Commun. Math. Phys. 182, 1 – 32 (1996)

Communications in
Mathematical
Physics

Global Aspects of Gauged Wess–Zumino–Witten Models

Kentaro Hori

Institute of Physics, University of Tokyo, Meguroku, Tokyo 153, Japan.
E-mail address: hori@danjuro.phys.s.u-tokyo.ac.jp

Received: 4 August 1994/Accepted: 20 December 1995

Abstract: A study of the gauged Wess–Zumino–Witten models is given focusing on the effect of topologically non-trivial configurations of gauge fields. A correlation function is expressed as an integral over a moduli space of holomorphic bundles with quasi-parabolic structure. Two actions of the fundamental group of the gauge group is defined: One on the space of gauge invariant local fields and the other on the moduli spaces. Applying these in the integral expression, we obtain a certain identity which relates correlation functions for configurations of different topologies. It gives an important information on the topological sum for the partition and correlation functions.

1. Introduction

The gauged Wess–Zumino–Witten model in two dimensions has two different aspects of interest. On the one hand, it is an exactly soluble quantum gauge theory and is interesting from the point of view of geometry of gauge fields. On the other hand, it is a conformally invariant quantum field theory (CFT): There are observations [1–5] that a wide class of solved CFTs such as unitary minimal models (bosonic [6] or supersymmetric [7]), parafermionic models [8], etc. are realized by gauged WZW models as lagrange field theories, up to a subtle point of field identification which will be addressed shortly.

In this paper, we focus on the former, the geometric aspects of the theory and propose a method to take into account the topologically non-trivial configurations of gauge fields. Then, we get an identity which shows that incorporation of non-trivial topology solves the problem of field identification, and which is therefore of vital importance from the point of view of the model building of CFTs.

A gauged WZW model is specified by a choice of the target group G, the gauge group H, and the level k. We concentrate on the case in which G is a compact, connected and simply connected Lie group and H is a connected, closed subgroup of G/Z_G, where Z_G is the center of G. For a closed Riemannian 2-manifold Σ, a map $g : \Sigma \to G$, and a one form $A \in \Omega^1(\Sigma, \mathfrak{h})$, the WZW action is given by

$$\begin{aligned} kI_\Sigma(A,g) &= -\frac{k}{8\pi}\int_\Sigma \mathrm{tr}(g^{-1}d_A g \wedge *g^{-1}d_A g) - \frac{ik}{12\pi}\int_{B_\Sigma} \mathrm{tr}(\tilde{g}^{-1}d\tilde{g})^3 \\ &\quad +\frac{ik}{4\pi}\int_\Sigma \mathrm{tr}(A(g^{-1}dg + dgg^{-1}) + Ag^{-1}Ag)\,. \end{aligned} \tag{1.1}$$

Here "tr" is the trace in a representation of G([1]), $g^{-1}d_A g = g^{-1}Ag + g^{-1}dg - A$, and $*$ is the Hodge operator which, acting on one forms, depends only on the complex structure of Σ. B_Σ is a compact three manifold bounding Σ and $\tilde{g} : B_\Sigma \to G$ is an extension of g. If k is an integer, the value $\mathrm{e}^{-kI_\Sigma(A,g)}$ which we call the *WZW weight* is independent of the choice of B_Σ and $\tilde{g}$, and hence may be used as the weight for the path integration over A and g. It is invariant under the gauge transformation $A \to A^h = h^{-1}Ah + h^{-1}dh$, $g \to h^{-1}gh$ and the resulting system is a quantum gauge theory, which has been extensively studied in [1, 3–5].

If $\pi_1(H)$ is non-trivial, one can also consider topologically non-trivial configurations of A and g: Let $\{U_0, U_\infty\}$ be an open covering of Σ such that U_0 contains a disc D_0 and $U_0 \cap U_\infty$ is an annular neighborhood of the boundary circle ∂D_0. Let $\{A_0, A_\infty\}$ and $\{g_0, g_\infty\}$ be gauge fields and maps defined on $\{U_0, U_\infty\}$ so that

$$A_0 = h_{\infty 0}^{-1}A_\infty h_{\infty 0} + h_{\infty 0}^{-1}dh_{\infty 0}, \qquad g_0 = h_{\infty 0}^{-1}g_\infty h_{\infty 0} \quad \text{on } U_0 \cap U_\infty\,, \tag{1.2}$$

where $h_{\infty 0}$ is a map to H. These determine a connection A of P and a section g of $P\times_H G$, where P is the principal H bundle determined by the transition function $h_{\infty 0}$. In Sect. 2, we define the WZW action $kI_{\Sigma,P}(A,g)$ for such a configuration. Note that any H-bundle admits such a description and the homotopy type of the loop $\gamma_{\infty 0} = h_{\infty 0}|_{\partial D_0}$ determines the topological type. Thus, $\pi_1(H)$ classifies the topological types of configurations. In Sects. 3 and 4, we give a method to calculate the correlation function $Z_{\Sigma,P}(\mathcal{O}_1 \cdots \mathcal{O}_s)$ of gauge invariant fields $\mathcal{O}_1, \ldots, \mathcal{O}_s$ for configurations associated with a bundle P.

The main purpose of the paper is to prove certain exact relationships of correlators for configurations of different topologies. Namely, in Sect. 4 we will see that the group $\pi_1(H)$, which acts on topological types of principal H-bundles $\gamma : P \mapsto P\gamma$ by multiplication on the transition functions $\gamma_{\infty 0} \mapsto \gamma_{\infty 0}\gamma$, acts also on the space of gauge invariant local fields $\gamma : \mathcal{O} \mapsto \gamma\mathcal{O}$ and that

$$Z_{\Sigma,P}(\mathcal{O}_1 \cdots \mathcal{O}_s \gamma\mathcal{O}) = Z_{\Sigma,P\gamma}(\mathcal{O}_1 \cdots \mathcal{O}_s \mathcal{O})\,. \tag{1.3}$$

We call this the *topological identity*. The proof is reduced to verifying a conjecture concerning the geometry of moduli spaces of holomorphic $H_{\mathbf{C}}$-bundles with quasi-parabolic structure. Verification is done for the cases $\Sigma =$ sphere with H general and $\Sigma =$ torus with $H = SO(3)$, in addition to the case of abelian gauge groups.

The significance of (1.3) can be seen if we take the sum over topologies; the fields $\mathcal{O}$ and $\gamma\mathcal{O}$ are then indistinguishable. For instance, consider the case of $G = SU(2)\times SU(2)$ with level $(k, 1)$, and $H = SO(3)$ diagonally embedded in $G/Z_G = SO(3)\times SO(3)$. The gauge invariant local fields can be classified by labels in $\{0, \frac{1}{2}, \ldots, \frac{k}{2}\} \times \{0, \frac{1}{2}, \ldots, \frac{k+1}{2}\}$. The space of fields labeled (j_1, j) is identified with

[1] Here is the expression for simple G, and "tr" is normalized by $\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}X\mathrm{ad}Y) = 2g^\vee\mathrm{tr}(XY)$ for $X, Y \in \mathfrak{g}$, where $g^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Generalization to the non-simple case is obvious.

the degenerate representation of the Virasoro algebra of central charge $1-\frac{6}{(k+2)(k+3)}$ and dimension $\frac{((k+3)j_1-(k+2)j+1)^2-1}{4(k+2)(k+3)}$ as is also the case for the label $(\frac{k}{2}-j_1,\frac{k+1}{2}-j)$. As we shall see in Sect. 4, this transformation $(j_1,j)\leftrightarrow(\frac{k}{2}-j_1,\frac{k+1}{2}-j)$ corresponds precisely to the transformation $O\leftrightarrow\gamma O$, where γ is the non-trivial element of $\pi_1(SO(3))=\mathbf{Z}_2$. Hence, only after the sum over topologies, the set of distinguishable fields coincides with that of the k^{th} unitary minimal model [6]. The situation is the same for general G and H. The space of local gauge invariant fields, acted on by the infinite conformal symmetry, is identified [3] with the direct sum of Virasoro modules by coset construction [9]. For each element $\gamma\in\pi_1(H)$, there is an isomorphism of coset Virasoro modules, known as the "field identification" [10–12], that corresponds to our transformation $O\mapsto\gamma O$. Hence, this identification of Virasoro mudules leads via the sum over topologies to a genuine identification of quantum fields. In Sect. 5, we give a support of this interpretation by a detailed calculation of torus partition function. We will see that topologically non-trivial configurations play an important role especially in the presence of "field identification fixed points" [12, 47].

2. Wess–Zumino–Witten Model

We describe some properties of the WZW model [13] which will be needed in the following sections. Throughout this section, $H=G/Z_G$, and a Riemann surface Σ is fixed with its open covering $\{U_0,U_\infty\}$ and $D_0\subset U_0$ as in Sect. 1. We choose a complex coordinate z on U_0 such that D_0 is a unit disc in the z-plane. This gives a parametrization $\theta\mapsto e^{i\theta}$ on the boundary circle $S=\partial D_0=-\partial\Sigma_\infty$, where $\Sigma_\infty:=\overline{\Sigma-D_0}$.

2.1. The WZW Action. Let P be a principal H-bundle with transition function $h_{\infty 0}:U_0\cap U_\infty\to H$, and let $A=\{A_0,A_\infty\}$ and $g=\{g_0,g_\infty\}$ (subject to (1.2)) be a connection of P and a section of $P\times_H G$. The WZW weight for this configuration is defined to factorize as a product

$$e^{-kI_{\Sigma,P}(A,g)}=\langle e^{-kI_{\Sigma_\infty}(A_\infty,g_\infty)},\mathrm{ad}\gamma_{\infty 0}\,e^{-kI_{D_0}(A_0,g_0)}\rangle\,. \tag{2.1}$$

Below, we define the ingredients in this expression; the WZW weights on surfaces with a circle boundary, adjoint transformation by a loop $\gamma_{\infty 0}=h_{\infty 0}|_S$, and the pairing.

Let us complete the disc D_0 by another disc D_∞ to a Riemann sphere $\mathbf{P}^1$. For a map $g:D_0\to G_{\mathbf{C}}$, we choose an extension $g:\mathbf{P}^1\to G_{\mathbf{C}}$. Following [14], we put an equivalence relation in $\mathrm{Map}(D_\infty,G_{\mathbf{C}})\times\mathbf{C}$ under which the class

$$\begin{aligned}e^{-kI_{D_0}(A,g)}=&\left[\left(g|_{D_\infty},\exp\left\{\frac{k}{8\pi}\int_{D_0}\mathrm{tr}(g^{-1}d_Ag\wedge *g^{-1}d_Ag)\right.\right.\right.\\&\left.\left.\left.+\frac{ik}{12\pi}\int_{B_{\mathbf{P}^1}}\mathrm{tr}(\tilde{g}^{-1}d\tilde{g})^3-k\Gamma_{D_0}(A,g)\right\}\right)\right]\,;\end{aligned} \tag{2.2}$$

$$\Gamma(A,g):=\frac{i}{4\pi}\int\mathrm{tr}(A(g^{-1}dg+dgg^{-1})+Ag^{-1}Ag), \tag{2.3}$$

is independent of the choice of the extension $g|_{D_\infty}$. This defines a line bundle $\mathscr{L}_{\rm WZ}^{\otimes k}$ over the loop group $LG_{\mathbf{C}}$, and (2.2) is in the line $\mathscr{L}_{\rm WZ}^{\otimes k}|_\gamma$ over the boundary loop $\gamma(\theta) = g(e^{i\theta})$. This is our definition of the WZW weight on D_0. A semi-group structure of $\mathscr{L}_{\rm WZ}^{\otimes k}$ is introduced by requiring the Polyakov–Wiegmann (PW) identity [15]

$$e^{-kI_{D_0}(A,h)}e^{-kI_{D_0}(A^h,g^h)}e^{-kI_{D_0}(A,h^*)} = e^{-kI_{D_0}(A,g)} \exp\left\{\frac{ik}{2\pi}\int_{D_0} \operatorname{tr}\left(h^*\partial_A h^{*-1} h^{-1}\bar{\partial}_A h\right)\right\}. \tag{2.4}$$

Here, A^h, g^h denote the *chiral gauge transform* $(A^h)^{01} = h^{-1}A^{01}h + h^{-1}\bar{\partial}h$, $(A^h)^{10} = h^*A^{10}h^{*-1} + h^*\partial h^{*-1}$, $g^h = h^{-1}gh^{*-1}$ by $h \in \mathrm{Map}(D_0, H_{\mathbf{C}})$. The non-zero elements of $\mathscr{L}_{\rm WZ}$ form a group which is isomorphic to the basic central extension $\widetilde{LG_{\mathbf{C}}}$ of $LG_{\mathbf{C}}$ [16].

The adjoint action of LH on $\mathscr{L}_{\rm WZ}^{\otimes k}$ is defined as follows: For $\gamma \in LH$, we choose an extension $h : D_\infty - \{\infty\} \to H$; $\gamma = h|_S$. Then

$$\mathrm{ad}\gamma^{-1}[(g,c)] := [(h^{-1}gh, c\exp\{-k\Gamma_{D_\infty}(hdh^{-1}, g)\})]\,. \tag{2.5}$$

Here $g : D_\infty \to G_{\mathbf{C}}$ is chosen to be $g = 1$ in a neighborhood of ∞ so that the $\Gamma_{D_\infty}(hdh^{-1}, g)$ to be well-defined.

The WZW weight on Σ_∞ is defined in the similar way as an element of the dual bundle $\mathscr{L}_{\rm WZ}^{*\otimes k}$. The pairing of $\mathscr{L}_{\rm WZ}^{*\otimes k}$ and $\mathscr{L}_{\rm WZ}^{\otimes k}$ is given by (we put $I(g) := I(0,g)$)

$$\langle e^{-kI_{\Sigma_\infty}(g)}, e^{-kI_{D_0}(g)}\rangle = e^{-kI_\Sigma(g)}, \quad g \in \mathrm{Map}(\Sigma, G_{\mathbf{C}}). \tag{2.6}$$

One may wonder whether our definition depends on the choice of the disc $D_0 \subset \Sigma$. To prove it does not, it is enough to show that we can replace D_0 by a larger disc $\tilde{D}_0 \supset D_0$. The problem is then reduced to proving the identity $\Gamma(A^h, g^h) = \Gamma(A,g) + \Gamma(h^{-1}dh, g^h)$ on the cylinder $\tilde{D}_0 - D_0$, which is a straightforward matter. The PW identity (2.4) together with its analog for weights on Σ_∞ leads to the global PW identity

$$I_{\Sigma,P}(A^h, g^h) = I_{\Sigma,P}(A,g) - I_{\Sigma,P}(A, hh^*)\,. \tag{2.7}$$

This ensures the gauge invariance of the action.

2.2. States and Fields. The factorization property (2.1) is inherited by the quantum theory: A correlation function in the gauge field A factorizes as

$$Z_{\Sigma,P}(A; \prod_i O_i(x_i)) = \left\langle Z_{\Sigma_\infty}\left(A_\infty; \prod_{x_i \in \Sigma_\infty} O_i(x_i)\right), \ \gamma_{\infty 0}\cdot Z_{D_0}\left(A_0; \prod_{x_j \in D_0} O_j(x_j)\right)\right\rangle. \tag{2.8}$$

Here, $Z_{D_0}(A_0; \prod_j O_j(x_j))$ denotes the wave function at $S = \partial D_0$ which results from the path-integration over the interior of D_0, and similarly for $Z_{\Sigma_\infty}(A_\infty; \prod_i O_i(x_i))$. $\gamma_{\infty 0}\cdot$ is the gauge transformation acting on sections of $\mathscr{L}_{\rm WZ}^{\otimes k}$ by

$$(\gamma_{\infty 0}\cdot\Phi)(\gamma) = \mathrm{ad}\gamma_{\infty 0}\Phi(\mathrm{ad}\gamma_{\infty 0}^{-1}\gamma)\,. \tag{2.9}$$

On the space of sections of $\mathscr{L}_{\rm WZ}^{\otimes k}$, there are the left-right representations of the group $\widetilde{LG_{\mathbf{C}}}$: $J(\tilde{\gamma}_1)\bar{J}(\tilde{\gamma}_2)\Phi(\gamma) = \tilde{\gamma}_1\Phi(\gamma_1^{-1}\gamma\gamma_2^{*-1})\tilde{\gamma}_2^*$. We identify the space of states with the direct sum $\mathscr{H}^{G,k} = \bigoplus_\Lambda \mathscr{H}_\Lambda^{G,k}$ of irreducible components of the representation $J\times\bar{J}$. Here, the sum is over the set $\mathrm{P}_+^{(k)}$ of weights of G, integrable at level k, and $\mathscr{H}_\Lambda^{G,k}$ is the irreducible $\widetilde{LG_{\mathbf{C}}}\times\widetilde{LG_{\mathbf{C}}}$-module isomorphic to $L_\Lambda^{G,k}\otimes\overline{L_\Lambda^{G,k}}$, where $L_\Lambda^{G,k}$ (resp. $\overline{L_\Lambda^{G,k}}$) is the holomorphic (resp. anti-holomorphic) irreducible representation of $\widetilde{LG_{\mathbf{C}}}$ with highest weight (Λ, k).

