $\mathscr{A}_q(AN)$ and the pairing between $\mathscr{A}_q(AN)$ and $\mathscr{A}_q(K)$ is non-degenerate, we conclude that $f = 0$, as required. $\square$

Proposition 5.11. *It holds true that*

$$(p_0 \otimes \mathrm{id})\, \Delta\langle \Gamma, u\rangle = \chi \otimes \langle \Gamma, u\rangle\,. \tag{5.38}$$

Particularly, putting $u = e_\lambda$, we have

$$(p_0 \otimes \mathrm{id})\, \Delta w_\lambda = \chi \otimes w_\lambda\,. \tag{5.39}$$

So the quantum function $w_\lambda^{-1}\langle \Gamma, u\rangle$ is left K_0-invariant and belongs to some completion of the algebra $\mathscr{A}_q(K_0\backslash K)$ obtained by allowing w_λ to be invertible.

Proof. First note that (5.34) can be rewritten dually as

$$(\mathrm{id} \otimes p_0)\mathscr{T}^\lambda \cdot (e_\lambda \otimes 1) = e_\lambda \otimes \chi\,.$$

Hence, using the unitarity of $\mathscr{T}^\lambda$ and χ, we have for any $u \in \mathscr{H}_\lambda$,

$$(\langle e_\lambda, (\,\cdot\,)u\rangle \otimes p_0)\mathscr{T}^\lambda = \langle e_\lambda, u\rangle\, \chi\,.$$

It follows that

$$\begin{aligned}(p_0 \otimes \mathrm{id})\, \Delta\langle \Gamma, u\rangle &= (\langle e_\lambda, (\,\cdot\,)u\rangle \otimes p_0 \otimes \mathrm{id})\mathscr{T}^\lambda_{12}\mathscr{T}^\lambda_{13} \\ &= (\langle e_\lambda, (\,\cdot\,)u\rangle\chi \otimes \mathrm{id})\mathscr{T}^\lambda = \chi \otimes \langle \Gamma, u\rangle\,. \qquad \square\end{aligned}$$

6. Canonical Element for the Double

The complex structure on the quantized homogeneous space $K_0\backslash K$ is introduced the same way as in the classical case. Namely, the subalgebra of $\mathscr{A}_q(K_0\backslash K)$ consisting of holomorphic functions coincides with $\mathscr{A}_q(P_0\backslash G)$. Let us make this statement more precise. Recall once more that $f \in \mathscr{A}_q(K_0\backslash K)$ means that $f \in \mathscr{A}_q(K)$ and $(p_0 \otimes \mathrm{id})\Delta f = 1 \otimes f$ and, similarly, $f \in \mathscr{A}_q(P_0\backslash G)$ iff $f \in \mathscr{A}_q(G) \equiv \mathscr{A}_q(K)$ and $(\tilde{p}_0 \otimes \mathrm{id})\Delta f = 1 \otimes f$. Furthermore, in addition to the $*$-algebra morphism $p_0 : \mathscr{A}_q(K) \to \mathscr{A}_q(K_0)$ and the algebra morphism $\tilde{p}_0 : \mathscr{A}_q(G) \equiv \mathscr{A}_q(K) \to \mathscr{A}_q(P_0)$ there exists also the algebra morphism $p_1 : \mathscr{A}_q(P_0) \to \mathscr{A}_q(K_0)$ given by $p_1(T_0) = U_0$. It clearly holds true that $p_0 = p_1 \circ \tilde{p}_0$. This implies

$$\mathscr{A}_q(P_0\backslash G) \subset \mathscr{A}_q(K_0\backslash K) \quad \text{(subalgebra)}$$

and we identify $\mathscr{A}_q(P_0\backslash G)$ with the subalgebra of holomorphic functions $\mathscr{A}_q^{\mathrm{hol}}(K_0\backslash K)$.

We claim that for every $u \in \mathscr{H}_\lambda$, $\langle u, \Gamma\rangle\, (w_\lambda^*)^{-1}$ is a holomorphic quantum function. This implies that one can represent vectors from $\mathscr{H}_\lambda$ by antiholomorphic functions,

$$u \mapsto \psi_u := {w_\lambda}^{-1}\langle \Gamma, u\rangle\,. \tag{6.1}$$

To verify this statement let us first rewrite w_λ^* since in $\mathscr{A}_q(G)$, though identical to $\mathscr{A}_q(K)$ as an algebra, we have no $*$-involution to our disposal. But since $\mathscr{T}^\lambda$ is unitary (as a corepresentation of $\mathscr{A}_q(K)$) we have

$$w_\lambda^* = \langle e_\lambda, \mathscr{T}^\lambda e_\lambda\rangle^* = \langle e_\lambda, (\mathscr{T}^\lambda)^{-1} e_\lambda\rangle$$

and consequently

$$\psi_u^* = \langle u, (\mathscr{T}^\lambda)^{-1} e_\lambda \rangle (\langle e_\lambda, (\mathscr{T}^\lambda)^{-1} e_\lambda \rangle)^{-1} .$$

But now the P_0-invariance is quite obvious since $\Delta(\mathscr{T}^\lambda)^{-1} = P((\mathscr{T}^\lambda)^{-1} \dot{\otimes} (\mathscr{T}^\lambda)^{-1})$, with $P =$ the permutation operator, and, in the same way as in the proof of Proposition 5.11, it holds true that

$$\tilde{p}_0 (\mathscr{T}^\lambda)^{-1} \cdot e_\lambda = e_\lambda \chi^{-1}$$

and hence

$$(\tilde{p}_0 \otimes \mathrm{id}) \Delta \langle u, (\mathscr{T}^\lambda)^{-1} e_\lambda \rangle = \chi^{-1} \otimes \langle u, (\mathscr{T}^\lambda)^{-1} e_\lambda \rangle .$$

Moreover, the mapping (6.1) is injective as one can show using the same reasoning as in the case of the symbol (Sect. 5). It is desirable to introduce quantum (non-commutative) local holomorphic coordinates z_j on $K_0 \backslash K$ and consequently to express $\psi_u = \psi_u(z_j^*)$ as a polynomial in z_j^*. To this end we shall employ the Gauss decomposition.

Denote by $\mathfrak{b}_\pm \subset \mathfrak{g}$ the Borel subalgebras and by $\mathfrak{h} = \mathfrak{b}_+ \cap \mathfrak{b}_-$ the Cartan subalgebra. It is known [11] that the Hopf algebras $\mathscr{U}_h(\mathfrak{b}_+)$ and $\mathscr{U}_h(\mathfrak{b}_-)^{op\Delta}$ are mutually dual and that the dual quantum double for $\mathscr{U}_h(\mathfrak{b}_+)$ can be identified as an algebra with $\mathscr{U}_h(\mathfrak{g}) \otimes \mathscr{U}_h(\mathfrak{h})$. To have this identification also for the coalgebras one has to twist, as shown in [36], the comultiplication in $\mathscr{U}_h(\mathfrak{g}) \otimes \mathscr{U}_h(\mathfrak{h})$ using the element

$$R_0 := \exp\left(-h \sum H_i^0 \otimes H_i^0\right),$$

with $\{H_i^0\}$ being any orthonormal basis in $\mathfrak{h}$. This means that there exists a coalgebra isomorphism

$$\check{\eta} : \mathscr{U}_h(\mathfrak{b}_-) \otimes \mathscr{U}_h(\mathfrak{b}_+) \to \mathscr{U}_h(\mathfrak{g}) \bigotimes_{\mathrm{twist}\Delta} \mathscr{U}_h(\mathfrak{h}),$$

where the twisted comultiplication is given by

$$\Delta_{\mathrm{twist}}(x \otimes y) := x^{(1)} \otimes R_0 (y^{(1)} \otimes x^{(2)}) R_0^{-1} \otimes y^{(2)},$$

with the usual notation $\Delta x = x^{(1)} \otimes x^{(2)}$, $\Delta y = y^{(1)} \otimes y^{(2)}$. According to the terminology we have adopted here the dual quantum double means twisted multiplication while the quantum double means twisted comultiplication.

On the dual level one should consider the corresponding algebras of quantum functions $\mathscr{A}_q(B_-) \otimes \mathscr{A}_q(B_+)$ and $\mathscr{A}_q(G) \bigotimes_{\mathrm{twist}\cdot} \mathscr{A}_q(A)$. A stands for the Abelian subgroup in G with the Lie algebra $\mathfrak{h}$. The vector corepresentations $L^{(\pm)}$ and J of the quantum groups $(B_\pm)_q$ and A_q, respectively, fulfill the corresponding RXX-equations and possibly also the deformed orthogonality condition. For a proper choice of the set Π of simple roots, $L^{(\pm)}$ is upper/lower triangular and J is diagonal.

Let us describe the dual form of the homomorphism $\check{\eta}$. The symbol τ_v still designates the irreducible representation of $\mathscr{U}_h(\mathfrak{g})$ corresponding to the vector corepresentation T of $\mathscr{A}_q(G)$, $T = (\tau_v \otimes \mathrm{id})\rho$.