The highest weight state $\Phi_\Lambda\in\mathscr{H}_\Lambda^{G,k}$ is explicitly expressed in the following way [14]. Let B be a Borel subgroup of $G_{\mathbf{C}}$, and $N\subset B$ be the maximal unipotent subgroup. Let B^+ and N^+ denote the subgroup of $LG_{\mathbf{C}}$ consisting of boundary loops of holomorphic maps $D_0\to G_{\mathbf{C}}$ such that the values at $z=0$ are in B and N respectively. Choosing such holomorphic maps $g_1(z)$ and $g_2(z)$ with $g_1(0)\in B$ and $g_2(0)\in N$, the state Φ_Λ is expressed as

$$\Phi_\Lambda(\gamma_1\gamma_2^*) = \mathrm{e}^{-\Lambda}(g_1(0))\mathrm{e}^{-kI_{D_0}(g_1g_2^*)}\,,\quad \gamma_1\gamma_2^* = g_1g_2^*|_S\,, \tag{2.10}$$

where e^Λ is the character of B corresponding to the highest weight Λ. As $B^+(N^+)^*$ is open dense in $LG_{\mathbf{C}}$, the above expression completely characterizes Φ_Λ.

A Ward identity follows from the PW identity (2.4):

$$Z_{D_0}(A_0^h;O) = J(\tilde{\gamma})\bar{J}(\tilde{\gamma})Z_{D_0}(A_0;hO)\exp\left\{\frac{ik}{2\pi}\int_{D_0}\mathrm{tr}\left(h^*\partial_A h^{*-1}h^{-1}\bar{\partial}_A h\right)\right\}\,, \tag{2.11}$$

where $\tilde{\gamma}^{-1} = \mathrm{e}^{-I_{D_0}(A_0,h)}$ and hO is defined by $hO(g) = O(h^{-1}gh^{*-1})$. Using this, we can identify the state Φ_Λ with the wave function for the disc D_0 with a flat connection $A_0 = 0$ and a field insertion O_Λ at $z=0$. Up to a renormalization, $O_\Lambda(g)$ is defined as the matrix element $(v_\Lambda, g^{-1}v_\Lambda)$, where v_Λ is a highest weight vector in the (finite dimensional) irreducible $G_{\mathbf{C}}$-module of highest weight Λ.

2.3. The Spectral Flow. We calculate the gauge transform $\gamma\cdot\Phi_\Lambda$ of the highest weight state Φ_Λ by a loop γ that represents an element of the group $\Gamma_{\widehat{\mathbf{C}}}\subset W'_{\rm aff}$ (see Appendix A). As $\mathrm{ad}\gamma$ preserves the subgroups B^+ and N^+ of $LG_{\mathbf{C}}$, the result should again be a highest weight state. It suffices to look at the behavior over the open dense subset $B^+(N^+)^*$.

The loop is expressed as

$$\gamma(\theta) = \mathrm{e}^{-i\mu\theta}n_w\,, \tag{2.12}$$

where μ is a minimal coweight and $n_w\in H$ represents an element w of the Weyl group W. Let $h_\gamma(z) = z^{-\mu}n_w$ be the meromorphic extension. Let g_1, g_2 and $\gamma_1\in B^+$, $\gamma_2\in N^+$ be holomorphic maps and boundary loops, as above. Since $\mathrm{ad}\gamma$ preserves the subgroups B^+ and N^+, holomorphic maps $h_\gamma^{-1}g_1h_\gamma$ and $h_\gamma^{-1}g_2h_\gamma$ are defined on D_0, and satisfy $(h_\gamma^{-1}g_1h_\gamma)(0)\in B$ and $(h_\gamma^{-1}g_2h_\gamma)(0)\in N$. Hence we have

$$\gamma\cdot\Phi_\Lambda(\gamma_1\gamma_2^*) = \mathrm{e}^{-\Lambda}((h_\gamma^{-1}g_1h_\gamma)(0))\mathrm{ad}\gamma(\mathrm{e}^{-kI_{D_0}((h_\gamma^{-1}g_1h_\gamma)(h_\gamma^{-1}g_2h_\gamma)^*)})\,. \tag{2.13}$$

If we put $g_1(0)\equiv\mathrm{e}^{t_0}\in T \bmod N$, we find that $(h_\gamma^{-1}g_1h_\gamma)(0)\equiv\mathrm{e}^{w^{-1}t_0}$, and hence

$$\mathrm{e}^{-\Lambda}((h_\gamma^{-1}g_1h_\gamma)(0)) = \mathrm{e}^{-w\Lambda(t_0)}\,. \tag{2.14}$$

Applying the transformation rule (2.5) of $\mathrm{ad}\gamma$, we find that

$$\mathrm{ad}\gamma\,(\mathrm{e}^{-kI_{D_0}(h_\gamma^{-1}g_1h_\gamma)}) = \mathrm{e}^{-k\,\mathrm{tr}(\mu t_0)}\mathrm{e}^{-kI_{D_0}(g_1)}\,, \tag{2.15}$$

$$\mathrm{ad}\gamma\,(\mathrm{e}^{-kI_{D_0}(h_\gamma^{-1}g_2h_\gamma)}) = \mathrm{e}^{-kI_{D_0}(g_2)}\,. \tag{2.16}$$

Combining these results, we obtain the expression

$$\gamma\cdot\Phi_\Lambda(\gamma_1\gamma_2^*) = \mathrm{e}^{-w\Lambda(t_0)-k\,\mathrm{tr}(\mu t_0)}\mathrm{e}^{-kI_{D_0}(g_1g_2^*)}\,. \tag{2.17}$$

Thus, the result is $\gamma\cdot\Phi_\Lambda = \Phi_{\gamma\Lambda}$, the state of highest weight

$$\gamma\Lambda = w\Lambda + k\breve{\mu}\,, \tag{2.18}$$

where $\breve{\mu}$ denotes the weight given by $\breve{\mu}(v) = \mathrm{tr}(\mu v)$.

Remark. The adjoint action (2.5) by the γ yields an automorphism of Kac–Moody algebra [17–19, 12], called the *spectral flow*. This induces the above map $\Lambda \mapsto \gamma\Lambda$ and permutes the elements of $\mathrm{P}_+^{(k)}$.

A simple consequence follows from this computation. Consider the gauge field configuration

$$A_{\varrho,\gamma} = \varrho\gamma^{-1}d\gamma = -\varrho(|z|)iw^{-1}\mu d\theta \quad \text{on } D_0\,, \tag{2.19}$$

where $0 \leqq \varrho(r) \leqq 1$ is a cut-off function such that $\varrho(r) = 0$ for $r < 1/4$ and $\varrho(r) = 1$ for $r > 3/4$. Since it is a pure gauge for $|z| > 3/4$, one can choose a horizontal frame $s(\theta) = s_0(\theta)\gamma(\theta)^{-1}$ along $S = \partial D_0$, where s_0 is the old frame over D_0. Note, furthermore, that $A_{\varrho,\gamma}$ can be made flat over D_0 by a chiral gauge transformation $h_{\varrho,\gamma}$ such that $h_{\varrho,\gamma}(0) = 1$ and $h_{\varrho,\gamma}|_S \equiv c_\varrho^{w^{-1}\mu}$ with c_ϱ being a constant. Let us insert the field O_Λ at $z = 0$ and look at the state at S. If we stand on the horizontal frame s, what we observe is

$$\gamma\cdot Z_{D_0}(A_{\varrho,\gamma}; O_\Lambda) = \mathrm{const}\,\gamma\cdot\Phi_\Lambda = \mathrm{const}\,\Phi_{\gamma\Lambda}\,. \tag{2.20}$$

The first equality is due to the Ward identity (2.11), where the constant is of the form $\mathrm{e}^{-k\,\mathrm{tr}(\mu^2)b_\varrho}|c_\varrho|^{2w\Lambda(\mu)}$. Let A be a connection of P which is flat on the disc D_0, and let σ_0 be a horizontal frame over D_0. Gluing the configurations $A_{\varrho,\gamma}$ and $A|_{\Sigma_\infty}$ along $S = D_0 \cap \Sigma_\infty$ by identifying $s(\theta)$ and $\sigma_0(\theta)$, we obtain another H-bundle $P\gamma$ with a connection A^γ. Taking the pairing of $Z_{\Sigma_\infty}(A|_{\Sigma_\infty}; O_1\cdots O_s)$ and $\gamma\cdot Z_{D_0}(A_{\varrho,a}; O_\Lambda)$ and using (2.20), we find that

$$Z_{\Sigma,P\gamma}(A^\gamma; O_1\cdots O_sO_\Lambda) = \mathrm{const}\cdot Z_{\Sigma,P}(A; O_1\cdots O_sO_{\gamma\Lambda})\,. \tag{2.21}$$

This may be considered as the prototype of (1.3). An equation of the same kind is already known in free abelian systems as the insertion theorem [22].

3. Integration Over Gauge Fields

We now turn to integration over gauge fields. In what follows, H is a closed connected subgroup of G/Z_G. For a principal H-bundle P over a Riemann surface Σ, we denote by $\mathscr{A}_P$ the set of connections of P, by $\mathscr{G}_P$ the group of gauge transformations (sections of the adjoint bundle $P\times_H H$), and by $\mathscr{G}_{P_{\mathbb{C}}}$ the group of chiral gauge transformations (sections of the complexified adjoint bundle $P\times_H H_{\mathbb{C}}$).

We consider the integration

$$Z_{\Sigma,P}(O_1\cdots O_s)=\frac{1}{\operatorname{vol}\mathscr{G}_P}\int_{\mathscr{A}_P}\mathscr{D}A\, Z_{\Sigma,P}(A\,;O_1\cdots O_s)\,, \tag{3.1}$$

with the integrand being the WZW correlator of gauge invariant fields. We take advantage of the chiral gauge symmetry

$$Z_{\Sigma,P}(A^h;O_1\cdots O_s)=\mathrm{e}^{kI_{\Sigma,P}(A,hh^*)}Z_{\Sigma,P}(A;hO_1\cdots hO_s)\,, \tag{3.2}$$

which is a consequence of the PW identity (2.7). Here, hO_i is the transform of O_i defined by $hO_i(g)=O_i(h^{-1}gh^{*-1})$. We will first integrate along the fibre of

$$\mathscr{A}_P\longrightarrow\mathscr{A}_P/\mathscr{G}_{P_{\mathbb{C}}}\,, \tag{3.3}$$

and then over the orbit space $\mathscr{A}_P/\mathscr{G}_{P_{\mathbb{C}}}$. A subtle problem here is that this orbit space, with the natural topology, is not in general a good space that admits an integration. However, there exists a submanifold $\mathscr{A}_{ss}$ of $\mathscr{A}_P$ with complement of codimension $\geqq 1$ such that a certain quotient of $\mathscr{A}_{ss}$ by $\mathscr{G}_{P_{\mathbb{C}}}$ is a compact complex manifold $\mathscr{N}_P$, possibly with orbifold points. This will be illustrated in Sect. 3.1. One can neglect the codimension $\geqq 1$ complement if the integrand of (3.1) is finite everywhere in $\mathscr{A}_P$. This is guaranteed when the fields $O_1,\ldots,O_s$ are matrix elements of representations of G that are integrable at level k, as is observed explicitly in [23, 24] for genus 0 and 1.

Following [1] or the route that is standard in string theory [25–27], we get a measure $\Omega_{\Sigma,P}(O_1\cdots O_s)$ of the moduli space $\mathscr{N}_P$ that gives the correlation function:

$$Z_{\Sigma,P}(O_1\cdots O_s)=\int_{\mathscr{N}_P}\Omega_{\Sigma,P}(O_1\cdots O_s)\,. \tag{3.4}$$

To express it, we choose a complex coordinate system $(u^1,\ldots,u^{d_{\mathscr{N}}})$ of an open subset U of $\mathscr{N}_P$ with a representative family $\{A_u\}_{u\in U}\subset\mathscr{A}_P$. Note that the differentials

$$v_a(u)=\frac{\partial}{\partial u^a}A_u^{01}\qquad a=1,\ldots,d_{\mathscr{N}}=\dim\mathscr{N}_P\,, \tag{3.5}$$

represent (1,0)-vector fields in U. The measure is then expressed as

$$\Omega_{\Sigma,P}(O_1\cdots O_s)=\prod_{a=1}^{d_{\mathscr{N}}}d^2u^a\mathscr{Z}_{\Sigma,P}\left(A_u\,;\frac{1}{|\operatorname{Aut}\bar{\partial}_{A_u}|}\prod_{a=1}^{d_{\mathscr{N}}}\left|\frac{1}{2\pi i}\int_{\Sigma}b\,v_a(u)\right|^2 hO_1\cdots hO_s\right)\,. \tag{3.6}$$

Here, $\mathscr{Z}_{\Sigma,P}(A_u\,;\cdots)$ denotes the correlation function for the combined system of the following three theories in a background gauge field A_u:

(i) level k WZW model based on the group G,
(ii) level $-\tilde{k}-2h^\vee$ WZW model based on the space $H_{\mathbb{C}}/H$([2]) with the action

$$(-\tilde{k}-2h^\vee)I^H_{\Sigma,P}(A_u,hh^*)\,,\quad h\in\mathscr{G}_{P_{\mathbb{C}}}\,,$$

[2]As the generalization is a trivial matter, we assume here that H is simple. $\tilde{k}$ is defined by $k\,\mathrm{tr}_G(XY)=\tilde{k}\,\mathrm{tr}_H(XY)$ for $X,Y\in\mathfrak{h}\subset\mathfrak{g}$, and $h^\vee$ is the dual Coxeter number of H.

(iii) spin (1,0) ghost system (b,c) in the adjoint representation of H

$$\frac{i}{2\pi}\int_{\Sigma} b\,\bar{\partial}_{A_u}c+\bar{b}\,\partial_{A_u}\bar{c}\,.$$

The factor $1/|\mathrm{Aut}\,\bar{\partial}_{A_u}|$ takes care of the case in which the $\mathscr{G}_{P_{\mathbf{C}}}$ action on $\mathscr{A}_P$ is not free. When the automorphism group $\mathrm{Aut}\,\bar{\partial}_{A_u}$ (the isotropy subgroup of $\mathscr{G}_{P_{\mathbf{C}}}$ at $A_u\in\mathscr{A}_P$) is a finite group, it is just to divide by the order of $\mathrm{Aut}\,\bar{\partial}_{A_u}$. When $\dim\mathrm{Aut}\,\bar{\partial}_{A_u}\geqq 1$, it stands for insertion of a suitable function of the h-field and c-ghosts, which will not be explicitly given in this paper, except for the special case considered in Sect. 5.

3.1. The Moduli Space. We illustrate some points of the structure of $\mathscr{G}_{P_{\mathbf{C}}}$-orbits in $\mathscr{A}_P$, presenting examples that will be used in Sects. 4 and 5. We start with some generalities. The set $\mathscr{A}_P/\mathscr{G}_{P_{\mathbf{C}}}$ is identified via $A\leftrightarrow\bar{\partial}_A$ with the set of isomorphism classes of holomorphic principal $H_{\mathbf{C}}$ bundles of topological type $P_{\mathbf{C}}$ (the complexification of P). If H is $U(1)$, irrespectively of the topological type, this set is a copy of the Jacobian variety of Σ which is a complex torus of dimension $g=$ genus of Σ. If H is non-abelian, the situation is quite different. For simplicity, we consider the case of simple group H. Atiyah and Bott [29] proved the stratification $\mathscr{A}_P=\bigcup_\mu\mathscr{A}_\mu$, where μ runs over a discrete subset of the closure of a Weyl chambre, and $\mathscr{A}_\mu$ is a $\mathscr{G}_{P_{\mathbf{C}}}$-invariant submanifold of $\mathscr{A}_P$ of codimension $d_\mu=\sum_{\alpha(\mu)>0}(\alpha(\mu)+g-1)$. For $g\geqq 1$, the unique solution to $d_\mu=0$ is $\mu=0$; this $\mathscr{A}_{\mu=0}$ is the space $\mathscr{A}_{ss}$ mentioned above, and the quotient $\mathscr{N}_P$ is the moduli space of semi-stable $H_{\mathbf{C}}$-bundles. By Narasimhan–Seshadri theorem [30, 32, 31], it is isomorphic to the space of flat connections of P modulo action of $\mathscr{G}_P$. For $g\geqq 2$, it has complex dimension $d_{\mathscr{N}}=\dim H(g-1)$.

Genus 0. Consider the case in which Σ is the Riemann sphere $\mathbf{P}^1$, covered by the z-plane U_0 and the w-plane U_∞ that are related by $zw=1$.