Proposition 6.1. *The non-degenerate pairing between the coalgebra* $\mathscr{U}_h(\mathfrak{g}) \bigotimes_{\mathrm{twist}\Delta} \mathscr{U}_h(\mathfrak{h})$ *and the algebra* $\mathscr{A}_q(G) \bigotimes_{\mathrm{twist}\cdot} \mathscr{A}_q(A)$ *fulfills the condition*

$$\langle x \otimes y, (f_1 \otimes g_1) \cdot (f_2 \otimes g_2) \rangle = \langle \Delta(x \otimes y), (f_1 \otimes g_1) \otimes (f_2 \otimes g_2) \rangle ,$$

for $\forall x \in \mathcal{U}_h(\mathfrak{g})$, $\forall y \in \mathcal{U}_h(\mathfrak{h})$, $\forall f_1, f_2 \in \mathcal{A}_q(G)$, $\forall g_1, g_2 \in \mathcal{A}_q(A)$, *if and only if the twisted multiplication in* $\mathcal{A}_q(G) \otimes \mathcal{A}_q(A)$ *is defined by*

$$\operatorname{diag}(R)\, T_1 J_2 = J_2 T_1 \operatorname{diag}(R)\,. \tag{6.2}$$

There exists an injective algebra homomorphism

$$\eta : \mathcal{A}_q(G) \underset{\text{twist}\cdot}{\bigotimes} \mathcal{A}_q(A) \to \mathcal{A}_q(B_-) \otimes \mathcal{A}_q(B_+)$$

unambiguously defined by

$$\langle x \otimes y, \eta(f \otimes g)\rangle := \langle \check{\eta}(x \otimes y), f \otimes g\rangle\,, \tag{6.3}$$

for $\forall x \in \mathcal{U}_h(\mathfrak{b}_-)$, $\forall y \in \mathcal{U}_h(\mathfrak{b}_+)$, $\forall f \in \mathcal{A}_q(G)$, $\forall g \in \mathcal{A}_q(A)$. *Its values on the generators are given by*

$$\eta(T \otimes 1) = L^{(-)} \dot{\otimes} L^{(+)}, \quad \eta(1 \otimes J) = (\operatorname{diag} L^{(-)})^{-1} \dot{\otimes} \operatorname{diag} L^{(+)} \tag{6.4}$$

$((L^{(-)} \dot{\otimes} L^{(+)})_{ij} := \sum_k L^{(-)}_{ik} \otimes L^{(+)}_{kj})$.

Proof. Concerning the twisted multiplication it is enough to show that

$$\begin{aligned}&\langle x \otimes y, (1 \otimes J_{j+1} \cdots J_{j+k}) \cdot (T_1 \cdots T_j \otimes 1)\rangle \\ &\quad = \langle R_0 (x \otimes y) R_0^{-1}, T_1 \cdots T_j \dot{\otimes} J_{j+1} \cdots J_{j+k}\rangle\,,\end{aligned} \tag{6.5}$$

for $\forall x \in \mathcal{U}_h(\mathfrak{g})$, $\forall y \in \mathcal{U}_h(\mathfrak{h})$, and $j, k = 0, 1, 2, \ldots$. In the second arguments on both sides, the symbol $\otimes$ stands for the tensor product of algebras while the tensor product of the vector spaces enumerated $1, \ldots, j+k$ is indicated by the leg notation. Using (6.2) one can evaluate the LHS of (6.5) as

$$\prod_{\substack{1 \leqq s \leqq j \\ j+1 \leqq t \leqq j+k}} \operatorname{diag}(R_{st}) \langle x, T_1 \cdots T_j\rangle \langle y, J_{j+1} \cdots J_{j+k}\rangle \prod_{\substack{1 \leqq s \leqq j \\ j+1 \leqq t \leqq j+k}} \operatorname{diag}(R_{st})^{-1}\,.$$

Using the pairing between $\mathcal{U}_h(\mathfrak{g})$ and $\mathcal{A}_q(G)$, $\mathcal{U}_h(\mathfrak{h})$ and $\mathcal{A}_q(A)$, and the facts that $\Delta T = T \dot{\otimes} T$, $\Delta J = J \dot{\otimes} J$, we find that the RHS of (6.5) equals

$$\begin{aligned}&\langle R_0, T_1 \cdots T_j \dot{\otimes} J_{j+1} \cdots J_{j+k}\rangle \langle x, T_1 \cdots T_j\rangle \langle y, J_{j+1} \cdots J_{j+k}\rangle \\ &\quad \times \langle R_0^{-1}, T_1 \cdots T_j \dot{\otimes} J_{j+1} \cdots J_{j+k}\rangle\,.\end{aligned}$$

Since for $\forall H \in \mathfrak{h}$, $\langle H, T\rangle = \langle H, J\rangle = \tau_v(H)$ and $\Delta H = H \otimes 1 + 1 \otimes H$, we have

$$\langle R_0^{\pm 1}, T_1 \cdots T_j \dot{\otimes} J_{j+1} \cdots J_{j+k}\rangle = \exp\left(\mp h \sum_i \sum_{\substack{1 \leqq s \leqq j \\ j+1 \leqq t \leqq j+k}} \tau_v(H_i^0)_s\, \tau_v(H_i^0)_t\right)\,.$$

The equality (6.5) now follows from

$$\operatorname{diag} R = \exp\left(-h \sum_i \tau_v(H_i^0) \otimes \tau_v(H_i^0)\right)\,.$$

To verify the action of η on the generators we should describe $\check{\eta}$ more closely.

It holds

$$\check{\eta}(X_i^- \otimes 1) = X_i^- \otimes \mathrm{e}^{-hH_i/2}, \qquad \check{\eta}(1 \otimes X_i^+) = X_i^+ \otimes \mathrm{e}^{hH_i/2},$$
$$\check{\eta}(H_i \otimes 1) = H_i \otimes 1 - 1 \otimes H_i, \qquad \check{\eta}(1 \otimes H_i) = H_i \otimes 1 + 1 \otimes H_i\,. \tag{6.6}$$

Furthermore, if the elements $x_1, \ldots, x_j$ belong to the set of generators of $\mathscr{U}_h(\mathrm{b}_-)$ and $y_1, \ldots, y_j$ belong to the set of generators of $\mathscr{U}_h(\mathrm{b}_+)$ then

$$\check{\eta}(x_1 \cdots x_j \otimes y_1 \cdots y_k) = \check{\eta}(x_1 \otimes 1) \cdots \check{\eta}(x_j \otimes 1)\check{\eta}(1 \otimes y_1) \cdots \check{\eta}(1 \otimes y_k)\,, \tag{6.7}$$

where the RHS should be evaluated using the standard multiplication in a tensor product of algebras.

Let us first consider $\eta(T \otimes 1)$. We have to show that

$$\langle x \otimes y, L^{(-)} \dot{\otimes} L^{(+)} \rangle = \langle \check{\eta}(x \otimes y), T \otimes 1 \rangle\,, \tag{6.8}$$

for $\forall x \in \mathscr{U}_h(\mathrm{b}_-)$, $\forall y \in \mathscr{U}_h(\mathrm{b}_+)$. The LHS equals $\langle x, L^{(-)}\rangle\langle y, L^{(+)}\rangle$. Using the explicit expression for $\check{\eta}$ (6.6), (6.7) and the fact that the pairing with 1 acts as the counit (and $\varepsilon(H_i) = 0$) we find that the RHS of (6.8) equals

$$\langle xy, T \rangle = \langle x \otimes y, \Delta T \rangle = \langle x, T \rangle \langle y, T \rangle\,.$$

But now the embeddings $\mathscr{U}_h(\mathrm{b}_\pm) \subset \mathscr{U}_h(\mathrm{g})$ mean that $\langle x, T\rangle = \langle x, L^{(-)}\rangle$, $\langle y, T\rangle = \langle y, L^{(+)}\rangle$, and the equality (6.8) follows.

Concerning $\eta(1 \otimes J)$, we have to show that

$$\langle x \otimes y, (\operatorname{diag} L^{(-)})^{-1} \dot{\otimes} \operatorname{diag} L^{(+)} \rangle = \langle \check{\eta}(x \otimes y), 1 \otimes J \rangle\,, \tag{6.9}$$

for $\forall x \in \mathscr{U}_h(\mathrm{b}_-)$, $\forall y \in \mathscr{U}_h(\mathrm{b}_+)$. The LHS equals $\langle x, (\operatorname{diag} L^{(-)})^{-1}\rangle\langle y, \operatorname{diag} L^{(+)}\rangle$. Notice that $\langle x_1 \cdots x_j, \operatorname{diag} L^{(-)}\rangle = 0$ whenever $x_1, \ldots, x_j$ belong to the set of generators of $\mathscr{U}_h(\mathrm{b}_-)$ and at least one of them belongs to $\{X_1^-, \ldots, X_l^-\}$. An analogous statement is valid for $\langle y_1 \cdots y_k, \operatorname{diag} L^{(+)}\rangle$. Moreover, $\langle x_1 \cdots x_j y_1 \cdots y_k, 1\rangle = 0$ whenever $x_s \in \{X_1^-, \ldots, X_l^-\}$ or $y_t \in \{X_1^+, \ldots, X_l^+\}$ for at least one s or t. Consequently, it is sufficient to verify the equality (6.9) for $x = H_{s_1} \cdots H_{s_j}$, $y = H_{t_1} \cdots H_{t_k}$. But now the result follows from the explicit form of $\check{\eta}$ (6.6), (6.7) and from the relations

$$\langle H_{\iota_1} \cdots H_{\iota_m}, (\operatorname{diag} L^{(-)})^{-1} \rangle = (-1)^m \tau_v(H_{\iota_1}) \cdots \tau_v(H_{\iota_m})\,,$$
$$\langle H_{\iota_1} \cdots H_{\iota_m}, \operatorname{diag} L^{(+)} \rangle = \tau_v(H_{\iota_1}) \cdots \tau_v(H_{\iota_m})\,,$$
$$\text{and} \quad \langle H_{\iota_1} \cdots H_{\iota_m}, J \rangle = \tau_v(H_{\iota_1}) \cdots \tau_v(H_{\iota_m})\,. \qquad \square$$