First, we consider the group $H=SU(n)/\mathbf{Z}_n$, where $\mathbf{Z}_n$ is the center of $SU(n)$. The Birkhoff factorization theorem [28, 16] states that any holomorphic $H_{\mathbf{C}}=PSL(n,\mathbf{C})$-bundle admits trivializations σ_0 and σ_∞ over U_0 and U_∞ respectively, that are related by

$$\sigma_0(z)=\sigma_\infty(z)z^{-a}\,,\quad z\in U_0\cap U_\infty\,,\tag{3.7}$$

where a is a matrix of the form

$$a=\begin{pmatrix}a_1&&\\&\ddots&\\&&a_n\end{pmatrix}\quad\text{with }a_i\in\mathbf{Z}-\frac{j}{n},\quad\sum_{i=1}^n a_i=0\,,\tag{3.8}$$

for some $j\in\{0,1,\dots,n-1\}$. The theorem also states that such a is unique up to permutations of $a_1,\dots,a_n$. Since the loop $\mathrm{e}^{i\theta}\mapsto\mathrm{e}^{-ia\theta}$ extends to a map on $U_0\to H$ if and only if all a_i are integers, the topological type of the bundle is determined by the number j. In other words, for each j, there is an H-bundle $P^{(j)}$ and its complexification admits holomorphic structures classified by $\mathrm{P}_j^\vee/W$ in which $\mathrm{P}_j^\vee$ is the set of matrices as (3.8), and W is the Weyl group acting by permutation of diagonal entries. Since the set $\overline{\mathrm{C}}$ of diagonal matrices t with $t_1^1\geqq\cdots\geqq t_n^n$ is a fundamental domain of W, we see that

$$\mathscr{A}_{P(j)} = \bigcup_{a \in \mathrm{P}_j^\vee \cap \bar{\mathrm{C}}} \mathscr{A}_a \,. \tag{3.9}$$

Here $\mathscr{A}_a$ is the $\mathscr{G}_{P^{(j)}_{\mathrm{c}}}$-orbit corresponding to the holomorphic $H_{\mathbf{C}}$-bundle with the transition rule (3.7). We denote such a holomorphic bundle by $\mathscr{P}_{[a]}$. One can estimate the codimension of $\mathscr{A}_a$ by counting the dimension of the group $\mathrm{Aut}\,\mathscr{P}_{[a]}$ of holomorphic automorphisms. An automorphism h is given by $H_{\mathbf{C}}$-valued holomorphic functions $h_0(z)$ and $h_\infty(w)$ that are related by $h_0(z) = z^a h_\infty(1/z) z^{-a}$, $z \neq 0, \infty$. One sees that the entry $h_0(z)^i_j$ is a span of $1, z, \cdots, z^{a_i - a_j}$ if $a_i \geqq a_j$ and is zero if $a_i < a_j$. The dimension of $\mathrm{Aut}\,\mathscr{P}_{[a]}$ is thus given by $n - 1 + \sum_{i<j}(\delta_{a_i, a_j} + 1 + |a_i - a_j|)$ and is minimized in $\mathrm{P}_j^\vee \cap \bar{\mathrm{C}}$ by the matrix $a = \mu_j$ given in (A.2). Hence $\mathscr{A}_{P(j)}$ contains an orbit $\mathscr{A}_{\mu_j}$ of maximal dimension, and any other orbit $\mathscr{A}_a$ has codimension $d_a = \sum_{a_i > a_j}(a_i - a_j - 1) > 0$. Thus, in this case $\mathscr{A}_{ss} = \mathscr{A}_{\mu_j}$, and $\mathscr{N}_{P(j)}$ is a single point.

For general group H, the story is the same. For an H-bundle $P^{(\gamma)}$ whose topological type is determined by $\gamma \in \pi_1(H)$, there is a single maximal orbit $\mathscr{A}_{ss}$. This is represented by a holomorphic bundle $\mathscr{P}_{[\mu]}$ described by the transition rule (3.7) with a being a minimal coweight $\mu \in \mathrm{M}_{\mathrm{C}}$ such that the loop $\mathrm{e}^{-i\mu\theta}$ represents $\gamma \in \pi_1(H)$.

Genus 1. We explicitly describe the moduli spaces of flat $SO(3)$-connections on the torus $\Sigma_\tau = \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$. Let $A(t) = t$ and $B(t) = \tau t$, $0 \leqq t \leqq 1$, be generators of the fundamental group of Σ_τ. A flat connection of an $SO(3)$-bundle P defines (up to conjugation) a holonomy representation $\rho : \pi_1\Sigma_\tau \to SO(3)$, which is determined by the commuting elements $a = \rho(A)$ and $b = \rho(B)$ of $SO(3)$.

If P is a trivial bundle, a and b are represented by commuting elements $\tilde{a}$ and $\tilde{b}$ of $SU(2)$. By conjugation, we can bring them to diagonal matrices

$$\tilde{a} = \begin{pmatrix} e^{2\pi i\phi} & 0 \\ 0 & e^{-2\pi i\phi} \end{pmatrix}, \qquad \tilde{b} = \begin{pmatrix} e^{2\pi i\psi} & 0 \\ 0 & e^{-2\pi i\psi} \end{pmatrix}. \tag{3.10}$$

Such holonomy is provided by the gauge field of the following form:

$$A_u = \left(\frac{\pi}{\tau_2} u\, d\bar{\zeta} - \frac{\pi}{\tau_2}\bar{u}\, d\zeta\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.11}$$

where $u = \psi - \tau\phi$. $A_{u'}$ is gauge equivalent to A_u if and only if $u' = \pm u - \frac{m}{2} + \tau\frac{n}{2}$ for some n, $m \in \mathbf{Z}$. Hence, the moduli space is given by

$$\mathscr{N}_{\mathrm{triv}} = \mathbf{C}/\{(\tfrac{1}{2}\mathbf{Z} + \tfrac{\tau}{2}\mathbf{Z})\tilde{\times}\{\pm 1\}\}\,. \tag{3.12}$$

It is an orbifold with four singularities $u \equiv 0, \frac{1}{4}, \frac{\tau}{4}, \frac{\tau+1}{4}$ of order 2.

If P is non-trivial, a and b are represented by elements $\tilde{a}$, $\tilde{b}$ of $SU(2)$ that do not commute but satisfy

$$\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.13}$$

There is only one such pair $(\tilde{a}, \tilde{b})$ modulo conjugation:

$$\tilde{a} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \tilde{b} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.14}$$

Hence,

$$\mathcal{N}_{\text{non-triv}} = \{\text{one point}\}\,. \tag{3.15}$$

In contrast with the abelian case, $\mathcal{N}_{\text{triv}}$ is not isomorphic to $\mathcal{N}_{\text{non-triv}}$ and even the dimensions are different.

For a general semi-simple group H, the moduli space of topologically trivial semi-stable $H_{\mathbf{C}}$-bundles over the torus Σ_τ is

$$\mathcal{N}_{\text{triv}} = \mathfrak{t}_{\mathbf{C}}/(\mathrm{P}^\vee + \tau\mathrm{P}^\vee)\tilde{\times} W\,, \tag{3.16}$$

where $\mathrm{P}^\vee$ is the coweight lattice, and hence of dimension $\operatorname{rank} H$. On the other hand, for each non-trivial H-bundle P, $\dim \mathcal{N}_P$ is strictly less than the rank of H.

3.2. Gauge Invariant Fields. We specify the set of gauge invariant local fields in the WZW model. One can translate the gauge invariance condition on a local field O to a condition on the state $\Phi_O \in \mathcal{H}_\Lambda^{G,k}$ at the boundary of the disc D_0 with the insertion of O at $z=0$:

$$\big(J_0(v) + \bar{J}_0(v)\big)\,\Phi_O = 0 \quad \text{for } v \in \mathfrak{h}, \tag{3.17}$$

$$J_n(v)\Phi_O = \bar{J}_n(v)\Phi_O = 0 \quad \text{for } v \in \mathfrak{h}_{\mathbf{C}} \text{ and } n = 1,2,\ldots. \tag{3.18}$$

Here, $J_n(v)$ and $\bar{J}_n(v)$ are infinitesimal generators of J and $\bar{J}$ corresponding to the element $z^n v$ of the loop algebra. We shall distinguish the space of states satisfying these conditions.

Let $\bar{H} \subset G$ be the group over $H \subset G/Z_G$. Following Goddard–Kent–Olive [9], we decompose $L_\Lambda^{G,k}$ into irreducible representations of the subgroup $\widetilde{L\bar{H}_{\mathbf{C}}}$ of $\widetilde{LG_{\mathbf{C}}}$:

$$L_\Lambda^{G,k} = \bigoplus_\lambda B_\Lambda^\lambda \otimes L_\lambda^{\bar{H},\tilde{k}}\,, \tag{3.19}$$

where B_Λ^λ is the subspace of $L_\Lambda^{G,k}$ consisting of highest weight vectors of weight $(\lambda,\tilde{k})$ with respect to $\widetilde{L\bar{H}_{\mathbf{C}}}$. We denote by $\mathcal{H}_\Lambda^\lambda$ the subspace of $\mathcal{H}_\Lambda^{G,k}$ corresponding to the subspace $B_\Lambda^\lambda \otimes \overline{B_\Lambda^\lambda}$ of $L_\Lambda^{G,k} \otimes \overline{L_\Lambda^{G,k}}$. Each $\Phi \in \mathcal{H}_\Lambda^\lambda$ generates an irreducible $J_0(\bar{H})\times \bar{J}_0(\bar{H})$-module in $\mathcal{H}_\Lambda^{G,k}$ which is isomorphic to $V_\lambda \otimes V_\lambda^*$, where V_λ is the irreducible $\bar{H}$-module of highest weight λ and V_λ^* is its dual. Choosing a base $\{e_m\}$ of V_λ and the dual base $\{e^m\} \subset V_\lambda^*$, we denote by $\Phi_m^{\bar{m}}$ the state corresponding to $e_m \otimes e^{\bar{m}} \in V_\lambda \otimes V_\lambda^*$. Then, the $\bar{H}$-invariant element $\sum_m \Phi_m^m$ satisfies (3.17) and (3.18). Thus, we identify the space of states corresponding to gauge invariant fields with

$$\mathcal{H}_{\text{hw}} := \bigoplus_{\Lambda,\lambda} \mathcal{H}_\Lambda^\lambda\,. \tag{3.20}$$

Let $O_{\Phi_m^{\bar{m}}}$ denote the field corresponding to the state $\Phi_m^{\bar{m}}$ and we consider it as a matrix element of a field $\mathbf{O}_\Phi$ valued in $\operatorname{End}(V_\lambda)$. Then, the gauge invariant field O_Φ corresponding to the state $\frac{1}{\dim V_\lambda}\sum_m \Phi_m^m$ is expressed as

$$O_\Phi = \frac{1}{\dim V_\lambda}\operatorname{tr}_{V_\lambda}(\mathbf{O}_\Phi)\,. \tag{3.21}$$

Since $J_0(\bar{h})\bar{J}_0(\bar{h})\Phi_m^{\bar{m}}$ with $\bar{h} \in \bar{H}_{\mathbf{C}}$ is expanded as $\sum_{r,s} \bar{h}_r^{*\bar{m}}\Phi_s^r \bar{h}_m^s$, the dressed field for O_Φ is given by

$$hO_\Phi = \frac{1}{\dim V_\lambda}\operatorname{tr}_{V_\lambda}(\mathbf{O}_\Phi hh^*)\,. \tag{3.22}$$

4. Topological Identity

In Sect. 3, we wrote down a formula (3.4) that expresses a correlation function as an integral over the moduli space of semi-stable bundles. If we are to use it to prove the topological identity (1.3), we must find some relation of the moduli spaces of bundles of distinct topologies. In general, however, they are not identical. In the former part of this section, we convert the formula into a new one in which a correlation function is expressed as an integral over a moduli space of bundles with a flag at the insertion point. We will see that the new moduli spaces for distinct topologies are isomorphic with each other, via the gauge transformation that defines the spectral flow. Applying the argument that leads to (2.21), we will get a proof of the topological identity.

4.1. The Flag Partner. The first step of reformulation is to express a gauge invariant field as a certain integral over the flag manifold of the gauge group.

The *flag manifold* $Fl(H)$ of H is the ensemble of choices of a maximal torus of H and a chambre. A choice $(T, \mathrm{C}) \in Fl(H)$ determines an identification

$$Fl(H) \cong H/T \cong H_{\mathbf{C}}/B\,, \tag{4.1}$$

where B is the Borel subgroup of $H_{\mathbf{C}}$ determined by (T, C). Thus, $Fl(H)$ is a compact homogeneous complex manifold. A weight $\lambda \in \mathrm{P}$ gives a character $\mathrm{e}^{\lambda} : T \to U(1)$ by $\mathrm{e}^{2\pi i v} \mapsto \mathrm{e}^{2\pi i \lambda(v)}$ and its extension $\mathrm{e}^{\lambda} : B \to \mathbf{C}^*$ defines a homogeneous line bundle

$$L_{-\lambda} = H_{\mathbf{C}} \times_B \mathbf{C} \longrightarrow Fl(H)\,, \tag{4.2}$$

by the relation $(hb, c) \sim (h, \mathrm{e}^{-\lambda}(b)c)$, where $h \in H_{\mathbf{C}}$, $b \in B$ and $c \in \mathbf{C}$. We denote by $h \cdot c \in L_{-\lambda}$ the class represented by (h, c). The Borel–Weil theorem states that the space $H^0(Fl(H), L_{-\lambda})$ of holomorphic sections is an irreducible $H_{\mathbf{C}}$-module V_{λ^*} of highest weight $\lambda^* = -w_0\lambda$ (see [33] and also [34, 35]). The line bundle $L_{-\lambda}$ is equipped with an H-invariant fibre metric $(h \cdot c_1, h \cdot c_2)_{-\lambda} = \bar{c}_1 c_2$, where $h \in H$. There also exists an H-invariant volume form Ω on $Fl(H)$. These induce the H-invariant inner product in $H^0(Fl(H), L_{-\lambda})$:

$$(\psi_1, \psi_2)_{Fl(H)} = \frac{1}{\mathrm{vol}\, Fl(H)} \int_{Fl(H)} (\psi_1, \psi_2)_{-\lambda}\Omega\,. \tag{4.3}$$

Let $\{e_m\,;\, m \in \tilde{\mathrm{P}}_\lambda\}$ be an orthonormal base of V_λ consisting of weight vectors where $\tilde{\mathrm{P}}_\lambda$ is an indexing set. The highest weight vector is denoted by e_λ. Writing the matrix element $(e_{m_1}, h e_{m_2})$ by $(h)^{m_1}_{m_2}$, we put

$$\psi^m(hB) = h \cdot (h)^m_\lambda\,. \tag{4.4}$$

Then, $\{\psi^m\,;\, m \in \tilde{\mathrm{P}}_\lambda\}$ forms an orthogonal base of $H^0(Fl(H), L_{-\lambda})$:

$$(\psi^{m_1}, \psi^{m_2})_{Fl(H)} = \frac{1}{\dim V_\lambda}\delta^{m_1, m_2}\,. \tag{4.5}$$

Integral Expression of Gauge Invariant Fields. Let O be the gauge invariant field (3.21) corresponding to a state $\Phi \in \mathscr{H}^{\lambda}_{\Lambda}$. We express the dressed field hO as an integral over the flag manifold.

Let us define a volume form $\Omega(hh^*)$ of $Fl(H)$ by the following: At the point $h_1B \in Fl(H)$ represented by $h_1 \in H$,

$$\Omega(hh^*)|_{h_1B} := h_1 O^\lambda_\lambda \, |e^{\lambda+2\rho}(b(h_1^{-1}h))|^2 \, \Omega|_{h_1B} , \tag{4.6}$$

where $b(h_1^{-1}h) \in B$ is the "Borel-part" of the decomposition $h_1^{-1}h = b(h_1^{-1}h)U(h_1^{-1}h)$; $b(h_1^{-1}h) \in B$, $U(h_1^{-1}h) \in H$. (This decomposition is unique up to T due to the Iwasawa decomposition.) Let $\ell_{h^{-1}} : Fl(H) \to Fl(H)$ be the left translation by h^{-1}. The relation

$$\ell^*_{h^{-1}} \Omega|_{h_1B} = |e^{2\rho}(b(h_1^{-1}h))|^2 \Omega|_{h_1B} , \tag{4.7}$$

(which we shall prove shortly) shows

$$\ell^*_h \Omega(hh^*)|_{h^{-1}h_1B} = h_1 O^\lambda_\lambda \, |e^{\lambda}(b(h_1^{-1}h))|^2 \, \Omega|_{h^{-1}h_1B} . \tag{4.8}$$

Putting $U = U(h_1^{-1}h)^{-1}$, we have $h^{-1}h_1 O^\lambda_\lambda = UO^\lambda_\lambda \, |e^{-\lambda}(b(h_1^{-1}h))|^2$. Hence we get

$$\ell^*_h \Omega_\lambda(hh^*)|_{UB} = hUO^\lambda_\lambda \, \Omega|_{UB} = \sum_{\bar{m},m} hO^{\bar{m}}_m (U^{-1})^\lambda_{\bar{m}} (U)^m_\lambda \, \Omega|_{UB} . \tag{4.9}$$

This amounts to the following identity of top differential forms:

$$\ell^*_h \Omega(hh^*) = \sum_{\bar{m},m} hO^{\bar{m}}_m (\psi^{\bar{m}}, \psi^m)_{-\lambda} \Omega , \tag{4.10}$$

where ψ^m is given in (4.4). Due to the orthogonality (4.5), it follows that

$$\frac{1}{\operatorname{vol} Fl(H)} \int_{Fl(H)} \Omega(hh^*) = \frac{1}{\dim V_\lambda} \operatorname{tr}_{V_\lambda} (\mathbf{O}hh^*) . \tag{4.11}$$

Proof of the relation (4.7). It is enough to prove $\ell^*_b \Omega|_B = |e^{-2\rho}(b)|^2 \Omega|_B$ for $b \in B$. Since the (1,0)-tangent space of $Fl(H)$ at B is isomorphic to $\mathfrak{h}_{\mathbf{C}}/\mathfrak{b}$, we have only to show that $e^{-2\rho}(b)$ is the determinant of $\operatorname{ad}(b) : \mathfrak{h}_{\mathbf{C}}/\mathfrak{b} \to \mathfrak{h}_{\mathbf{C}}/\mathfrak{b}$. In view of $2\rho = \sum_{\alpha>0} \alpha$, the proof is trivial since we can order the base of $\mathfrak{h}_{\mathbf{C}}/\mathfrak{b}$ consisting of negative root vectors so that $\operatorname{ad}(b)$ is represented by an upper triangular matrix.