Remarks. In what follows, we shall identify T with $\eta(T \otimes 1)$ whenever convenient. The range of η doesn't cover the algebra $\mathscr{A}_q(B_-) \otimes \mathscr{A}_q(B_+)$ completely but, on the other hand, $\mathscr{A}_q(B_-) \otimes \mathscr{A}_q(B_+)$ can be regarded as some kind of completion of $\mathscr{A}_q(G) \bigotimes_{\text{twist}} \mathscr{A}_q(A)$. To see what kind of completion one has to invert η. This means to decompose T as

$$T = T_{(-)} D T_{(+)}\,,$$

where $T_{(\pm)}$ is upper/lower triangular with units on the diagonal and D is diagonal. This is possible provided one allows some elements of $\mathscr{A}_q(G)$ (the q-minors of T) to be invertible. Then η^{-1} is given by

$$\eta^{-1}(L^{(-)}) = T_{(-)}(DJ^{-1})^{1/2}, \qquad \eta^{-1}(L^{(+)}) = (JD)^{1/2} T_{(+)}\,.$$

So one has to incorporate the square roots of the entries of the diagonal matrices DJ^{-1} and JD, too. Let us just note that from the relation (6.2) one can derive easily that

$$\begin{aligned}
&\operatorname{diag}(R)\,T_{(-)1}\operatorname{diag}(R)^{-1} = J_2\,T_{(-)1}\,J_2^{-1}\,,\\
&\operatorname{diag}(R)^{-1}\,T_{(+)1}\operatorname{diag}(R) = J_2^{-1}T_{(+)1}\,J_2\,,\\
&D_1J_2 = J_2D_1\,.
\end{aligned}$$

This just described structure has turned out to be very helpful in construction of the universal R-matrix $R^u \in \mathscr{U}_h(\mathfrak{g}) \otimes \mathscr{U}_h(\mathfrak{g})$ [20, 27]. As explained in the paragraph preceding the proof of Proposition 5.7, by fixing a reduced decomposition of the maximal Weyl element one orders the set Δ^+ of positive roots as $(\beta_1, \ldots, \beta_d)$, $d = |\Delta^+|$. To each root β_j there are related elements $E(j) := X^+_{\beta_j} \in \mathscr{U}_h(\mathfrak{b}_+)$ and $F(j) := X^-_{\beta_j} \in \mathscr{U}_h(\mathfrak{b}_-)$ so that the elements

$$E(d)^{n_d} \cdots E(1)^{n_1} H_l^{m_l} \cdots H_1^{m_1}\,, \tag{6.10}$$

$n_i, m_i \in \mathbb{Z}_+$, form a basis in $\mathscr{U}_h(\mathfrak{b}_+)$. The vectors H_i can be replaced by any elements forming a basis in $\mathfrak{h}$. A similar assertion is valid also for $\mathscr{U}_h(\mathfrak{b}_-)$. In the nondeformed case, i.e., specializing h to 0, the elements $E(j)$ and $F(j)$ become the root vectors $X_{\beta_j} \in \mathfrak{n}_+$ and $X_{-\beta_j} \in \mathfrak{n}_-$, respectively. We recall that the universal R-matrix, originally derived in [20, 27], can be also written in the form [22]

$$R^u = \exp_{q_d}(\mu_d\,F(d) \otimes E(d)) \cdots \exp_{q_1}(\mu_1\,F(1) \otimes E(1)) \exp(\kappa)\,, \tag{6.11}$$

where $\exp_q$ are the q-deformed exponential functions, μ_j are some coefficients depending on the parameter h and κ is some element from $\mathscr{U}_h(\mathfrak{h}) \otimes \mathscr{U}_h(\mathfrak{h})$.

Equipped with this knowledge we are able to reveal the structure of the canonical element for the double $\mathscr{A}_q(AN) \otimes \mathscr{A}_q(K)$. We make use of the fact that $\mathscr{A}_q(AN) \simeq \mathscr{U}_h(\mathfrak{g})^{op\Delta}$ is a factor algebra of $\mathscr{U}_h(\mathfrak{b}_-)^{op\Delta} \bigotimes_{\mathrm{twist}} \mathscr{U}_h(\mathfrak{b}_+)^{op\Delta}$ and $\mathscr{A}_q(K) \simeq \mathscr{U}_h(\mathfrak{g})^*$ is a subalgebra in $\mathscr{U}_h(\mathfrak{b}_+)^{op\cdot} \otimes \mathscr{U}_h(\mathfrak{b}_-)^{op\Delta}$. The canonical element $\tilde{\rho}$ in

$$\Big(\mathscr{U}_h(\mathfrak{b}_-)^{op\Delta} \bigotimes_{\mathrm{twist}} \mathscr{U}_h(\mathfrak{b}_+)^{op\Delta}\Big) \otimes (\mathscr{U}_h(\mathfrak{b}_+)^{op\cdot} \otimes \mathscr{U}_h(\mathfrak{b}_-)^{op\Delta}) \tag{6.12}$$

can be decomposed as follows [12]:

$$\begin{aligned}
\tilde{\rho} &= \textstyle\sum (e_j \otimes e^k) \otimes (f^j \otimes f_k)\\
&= \textstyle\sum (e_j \otimes 1 \otimes f^j \otimes 1) \cdot (1 \otimes e^k \otimes 1 \otimes f_k)\\
&= \tilde{R}_{13}\tilde{R}'_{24}\,.
\end{aligned} \tag{6.13}$$

Here $\{e_j\}$, $\{e^k\}$, $\{f^j\}$ and $\{f_k\}$ stand for bases in the corresponding factors, $\{e_j\}$ and $\{f^j\}$ are dual and the same is assumed about $\{e^k\}$ and $\{f_k\}$, the dot in the third member of equalities (6.13) indicates multiplication in the double and $\tilde{R}'$ is obtained from $\tilde{R}$ by reversing the order of multiplication ($\tilde{R}$ and $\tilde{R}'$ are defined by the last equality in (6.13)). To express ρ we shall use again bases of the type (6.10). In our notation the elements $F(j)$, $E(j)$, $\tilde{E}(j)$ and $\tilde{F}(j)$ belong in this order to the individual factors in (6.12) (the tildes here have been used just to distinguish $E(j)$ from $\tilde{E}(j)$ etc.). Factorizing off the redundant Cartan elements we obtain finally

Proposition 6.2. *The canonical element for the quantum double $\mathscr{A}_q(AN) \otimes \mathscr{A}_q(K)$ has the form*

$$\rho = \exp_{q_d}(\mu_d F(d) \otimes \tilde{E}(d)) \cdots \exp_{q_1}(\mu_1 F(1) \otimes \tilde{E}(1)) \exp(\kappa)$$
$$\times \exp_{q_1}(\mu_1 E(1) \otimes \tilde{F}(1)) \cdots \exp_{q_d}(\mu_d E(d) \otimes \tilde{F}(d)) . \qquad (6.14)$$

To proceed further in this analysis we note that the maximal Weyl element can be chosen so that there exists $p \in \mathbb{Z}_+$, $p \leqq d$, such that the vectors $X_{-\beta_1}, \dots, X_{-\beta_d}$, $H_1, \dots, H_l$, $X_{\beta_1}, \dots, X_{\beta_p}$ form a basis of $\mathfrak{p}_0$. Then $X_{\beta_{p+1}}, \dots, X_{\beta_d}$ form a basis of a nilpotent subalgebra $\mathfrak{n}_0$ and $\mathfrak{g} = \mathfrak{p}_0 \oplus \mathfrak{n}_0$ (remark: in the generic case $\Pi_0 = \emptyset$ and hence $p = 0, \mathfrak{p}_0 = \mathfrak{b}_-$ and $\mathfrak{n}_0 = \mathfrak{n}_+$). This means that all elements $F(j)$ belong to $\mathscr{U}_h(\mathfrak{p}_0)$ while $E(j)$ belongs to $\mathscr{U}_h(\mathfrak{p}_0)$ only for $j = 1, \dots, p$. Consequently,

$$\tau_\lambda(F(j))\, e_\lambda = 0, \quad \text{for } j = 1, \dots, d,$$
$$\tau_\lambda(E(j))\, e_\lambda = 0, \quad \text{for } j = 1, \dots, p. \qquad (6.15)$$

Corollary 6.3. *T can be written as a product,*

$$T = \Lambda_{(-)} Z, \qquad (6.16)$$

where

$$\Lambda_{(-)} = (\tau_v \otimes \mathrm{id}) \exp_{q_d}(\mu_d F(d) \otimes \tilde{E}(d)) \cdots \exp_{q_1}(\mu_1 F(1) \otimes \tilde{E}(1)) \exp(\kappa)$$
$$\times \exp_{q_1}(\mu_1 E(1) \otimes \tilde{F}(1)) \cdots \exp_{q_p}(\mu_p E(p) \otimes \tilde{F}(p)),$$
$$Z = (\tau_v \otimes \mathrm{id}) \exp_{q_{p+1}}(\mu_{p+1} E(p+1) \otimes \tilde{F}(p+1)) \cdots \exp_{q_d}(\mu_d E(d) \otimes \tilde{F}(d)) .$$

The matrix $\Lambda_{(-)}$ is block lower triangular, Z is block upper triangular and the blocks on the diagonal of Z are unit matrices.

Remark. The splitting into the blocks is determined by decomposition of $\mathfrak{g}_0 =$ "complexification of $\mathfrak{k}_0$" into the direct sum of simple subalgebras and an Abelian subalgebra and it will be described more explicitly in the next section. In the generic case of $\Pi_0 = \emptyset$, $\mathfrak{g}_0 = \mathfrak{h}$ and the matrices $\Lambda_{(-)}$ and Z are simply lower and upper triangular.

Notice that the entries of Z are expressed as polynomials in $d - p = \dim_{\mathbb{C}}(P_0 \backslash G)$ noncommutative variables $\tilde{F}(p+1), \dots, \tilde{F}(d)$ and can be considered as local holomorphic coordinates on the orbit. Next we are going to derive explicit commutation relations for them.

Recalling the Definition (5.1) of the coherent state Γ and using the relations (6.14), (6.15), we obtain

$$\Gamma = \exp_{q_d}^{-1}(\mu_d\, \tau_\lambda(E(d)) \otimes \tilde{F}(d)) \cdots$$
$$\times \exp_{q_{p+1}}^{-1}\big(\mu_{p+1}\, \tau_\lambda(E(p+1)) \otimes \tilde{F}(p+1)\big) \cdot (e_\lambda \otimes w_\lambda^*), \qquad (6.17)$$

since

$$w_\lambda^* = (\langle e_\lambda, \tau_\lambda(\,\cdot\,) e_\lambda \rangle \otimes \mathrm{id})\, \rho^{-1} = \exp((\langle e_\lambda, \tau_\lambda(\,\cdot\,) e_\lambda \rangle \otimes \mathrm{id})\, \kappa) .$$

Thus we find again that, for every $u \in \mathscr{H}_\lambda$, ψ_u given by (6.1) is an antiholomorphic quantum function and should be expressible in the variables z^*.

7. Quantum Holomorphic Coordinates on a General Dressing Orbit

We start from the decomposition $T = \Lambda_{(-)}Z$. The symbol $\tilde{p}_0$ still stands for the "restriction" morphism $\mathscr{A}_q(G) \to \mathscr{A}_q(P_0)$. First we shall verify that the entries of Z are left P_0-invariant quantum functions. We have

$$(\tilde{p}_0 \otimes \mathrm{id})\, \Delta T = (\tilde{p}_0 \otimes \mathrm{id})\, \Delta \Lambda_{(-)} \cdot (\tilde{p}_0 \otimes \mathrm{id})\, \Delta Z .$$

At the same time,

$$(\tilde{p}_0 \otimes \mathrm{id})\, \Delta T = \tilde{p}_0(T) \dot{\otimes} T = (\tilde{p}_0(T) \dot{\otimes} \Lambda_{(-)})(\mathbf{I} \dot{\otimes} Z) .$$

Since the decomposition into a product of block lower triangular and block upper triangular matrices, the latter having unit diagonal blocks, is unambiguous we find by comparing that

$$(\tilde{p}_0 \otimes \mathrm{id})\, \Delta Z = \mathbf{I} \dot{\otimes} Z . \tag{7.1}$$

To derive commutation relations for the matrix elements of Z one can again employ the Gauss decomposition. This time we have in mind the injective morphism η (Proposition 6.1) and the identification $T \equiv \eta(T \otimes 1)$. We are going to enumerate the matrix elements in the vector representation by weights. This is possible since for all four principal series A, B, C, D, the weights of the vector representations are simple. Every weight belongs either to the Weyl group orbit of the corresponding fundamental weight or is zero (only for the series B). We shall use the standard ordering on the set of weights: $\sigma > \nu$ iff $\sigma \neq \nu$ and $\sigma - \nu = \sum m_i \alpha_i$, with $m_i \in \mathbb{Z}_+$ ($0 \in \mathbb{Z}_+$). Set

$$\mathscr{W}_0 = \bigoplus_{\alpha_i \in \Pi_0} \mathbb{Z}_+ \alpha_i \,; \tag{7.2}$$

for $\Pi_0 = \emptyset$ we have $\mathscr{W}_0 = \{0\}$ by definition. We shall write simply $L = (L_{\sigma\nu})$ instead of $L^{(+)}$. Thus $L_{\sigma\nu} = 0$ whenever $\sigma < \nu$ (pay attention, the ordering on weights is reversed in comparison with the standard enumeration of weights and weight vectors in the vector representation). Further we introduce a matrix A by

$$\begin{aligned} A_{\sigma\nu} &= L_{\sigma\nu}, \quad \text{if } \sigma - \nu \in \mathscr{W}_0 ,\\ &= 0, \quad \text{otherwise} . \end{aligned} \tag{7.3}$$

Comparing (6.4) and (6.16) we obtain

$$Z = A^{-1} L . \tag{7.4}$$

Next we recall a useful property of the R-matrix. Namely, $R_{\sigma\tau,\mu\nu} \neq 0$ implies $\sigma - \mu = \nu - \tau$, $\sigma \leqq \mu$, $\tau \geqq \nu$, and one of the following three possibilities happens:

(i) $\sigma = \mu$, $\tau = \nu$,
(ii) $\sigma = \nu < \tau = \mu$,
(iii) $\sigma = -\tau < \mu = -\nu$.

We continue by deriving some auxiliary relations. The first one is

Lemma 7.1. *It holds*

$$\Delta A = A \dot{\otimes} A , \quad \text{in } \mathscr{A}_q(B_+) , \tag{7.5}$$

and consequently

$$R A_1 A_2 = A_2 A_1 R . \tag{7.6}$$

Proof. In the equality

$$\Delta L_{\sigma\nu} = \sum_{\xi} L_{\sigma\xi} \otimes L_{\xi\nu} ,$$

the nonzero summands should fulfill $\sigma \geqq \xi \geqq \nu$. To obtain (7.5) it is enough to notice that then $\sigma - \nu \in \mathscr{W}_0$ implies $\sigma - \xi$, $\xi - \nu \in \mathscr{W}_0$.

The relation (7.6) is the same as

$$\langle Y, R A_1 A_2 - A_2 A_1 R \rangle = 0 , \quad \text{for all } Y \in \mathscr{U}_h(\mathfrak{b})_+ .$$

The last equality can be deduced from the following facts. This relation is valid provided A is replaced by L. Clearly $\langle X_i^+, A\rangle = 0$ whenever $\alpha_i \notin \Pi_0$ and so

$$\langle Y_1 X_i^+ Y_2, A\rangle = 0 , \quad \text{for } \alpha_i \notin \Pi_0 \text{ and any } Y_1, Y_2 \in \mathscr{U}_h(\mathfrak{b}_+) .$$

Finally,

$$\begin{aligned} \langle H_i, A\rangle &= \langle H_i, L\rangle, \quad \text{for all } i , \\ \langle X_i^+, A\rangle &= \langle X_i^+, L\rangle, \quad \text{provided } \alpha_i \in \Pi_0 . \qquad \square \end{aligned}$$

By annulating some entries of the R-matrix we define another matrix $Q = Q_{12}$,

$$\begin{aligned} Q_{\sigma\tau,\mu\nu} &= R_{\sigma\tau,\mu\nu} , \quad \text{provided } \tau - \nu = \mu - \sigma \in \mathscr{W}_0 , \\ &= 0 , \quad \text{otherwise}. \end{aligned} \tag{7.7}$$

Lemma 7.2. *It holds*

$$Q L_1 A_2 = A_2 L_1 Q \tag{7.8}$$

and

$$Q A_1 A_2 = A_2 A_1 Q . \tag{7.9}$$

Proof. To show (7.8) assume in the equality

$$\sum_{\xi\eta} R_{\sigma\tau,\xi\eta} L_{\xi\mu} L_{\eta\nu} = \sum_{\xi\eta} L_{\tau\eta} L_{\sigma\xi} R_{\xi\eta,\mu\nu} ,$$

that $\tau - \nu \in \mathscr{W}_0$. The nonzero summands on both sides should fulfill $\tau \geqq \eta \geqq \nu$, whence $\tau - \eta$, $\eta - \nu \in \mathscr{W}_0$. Thus we obtain

$$\sum_{\xi\eta} Q_{\sigma\tau,\xi\eta} L_{\xi\mu} A_{\eta\nu} = \sum_{\xi\eta} A_{\tau\eta} L_{\sigma\xi} Q_{\xi\eta,\mu\nu} . \tag{7.10}$$

It remains to verify validity of (7.10) also for $\tau - \nu \notin \mathscr{W}_0$. Again, the nonzero summands on both sides of (7.10) should satisfy $\tau - \eta$, $\eta - \nu \in \mathscr{W}_0$. But $\mathscr{W}_0$ is additive and so this can never happen.