The Flag Partner. Suppose that hO is inserted at the center x of a disc $D_0 \subset \Sigma$ in a correlator $\mathscr{Z}_{\Sigma,P}(A; \cdots)$, where the gauge field A is chosen to be flat over D_0. The fibre P_x is a copy of H, and accordingly, one can consider a copy $Fl(P_x)$ of $Fl(H)$. Let us choose $f \in Fl(P_x)$ with its representative $s(x) \in P_x$. $s(x)$ extends to a horizontal frame $s : D_0 \to P$ which determines a correspondence of fields and states. The field corresponding to $\Phi \in \mathscr{H}^\lambda_A$ can be denoted as $O^\lambda_\lambda(f)$. We also denote by $b_f(h)$ the "Borel-part" of the decomposition of $h(x)$ with respect to $s(x)$. Let us choose a local coordinate $f^1, \ldots, f^{|\Delta_+|}$ of $Fl(P_x)$ and a family $\{\sigma_f\}$ of holomorphic sections of $P_{\mathbf{C}}|_{D_0}$ with respect to $\bar{\partial}_A$. Then, the expression $(\partial\sigma_f/\partial f^\alpha)\sigma_f^{-1}$ determines a holomorphic section $v_\alpha(f)$ of $\operatorname{ad}P_{\mathbf{C}} = P \times_H \mathfrak{h}_{\mathbf{C}}$ over D_0. Using the standard OPE of b and c ghosts, from (4.11) we obtain

$$hO(x) = \frac{1}{\operatorname{vol} Fl(H)} \int_{Fl(P_x)} \prod_{\beta=1}^{|\Delta_+|} d^2 f^\beta \frac{1}{2\pi i} \oint_x b \, v_\beta(f) \frac{1}{2\pi i} \oint_x \bar{b} \, \bar{v}_\beta(f) \, \widehat{O}(f) , \tag{4.12}$$

where

$$\widehat{O}(f) = O^\lambda_\lambda(f) \, |e^{\lambda+2\rho}(b_f(h))|^2 \prod_{-\alpha \in \Delta_-} c^{-\alpha}(x) \bar{c}^{-\alpha}(x) . \tag{4.13}$$

In the above expression, $c^{-\alpha}(x)$ are the coefficients of negative root vectors in the expansion of the ghost $c(x)$ with respect to the frame of $\mathrm{ad}P_{\mathbf{C}x}$ determined by $s(x)$. We call this field $\widehat{O}(f)$ the *flag partner* of O associated to $f \in Fl(P_x)$.

The state corresponding to the flag partner $\widehat{O}(f)$ with respect to the frame s is expressed as

$$\Phi \otimes \Phi_{-\lambda-2\rho} \otimes |\Omega\rangle \,. \tag{4.14}$$

Here, $\Phi_{-\lambda-2\rho}$ is the state corresponding to the field $|\mathrm{e}^{\lambda+2\rho}(b_f(h))|^2$ of the WZW model based on $H_{\mathbf{C}}/H$. As in WZW models based on compact groups, the state space of this system has left-right representation of the affine Lie algebra $L\widetilde{\mathfrak{h}}_{\mathbf{C}}$. The state $\Phi_{-\lambda-2\rho}$ is a highest weight state of left-right equal weight $(-\lambda-2\rho, -\tilde{k}-2h^\vee)$. The $|\Omega\rangle$ is the state $\prod_{-\alpha<0} c_0^{-\alpha}\bar{c}_0^{-\alpha}|0\rangle$ in the ghost Fock space. It has ghost number $|\Delta|$. In view of the identification (3.20), we see that the space of states corresponding to the flag partners of gauge invariant fields is given by

$$\widehat{\mathscr{H}_{\mathrm{inv}}} = \bigoplus_{\Lambda,\lambda} \mathscr{H}_\Lambda^\lambda \otimes \Phi_{-\lambda-2\rho} \otimes |\Omega\rangle \,. \tag{4.15}$$

Remark. In the literature (see [36] and references therein), the state of the form (4.14) is identified as a non-trivial element of the BRST cohomology that seems to correspond to the physical state space of the gauged WZW model.

4.2. A New Integral Expression. Let us consider the correlation function $Z_{\Sigma,P}(O_1 \cdots O_s O(x))$, which is expressed as the integral of a measure $\Omega_{\Sigma,P}(O_1 \cdots O_s O(x))$ on the moduli space $\mathscr{N}_P$. Let U be an open subset of $\mathscr{N}_P$ with a representative family $\{A_u\}_{u\in U}$. The result (4.12) shows that the measure

$$\begin{aligned}\tilde{\Omega}_{\Sigma,P,x}(O_1 \cdots O_s\widehat{O})_U &= \prod_{a=1}^{d_{\mathscr{N}}} d^2u^a \prod_{\alpha=1}^{|\Delta_+|} d^2 f^\alpha \\ &\times \mathscr{Z}_{\Sigma,P}\left(A_u;\, \frac{1}{|\mathrm{Aut}\,\bar{\partial}_{A_u}|} hO_1 \cdots hO_s \prod_{a=1}^{d_{\mathscr{N}}} \left|\frac{1}{2\pi i}\int_\Sigma b\,\nu_a(u)\right|^2 \prod_{\alpha=1}^{|\Delta_+|} \left|\frac{1}{2\pi i}\oint_x b\,\nu_\alpha(f)\right|^2 \widehat{O}(f)\right),\end{aligned} \tag{4.16}$$

on $U\times Fl(P_x)$ reproduces $\Omega_{\Sigma,P}(O_1 \cdots O_s O(x))$ by integration along $Fl(P_x)$. Let us see what happens when the family A_u is replaced by its chiral gauge transform $A_u^{h_c}$ by $h_c \in \mathscr{G}_{P_{\mathbf{C}}}$. Absence of chiral anomaly in the combined system of the three CFTs shows

$$\begin{aligned}&\mathscr{Z}_{\Sigma,P}\left(A_u^{h_c};\, \frac{1}{|\mathrm{Aut}\,\bar{\partial}_{A_u^{h_c}}|} hO_1 \cdots hO_s \mathscr{F}(b;\bar{b})\,\widehat{O}(f)\right) \\ &= \mathscr{Z}_{\Sigma,P}\left(A_u;\, \frac{1}{|\mathrm{Aut}\,\bar{\partial}_{A_u}|} hO_1 \cdots hO_s \mathscr{F}(h_c^{-1}bh_c; h_c^*\bar{b}h_c^{*-1})\, h_c\widehat{O}(f)\right),\end{aligned} \tag{4.17}$$

where $\mathscr{F}(b;\bar{b})$ is an arbitrary function of b and $\bar{b}$. By a simple argument, one can show that $h_c\widehat{O}(f) = \widehat{O}(h_c f)$, where h_c acts on flags by evaluation at x. Since $\delta(A_u^{h_c})^{01} = h_c^{-1}\delta A_u^{01} h_c$, the measure (4.16) is invariant under the replacement

$$(A_u, f) \mapsto (A_u^{h_c}, h_c^{-1} f)\,. \tag{4.18}$$

In other words, (4.16) defines a $\mathscr{G}_{P_{\mathbf{C}}}$-invariant form of degree $2d_{\mathscr{N}}+|\Delta|$ on

$$\mathscr{A}_{P,x}=\mathscr{A}_P\times Fl(P_x). \tag{4.19}$$

Let us see whether this form is null along the direction of $\mathscr{G}_{P_{\mathbf{C}}}$-orbits $(\delta A^{01},\delta f)=(\bar{\partial}_A\varepsilon,-\varepsilon f)$. When $\operatorname{Aut}\bar{\partial}_A$ is trivial, the b-ghost is holomorphic over $\Sigma-\{x\}$:

$$\int_\Sigma b\,\bar{\partial}_A\varepsilon=\oint_x b\varepsilon\,. \tag{4.20}$$

In this case, the b-insertions from $\delta A^{01}=\bar{\partial}_A\varepsilon$ and from $\delta f=-\varepsilon f$ cancel with each other, and (4.16) is null along $\mathscr{G}_{P_{\mathbf{C}}}$-orbits. The story is the same when every automorphism of $\bar{\partial}_A$ preserves any flag f; $\operatorname{Aut}\bar{\partial}_A=\operatorname{Aut}(\bar{\partial}_A,f)$, since in such a case $\widehat{O}(f)$ does not carry c-ghost zero modes so that b-ghosts are holomorphic over $\Sigma-\{x\}$. However, the situation is involved when $\operatorname{Aut}\bar{\partial}_A\neq\operatorname{Aut}(\bar{\partial}_A,f)$. In this case, b-ghosts have poles at points of the c-ghost insertion associated with $|\operatorname{Aut}\bar{\partial}_A|^{-1}$: If ε is a holomorphic section of $\mathrm{ad}P_{\mathbf{C}}$ such that $\varepsilon f\neq 0$, the $b\varepsilon$ is a meromorphic differential which has poles also in $\Sigma-\{x\}$. Let $\sim$ be an equivalence relation in $\mathscr{A}_{P,x}$ defined by $(A,f)\sim(A,h_c^{-1}f)$; $h_c\in\operatorname{Aut}\bar{\partial}_A$. Then, integration along the fibre of $\mathscr{A}_{P,x}\to\mathscr{A}_{P,x}/\sim$ yields a form on $\mathscr{A}_{P,x}/\sim$ which is null along the fibre of $\mathscr{A}_{P,x}/\sim\ \to\mathscr{A}_{P,x}/\mathscr{G}_{P_{\mathbf{C}}}$.

In any case, after a certain integration if necessary, the form (4.16) descends to a measure of a quotient of $\mathscr{A}_{P,x}$ by $\mathscr{G}_{P_{\mathbf{C}}}$. Although taking a suitable quotient is a subtle problem, we proceed by assuming that there exists a good one which we denote by $\mathscr{N}_{P,x}$. This will be specified in the next subsection in certain cases.

To be explicit, let us choose a coordinate system $(v^1,\dots,v^{\hat{d}_{\mathscr{N}}})$ of an open subset V of $\mathscr{N}_{P,x}$ with a representative family $\{(A_v,f_v)\}_{v\in V}$. We choose a family of holomorphic sections $\sigma_0(v)$ and $\sigma_\infty(v)$ of $P_{\mathbf{C}}$ over a neighborhood U_0 of x and $U_\infty=\Sigma-\{x\}$, that are related by holomorphic transition functions $h_{\infty 0}(v)$. The $\sigma_0(v)$ is chosen to represent f_v at x. Let $v_{\mathrm{A}}(v)$ be the holomorphic sections of $\mathrm{ad}P_{\mathbf{C}}$ over $U_0\cap U_\infty$ defined by

$$v_{\mathrm{A}}(v)=\sigma_0(v)\cdot h_{\infty 0}(v)^{-1}\frac{\partial}{\partial v^{\mathrm{A}}}h_{\infty 0}(v),\quad \mathrm{A}=1,\dots,\hat{d}_{\mathscr{N}}\,. \tag{4.21}$$

Then, the new measure $\Omega_{\Sigma,P,x}(O_1\cdots O_s\widehat{O})$ of $\mathscr{N}_{P,x}$ is expressed on V as

$$\begin{aligned}&\Omega_{\Sigma,P,x}(O_1\cdots O_s\widehat{O})\\&=\prod_{\mathrm{A}=1}^{\hat{d}_{\mathscr{N}}}d^2v^{\mathrm{A}}\,\mathscr{Z}_{\Sigma,P}\left(A_v\,;\,\frac{1}{|\operatorname{Aut}(\bar{\partial}_{A_v},f_v)|}hO_1\cdots hO_s\prod_{\mathrm{A}=1}^{\hat{d}_{\mathscr{N}}}\left|\frac{1}{2\pi i}\oint_x b\,v_{\mathrm{A}}(v)\right|^2\widehat{O}(f_v)\right).\end{aligned} \tag{4.22}$$

By construction, integration of this measure reproduces the correlation function under study. Thus, we get a new formula:

$$Z_{\Sigma,P}(O_1\cdots O_s\,O(x))=\frac{1}{\operatorname{vol}Fl(H)}\int_{\mathscr{N}_{P,x}}\Omega_{\Sigma,P,x}(O_1\cdots O_s\,\widehat{O})\,. \tag{4.23}$$

4.3. The Moduli Space of Parabolic Bundles. In this subsection, we explicitly construct the quotient space $\mathscr{N}_{P,x}$ for the cases: $\Sigma=\mathbf{P}^1$ and H is general, and Σ is torus and $H=SO(3)$.

For our purpose, it is enough to find a submanifold of $\mathscr{A}_{P,x}$ with complement of codimension $\geqq 1$, having a good $\mathscr{G}_{P_{\mathbf{C}}}$-quotient. As a candidate for the cases at hand, we propose to consider the set of (A,f) whose automorphism group $\mathrm{Aut}\,(\bar{\partial}_A,f)$ is of minimal dimension. We denote this set by $\mathscr{A}_{P,x}^{\min}$. Note that $\mathscr{A}_{P,x}/\mathscr{G}_{P_{\mathbf{C}}}$ is naturally identified with the set of isomorphism classes of holomorphic principal $H_{\mathbf{C}}$-bundles of topological type $\mathrm{P}_{\mathbf{C}}$ with a choice of flag at x. In [37], such objects are called bundles with *quasi-parabolic structure* at x. By abuse of language, we shall call them *parabolic bundles* over (Σ,x). Two parabolic bundles $(\mathscr{P}_1,f_1)$ and $(\mathscr{P}_2,f_2)$ are said to be isomorphic when there is an isomorphism $\mathscr{P}_1 \to \mathscr{P}_2$ which sends f_1 to f_2. If $(A,f) \in \mathscr{A}_{P,x}$ corresponds to $(\mathscr{P},f)$, the groups of automorphisms coincide: $\mathrm{Aut}\,(\bar{\partial}_A,f) \cong \mathrm{Aut}\,(\mathscr{P},f)$.

Genus 0. We start with $H = SU(n)/\mathbf{Z}_n$. We choose a Borel subgroup B that is represented by the group of upper triangular matrices. Let us take a parabolic $H_{\mathbf{C}}$-bundle over $(\mathbf{P}^1, z=0)$ ($\mathbf{P}^1$ is covered by the z-plane U_0 and w-plane U_∞, $zw=1$). By the Birkhoff theorem [16], there is a unique coweight $a \in \mathrm{P}^\vee$ so that we can choose a section σ_0 on U_0 representing the flag at $z=0$ and a section σ_∞ on U_∞ that are related by $\sigma_0(z) = \sigma_\infty(z) z^{-a}$. We denote this bundle by $\widehat{\mathscr{P}}_a = (\mathscr{P}_{[a]}, f_a)$. Its automorphism group $\mathrm{Aut}\,\widehat{\mathscr{P}}_a$ is the subgroup of $\mathrm{Aut}\,\mathscr{P}_{[a]}$ that preserves f_a. Recall that an element h of $\mathrm{Aut}\,\mathscr{P}_{[a]}$ is represented with respect to σ_0 by a matrix-valued function $h_0(z)$ whose i-j^{th} entry $(h_0)^i_j(z)$ is a span of $1, z, \ldots, z^{a_i - a_j}$ if $a_i \geqq a_j$ and zero if $a_i < a_j$. It belongs to $\mathrm{Aut}\,\widehat{\mathscr{P}}_a$ if $(h_0)^i_j(0) = 0$ for $i > j$. Thus,

$$\dim \mathrm{Aut}\,\widehat{\mathscr{P}}_a = \dim \mathrm{Aut}\,\mathscr{P}_{[a]} - \sum_{\substack{i>j \\ a_i \geqq a_j}} 1 = n - 1 + \sum_{i<j} \left(|a_i - a_j| + \theta_{a_i, a_j} \right), \tag{4.24}$$

where $\theta_{x,y} = 0$ if $x < y$ and $\theta_{x,y} = 1$ if $x \geqq y$. The lowest value $(n+2)(n-1)/2$ is saturated by n elements; $a = (w_j w_0)^{-1} \mu_j$, $j = 0, 1, \ldots, n-1$, where μ_j and $w_j w_0$ are the minimal coweight and the Weyl group element given in (A.2). They correspond to distinct topological types. Thus, for each H-bundle $P = P^{(j)}$, our submanifold $\mathscr{A}_{P,x}^{\min}$ is the unique orbit of maximal dimension corresponding to the parabolic bundle $\widehat{\mathscr{P}}_j = \widehat{\mathscr{P}}_{(w_j w_0)^{-1} \mu_j}$. Therefore, the quotient $\mathscr{N}_{P^{(j)},x}$ is a one point set. Note that $\widehat{\mathscr{P}}_j$ admits the transition rule $\sigma_0(z) = \sigma_\infty(z) z^{-\mu_j} n_{w_j w_0}$ with $\sigma_0(0)B$ being the flag, where the loop $\mathrm{e}^{-i\mu_j\theta} n_{w_j w_0}$ represents an element of $\Gamma_{\widehat{\mathbf{C}}}$. Note also that $\dim \mathrm{Aut}\,\widehat{\mathscr{P}}_j$ is independent of the topological type. In fact, they are all isomorphic. We revisit these points in the next subsection.

For a general compact group H, the story is essentially the same. Each $a \in \mathrm{P}^\vee$ indexes an isomorphism class represented by a parabolic bundle whose automorphism group has dimension $l + \sum_{\alpha>0} (|\alpha(a)| + \theta_{\alpha(a),0})$ which is minimized by $a = w^{-1}\mu \in -\overline{\mathrm{C}}$, where μ is a minimal coweight. Thus, for each topological type $\gamma \in \pi_1(H)$, the quotient $\mathscr{N}_{P^{(\gamma)},x}$ is a one point set represented by a parabolic bundle $\widehat{\mathscr{P}}_\gamma$ described by the transition rule

$$\sigma_0(z) = \sigma_\infty(z) z^{-\mu} n_w, \tag{4.25}$$

with $\sigma_0(0)B$ being the flag. Here, the loop $\gamma(\theta) = \mathrm{e}^{-i\mu\theta} n_w$ represents an element of $\Gamma_{\widehat{\mathbf{C}}}$ that corresponds to $\gamma \in \pi_1(H)$.

Genus 1 and $H = SO(3)$. Here we consider parabolic $H_{\mathbf{C}} = PSL(2,\mathbf{C})$-bundles over the torus $\Sigma_\tau = \mathbf{C}^*/q^{\mathbf{Z}}$ with a marked point x with $z(x) \equiv 1$, where $q^{\mathbf{Z}}$ is the subgroup of $\mathbf{C}^*$ generated by $q = e^{2\pi i\tau}$. (This torus is identical with the one considered in Sect. 3 under $e^{-2\pi i\zeta} = z$.)

A holomorphic $H_{\mathbf{C}}$-bundle $\mathscr{P}$ over Σ_τ is described by the transition rule

$$\sigma(qz) = \sigma(z)h(q;z) \tag{4.26}$$

for a multivalued holomorphic section $\sigma(z)$. The bundle is topologically trivial if and only if $e^{i\theta} \mapsto h(q;e^{i\theta})$ is single valued as a map $S^1 \to SL(2,\mathbf{C})$. Below, we list up some bundles by exhibiting their transition matrices $h(q;z)$:

$$\begin{array}{llll} \text{trivial} & & \text{non-trivial} & \\ \mathscr{P}^{(0)}_u : & \begin{pmatrix} t_u & 0 \\ 0 & t_u^{-1} \end{pmatrix} & \mathscr{P}^{(1)}_F : & \begin{pmatrix} 0 & q^{-\frac14}z^{-\frac12} \\ -q^{\frac14}z^{\frac12} & 0 \end{pmatrix} \\ \mathscr{P}^{(0)}_{00} : & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \mathscr{P}^{(1)}_u : & \begin{pmatrix} it_u^{-1}z^{-\frac12} & 0 \\ 0 & -it_u z^{\frac12} \end{pmatrix} . \end{array}$$

Here, $t_u = e^{-2\pi i u}$. Note the identifications

$$\mathscr{P}^{(0)}_u \cong \mathscr{P}^{(0)}_{u'} \Leftrightarrow u \equiv \pm u' \bmod \tfrac12\mathbf{Z} + \tfrac{\tau}{2}\mathbf{Z} \overset{\text{def}}{\Longleftrightarrow} u \sim u',$$
$$\mathscr{P}^{(1)}_u \cong \mathscr{P}^{(1)}_{u'} \Leftrightarrow u \equiv u' \bmod \tfrac12\mathbf{Z} + \tfrac{\tau}{2}\mathbf{Z}\,.$$

Atiyah's classification [38] shows that other bundles have $h(q;z) = \mathrm{diag}\cdot(t_u z^{\frac{n}{2}}, t_u^{-1}z^{-\frac{n}{2}})$ with $n \geqq 2$. They have automorphism groups of dimension $\geqq 3$ and hence are irrelevant in our story. $\{\mathscr{P}^{(0)}_u, \mathscr{P}^{(0)}_{00}, \mathscr{P}^{(1)}_F\}$ is the collection of semi-stable bundles. Note that $\mathscr{P}^{(0)}_u$ and $\mathscr{P}^{(1)}_F$ are the flat $SO(3)$ bundles with the holonomies (3.10) and (3.14) respectively.

An automorphism of $\mathscr{P}$ is described by $\sigma(z) \mapsto \sigma(z)h(z)$, where $h(z)$ is a holomorphic map $\mathbf{C}^* \to PSL(2,\mathbf{C})$ satisfying

$$h(q;z)h(zq) = h(z)h(q;z)\,. \tag{4.27}$$

The following is the list of the automorphism groups with their typical elements:

$$\operatorname{Aut}\mathscr{P}^{(0)}_u \cong \begin{cases} \mathbf{C}^* \quad \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} & \text{if } u \not\sim 0, \frac14, \frac{\tau}{4}, \frac{1+\tau}{4} \\ PSL(2,\mathbf{C}) & \text{if } u \sim 0 \\ \mathbf{C}^*\tilde{\times}\mathbf{Z}_2 \quad \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} & \text{if } u \sim \frac14 \\ \mathbf{C}^*\tilde{\times}\mathbf{Z}_2 \quad \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \begin{pmatrix} 0 & cz^{\frac12} \\ -c^{-1}z^{-\frac12} & 0 \end{pmatrix} & \text{if } u \sim \frac{\tau}{4}, \frac{1+\tau}{4}, \end{cases} \tag{4.28}$$

$$\operatorname{Aut}\mathscr{P}^{(0)}_{00} \cong \mathbf{C} \quad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \tag{4.29}$$

$$\operatorname{Aut}\mathscr{P}_F^{(1)} \cong \mathbf{Z}_2\times\mathbf{Z}_2 = \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}i&0\\0&-i\end{pmatrix},\begin{pmatrix}0&i\\i&0\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right\}, \tag{4.30}$$

$$\operatorname{Aut}\mathscr{P}_u^{(1)} \cong B_0^- \qquad \begin{pmatrix}c&0\\x\vartheta_{\tau,u}(z)&c^{-1}\end{pmatrix}. \tag{4.31}$$

Here, $\vartheta_{\tau,u}(z)=\vartheta(\zeta+2u+\frac{1+\tau}{2},\tau)$, where $\vartheta(\zeta,\tau)$ is the Riemann's theta function $\sum_{n\in\mathbf{Z}}q^{\frac12 n^2}e^{2\pi i n\zeta}$. (Recall $q=e^{2\pi i\tau}$ and $z=e^{-2\pi i\zeta}$.) Note that $\vartheta_{\tau,u}(1)=0$ if and only if $u\equiv 0 \bmod \frac12\mathbf{Z}+\frac{\tau}{2}\mathbf{Z}$.

Looking at the action of $\operatorname{Aut}\mathscr{P}$ on the flags over $z\equiv 1$ we can classify the parabolic bundles. Below is the list of isomorphism classes together with the automorphism groups. Here, a flag $\sigma(1)\begin{pmatrix}a&b\\c&d\end{pmatrix}B$ is represented by a number $c/a\in\mathbf{C}\cup\infty$.

trivial

$$\left.\begin{array}{ll}(\mathscr{P}_u^{(0)},1) & 1\\ (\mathscr{P}_u^{(0)},0) & \mathbf{C}^*\\ (\mathscr{P}_u^{(0)},\infty) & \mathbf{C}^*\end{array}\right\} u\not\sim 0,\tfrac14,\tfrac{\tau}{4},\tfrac{\tau+1}{4}$$

$$\left.\begin{array}{ll}(\mathscr{P}_u^{(0)},1) & \mathbf{Z}_2\\ (\mathscr{P}_u^{(0)},0) & \mathbf{C}^*\end{array}\right\} u\sim \tfrac14,\tfrac{\tau}{4},\tfrac{\tau+1}{4}$$

$$(\mathscr{P}_0^{(0)},1) \quad B$$

$$\left.\begin{array}{ll}(\mathscr{P}_{00}^{(0)},\infty) & 1\\ (\mathscr{P}_{00}^{(0)},0) & \mathbf{C}\end{array}\right\}$$

non-trivial

$$\left.\begin{array}{lll}(\mathscr{P}_F^{(1)},y) & 1 & y\not\approx 0,1,i\\ (\mathscr{P}_F^{(1)},y) & \mathbf{Z}_2 & y\approx 0,1,i\end{array}\right\},$$

$$y\approx -y\approx y^{-1}\approx -y^{-1}$$

$$\left.\begin{array}{ll}(\mathscr{P}_u^{(1)},0) & \mathbf{C}^*\\ (\mathscr{P}_u^{(1)},\infty) & B\end{array}\right\} u\not\equiv 0,$$

$$\left.\begin{array}{ll}(\mathscr{P}_0^{(1)},1) & \mathbf{C}\\ (\mathscr{P}_0^{(1)},0) & B\\ (\mathscr{P}_0^{(1)},\infty) & B\end{array}\right\}.$$

Note that

$$(\mathscr{P}_F^{(1)},y)\cong(\mathscr{P}_F^{(1)},y')\Leftrightarrow y'=y,-y,y^{-1},\text{or }-y^{-1}\overset{\text{def}}{\Longleftrightarrow} y\approx y'.$$

By looking at the dimension of the automorphism groups, we see that our candidate $\mathscr{A}_{P,x}^{\min}$ consists of the $\mathscr{G}_{P_{\mathbf{C}}}$-orbits represented by $\{(\mathscr{P}_u^{(0)},1)\}_{u\not\sim 0}$ and $(\mathscr{P}_{00}^{(0)},\infty)$ for the trivial bundle, and by $\{(\mathscr{P}_F^{(1)},y)\}_y$ for the non-trivial bundle. A more careful look at the dimension shows that this $\mathscr{A}_{P,x}^{\min}\subset\mathscr{A}_{P,x}$ indeed has a complement of codimension $\geqq 1$. Does $(\mathscr{P}_{00}^{(0)},\infty)$ represent a separated orbit? The answer is no: As we will see in Sect. 4.4, there is a holomorphic family $\{\widehat{\mathscr{P}}_v^{(0)}\}_v$ of parabolic bundles such that $\widehat{\mathscr{P}}_0^{(0)}\cong(\mathscr{P}_{00}^{(0)},\infty)$ and $\widehat{\mathscr{P}}_{u^2}^{(0)}\cong(\mathscr{P}_u^{(0)},1)$ for $u\not\equiv 0$. So, the quotient space should be smooth around the orbit of $(\mathscr{P}_{00}^{(0)},\infty)$. We thus get the quotients

$$\mathscr{N}_{\text{triv},x}\cong\widehat{\mathbf{C}/\sim}, \tag{4.32}$$

$$\mathscr{N}_{\text{non-triv},x}\cong\mathbf{P}^1/\approx. \tag{4.33}$$

Here $\widehat{\mathbf{C}/\sim}$ is the desingularization of $\mathbf{C}/\sim$ at the $\mathbf{Z}_2$-orbifold point $u\sim 0$ so that $v=u^2$ is a good complex coordinate, with $v=0$ being identified with $(\mathscr{P}^{(0)}_{00},\infty)$. Both spaces are complex orbifolds with three $\mathbf{Z}_2$-orbifold points; $u\sim\frac{1}{4},\frac{\tau}{4},\frac{\tau+1}{4}\in\mathscr{N}_{\mathrm{triv},x}$, and $y\approx 0,1,\ i\in\mathscr{N}_{\mathrm{non-triv},x}$. We will see in the next subsection that these are isomorphic.

4.4. Action of $\pi_1(H)$ on the Moduli Spaces. Let (Σ,x) be a closed Riemann surface with a marked point. We choose a neighborhood U_0 of x with a coordinate z such that $z(x)=0$. A holomorphic principal $\mathbf{C}^*$-bundle admits trivializations over U_0 and $U_\infty=\Sigma-x$ that are related by a holomorphic transition function $h_{\infty 0}:U_0\cap U_\infty\to\mathbf{C}^*$. For each $a\in\mathbf{Z}$, the replacement

$$h_{\infty 0}(z)\mapsto h_{\infty 0}(z)z^{-a} \tag{4.34}$$

of transition function induces the translation of the Picard group of Σ by an element of degree a. This defines an action of $\pi_1(U(1))\cong\mathbf{Z}$ on the Picard group that covers the natural action on the set $H^2(\Sigma,\mathbf{Z})\cong\mathbf{Z}$ of topological types of $U(1)$-bundles. This action depends on x but not on the choice of coordinate z.

We ask whether such an action exists for a general compact connected group H: Does the natural action of $\pi_1(H)$ on the set of topological types of principal H-bundles lift to an action on the set of isomorphism classes of holomorphic principal $H_{\mathbf{C}}$-bundles? The answer is no. Instead, as we will see, $\pi_1(H)$ acts on the set of isomorphism classes of parabolic $H_{\mathbf{C}}$-bundles over (Σ,x). We denote this set by $\mathscr{N}_{H_{\mathbf{C}}}(\Sigma,x)$.

Action of $\pi_1(H)$ on $\mathscr{N}_{H_{\mathbf{C}}}(\Sigma,x)$. For a parabolic bundle $(\mathscr{P},f)$ over (Σ,x), we say that a section of $\mathscr{P}|_{U_0}$ is *admissible* with respect to f when it represents f over x. Let $h:(\mathscr{P},f)\to(\mathscr{P}',f')$ be an isomorphism. Under a choice of trivializations $\{\sigma_0,\sigma_\infty\}$ of $\mathscr{P}$ and $\{\sigma_0',\sigma_\infty'\}$ of $\mathscr{P}'$ over $\{U_0,U_\infty\}$ such that σ_0 and σ_0' are admissible with repect to f and f', h is represented by $H_{\mathbf{C}}$-valued holomorphic functions $\{h_0,h_\infty\}$ on $\{U_0,U_\infty\}$ with $h_0(x)\in B:\sigma_I\mapsto\sigma_I'h_I$ $(I=0,\infty)$. Then, the transition functions $h_{\infty 0}$ and $h'_{\infty 0}$ are subject to the relation

$$h'_{\infty 0}(z)=h_\infty(z)h_{\infty 0}(z)h_0(z)^{-1},\qquad z\in U_0\cap U_\infty. \tag{4.35}$$

For an open Riemann surface U, we denote by $L^U H_{\mathbf{C}}$ the group of holomorphic maps $U\to H_{\mathbf{C}}$. Pulling back by inclusions $U_{\infty 0}=U_0\cap U_\infty\hookrightarrow U_0,U_\infty$, the groups $L^{U_0}H_{\mathbf{C}}$ and $L^{U_\infty}H_{\mathbf{C}}$ may be considered as subgroups of $L^{U_{\infty 0}}H_{\mathbf{C}}$. We denote by B^{U_0} the subgroup of $L^{U_0}H_{\mathbf{C}}$ consisting of maps with values at x being in B. By the above argument,

$$\mathscr{N}_{H_{\mathbf{c}}}(\Sigma,x)\cong L^{U_\infty}H_{\mathbf{C}}\backslash L^{U_{\infty 0}}H_{\mathbf{C}}/B^{U_0}\,. \tag{4.36}$$

The fundamental group $\pi_1(H)$ is isomorphic to the subgroup $\Gamma_{\widehat{\mathrm{C}}}$ of the affine Weyl group W'_{aff} consisting of elements that preserve the alcôve $\widehat{\mathrm{C}}$ (see Appendix A). For each $\gamma\in\Gamma_{\widehat{\mathrm{C}}}$, there is a holomorphic extension $h_\gamma:\mathbf{C}^*\to H_{\mathbf{C}}$. We identify h_γ as an element of $L^{U_{\infty 0}}H_{\mathbf{C}}$ via the coordinate $z:U_{\infty 0}\to\mathbf{C}^*$. Since the adjoint action of h_γ on $L^{U_{\infty 0}}H_{\mathbf{C}}$ preserves the subgroup $B^{U_0}\subset L^{U_{\infty 0}}H_{\mathbf{C}}$, we find that the replacement

$$h_{\infty 0}(z)\mapsto h_{\infty 0}(z)h_\gamma(z) \tag{4.37}$$

of transition function induces a transformation

$$\gamma_x : \mathcal{N}_{H_{\mathrm{c}}}(\Sigma, x) \to \mathcal{N}_{H_{\mathrm{c}}}(\Sigma, x) . \tag{4.38}$$

This changes the homotopy type of the transition function by $\gamma \in \pi_1(H)$ and hence permutes the subsets $\mathcal{A}_{P,x}/\mathcal{G}_{P_{\mathrm{C}}} \subset \mathcal{N}_{H_{\mathrm{c}}}(\Sigma, x)$ as

$$\gamma_x : \mathcal{A}_{P,x}/\mathcal{G}_{P_{\mathrm{C}}} \to \mathcal{A}_{P\gamma,x}/\mathcal{G}_{P\gamma_{\mathrm{c}}} . \tag{4.39}$$

Thus $\gamma \mapsto \gamma_x$ is the desired action of $\pi_1(H)$ on $\mathcal{N}_{H_{\mathrm{C}}}(\Sigma, x)$.

The Conjecture. Note that this action preserves the automorphism groups. Namely, if the class of $(\mathcal{P}, f)$ is mapped by γ_x to a class represented by $(\mathcal{P}^\gamma, f^\gamma)$, we have

$$\mathrm{Aut}(\mathcal{P}, f) \cong \mathrm{Aut}(\mathcal{P}^\gamma, f^\gamma) . \tag{4.40}$$

This can be seen by multiplying $h_\gamma(z)$ on the right of Eq. (4.35) with $h'_{\infty 0} = h_{\infty 0}$. Note also that a holomorphic family representing a subset of $\mathcal{A}_{p,x}/\mathcal{G}_{P_{\mathrm{C}}}$ is mapped holomorphically by γ_x to another holomorphic family representing a subset of $\mathcal{A}_{P\gamma,x}/\mathcal{G}_{P\gamma_{\mathrm{c}}}$.