Let us show (7.9). Assume in (7.10) that $\mu \geqq \sigma$. The nonzero summands on the LHS should fulfill $\xi - \sigma \in \mathscr{W}_0$ and $\xi \geqq \mu \geqq \sigma$ whence $\xi - \mu \in \mathscr{W}_0$. Analogously for the RHS we have $\mu - \xi \in \mathscr{W}_0$ and $\mu \geqq \sigma \geqq \xi$ whence $\sigma - \xi \in \mathscr{W}_0$. Thus we obtain in this case

$$\sum_{\xi\eta} Q_{\sigma\tau,\xi\eta} A_{\xi\mu} A_{\eta\nu} = \sum_{\xi\eta} A_{\tau\eta} A_{\sigma\xi} Q_{\xi\eta,\mu\nu} . \tag{7.11}$$

Next assume in (7.10) that $\sigma - \mu \in \mathscr{W}_0$. The nonzero summands on the LHS should fulfill $\xi - \sigma \in \mathscr{W}_0$ whence, owing to the additivity, $\xi - \mu \in \mathscr{W}_0$. Analogously for the RHS we have $\mu - \xi \in \mathscr{W}_0$ and hence $\sigma - \xi \in \mathscr{W}_0$. Also in this case we arrive at (7.11). It remains to verify (7.11) for $\sigma > \mu$ but $\sigma - \mu \notin \mathscr{W}_0$. Now the nonzero summands on the LHS of (7.11) should fulfill $\xi - \sigma, \xi - \mu \in \mathscr{W}_0$. But this can never happen since then $\mu < \sigma \leqq \xi$ and $\sigma - \mu$ would belong to $\mathscr{W}_0$. Analogously on the RHS, it never happens that, at the same time, $\mu - \xi$ and $\sigma - \xi$ belong to $\mathscr{W}_0$. □

The final relation we shall need is

Lemma 7.3. *It holds*

$$A_2^{-1} Z_1 A_2 = Q^{-1} Z_1 Q \,. \tag{7.12}$$

Proof. One can verify (7.12) by using in (7.8) the substitution $L = AZ$ and the equality (7.9),

$$A_2 A_1 Z_1 Q = Q A_1 A_2 A_2^{-1} Z_1 A_2 = A_2 A_1 Q A_2^{-1} Z_1 A_2 \,. \qquad \square$$

Now we are able to state the desired commutation relation.

Proposition 7.4. *The matrix Z obeys the equality*

$$R Q_{12}^{-1} Z_1 Q_{12} Z_2 = Q_{21}^{-1} Z_2 Q_{21} Z_1 R \,. \tag{7.13}$$

Proof. To prove (7.13) use the substitution $L = AZ$ in the *RLL*-equation,

$$R A_1 A_2 (A_2^{-1} Z_1 A_2) Z_2 = A_2 A_1 (A_1^{-1} Z_2 A_1) Z_1 R \,,$$

and apply (7.6) and (7.12),

$$A_2 A_1 R Q_{12}^{-1} Z_1 Q_{12} Z_2 = A_2 A_1 Q_{21}^{-1} Z_2 Q_{21} Z_1 R \,. \qquad \square$$

This result should be completed by the relations following from the q-deformed orthogonality condition.

Proposition 7.5. *For the series B, C and D, the matrix Z fulfills also*

$$\delta_{jk} = \sum_s (Z_2 C_2 Q Z_2^t Q^{-1} C_2^{-1})_{kj,ss} \,. \tag{7.14}$$

Proof. Since $C L^t C^{-1} = L^{-1}$, $C A^t C^{-1} = A^{-1}$, we have

$$C(AZ)^t C^{-1} = Z^{-1} A^{-1} = Z^{-1} C A^t C^{-1} \,. \tag{7.15}$$

Furthermore, multiplying (7.12) by C_2^{-1} from the left and by C_2 from the right one obtains

$$A_2^t Z_1 (A_2^t)^{-1} = \tilde{Q}^{-1} Z_1 \tilde{Q} \,, \quad \text{where } \tilde{Q} = C_2^{-1} Q C_2 \,.$$

Using this relation one can derive for the matrix elements

$$[(AZ)^t (A^t)^{-1}]_{jk} = \sum_s [A_2^t Z_1 (A_2^t)^{-1}]_{ss,jk} = \sum_s (\tilde{Q}^{-1} Z_1 \tilde{Q})_{ss,jk} \,.$$

Consequently,

$$\begin{aligned}
[ZC(AZ)^t (A^t)^{-1} C^{-1}]_{jk} &= \sum_{stu} (ZC)_{js} [\tilde{Q}^t Z_1^t (\tilde{Q}^t)^{-1}]_{st,uu} (C^{-1})_{tk} \\
&= \sum_s (Z_2 C_2 Q Z_2^t Q^{-1} C_2^{-1})_{kj,ss} \,.
\end{aligned}$$

In view of (7.15) we have arrived at the sought relations. □

In the generic case ($\Pi_0 = \emptyset$) the dressing orbit is nothing but the flag manifold. In this case $Q_{12} = Q_{21} = \operatorname{diag} R$ and Z is an upper triangular matrix with units on the diagonal. The relation (7.13) can be simplified since $\operatorname{diag} R$ commutes with R,

$$R Z_1 \operatorname{diag}(R) Z_2 = Z_2 \operatorname{diag}(R) Z_1 R\,. \tag{7.16}$$

For the series A, i.e., $K = SU_q(N)$ we have

$$\begin{aligned} R_{jk,st} &= \delta_{js}\delta_{kt} + (q - q^{\operatorname{sgn}(k-j)})\,\delta_{jt}\delta_{ks}\,, \\ Q_{jk,st} &= q^{\delta_{jk}}\,\delta_{js}\delta_{kt}\,, \end{aligned}$$

and the relation (7.16) can be rewritten for the individual matrix entries as

$$q^{\delta_{ks}} z_{js} z_{kt} - q^{\delta_{jt}} z_{kt} z_{js} = (q^{\operatorname{sgn}(k-j)} - q^{\operatorname{sgn}(s-t)}) q^{\delta_{js}} z_{ks} z_{jt}\,. \tag{7.17}$$

The relations (7.17) are already known [52, 3]. Originally they were obtained by expressing the entries z_{jk} by means of the q-minors ($j < k$),

$$z_{jk} = \left| T^{1\cdots j}_{1\cdots j} \right|_q^{-1} \left| T^{1\cdots j}_{1\cdots j-1,k} \right|_q\,.$$

But this derivation seems to be rather tedious and doesn't suggest the compact form (7.16).

8. Representation Acting in a Space of Antiholomorphic Functions

Let us denote by $\mathscr{C}_\lambda$ the algebra of quantum holomorphic functions living on the cell. This means that $\mathscr{C}_\lambda$ is generated by the entries of Z fulfilling (7.13) and in case $B-C-D$ also relations following from the deformed orthogonality condition. $\mathscr{C}_\lambda^*$ stands for the algebra of antiholomorphic functions determined by the adjoint relations. We know that every vector $u \in \mathscr{H}_\lambda$ is represented by an element $\psi_u \equiv \psi_u(z^*)$ from $\mathscr{C}_\lambda^*$ (cf. (6.1)), the mapping $u \mapsto \psi_u$ is linear and injective and the lowest weight vector is sent to the unit. Denote by $\mathscr{M}_\lambda \subset \mathscr{C}_\lambda^*$ the image of $\mathscr{H}_\lambda$. We wish to transcribe the representation τ_λ as acting in $\mathscr{M}_\lambda$, but without introducing a special symbol for this new realization. We recall that both $\mathscr{A}_q(K)$ and $\mathscr{A}_q(K_0\backslash K)$ become left $\mathscr{U}_h(\mathfrak{k})$-modules provided one relates to every element $Y \in \mathscr{U}_h(\mathfrak{k})$ the left–invariant map ξ_Y on K_q,

$$\xi_Y \cdot f = (\mathrm{id} \otimes \langle Y, \cdot\,\rangle)\Delta f\,, \quad f \in \mathscr{A}_q(K)\,. \tag{8.1}$$

Then $\mathscr{C}_\lambda^*$ becomes a left $\mathscr{U}_h(\mathfrak{k})$-module with respect to the action

$$(Y, f) \mapsto w_\lambda^{-1}\,\xi_Y \cdot (w_\lambda f)\,. \tag{8.2}$$

Proposition 8.1. *$\mathscr{M}_\lambda$ is the cyclic $\mathscr{U}_h(\mathfrak{k})$-submodule in $\mathscr{C}_\lambda^*$ with the cyclic vector* 1, *i.e.,*

$$\tau_\lambda(Y)\psi = w_\lambda^{-1}\,\xi_Y \cdot (w_\lambda \psi), \quad \textit{for } \forall Y \in \mathscr{U}_h(\mathfrak{k}),\ \forall \psi \in \mathscr{M}_\lambda\,. \tag{8.3}$$