Having these in mind, we conjecture that the following holds: *There exists a method to take the quotient $\mathcal{N}_{P,x}$ so that $\mathcal{N}_{P,x}$ is mapped isomorphically onto $\mathcal{N}_{P\gamma,x}$ by γ_x.* If, furthermore, $\mathcal{N}_{P,x}$ is mapped onto $\mathcal{N}_P$ by forgetting the flags, we have the following double fibration:

$$\begin{tikzcd}[column sep=small] & \mathcal{N}_{P,x} \cong \mathcal{N}_{P\gamma,x} \arrow[dl] \arrow[dr] & \\ \mathcal{N}_P & & \mathcal{N}_{P\gamma} \end{tikzcd} \tag{4.41}$$

In this way, we can relate moduli spaces of bundles of distinct topological types. This seems to be what mathematicians call the *Hecke correspondence* [39].

Verification for Genus 0. In Sect. 4.3, we already defined the moduli space $\mathcal{N}_{P(\gamma'),x}$ for $(\mathbf{P}^1, z(x) = 0)$. This is a one point set represented by $\widehat{\mathcal{P}}_{\gamma'}$ that is described by the transition relation $\sigma_0(z) = \sigma_\infty(z) h_{\gamma'}(z)$, where σ_0 is admissible. The γ_x-transform of $\widehat{\mathcal{P}}_{\gamma'}$ is then described by $\sigma_0^\gamma(z) = \sigma_\infty^\gamma(z) h_{\gamma'}(z) h_\gamma(z)$, where σ_0^γ is now admissible. Since $h_{\gamma'}(z) h_\gamma(z) = h_{\gamma'\gamma}(z)$, it is $\widehat{\mathcal{P}}_{\gamma'\gamma}$ and we see that

$$\gamma_x : \mathcal{N}_{P(\gamma'),x} \longrightarrow \mathcal{N}_{P(\gamma'\gamma),x} . \tag{4.42}$$

The conjecture is thus verified on the sphere.

Verification for Genus 1 and $H = SO(3)$. For $H = SO(3)$, the non-trivial element γ of $\Gamma_{\widehat{\mathrm{C}}} \cong \mathbf{Z}_2$ is represented by a path

$$\gamma(\theta) = \begin{pmatrix} 0 & -\mathrm{e}^{-\frac{i}{2}\theta} \\ \mathrm{e}^{\frac{i}{2}\theta} & 0 \end{pmatrix}, \quad 0 \leqq \theta \leqq 2\pi \tag{4.43}$$

in $SU(2)$. We apply γ_x to the topologically trivial parabolic bundles over (Σ_τ, x) with $z(x) \equiv 1$. An H_{C}-bundle $\mathcal{P}$ we consider is described by the transition relation

$\sigma(zq) = \sigma(z)h(q;z)$ and a flag is parametrized by $y \in \mathbf{C} \cup \{\infty\}$. If a matrix $h_f \in SL(2,\mathbf{C})$ obeys $(h_f)^2_1/(h_f)^1_1 = y$, then, $\sigma_0(z) = \sigma(z)h_f$ is an admissible section on a small neighborhood U_0 of $z = 1$. Hence, the γ_x-transform of $(\mathscr{P}, y)$ is represented by a bundle $\mathscr{P}^\gamma$ with an admissible section σ_0^γ on U_0 and a section σ' on $\mathbf{C}^* - q^{\mathbf{Z}}$ that are related by

$$\sigma_0^\gamma(z) = \sigma'(z)h_f\, h_\gamma(z-1)\,, \quad z \in U_0 - \{1\}, \tag{4.44}$$

$$\sigma'(zq) = \sigma'(z)h(q;z)\,, \quad z \not\equiv 1 \bmod q^{\mathbf{Z}}. \tag{4.45}$$

The conservation $\mathrm{Aut}(\mathscr{P}, f) \cong \mathrm{Aut}(\mathscr{P}^\gamma, f^\gamma)$ of automorphism groups enables us to guess how γ_x transforms the parabolic bundles listed in Sect. 4.3. After a calculation, we find the following solution (see Appendix C for the proof):

$$\begin{aligned} &(\mathscr{P}^{(0)}_u, 1) \to (\mathscr{P}^{(1)}_F, y_u) \quad u \not\sim 0\,,\\ &(\mathscr{P}^{(0)}_{00}, \infty) \to (\mathscr{P}^{(1)}_F, y_0)\,, \end{aligned} \tag{4.46}$$

where

$$y_u = iq^{\frac{1}{4}} \mathrm{e}^{2\pi i u} \frac{\vartheta(2u+\tau, 2\tau)}{\vartheta(2u, 2\tau)}\,. \tag{4.47}$$

This function satisfies $y_{-u} = y_u$, $y_{u+\frac{1}{2}} = -y_u$ and $y_{u+\frac{\tau}{2}} = -y_u^{-1}$, and hence determines a map $\mathbf{C}/\!\sim\; \to \mathbf{P}^1/\!\approx$. The $\mathbf{Z}_2$-orbifold points $u \sim \frac{1}{4}, \frac{\tau}{4}, \frac{\tau+1}{4}$ of $\mathbf{C}/\!\sim$ are mapped to the $\mathbf{Z}_2$-orbifold points $y \approx 0, i, 1$ of $\mathbf{P}^1/\!\approx$ respectively. In a neighborhood of $u = 0$, it behaves as $y_u = y_0 + cu^2 + \cdots$, with c being a non-zero constant. Applying γ_x^{-1} to a holomorphic family $(\mathscr{P}^{(1)}_F, y)$ of parabolic bundles around $(\mathscr{P}^{(1)}_F, y_0)$, we get a holomorphic family $\widehat{\mathscr{P}}^{(0)}_v$ around $\widehat{\mathscr{P}}^{(0)}_0 = (\mathscr{P}^{(0)}_{00}, \infty)$ parametrized by $v = u^2$ such that $\widehat{\mathscr{P}}^{(0)}_{u^2} \cong (\mathscr{P}^{(0)}_u, 1)$ for $u \not\sim 0$. This has been the basis of the construction of the moduli space $\mathscr{N}_{\mathrm{triv},x}$. Now, we see that γ_x yields an isomorphism

$$\gamma_x : \mathscr{N}_{\mathrm{triv},x} = \widehat{\mathbf{C}/\!\sim} \longrightarrow \mathscr{N}_{\text{non-triv},x} = \mathbf{P}^1/\!\approx\,. \tag{4.48}$$

The conjecture is also verified in this case.

4.5. The Topological Identity. We define an action of $\pi_1(H)$ on the space of gauge invariant fields and prove the topological identity (1.3) using the results of the previous sections.

Action of $\pi_1(H)$ on Gauge Invariant Local Fields. Let γ be a loop in H representing an element of $\Gamma_{\widehat{\mathbf{C}}} \cong \pi_1(H)$. Let O be the gauge invariant field corresponding to a state $\Phi \in \mathscr{H}^\lambda_\Lambda$. Since $\mathrm{ad}\gamma$ preserves the subgroup N^+ of $LH_{\mathbf{C}}$, the gauge transformation by γ preserves the highest weight condition with respect to $\widetilde{L\mathfrak{h}}_{\mathbf{C}}$. So, the gauge transform $\gamma \cdot \Phi$ of Φ is in $\mathscr{H}^{\gamma\lambda}_{\gamma_G\Lambda}$, where γ_G is the image of γ under the natural map $\pi_1(H) \to \pi_1(G/Z_G)$, and $\gamma\lambda$ and $\gamma_G\Lambda$ are defined as in (2.18). We denote the corresponding gauge invariant field by γO. As it is independent of the choice of a loop representing an element of $\Gamma_{\widehat{\mathbf{C}}} \cong \pi_1(H)$, this $O \mapsto \gamma O$ gives rise to a $\pi_1(H)$-action on the set of gauge invariant fields.

Gauge Transformation of the States Corresponding to Flag Partners. Let us recall from Sect. 2.3 the configuration $A_{\varrho,\gamma}$ of gauge field on the unit disc D_0. We consider it as a connection of the trivial bundle $D_0 \times H$. As noted before, $A_{\varrho,\gamma}$ can be made flat by a chiral gauge transformation $h_{\varrho,\gamma}$ such that $h_{\varrho,\gamma}(0) = 1$ and $h_{\varrho,\gamma}|_{\partial D_0}$ is a constant loop in the Cartan subgroup $T_{\mathbf{C}}$ of $H_{\mathbf{C}}$. Let us insert the flag partner of O associated with the flag $f_0 = (0, B)$. Since chiral anomaly is absent in the combined system, and since $\mathscr{Z}_{D_0}(0; \widehat{O}(f_0)) = \Phi \otimes \Phi_{-\lambda-2\rho} \otimes |\Omega\rangle$ has weight zero with respect to T, the state we observe at $S = \partial D_0$ is still

$$\mathscr{Z}_{D_0}(A_{\varrho,\gamma}; \widehat{O}(f_0)) = \Phi \otimes \Phi_{-\lambda-2\rho} \otimes |\Omega\rangle\,. \tag{4.49}$$

If we look at the same state standing on the horizontal frame $s(\theta) = (\mathrm{e}^{i\theta}, \gamma(\theta)^{-1})$, what we observe is its gauge transform $\gamma \cdot \Phi \otimes \gamma \cdot \Phi_{-\lambda-2\rho} \otimes \gamma \cdot |\Omega\rangle$. We determine what this is. First, note that a wave function of the WZW model based on $H_{\mathbf{C}}/H$ is a section of a certain line bundle over the loop space of $H_{\mathbf{C}}/H$. It is a pull back of the line bundle $\mathscr{L}_{\mathrm{WZ}}^{-\tilde{k}-2h^\vee}$ over $L\tilde{H}_{\mathbf{C}}$ by the map of loop spaces induced by $hH \in H_{\mathbf{C}}/H \mapsto hh^* \in \tilde{H}_{\mathbf{C}}$. (Here, $\tilde{H}_{\mathbf{C}}$ is the universal cover of $H_{\mathbf{C}}$, if $H_{\mathbf{C}}$ is simple. Extension to a general case is obvious.) Just as in Sect. 2.3, a computation proves $\gamma \cdot \Phi_{-\lambda-2\rho} = \Phi_{-\gamma\lambda-2\rho}$. As for the ghost part, $|\Omega\rangle$ can be considered in a certain sense [16, 40] as a "volume form" of the infinite dimensional space $L\mathfrak{h}_{\mathbf{C}}/\mathfrak{b}^+$, where $\mathfrak{b}^+$ is the Lie algebra of the group $B^+ \subset LH_{\mathbf{C}}$. Since $\mathrm{ad}\gamma$ induces an orthogonal transformation of $L\mathfrak{h}_{\mathbf{C}}$ and preserves the subalgebra $\mathfrak{b}^+$, $\gamma \cdot |\Omega\rangle = |\Omega\rangle$. Thus, the gauge transform is $\gamma \cdot \Phi \otimes \Phi_{-\gamma\lambda-2\rho} \otimes |\Omega\rangle$. By the above definition of the $\pi_1(H)$ action, we get

$$\mathscr{Z}_{D_0}^{(s)}(A_{\varrho,\gamma}; \widehat{O}(f_0)) = \mathscr{Z}_{D_0}(0; \widehat{\gamma O}(f_0))\,, \tag{4.50}$$

where (s) signifies that $\mathscr{Z}^{(s)}$ is the state observed on the horizontal frame s.

Proof of (1.3). We are now in a position to prove (1.3). We make use of the new integral expression (4.23). Let V be an open subet of $\mathscr{N}_{P,x}$ with a holomorphic family $\{(A_v, f_v)\}_{v\in V}$ of representatives. Absence of chiral anomaly in the combined system enables us to take the representatives so that there is a family $\{\sigma_0(v)\}_{v\in V}$ of *horizontal* and *admissible* sections on a neighborhood U_0 of x.

Let us choose a complex coordinate z on U_0 such that $z(x) = 0$ and $z(U_0)$ includes the unit disc D_0, and put $\Sigma_\infty = \Sigma - D_0$. If we glue along $S = \Sigma_\infty \cap D_0$ the configurations $A_v|_{\Sigma_\infty}$ and $A_{\varrho,\gamma}$ by identifying $\sigma_0(v, \mathrm{e}^{i\theta})\gamma(\theta)$ and $s_0(\mathrm{e}^{i\theta}) := (\mathrm{e}^{i\theta}, 1)$, we obtain another H-bundle $P\gamma$ over Σ with a smooth connection A_v^γ. We denote by $\sigma(v)$ and $s_0^\gamma(v)$ the sections of $P\gamma$ over $\Sigma_\infty \cap U_0$ and D_0 respectively which had been $\sigma_0(v)|_{\Sigma_\infty\cap U_0}$ and s_0 before the gluing. Consider the flag $f_v^\gamma = s_0^\gamma(v,x)B$. We can find a section $\sigma_0^\gamma(v)$ over U_0, holomorphic and admissible with respect to (A_v^γ, f_v^γ), such that $\sigma_0^\gamma(v,z) = \sigma(v,z)h_\gamma(z)$ on a neighborhood of S. Thus the replacements

$$(A_v, f_v) \mapsto (A_v^\gamma, f_v^\gamma)\,, \quad v \in V\,, \tag{4.51}$$

represent the transformation $\gamma_x : V \to \gamma_x(V)$. The image $\gamma_x(V)$ is an open subset of $\mathscr{N}_{P\gamma,x}$, under the assumption that the conjecture $\gamma_x : \mathscr{N}_{P,x} \cong \mathscr{N}_{P\gamma,x}$ holds.

Now, we have the identity modelled after (4.50),

$$\mathscr{Z}_{D_0}^{(\sigma(v))}(A_v^\gamma; \widehat{O}(f_v^\gamma)) = \mathscr{Z}_{D_0}^{(\sigma_0(v))}(A_v; \widehat{\gamma O}(f_v))\,. \tag{4.52}$$

Since the automorphism groups are naturally isomorphic, this yields

$$\mathscr{Z}_{\Sigma,P}\left(A_v;\ \frac{1}{|\mathrm{Aut}\,(\bar{\partial}_{A_v},f_v)|}hO_1\cdots hO_s\prod_{\mathrm{A}=1}^{\hat{d}_{\mathscr{N}}}\left|\frac{1}{2\pi i}\oint_x b\,v_{\mathrm{A}}(v)\right|^2\widehat{\gamma O}(f_v)\right)$$

$$=\mathscr{Z}_{\Sigma,P\gamma}\left(A_v^\gamma;\ \frac{1}{|\mathrm{Aut}(\bar{\partial}_{A_v^\gamma},f_v^\gamma)|}hO_1\cdots hO_s\prod_{\mathrm{A}=1}^{\hat{d}_{\mathscr{N}}}\left|\frac{1}{2\pi i}\oint_x b\,v_{\mathrm{A}}^\gamma(v)\right|^2\widehat{O}(f_v^\gamma)\right), \quad (4.53)$$

where $v_{\mathrm{A}}^\gamma(v)$ is given by

$$v_{\mathrm{A}}^\gamma(v)=\sigma(v)\cdot h_{\infty 0}(v)^{-1}\frac{\partial}{\partial v^{\mathrm{A}}}h_{\infty 0}(v)=\sigma_0^\gamma(v)\cdot h_{\infty 0}^\gamma(v)^{-1}\frac{\partial}{\partial v^{\mathrm{A}}}h_{\infty 0}^\gamma(v), \quad (4.54)$$

in which $h_{\infty 0}^\gamma(v)=h_{\infty 0}(v)h_\gamma$. This amounts to the identity

$$\Omega_{\Sigma,P,x}(O_1\cdots O_s\widehat{\gamma O})=\gamma_x^*\Omega_{\Sigma,P\gamma,x}(O_1\cdots O_s\widehat{O}), \quad (4.55)$$

which shows (1.3).

5. Sum Over Topologies

The full correlation function $Z_\Sigma(O_1\cdots O_s)$ of gauge invariant fields $O_1\cdots O_s$ is given by the sum $\sum_P Z_{\Sigma,P}(O_1\cdots O_s)$ over all topological types of principal H-bundles over Σ. If we use the topological identity (1.3), we have

$$Z_\Sigma(O_1\cdots O_s O)=\sum_{\gamma\in\pi_1(H)}Z_{\Sigma,P}(O_1\cdots O_s\gamma O), \quad (5.1)$$

where P is any H-bundle. This shows that O and O' are indistinguishable in any full correlator if $\sum\gamma O=\sum\gamma O'$. This motivates us to consider the quotient of the space of gauge invariant local fields by the kernel of the operator $\sum_{\gamma\in\pi_1(H)}\gamma$, or equivalently, the quotient $\dot{\mathscr{H}}_{\mathrm{hw}}$ of $\mathscr{H}_{\mathrm{hw}}$ by the kernel of $\sum_{\gamma\in\Gamma_{\widehat{\mathrm{C}}}}\gamma\,\cdot$. By the general principle of CFT, we expect that the torus partition funtion $Z_{\Sigma_\tau}(1)$ satisfies

$$Z_{\Sigma_\tau}(1)=\mathrm{tr}_{\dot{\mathscr{H}}_{\mathrm{hw}}}(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}), \quad (5.2)$$

where $c=c_{G,k}-c_{H,\tilde{k}}$. Here, L_0 and $\bar{L}_0$ are GKO generators [9] which commute with the operators $\gamma\,\cdot$ for $\gamma\in\Gamma_{\widehat{\mathrm{C}}}$ and hence can act on the quotient space $\dot{\mathscr{H}}_{\mathrm{hw}}$.