Proof. The proof is done by the following chain of equalities,

$$\begin{aligned} w_\lambda \tau_\lambda(Y)\psi_u &= \langle \Gamma, \tau_\lambda(Y)u\rangle = (\langle e_\lambda, \tau_\lambda(\,\cdot\,)\tau_\lambda(Y)u\rangle \otimes \mathrm{id})\rho \\ &= (\langle e_\lambda, \tau_\lambda(\,\cdot\,)u\rangle \otimes \mathrm{id} \otimes \langle Y, \,\cdot\,\rangle)\rho_{12}\rho_{13} \\ &= (\mathrm{id} \otimes \langle Y, \,\cdot\,\rangle)\, \Delta\,(\langle e_\lambda, \tau_\lambda(\,\cdot\,)u\rangle \otimes \mathrm{id})\,\rho \\ &= \xi_Y \cdot (w_\lambda \psi)\,. \end{aligned}$$

In the third equality we have used the identity

$$\tau_\lambda(Y) = (\tau_\lambda(\,\cdot\,) \otimes \langle Y, \,\cdot\,\rangle)\rho\,. \qquad \square$$

Remark. With the same success we could use for the mapping $\mathscr{H}_\lambda \to \mathscr{C}_\lambda^*$, instead of (6.1), the prescription $u \mapsto \psi'_u := \langle \Gamma, u\rangle w_\lambda^{-1}$. This would lead to an equivalent representation τ'_λ acting in a cyclic submodule $\mathscr{M}'_\lambda$, again with the cyclic vector 1, according to the formula

$$\tau'_\lambda(Y)\psi = (\xi_Y \cdot (\psi w_\lambda))w_\lambda^{-1}\,.$$

Finally we are going to show that the reproducing kernel can be introduced also in the quantum case and the scalar product in $\mathscr{M}_\lambda$can be expressed with its help. Let η designate the Haar measure on $\mathscr{A}_q(K)$. We have the orthogonality relations [54]

$$\eta(\langle u_1, \mathscr{T}^\lambda v_1\rangle^* \langle u_2, \mathscr{T}^\lambda v_2\rangle) = M_\lambda^{-1}\,\langle v_1, v_2\rangle\langle u_1, \tau_\lambda(\gamma^{-1})\,u_2\rangle\,, \tag{8.4}$$

where

$$\gamma = \exp\Big(-\frac{h}{2}\sum_{\alpha>0} H_\alpha\Big), \qquad M_\lambda = \mathrm{tr}\,\tau_\lambda(\gamma^2)\,.$$

Letting $u_1 = u_2 = e_\lambda$ in (8.4) we get

$$\langle u, v\rangle = c_\lambda\,\eta(\langle e_\lambda, \mathscr{T}^\lambda u\rangle^* \langle e_\lambda, \mathscr{T}^\lambda v\rangle)\,, \tag{8.5}$$

where $c_\lambda = M_\lambda / \langle e_\lambda, \tau_\lambda(\gamma^{-1})e_\lambda\rangle$. Consequently,

$$\langle u, v\rangle = c_\lambda\,\eta(\psi_u^* w_\lambda^* w_\lambda \psi_v)\,. \tag{8.6}$$

Set now

$$\Psi(z) = \Gamma\,(w_\lambda^*)^{-1} \in \mathscr{H}_\lambda \otimes \mathscr{C}_\lambda\,, \tag{8.7}$$

and define the reproducing kernel as

$$K(\zeta^*, z) := \langle \Psi(\zeta), \Psi(z)\rangle \in \mathscr{C}_\lambda^* \otimes \mathscr{C}_\lambda\,. \tag{8.8}$$

Here ζ^* stands for the generators in $\mathscr{C}_\lambda^*$ and z for those in $\mathscr{C}_\lambda$.

It holds

$$\langle u, v\rangle = c_\lambda\,\eta(\psi_u(z^*)^* K(z^*, z)^{-1}\psi_v(z^*))\,. \tag{8.9}$$

It is enough to notice that $K(z^*, z) \in \mathscr{C}_\lambda^* \cdot \mathscr{C}_\lambda$ is equal to $(w_\lambda^* w_\lambda)^{-1}$,

$$K(z^*, z) = w_\lambda^{-1}\,\langle e_\lambda, \mathscr{T}^\lambda(\mathscr{T}^\lambda)^{-1}e_\lambda\rangle\,(w_\lambda^*)^{-1} = w_\lambda^{-1}(w_\lambda^*)^{-1}\,.$$

Furthermore, substituting $\Psi(\zeta)$ for u in (8.9) we obtain

$$\psi(\zeta^*) = c_\lambda\,\eta_z(K(\zeta^*, z)K(z^*, z)^{-1}\psi(z^*)), \text{ for every } \psi \in \mathscr{C}_\lambda^*\,. \tag{8.10}$$

9. Representations and Non-Commutative Differential Geometry

We shall use the summation rule through this section. All indices are running from 1 to N, N being the dimension of the vector representation. With some abuse of notation we shall no longer distinguish between the element $X \in \mathscr{U}_h(\mathfrak{k})$ and the corresponding left–invariant mapping ξ_X (8.1). We keep only the $\cdot$ to indicate the action of $\mathscr{U}_h(\mathfrak{k})$ on $\mathscr{A}_q(K)$. The following notions and facts concerning the differential calculus on $\mathscr{A}_q(K)$ will be useful [55, 18, 9]. Let us denote as M_{ijkl} the following family of quantum functions on K:

$$M_{ijkl} = S^{-1}(U_{lj})U_{ik} \, .$$

Let also

$$f_{ijkl} = S^2(L^+_{jl})S(L^-_{ki})$$

be a family of elements of $\mathscr{U}_h(\mathfrak{k})$. We shall denote by $\mathscr{E}$ the free left module over $\mathscr{A}_q(K)$ with generators denoted by Ω_{ij}. Let us introduce the right multiplication, the right coaction δ_R and the left coaction δ_L of $\mathscr{A}_q(K)$ on $\mathscr{E}$ by

$$\begin{aligned} a_{ij}\Omega_{ij}b &= a_{ij}f_{ijkl} \cdot b\Omega_{kl} \, , \\ \delta_R(a_{ij}\Omega_{ij}) &= \Delta a_{ij}(\Omega_{kl} \otimes M_{klij}) \, , \\ \delta_L(a_{ij}\Omega_{ij}) &= \Delta a_{ij}(1 \otimes \Omega_{ij}) \, , \end{aligned}$$

for $a_{ij},\ b \in \mathscr{A}_q(K)$. Then the triple $(\mathscr{E}, \delta_R, \delta_L)$ is an $\mathscr{A}_q(K)$-bicovariant bimodule in the sense of [55]. If we introduce quantum functionals $\chi_{ij} \in \mathscr{U}_h(\mathfrak{k})$ by

$$\chi_{ij} = \delta_{ij} - L^-_{im}S(L^+_{mj}) \, , \tag{9.1}$$

then the mapping $d \colon \mathscr{A}_q(K) \to \mathscr{E}$

$$da = \Omega_{ij}\chi_{ij} \cdot a, \quad a \in \mathscr{A}_q(K) \tag{9.2}$$

defines a bicovariant first-order differential calculus on $\mathscr{A}_q(K)$, which extends uniquely to the exterior differential calculus on $\mathscr{A}_q(K)$. The linear space ${}_{\mathrm{inv}}\mathscr{E}$ spanned by Ω_{ij}'s is the space of left invariant one-forms. Let us denote as ${}_{\mathrm{inv}}\mathscr{X}$ the dual linear space of left-invariant vector fields spanned by χ_{ij}'s. The linear space ${}_{\mathrm{inv}}\mathscr{X}$ is closed under the q-commutator

$$[X, Y]_q = \mathrm{ad}_X\, Y$$

and the comultiplication on χ_{ij} reads

$$\begin{aligned} \Delta\chi_{ij} &= \chi_{ij} \otimes 1 + O_{ijkl} \otimes \chi_{kl} \, , \\ O_{ijkl} &= L^-_{ik}S(L^+_{lj}) \, . \end{aligned} \tag{9.3}$$

In the following we shall use the Cartan calculus on quantum groups developed in [42, 2], where the inner derivation $\mathfrak{i}_\xi$ and the Lie derivative $\mathscr{L}_\xi$ of a general n-form along a general vector field ξ have been introduced. Let us mention that the linear space of left-invariant vector fields ${}_{\mathrm{inv}}\mathscr{X}$ can be used to freely generate an $\mathscr{A}_q(K)$-bicovariant bimodule $\mathscr{X}$ of general vector fields on $\mathscr{A}_q(K)$ [2]. The right coaction of $\mathscr{A}_q(K)$ then coincides on ${}_{\mathrm{inv}}\mathscr{X}$ with the right dressing action $\mathscr{R}$ (3.3).