In this section, we calculate the full partition function on the torus Σ_τ and see whether (5.2) holds. For simplicity, we assume H is semi-simple. In this case, $\pi_1(H)$ is a finite group and the quotient $\dot{\mathscr{H}}_{\mathrm{hw}}$ is mapped by $\frac{1}{|\pi_1(H)|}\sum_\gamma\gamma\,\cdot$ isomorphically onto the subspace of $\mathscr{H}_{\mathrm{hw}}$ of $\Gamma_{\widehat{\mathrm{C}}}$-invariant elements. We denote by $\tilde{H}$ the universal cover of H.

5.1. Torus Partition Function for the Trivial Topology. Let P_{triv} be the trivial bundle $\Sigma_\tau\times H$. Recall that the moduli space $\mathscr{N}_H=\mathscr{N}_{P_{\mathrm{triv}}}$ is parametrized by $u\in\mathrm{t}_{\mathrm{C}}$

with the representative flat gauge fields

$$A_u = \frac{\pi}{\tau_2} u d\bar{\zeta} - \frac{\pi}{\tau_2} \bar{u} d\zeta . \tag{5.3}$$

$A_{u'}$ is gauge equivalent to A_u if and only if $u' = wu + n + \tau m$ for some $w \in W$ and $n, m \in \mathrm{P}^\vee$. The WZW model in the background A_u has the partition function

$$Z_{\Sigma_\tau, P_{\mathrm{triv}}}(A_u; 1) = \mathrm{e}^{\frac{\pi}{2\tau_2} k \operatorname{tr}_G (u-\bar{u})^2} \sum_{\Lambda \in \mathrm{P}_+^{(k)}(G)} |\chi_\Lambda^{G,k}(\tau, u)|^2 , \tag{5.4}$$

(see [41, 42, 1]) in which $\chi_\Lambda^{G,k}$ is the character of the representation $L_\Lambda^{G,k}$ of $\widetilde{LG}$:

$$\chi_\Lambda^{G,k}(\tau, u) = \operatorname{tr}_{L_\Lambda^{G,k}}(q^{L_0 - \frac{c_{G,k}}{24}} \mathrm{e}^{2\pi i J_0(u)}), \tag{5.5}$$

where $u \in \mathfrak{t}_{\mathbf{C}}$ is considered as an element of $\mathfrak{g}_{\mathbf{C}}$. As it should be, (5.4) is invariant under the gauge transformation $u \mapsto wu + n + \tau m$. It is also invariant [44] under the modular transformations $(\tau, u) \mapsto (\tau + 1, u)$ and $(-\frac{1}{\tau}, \frac{u}{\tau})$.

We calculate the partition function in the trivial sector

$$Z_{\Sigma_\tau, P_{\mathrm{triv}}}(1) = \int_{\mathcal{N}_H} \prod_{i=1}^{l} d^2u^i \ \mathscr{Z}_{\Sigma_\tau, P_{\mathrm{triv}}} \left(A_u ; \frac{1}{|\mathrm{Aut}\, \bar{\partial}_{A_u}|} \prod_{j=1}^{l} \left| \frac{1}{2\pi i} \int_{\Sigma_\tau} b \frac{\partial A_u^{01}}{\partial u^j} \right|^2 \right) . \tag{5.6}$$

The automorphism group of $\bar{\partial}_{A_u}$ for generic u is the group of constant gauge transformations by elements of $T_{\mathbf{C}}$. Parametrizing hh^* as $n_+ \mathrm{e}^{\varphi} n_+^*$, where n_+ is N-valued and φ is $i\mathfrak{t}$-valued, division by automorphism group is implemented by

$$\frac{1}{|\mathrm{Aut}\, \bar{\partial}_{A_u}|} = \frac{\delta^{(l)}(\varphi(x_0))}{\mathrm{vol}(T)} \prod_{i=1}^{l} c^i(x_0) \bar{c}^i(x_0) , \tag{5.7}$$

where x_0 is a point of Σ_τ. As calculated in [1], the partition function of the ghost system with the insertion of $\prod_{j=1}^{l} |\frac{c^j(x_0)}{2\pi i} \int b_j \frac{\pi}{\tau_2} d\bar{\zeta}|^2$, and of the $H_{\mathbf{C}}/H$-WZW model with the insertion of $\delta^{(l)}(\varphi(x_0))/\mathrm{vol}(T)$ are given respectively by

$$\left(\frac{\pi}{\tau_2}\right)^{2l} \det{}'_{\mathrm{ad}}\left(\bar{\partial}_{A_u}^\dagger \bar{\partial}_{A_u}\right), \quad \text{and} \quad \frac{(2\tau_2(\tilde{k} + h^\vee))^{\frac{l}{2}}}{(2\pi)^l \mathrm{vol}(T)} \Big/ \sqrt{\det{}'_{\mathrm{ad}}\left(\bar{\partial}_{A_u}^\dagger \bar{\partial}_{A_u}\right)} . \tag{5.8}$$

Here, $\det{}'_{\mathrm{ad}}(\bar{\partial}_{A_u}^\dagger \bar{\partial}_{A_u})$ is the ζ-regularized determinant of the Laplace operator $\bar{\partial}_{A_u}^\dagger \bar{\partial}_{A_u}$ acting on sections of the adjoint bundle. This is calculated in [45]:

$$\det{}'_{\mathrm{ad}}\left(\bar{\partial}_{A_u}^\dagger \bar{\partial}_{A_u}\right) = (2\tau_2)^{2l} \mathrm{e}^{\frac{\pi}{2\tau_2} 2h^\vee \operatorname{tr}(u-\bar{u})^2} |\Pi_{\tilde{H}}(\tau, u)|^4 , \tag{5.9}$$

where $\Pi_{\tilde{H}}(\tau, u)$ is the Weyl–Kac denominator. Thus, $Z_{\Sigma_\tau, P_{\mathrm{triv}}}(1)$ is equal to

$$\left(\frac{\tilde{k} + h^\vee}{2\tau_2}\right)^{\frac{l}{2}} \frac{(2\pi)^l}{\mathrm{vol}(T)} \int_{\mathcal{N}_H} \prod_{i=1}^{l} d^2u^i \mathrm{e}^{\frac{\pi}{2\tau_2}(\tilde{k}+h^\vee)\operatorname{tr}(u-\bar{u})^2} \sum_{\Lambda} |\chi_\Lambda^{G,k}(\tau, u) \Pi_{\tilde{H}}(\tau, u)|^2 . \tag{5.10}$$

The branching rule (3.19) leads to the expansion

$$\chi_\Lambda^{G,k}(\tau,u) = \sum_\lambda b_\Lambda^\lambda(\tau)\chi_\lambda^{\tilde{H},\tilde{k}}(\tau,u), \tag{5.11}$$

in which the branching function b_Λ^λ is defined by $b_\Lambda^\lambda(\tau) = \mathrm{tr}_{B_\Lambda^\lambda}(q^{L_0-\frac{c}{24}})$. As the GKO operators L_0 and $\tilde{L}_0$ commute with the $\gamma\cdot$, we have $b^{\gamma\lambda}_{\gamma_G\Lambda} = b_\Lambda^\lambda$, which enables us to replace the integration (5.10) over $\mathcal{N}_H$ by an integration over $\mathcal{N}_{\tilde{H}}$ divided by $|\mathrm{P}^\vee/\mathrm{Q}^\vee|^2$. Using $\mathrm{vol}(T) = (2\pi)^l\mathrm{vol}(it/\mathrm{P}^\vee)$ and the orthogonality of characters;

$$\int_{\mathcal{N}_{\tilde{H}}} \prod_{i=1}^l d^2u^i e^{\frac{\pi}{2\tau_2}(\tilde{k}+h^\vee)\mathrm{tr}(u-\bar{u})^2}\chi_\lambda^{\tilde{H},\tilde{k}}(\tau,u)\overline{\chi_{\lambda'}^{\tilde{H},\tilde{k}}(\tau,u)}|\Pi(\tau,u)|^2 = \mathrm{vol}(it/\mathrm{Q}^\vee)\left(\frac{2\tau_2}{\tilde{k}+h^\vee}\right)^{\frac{l}{2}}\delta_{\lambda,\lambda'}, \tag{5.12}$$

we get

$$Z_{\Sigma_\tau,P_{\mathrm{triv}}}(1) = \frac{1}{|\pi_1(H)|}\sum_{\Lambda,\lambda}|b_\Lambda^\lambda(\tau)|^2, \tag{5.13}$$

where (Λ,λ) runs over $\mathrm{P}_+^{(k)}(G)\times\mathrm{P}_+^{(\tilde{k})}(\tilde{H})$. Since $b^{\gamma\lambda}_{\gamma_G\Lambda} = b_\Lambda^\lambda$, it can also be expressed as

$$Z_{\Sigma_\tau,P_{\mathrm{triv}}}(1) = \sum_{[\Lambda,\lambda]}\frac{1}{|\mathrm{A}_\Lambda^\lambda|}|b_\Lambda^\lambda(\tau)|^2, \tag{5.14}$$

where the sum is over the quotient $(\mathrm{P}_+^{(k)}(G)\times\mathrm{P}_+^{(\tilde{k})}(\tilde{H}))/\Gamma_{\widehat{\mathrm{C}}}$ and $\mathrm{A}_\Lambda^\lambda$ is the isotropy subgroup of $\Gamma_{\widehat{\mathrm{C}}}$ at (Λ,λ).

If $\mathrm{A}_\Lambda^\lambda = 1$ for every (Λ,λ), obviously we have

$$Z_{\Sigma_\tau,P_{\mathrm{triv}}}(1) = \mathrm{tr}_{\mathscr{H}_{\mathrm{hw}}}(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}). \tag{5.15}$$

As we shall see shortly, in this case, topologically non-trivial bundles do not contribute to the partition function and hence $Z_{\Sigma_\tau,P_{\mathrm{triv}}}(1)$ is itself the full partition function. Thus, (5.2) holds when $\pi_1(H)$ acts freely on $\mathrm{P}_+^{(k)}(G)\times\mathrm{P}_+^{(\tilde{k})}(\tilde{H})$.

5.2. Field Identification Fixed Points. To each $\gamma\in\pi_1(H)$ is associated a principal H-bundle $P^{(\gamma)} = P_{\mathrm{triv}}\gamma$ over Σ_τ. Due to the topological identity (1.3), the partition function for $P^{(\gamma)}$ is the one point function for P_{triv} of the field $\gamma(1)$ corresponding to a state $\gamma\cdot\Phi_0$ in $\mathscr{H}^{\gamma 0}_{\gamma_G 0}$:

$$Z_{\Sigma_\tau,P^{(\gamma)}}(1) = Z_{\Sigma_\tau,P_{\mathrm{triv}}}(\gamma(1)). \tag{5.16}$$

This is expressed as an integral over $\mathcal{N}_H$ whose integrand contains a factor $Z_{\Sigma_\tau,P_{\mathrm{triv}}}(A_u;\mathbf{O}_{\gamma\cdot\Phi_0})$. For this to be non-vanishing, the fusion rule [41, 46, 24] requires

$$\sum_{\Lambda\in\mathrm{P}_+^{(k)}(G)} N^\Lambda_{\gamma_G 0\,\Lambda} \neq 0 \quad\text{and}\quad \sum_{\lambda\in\mathrm{P}_+^{(\tilde{k})}(\tilde{H})} N^\lambda_{\gamma 0\,\lambda} \neq 0. \tag{5.17}$$

Here $N^{\Lambda''}_{\Lambda\Lambda'}$ (resp. $N^{\lambda''}_{\lambda\lambda'}$) is the fusion coefficient of the WZW model with target G and level k (resp. target $\tilde{H}$ and level $\tilde{k}$). The Verlinde fomula expresses them in terms of the modular transformation matrix: $N_{\lambda_1\lambda_2}^{\lambda_3} = \sum_{\lambda} S_{\lambda_1}^{\lambda} S_{\lambda_2}^{\lambda} S_{\lambda_3}^{\lambda *}/S_0^{\lambda}$ [46]. From Gepner's observation [10] $S_{\lambda}^{\gamma\lambda'} = (-1)^{l(w)} \mathrm{e}^{-2\pi i(\lambda+\rho)(\mu)} S_{\lambda}^{\lambda'}$; $\gamma(\theta) = \mathrm{e}^{-i\mu\theta} w$, it follows that $N_{\gamma_1\lambda\,\gamma_2\lambda'}^{\gamma_1\gamma_2\lambda''} = N_{\lambda\,\lambda'}^{\lambda''}$. Since $N_{0\,\lambda}^{\lambda'} = \delta_{\lambda}^{\lambda'}$, (5.17) is equivalent to the condition that there exist $\Lambda \in \mathrm{P}_+^{(k)}(G)$ and $\lambda \in \mathrm{P}_+^{(\tilde{k})}(\tilde{H})$ such that $\gamma_{_G}\Lambda = \Lambda$ and $\gamma\lambda = \lambda$.

If there is a pair (Λ, λ) at which the isotropy $\mathrm{A}_{\Lambda}^{\lambda} \subset \pi_1(H)$ is not $\{1\}$ (such a pair is called the *field identification fixed point* in the literature), the partition function for the trivial topology (5.14) has fractional coefficients in the $q, \bar{q}$-expansion and we can hardly expect that it is expressed as a trace of $q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}$ in a Virasoro module. In algebraic treatments of coset models [12, 47], this was recognized as a problem of field identification fixed points. We expect that a natural resolution is provided by the sum over topologies: If $\mathrm{A}_{\Lambda}^{\lambda} \neq \{1\}$, the contribution $Z_{\Sigma_\tau, P(\gamma)}(1)$ for $\gamma \in \mathrm{A}_{\Lambda}^{\lambda} - \{1\}$ may be non-vanishing and the integrality of the coefficients may be restored for the *full* partition function.[3] In the next subsection, we examine whether this happens in a specific example.

The partition function (5.13) for the trivial topology is manifestly modular invariant. It should also hold for non-trivial topologies, since P and f^*P are topologically isomorphic for any diffeomorphism f. This is indeed the case for the example below.

5.3. Models with $G = SU(2)\times SU(2)$ and $H = SO(3)$. We consider the case of $G = SU(2)\times SU(2)$ and $H = SO(3)$ the diagonal subgroup of $G/Z_G = SO(3)\times SO(3)$. The level induced from (k_1, k_2) is $\tilde{k} = k_1 + k_2$. Since a highest weight representation of $SU(2)$ is conventionally labeled by the spin $\in \frac{1}{2}\mathbf{Z}$, we identify $\mathrm{P}_+^{(k)}(SU(2)) = \{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\}$. The non-trivial element of $\pi_1(H) = \mathbf{Z}_2$ induces the involution $((j_1, j_2), j) \leftrightarrow ((\frac{k_1}{2} - j_1, \frac{k_2}{2} - j_2), \frac{\tilde{k}}{2} - j)$ in $\mathrm{P}_+^{(k_1,k_2)}(G)\times\mathrm{P}_+^{(\tilde{k})}(H)$. If k_1 or k_2 is an odd integer, there is no fixed point and the full partition function is given by $\frac{1}{2}\sum_{j_1,j_2,j} |b_{(j_1,j_2)}^{j}(\tau)|^2$. For the case $k_2 = 1$, it is the diagonal modular invariant partition function of the k_1^{th} unitary minimal model.