Here we shall give a short account of the results contained in [42]. We shall need only the specifications of these in the case of left-invariant vector fields ${}_{\mathrm{inv}}\mathscr{X}$ and we shall restrict ourselves to this case. This will save us also from introducing more cumbersome notations and from a considerable extension of the paper. The interested reader is referred to the above-mentioned papers [42, 2] for the more general case. The space of all forms over $\mathscr{A}_q(K)$ will be denoted as $\mathscr{E}^\wedge$ and the space of p-forms as $\mathscr{E}^{\wedge p}$. A general p-form $\alpha \in \mathscr{E}^{\wedge p}$ can be written with the help of the left-invariant one-forms Ω_{ij} as

$$\alpha = \Omega_{i_1 j_1} \wedge \cdots \wedge \Omega_{i_p j_p} a_{i_1 \cdots i_p, j_1 \cdots j_p} \, . \tag{9.4}$$

The differential d is extended to the whole $\mathscr{E}^\wedge$ as in the classical case by the graded Leibniz rule [55]. Let us employ the short-hand notation $\delta_R(\Omega_{ij}) = \Omega_{ij}^{(1)} \otimes a_{ij}^{(2)} \in \mathscr{E} \otimes \mathscr{A}_q(K)$ for the result of application of the right coaction δ_R to the left–invariant one form Ω_{ij}. Then the right and left coactions δ_R, δ_L are extended to the whole $\mathscr{E}^\wedge$ by (we assume α is of the form (9.4))

$$\delta_R : \mathscr{E}^\wedge \longrightarrow \mathscr{E}^\wedge \otimes \mathscr{A}_q(K)$$

$$\alpha \mapsto \Omega_{i_1 j_1}^{(1)} \wedge \cdots \wedge \Omega_{i_p j_p}^{(1)} a_{i_1 \cdots i_p, j_1 \cdots j_p}^{(1)} \otimes a_{i_1 j_1}^{(2)} \cdots a_{i_p j_p}^{(2)} a_{i_1 \cdots i_p, j_1 \cdots j_p}^{(2)}$$

$$\delta_L : \mathscr{E}^\wedge \longrightarrow \mathscr{A}_q(K) \otimes \mathscr{E}^\wedge$$

$$\alpha \mapsto a_{i_1 \cdots i_p, j_1 \cdots j_p}^{(1)} \otimes \Omega_{i_1 j_1} \wedge \cdots \wedge \Omega_{i_p j_p} a_{i_1 \cdots i_p, j_1 \cdots j_p}^{(2)}$$

and by linearity.

Now the Lie derivative $\mathscr{L}_X \alpha$ of a form $\alpha \in \mathscr{E}^\wedge$ along a left-invariant vector field $X \in_{\mathrm{inv}} \mathscr{X}$ can be introduced as

$$\mathscr{L}_X : \mathscr{E}^\wedge \longrightarrow \mathscr{E}^\wedge : \alpha \mapsto (\mathrm{id} \otimes \langle X, \cdot \rangle) \delta_R \alpha \, . \tag{9.5}$$

As in the classical case, the Lie derivative preserves the degree of a form.

The inner derivation $\mathfrak{i}_{\chi_{mn}} \alpha$ of a form $\alpha \in \mathscr{E}$ with respect to a left–invariant vector field $\chi_{mn} \in {}_{\mathrm{inv}}\mathscr{X}$ is defined on the general p-form α recursively as

$$\mathfrak{i}_{\chi_{mn}} : \mathscr{E}^{\wedge p} \longrightarrow \mathscr{E}^{\wedge p-1}$$

$$\Omega_{i_1 j_1} \wedge \cdots \wedge \Omega_{i_p j_p} a_{i_1 \cdots i_p, j_1 \cdots j_p} \mapsto \langle \chi_{mn}, \Omega_{i_1 j_1} \rangle \Omega_{i_2 j_2} \wedge \cdots \wedge \Omega_{i_p j_p} a_{i_1 \cdots i_p, j_1 \cdots j_p}$$

$$+ (-1)^p \langle O_{mnop}, M_{kli_1 j_1} \rangle \Omega_{kl} \wedge (\mathfrak{i}_{\chi_{op}} \Omega_{i_2 j_2} \wedge \cdots \wedge \Omega_{i_p j_p}) a_{i_1 \cdots i_p, j_1 \cdots j_p} \tag{9.6}$$

and is extended by linearity to an inner derivation $\mathfrak{i}_X \alpha$ of a general form $\alpha \in \mathscr{E}$ with respect to a general left-invariant vector field $X \in_{\mathrm{inv}} \mathscr{X}$. With these definitions the Cartan identity from the classical case remains unchanged

$$\mathscr{L}_X = \mathfrak{i}_X d + d \mathfrak{i}_X, \quad X \in_{\mathrm{inv}} \mathscr{X} \, . \tag{9.7}$$

For any quantum function $a \in \mathscr{A}_q(K)$ let us introduce the left-invariant one-form

$$\Theta_L^a = da^{(2)} S^{-1}(a^{(1)}) = \Omega_{ij} \langle \chi_{ij}, a \rangle \tag{9.8}$$

as well as the right invariant form

$$\Theta_R^a = da^{(1)}S(a^{(2)}) = \Theta_L^{a^{(2)}} a^{(1)}S(a^{(3)}) = \Omega_{kl}S(U_{ki})U_{jl}\langle\chi_{ij}, a\rangle \,. \tag{9.9}$$

Let $X \in_{\rm inv} \mathscr{X}$, then we have for its symbol $\sigma(X)$,

$$\sigma(X) = \mathrm{i}_X \Theta_R^{w_\lambda} \,. \tag{9.10}$$

This equality is a consequence of a chain of simple identities

$$\sigma(X) = \langle X, w_\lambda^{(2)}\rangle w_\lambda^{(1)} S(w_\lambda^{(3)}) = (\mathrm{i}_X \Theta_L^{w_\lambda^{(2)}}) w_\lambda^{(1)} S(w_\lambda^{(3)}) = \mathrm{i}_X \Theta_R^{w_\lambda} \,.$$

Applying the differential d to the equality (9.10), making use of the Cartan identity (9.7) and using the fact that

$$\mathscr{L}_X \omega = 0$$

for X left-invariant and ω right-invariant we obtain immediately

$$d\sigma(X) = -\mathrm{i}_X d\Theta_R^{w_\lambda} \,. \tag{9.11}$$

Let $Y \in \mathscr{X}$ be now another left-invariant vector field. We have

$$-\,\mathrm{i}_Y \mathrm{i}_X d\Theta_R^{w_\lambda} = \sigma([Y,X]_q) \,, \tag{9.12}$$

which follows from an application of i_Y to the equality (9.11):

$$-\mathrm{i}_Y \mathrm{i}_X d\Theta_R^{w_\lambda} = \mathrm{i}_Y d\sigma(X) = Y \cdot \sigma(X) = \sigma(\mathrm{Ad}_Y X) \,.$$

The third equality in the above chain is a direct consequence of the definition of the inner derivation adopted above.

Here we would like to note the following. Let us assume the image $\sigma(\mathscr{U}_h(\mathfrak{k}) \subset \mathscr{A}_q(K_0\backslash K)$ under symbol mapping σ equipped with a new product $*$, which respects the algebra structure of $\mathscr{U}_h(k)$

$$\sigma(X) * \sigma(Y) = \sigma(XY) \quad \text{for } X,\, Y \in \mathscr{U}_h(k),$$

which is just the Berezin quantization prescription for the symbols in the classical case. Then from (5.4) it follows immediately that the mapping σ is a quantum momentum map in the sense of [30] and we can rewrite (9.12) in the form

$$-\,\mathrm{i}_Y \mathrm{i}_X d\Theta_R^{w_\lambda} = \sigma(Y^{(1)}) * \sigma(X) * \sigma(S(Y^{(2)})) \,. \tag{9.13}$$

Using the expression of the right invariant one-form $\Theta_R^{w_\lambda}$ with the help of the left invariant one-forms Ω_{ij} following from (9.9) we obtain an alternative definition of the isotropy subgroup K_0. Instead of (5.7) we may equivalently require the invariance of $\Theta_R^{w_\lambda}$ with respect to the left coaction of K_0,

$$(p_0 \otimes \mathrm{id})\delta_L \Theta_R^{w_\lambda} = 1 \otimes \Theta_R^{w_\lambda} \in \mathscr{A}_q(K_0) \otimes \mathscr{E} \,. \tag{9.14}$$

Let us denote for convenience by $\mathscr{Z} \in \mathscr{H}_\lambda \otimes \mathscr{A}_q(K_0\backslash K)$ the unnormalized coherent state $\mathscr{Z} = \Gamma(w_\lambda^*)^{-1}$ and let the expressions $d\mathscr{Z} \in \mathscr{H}_\lambda \otimes \mathscr{E}$ and $d\Gamma \in \mathscr{H}_\lambda \otimes \mathscr{E}$ have the obvious meaning of differentiating with respect to the second factor in $\mathscr{H}_\lambda \otimes \mathscr{A}_q(K)$. Let us also introduce a new one-form $\Theta^{w_\lambda} \equiv \Theta_R^{w_\lambda} - dw_\lambda(w_\lambda)^{-1} \in \mathscr{E}$. Like in the classical case the one-forms $\Theta_R^{w_\lambda}$ and Θ^{w_λ} can be expressed through the coherent states Γ and $\mathscr{Z}$ as

$$\Theta_R^{w_\lambda} = \langle d\Gamma | \Gamma \rangle$$

and

$$\Theta^{w_\lambda} = w_\lambda \langle d\mathscr{Z} | \mathscr{Z} \rangle w_\lambda^* ,$$

respectively.

Now we are prepared to give a formula for the action of the elements χ_{ij} in the irreducible $*$-representation τ_λ of $\mathscr{U}_h(\mathfrak{k})$, which directly generalizes the geometric quantization prescription for the action of generators of $U(\mathfrak{k})$ in the irreducible representation of K corresponding to a minimal weight λ. Starting from formula (8.3) and using (9.3) we have for $\psi \in \mathscr{M}_\lambda$,

$$\tau_\lambda(\chi_{ij})\psi = w_\lambda^{-1}(\chi_{ij} \cdot w_\lambda)\psi + w_\lambda^{-1}(O_{ijkl} \cdot w_\lambda)\chi_{kl} \cdot \psi ,$$

which can be finally rewritten making use of the following identities:

$$(\chi_{ij} \cdot w_\lambda) w_\lambda^{-1} = \mathrm{i}_{\chi_{ij}} d w_\lambda w_\lambda^{-1} = \sigma(\chi_{ij}) - \mathrm{i}_{\chi_{ij}} \Theta^{w_\lambda}$$

in the form

$$\tau_\lambda(\chi_{ij})\psi = w_\lambda^{-1}(O_{ijkl} \cdot w_\lambda)\chi_{kl} \cdot \psi + w_\lambda^{-1}(\sigma(\chi_{ij}) - \mathrm{i}_{\chi_{ij}} \Theta^{w_\lambda}) w_\lambda \psi . \tag{9.15}$$

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References

1. Arik, M., Coon, D.D.: Hilbert space of analytic functions and generalized coherent states. J. Math. Phys. **17**, 524 (1976)
2. Aschieri, P.: The space of vector fields on quantum groups. Preprint UCLA/93/TEP/25
3. Awata, H., Noumi, M., Odake, S.: Heisenberg realization for $U_q(sl_n)$ on the flag manifold. Lett. Math. Phys. **30**, 35 (1994)
4. Babelon, O., Bernard, D.: Dressing transformations and the origin of the quantum group symmetries. Phys. Lett. B **260**, 81 (1991)
5. Berezin, F.A.: General concept of quantization. Commun. Math. Phys. **40**, 153 (1975)
6. Berger, R.: The quantum Poincaré–Birkhoff–Witt theorem. Commun. Math. Phys. **143**, 215 (1992)
7. Biedenharn, L.C., Lohe, M.H.: An extension of the Borel–Weil construction to the quantum group $U_q(n)$. Commun. Math. Phys. **146**, 483 (1992)
8. Breziński, T., Majid, S.: Quantum group gauge theory on quantum spaces. Commun. Math. Phys. **157**, 501 (1993)
9. Carow–Watamura, U., Schlieker, M., Watamura, S., Weich, W.: Bicovariant differential calculus on quantum groups $SU_q(n)$ and $SO_q(n)$. Commun. Math. Phys. **142**, 605 (1991)
10. Chari, V., Pressley, A.: A guide to quantum groups. Cambridge: Cambridge University Press, 1994
11. Drinfeld, V.G.: Quantum groups. In: Proc. ICM Berkeley 1986, Providence, RI: AMS, 1987, p. 798
12. Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Quantum groups. In: Yang, C.N., Ge, M.L. (eds.) Braid groups, knot theory and statistical mechanics. Singapore: World Scientific, 1989
13. Fronsdal, C., Galindo, A.: The universal T-matrix. Lett. Math. Phys. **27**, 59 (1993)

14. Grosse, H., Prešnajder, P.: The construction of non-commutative manifolds using coherent states. Lett. Math. Phys. **28**, 239 (1993)
15. Jimbo, M.: A q-difference analogue of $U(g)$ and the Yang–Baxter equation. Lett. Math. Phys. **10**, 63 (1985)
16. Jimbo, M.: Quantum R-matrix for the generalized Toda system. Commun. Math. Phys. **102**, 537 (1986)
17. Jurčo, B.: On coherent states for the simplest quantum groups. Lett. Math. Phys. **21**, 51 (1991)
18. Jurčo, B.: Differential calculus on quantized simple Lie groups. Lett. Math. Phys. **22**, 177 (1991)
19. Jurčo, B., Šťovíček, P.: Quantum dressing orbits on compact groups. Commun. Math. Phys. **152**, 97 (1993)
20. Kirillov, A.N., Reshetikhin, N.Yu.: q-Weyl group and a multiplicative formula for universal R-matrices. Commun. Math. Phys. **134**, 421 (1990)
21. Klauder, J.R., Skagerstam, B.-S. (eds.): Coherent states. Singapore: World Scientific, 1986
22. Khoroshkin, S.M., Tolstoy, V.N.: Universal R-matrix for quantized (super)algebras. Commun. Math. Phys. **141**, 599 (1991)
23. Lakshmibai, V., Reshetikhin, N.: Quantum deformations of flag and Schubert schemes. C.R. Acad. Sci. Paris **313**, Série I, 121 (1991)
24. Levendorskii, S.Z.: On PWB bases for Yangians. Lett. Math. Phys. **27**, 37 (1993)
25. Levendorskii, S., Soibelman, Ya.: Algebras of functions on compact quantum groups, Schubert cells and quantum tori. Commun. Math. Phys. **139**, 141 (1991)
26. Levendorskii, S.Z., Soibelman, Ya. S.: Quantum Weyl group and multiplicative formula for the R-matrix of a simple Lie algebra. Funct. Anal. Appl. **25**, 143 (1991)
27. Levendorskii, S.Z., Soibelman, Ya.S.: Some application of quantum Weyl groups. The multiplicative formula for universal R-matrix for simple Lie algebra. J. Geom. Phys. **7**(4), 1 (1990)
28. Lu, J.-H.: Moment maps at quantum level. Commun. Math. Phys. **157**, 389 (1993)
29. Lu, J.H., Weinstein, A.: Poisson Lie groups, dressing transformations and Bruhat decompositions. J. Diff. Geom. **31**, 501 (1990)
30. Lusztig, G.: Quantum deformations of certain simple modules over enveloping algebras. Adv. Math. **70**, 237 (1988)
31. Lusztig, G.: Quantum groups at roots of 1, Geom. Dedicata **35**, 89 (1990)
32. Onofri, E.: A note on coherent state representaions of Lie groups. J. Math. Phys. **16**, 1088 (1975)
33. Parshall, B., Wang, J.: Quantum linear groups. Providence, Rhode Island: AMS, 1991
34. Perelomov, A.M.: Generalized coherent states and their applications. Berlin, Heidelberg, New York: Springer, 1986
35. Podles, P., Woronowicz, S.L.: Quantum deformation of Lorentz group. Commun. Math. Phys. **130**, 381 (1990)
36. Reshetikhin, N. Yu., Semenov–Tian–Shansky, M.A.: Quantum R-matrices and factorization problems. J. Geom. Phys. **5**, 533 (1988)
37. Rawnsley, J., Cahen, M., Gutt, S.: Quantization of Kähler manifolds I: Geometric interpretation of Berezin's quantization J. Geom. Phys. **7**(1), 45 (1990) and Quantization of Kähler manifolds II. Preprint (1990)
38. Reshetikhin, N. Yu., Takhtajan, L.A., Faddeev, L.D.: Quantizaton of Lie groups and Lie algebras. Algebra i analiz **1**, 178 (1989) (in Russian) and Leningrad Math. J. **1**, 193 (1990)
39. Rosso, M.: An analogue of P.B.W. theorem and the universal R-matrix for $U_h sl(N+1)$. Commun. Math. Phys. **124**, 307 (1989)
40. Rosso, M.: Algèbres enveloppantes quantifiées, groupes quantiques compacts et calcul différentiel non commutatif. Duke Math. J. **61**, 11–40 (1990)
41. Semenov–Tian–Shansky, M.A.: Dressing transformation and Poisson Lie group actions. Publ. RIMS, Kyoto University **21**, 1237 (1985)
42. Schupp, P., Watts, P., Zumino, B.: Cartan calculus for Hopf algebras and quantum groups. Preprint LBL-34215
43. Simon, B.: The classical limit of quantum partition functions. Commun. Math. Phys. **71**, 247 (1980)

44. Sklyanin, E.K.: Some algebraic structure connected with the Yang–Baxter equation. Representations of quantum algebras. Funct. Anal. Appl. **17**, 273 (1983)
45. Soibelman, Y.: Irreducible representations of the function algebra on the quantum group $SU(n)$ and Schubert cells. Soviet. Math. Dokl. **40**, 34 (1990)
46. Soibelman, Y.: Algebra of functions on compact quantum group and its representations. Leningrad Math. J. **2**, 1 (1991)
47. Soibelman, Y.: On quantum flag manifolds. Funct. Anal. Appl. **26**, 225 (1992)
48. Soibelman, Y.: Orbit method for algebras of functions on quantum groups and coherent states. Int. Math. Res. Notes **5**, 151 (1993)
49. Vaksman, L.L., Soibelman, Y.S.: Function algebra on the quantum group $SU(2)$. Funct. Anal. Appl. **22**, 3 (1988)
50. Šťovíček, P.: Quantum Grassmann manifolds. Commmun. Math. Phys. **158**, 135 (1993)
51. Šťovíček, P.: Quantum line bundles on S^2 and the method of orbits for $SU_q(2)$. J. Math. Phys. **34**, 1606 (1993)
52. Taft, E., Towber, J.: Quantum deformation of flag schemes and Grassmann schemes I. A q-deformation of the shape-algebra for $GL(n)$. J. Algebra **142**, No. 1 (1991)
53. Woodhouse, N.: Geometric quantization. Oxford: Clarendon Press, 1980
54. Woronowicz, S.L.: Compact matrix pseudogroups. Commun. Math. Phys. **111**, 613 (1987)
55. Woronowicz, S.L.: Differential calculus on quantum matrix pseudogroups (quantum groups). Commun. Math. Phys. **122**, 125 (1989)

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