Partition Function for the Non-trivial Topology. In the following, we assume that k_1 and k_2 are both even integers. Then, there is a unique fixed point $((\frac{k_1}{4}, \frac{k_2}{4}), \frac{\tilde{k}}{4})$ and the topologically non-trivial configurations contribute to the partition function. Recall that there is a unique flat $SO(3)$ connection A_F of non-trivial topological type, which corresponds to the semi-stable bundle $\mathscr{P}_F^{(1)}$ studied in Sect. 4.3. With respect to the multi-valued section $\sigma(z)$, the connection form of A_F is expressed as

$$A_F^{\sigma} = \frac{1}{4}\frac{dz}{z}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.18}$$

As $\mathscr{P}_F^{(1)}$ has the automorphism group $\mathbf{Z}_2\times\mathbf{Z}_2$ of order 4, the partition function $Z_{\Sigma_\tau,\text{non-triv}}(1)$ is $\mathscr{Z}_{\Sigma_\tau,\text{non-triv}}(A_F; \frac{1}{4})$. It factorizes as the product

$$\frac{1}{4}\prod_{i=1}^{2} Z_{\Sigma_\tau,\text{non-triv}}^{SU(2),k_i}(A_F; 1) Z_{\Sigma_\tau,\text{non-triv}}^{H_c/H,\,-\tilde{k}-4}(A_F; 1) Z_{\Sigma_\tau,\text{non-triv}}^{\text{ghost}}(A_F; 1) \tag{5.19}$$

[3] In [47], a method for "fixed point resolution" is presented. Characters of the "fixed point CFTs" in that reference may be related to the partition functions for non-trivial topologies.

of partition functions of the four constituent systems. We show that this is independent on τ, by proving that the one point function of the energy-momentum tensor vanishes for each system. Let $\sigma_{\rm ad}(z) : \mathfrak{h}_{\mathbf{C}} \xrightarrow{\cong} {\rm ad}P_{\mathbf{C}}|_z$ be the frame associated to $\sigma(z) \in P_{\mathbf{C}}|_z$. The Green function of the operator $\bar{\partial}_{A_F}$ is expressed as $G_w(z) = \sigma_{\rm ad}(w) g(w,z) \sigma_{\rm ad}(z)^{-1} \otimes dz$, where $g(w,z) \in {\rm End}(\mathfrak{h}_{\mathbf{C}})$ is represented by the matrix

$$g(w,z) = \begin{pmatrix} \sum_{n\in\mathbf{Z}} \frac{q^n}{z-q^{2n}w} & 0 & -\sum_{n\in\mathbf{Z}} \frac{z^{-1}q^{n-\frac{1}{2}}}{z-q^{2n-1}w} \\ 0 & f(w,z) & 0 \\ -\sum_{n\in\mathbf{Z}} \frac{wq^{n-\frac{1}{2}}}{z-q^{2n-1}w} & 0 & \sum_{n\in\mathbf{Z}} \frac{wz^{-1}q^n}{z-q^{2n}w} \end{pmatrix}, \tag{5.20}$$

with respect to the base $(\sigma_+, \sigma_3, \sigma_-)$ of $\mathfrak{h}_{\mathbf{C}} = \mathfrak{sl}(2,\mathbf{C})$. ($\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$; σ_i are Pauli matrices.) $f(w,z)$ is expressed by the theta function ϑ and its derivative $\vartheta' = \frac{\partial}{\partial\zeta}\vartheta$ as

$$f({\rm e}^{-2\pi i\xi}, {\rm e}^{-2\pi i\zeta}) = \frac{1}{2\pi i z} \frac{\vartheta(\xi - \zeta + \frac{\tau}{2}, \tau)}{\vartheta(\xi - \zeta + \frac{\tau+1}{2}, \tau)} \frac{\vartheta'(\frac{\tau+1}{2}, \tau)}{\vartheta(\frac{\tau}{2}, \tau)}. \tag{5.21}$$

The level k $SU(2)$-WZW model in the background A_F enjoys the chiral Ward identities $\langle J \rangle = 0$ and

$$\langle J \cdot \varepsilon(z) J \cdot \varepsilon'(w) \rangle = k \, {\rm tr}_P(\partial_{A_F}^{(w)} G_w \varepsilon(z) \varepsilon'(w)). \tag{5.22}$$

Plugging these into the Sugawara form (B.6) of the energy momentum tensor, we find

$$\frac{\partial}{\partial\tau} Z^{SU(2),k}_{\Sigma_\tau, \text{non-triv}}(A_F; 1) = 0. \tag{5.23}$$

This also holds for the $H_{\mathbf{C}}/H$-WZW model. As for the ghost system, the identity

$$\langle c(w) b(z) \rangle = G_w(z) \tag{5.24}$$

yields, through the expression (B.3) of the energy momentum tensor,

$$\frac{\partial}{\partial\tau} Z^{\text{ghost}}_{\Sigma_\tau, \text{non-triv}}(A_F; 1) = 0. \tag{5.25}$$

Thus, the partition function is a constant:

$$Z_{\Sigma_\tau, \text{non-triv}}(1) = C_{\text{non-triv}}. \tag{5.26}$$

The Full Partition Function. The partition function for topologically trivial configurations is given by

$$Z_{\Sigma_\tau, \text{triv}}(1) = \dot{\sum_{[(j_1,j_2),j]}}^{\circ} |b^{j}_{(j_1,j_2)}(\tau)|^2 + \frac{1}{2} |b^{\tilde{k}/4}_{(k_1/4,\, k_2/4)}(\tau)|^2, \tag{5.27}$$

where the sum $\dot{\sum}^\circ$ is over the $\mathbf{Z}_2$-quotient of ${\rm P}_+^{(k_1,k_2)}(G) \times {\rm P}_+^{(\tilde{k})} - \{((\frac{k_1}{4}, \frac{k_2}{4}), \frac{\tilde{k}}{4})\}$. For the non-trivial topology, we have (5.26). The term $\dot{\sum}^\circ$ in (5.27) is the trace of

$q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}$ on the space

$$\mathscr{H}^{\circ}_{\mathrm{hw}} = \mathscr{H}_{\mathrm{hw}} \ominus \mathscr{H}^{\mathrm{f}}; \qquad \mathscr{H}^{\mathrm{f}} := \mathscr{H}^{\tilde{k}/4}_{(k_1/4, k_2/4)} . \tag{5.28}$$

The question is whether there is a constant $C_{\text{non-triv}}$ such that

$$\frac{1}{2}|\, b^{\tilde{k}/4}_{(k_1/4,k_2/4)}(\tau)\,|^2 + C_{\text{non-triv}} = \mathrm{tr}_{\dot{\mathscr{H}}^{\mathrm{f}}}\,(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\,). \tag{5.29}$$

We answer this in the case $k_2 = 2$. The Virasoro modules by the GKO construction $SU(2)\times SU(2)/SU(2)$ at level $(k_1, 2)$ are known [9] to be the ones appearing in the k_1^{th} $N = 1$ superconformal minimal model. In particular, $B^{\mathrm{f}} := B^{\tilde{k}/4}_{(k_1/4,k_2/4)}$ is in the Ramond sector with a supercharge G_0 ($G_0^2 = L_0 - \frac{c}{24}$), and contains a unique ground state with $L_0 = \frac{c}{24}$. One can show that $\gamma\cdot\, : \mathscr{H}^{\mathrm{f}} \to \mathscr{H}^{\mathrm{f}}$ induces an involution U_γ of the Virasoro module B^{f} such that

$$G_0U_\gamma + U_\gamma G_0 = 0 . \tag{5.30}$$

Let B_n be the L_0-eigenspace with $G_0^2 = n$. We may put $U_\gamma = 1$ on $B_0 \cong \mathbf{C}$ and the anti-commuting relation (5.30) shows that

$$B_n = B_n^{(+)} \oplus B_n^{(-)}, \qquad B_n^{(+)} \underset{G_0}{\overset{G_0}{\rightleftarrows}} B_n^{(-)} \text{ (isomorphic)} \tag{5.31}$$

for $n \geqq 1$, where $B_n^{(\pm)}$ is the subspace of B_n on which $U_\gamma = \pm 1$. Thus, we have $\mathscr{H}^{\mathrm{f}} = \mathscr{H}^{(+)} \oplus \mathscr{H}^{(-)}$, where

$$\mathscr{H}^{(+)} \;\cong\; \bigoplus_{n,m=0}^{\infty} B_n^{(+)} \otimes \overline{B_m^{(+)}} \oplus \bigoplus_{n,m=1}^{\infty} B_n^{(-)} \otimes \overline{B_m^{(-)}} \tag{5.32}$$

and

$$\mathscr{H}^{(-)} \;\cong\; \bigoplus_{n\geqq 0,\, m\geqq 1}^{\infty} \{B_n^{(+)} \otimes \overline{B_m^{(-)}} \oplus B_m^{(-)} \otimes \overline{B_n^{(+)}}\} \tag{5.33}$$

are subspaces on which $\gamma\cdot\, = 1$ and $\gamma\cdot\, = -1$ respectively. Since $\dot{\mathscr{H}}^{\mathrm{f}}$ is isomorphic to $\mathscr{H}^{(+)}$, we see that (5.29) and hence (5.2) hold if we tune $C_{\text{non-triv}} = \frac{1}{2}$.

6. Concluding Remarks

Our argument is based on the definition (2.1) of the WZW action. However, we could have started with another one generalizing (1.1). There is a way to define topological lagrangians subject to conditions such as locality, unitarity, gluing property, etc. It is to use the (equivariant) differential character. Construction of an action in terms of Cheeger–Simons differential character was initiated by Dijkgraaf and Witten in Chern–Simons gauge theory [48] and the method was elaborated in Ref. [49] (see also [50]). According to it, WZW actions with the target G and the gauge group H are classified by the equivariant cohomology group $H^3_H(G;\mathbf{Z}) := H^3(EH\times_H G;\mathbf{Z})$, where H acts on G via adjoint transformations.

For a semi-simple group H, we have

$$H^3_H(G;\mathbf{Z}) = H^3(G;\mathbf{Z}) \oplus \mathrm{Hom}(\pi_1(H), \mathbf{R}/\mathbf{Z})\,. \tag{6.1}$$

The levels are classified by $H^3(G;\mathbf{Z})$. In the quantum theory, a term from the torsion part $\mathrm{Hom}(\pi_1(H), \mathbf{R}/\mathbf{Z})$ would modify the $\pi_1(H)$-actions on gauge invariant local fields. In a theory with fixed points, it would modify the partition function as well. For example, when $G = SU(2)\times SU(2)$ and $H = SO(3)$, the theory corresponding to $(k_1, k_2, \pm 1) \in \mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}_2 \cong H^3_H(G;\mathbf{Z})$ with even k_1, k_2 would have the full partition function

$$Z_{\Sigma_\tau,\mathrm{triv}}(1) \pm C_{\text{non-triv}} = \frac{1}{2}|\, b^{\tilde{k}/4}_{(k_1/4,\,k_2/4)}(\tau)\,|^2 \pm C_{\text{non-triv}} + \cdots\,. \tag{6.2}$$

For $k_2 = 2$, both have positive integral coefficients in the $q,\bar{q}$-expansions if and only if $C_{\text{non-triv}} = \pm\frac{1}{2}$. If $C_{\text{non-triv}} = \frac{1}{2}$, (5.2) holds in each theory. Due to the relation (5.30), the involution $\gamma\,\cdot$ can be identified with the mod two fermion number $(-1)^F$ and the theory for $(k_1, 2, \pm 1)$ is the *spin model* [7, 51] with the projection $(-1)^F = \pm 1$ on the Ramond sector. We expect in a general model that adding a torsion term has such a simple and significant consequence in physics.

In this paper, we have been concentrated on the model whose matter theory is the WZW model with a compact simply connected target group. However, our argument is applicable to the models with non-simply connected target, the study of which may be important for the classification of rational CFTs. Another interesting class of theories is the $N = 2$ coset conformal field theory (Kazama–Suzuki model) [52, 12]. Algebraic structure of the spectral flows of such a model has been studied by many authors [12, 53–55]. In ref. [56], a geometric interpretation of field identification is attempted along the line similar to ours, though the argument uses the old expression (3.4) and hence is applicable only for abelian gauge groups. The fixed point resolution in these systems (see [55, 57] for algebraic approaches) by the topological sum with torsion terms will be interesting and perhaps of some importance in superstring theory.

Appendix A

In order to fix the notation and terminology, here we describe some facts on affine Weyl groups. See [21] for the proofs. Let H be a compact connected simple Lie group without center and let $\pi : \tilde{H} \to H$ be the universal cover. We choose a maximal torus T of H and put $\tilde{T} = \pi^{-1}(T)$. The Lie algebras of T and $\tilde{T}$ are identified and denoted by $\mathfrak{t}$. We introduce lattices $\mathrm{Q}^\vee \subset \mathrm{P}^\vee$ in $i\mathfrak{t}$ so that the exponential maps induce isomorphisms $\mathfrak{t}/2\pi i\mathrm{P}^\vee \cong T$ and $\mathfrak{t}/2\pi i\mathrm{Q}^\vee \cong \tilde{T}$. Note that $\pi_1(H) \cong \mathrm{P}^\vee/\mathrm{Q}^\vee$. The dual lattices of $\mathrm{P}^\vee$ and $\mathrm{Q}^\vee$ are the root lattice Q and the weight lattice P respectively. In this paper, we call $\mathrm{Q}^\vee$ the coroot lattice and $\mathrm{P}^\vee$ the coweight lattice. A choice of a chambre C determines a decomposition of the root system Δ into positive and negative parts $\Delta = \Delta_+ \cup \Delta_-$. An element $\mu \in \mathrm{P}^\vee$ is called a *minimal coweight* if $\alpha(\mu) = 0$ or 1 for any $\alpha \in \Delta_+$. We denote by M_C the set of minimal coweight. $\mathrm{M}_\mathrm{C} \subset \mathrm{P}^\vee$ is a section of the projection $\mathrm{P}^\vee \to \mathrm{P}^\vee/\mathrm{Q}^\vee$.

The affine Weyl group of $\tilde{H}$ and H are defined by $W_{\mathrm{aff}} = \mathrm{Hom}(U(1), \tilde{T})\tilde{\times} W \cong \mathrm{Q}^\vee\tilde{\times} W$ and $W'_{\mathrm{aff}} = \mathrm{Hom}(U(1), T)\tilde{\times} W \cong \mathrm{P}^\vee\tilde{\times} W$ respectively, where W is the Weyl group of (H, T). We consider W_{aff} as a subgroup of W'_{aff} by the inclusion $\mathrm{Q}^\vee \subset \mathrm{P}^\vee$. W'_{aff} acts on the set of alcôves whereas W_{aff} acts simply transitively. For an alcôve

$\widehat{\mathrm{C}}$, we denote by $\Gamma_{\widehat{\mathrm{C}}}$ the isotropy subgroup of W'_{aff}. Then, $W'_{\mathrm{aff}} \cong W_{\mathrm{aff}} \tilde{\times} \Gamma_{\widehat{\mathrm{C}}}$, and we have the isomorphisms

$$\pi_1(H) \cong \mathrm{P}^\vee/\mathrm{Q}^\vee \cong W'_{\mathrm{aff}}/W_{\mathrm{aff}} \cong \Gamma_{\widehat{\mathrm{C}}} . \tag{A.1}$$

A choice $\widehat{\mathrm{C}}$ determines the decomposition $\Delta_{\mathrm{aff}} = \Delta_{\mathrm{aff}+} \cup \Delta_{\mathrm{aff}-}$ of affine roots, which is preserved by $\Gamma_{\widehat{\mathrm{C}}}$. In other words, $\Gamma_{\widehat{\mathrm{C}}}$ permutes the simple affine roots and can be considered as an automorphism group of the extended Dynkin diagram.

Each element of $\Gamma_{\widehat{\mathrm{C}}}$ has a representative loop of the form $\gamma(\theta) = \mathrm{e}^{-i\mu\theta} n_w$, where $\mu \in \mathrm{M}_{\mathrm{C}}$, and n_w represents a certain $w \in W$. To be more precise, let $\alpha_1, \ldots, \alpha_l \in \Delta_+$ be the simple roots and $\mu_1, \ldots, \mu_l \in \mathrm{P}^\vee$ be the dual base; $\alpha_i(\mu_j) = \delta_{i,j}$. Let $\tilde{\alpha}$ be the highest root. Then, $\mu \in \mathrm{M}_{\mathrm{C}}$ iff $\mu = 0$ or $\mu = \mu_j$ with $\tilde{\alpha}(\mu_j) = 1$. For such j, let W_j be the subgroup of W generated by the reflections $\{s_{\alpha_i};\ i \neq j\}$. Let w_j be the longest element of W_j and let w_0 be the longest element of W. Then, $w_j w_0$ is the element of W corresponding to $\mu_j \in \mathrm{M}_{\mathrm{C}}$ so that $\mathrm{e}^{-i\mu_j\theta} n_{w_j w_0}$ represents an element of $\Gamma_{\widehat{\mathrm{C}}}$. For the case of $H = SU(n)/\mathbf{Z}_n$, all the base elements μ_j $(j = 1, \ldots, n-1)$ are minimal. They are expressed together with $n_{w_j w_0}$ as

$$\mu_j = \begin{pmatrix} \mathbf{1}_j & 0 \\ 0 & 0 \end{pmatrix} - \frac{j}{n}\mathbf{1}_n , \qquad n_{w_j w_0} = \begin{pmatrix} 0 & \mathbf{1}_j \\ \mathbf{1}_{n-j} & 0 \end{pmatrix} (-1)^{\frac{j(n-j)}{n}} , \tag{A.2}$$

where $\mathbf{1}_j$ is the unit matrix of size j.

For a general compact Lie group H, the coweight lattice $\mathrm{P}^\vee$ can be defined in the same way, though the dual lattice is no longer the root lattice. Under appropriate definitions, the isomorphisms (A.1) holds.

Appendix B

In this appendix, we give expressions of the energy-momentum tensor T_{zz} and the current J_z for the adjoint ghost system and for the level k WZW model based on a compact simple Lie group H.

We fix a metric g and an H-connection A. Choose a local complex coordinate z and a local holomorphic section σ with respect to $\bar{\partial}_A$. To a base $\{e_{\mathrm{a}}\}$ of $\mathfrak{h}_{\mathrm{C}}$, σ associates a local holomorphic frame $\{\sigma_{\mathrm{a}}\}$ of the adjoint bundle and the dual frame $\{\sigma^{\mathrm{a}}\}$ of the coadjoint bundle. We denote by $\omega_z dz = g^{z\bar{z}}\partial g_{z\bar{z}}$ and $A_z^\sigma dz$ the 1-forms for the Levi–Civita connection and A respectively with respect to the holomorphic frames $\frac{\partial}{\partial z}$ and σ.

Ghost System. We put $c^\sigma(z) = \sum_{\mathrm{a}} e_{\mathrm{a}}\sigma^{\mathrm{a}} \cdot c(z) \in \mathfrak{h}_{\mathrm{C}}$ and $b_z^\sigma(z) = \sum_{\mathrm{a}} e^{\mathrm{a}} b_z \cdot \sigma_{\mathrm{a}}(z) \in \mathfrak{h}_{\mathrm{C}}^*$. Define the regularized product $: b_z^\sigma(z) c^\sigma(w) :$ by

$$b_z^\sigma(z) \otimes c^\sigma(w) = \frac{\sum_{\mathrm{a}} e^{\mathrm{a}} \otimes e_{\mathrm{a}}}{z - w} + : b_z^\sigma(z) \otimes c^\sigma(w) : . \tag{B.1}$$

Then, we have

$$J_z \cdot \sigma X = : b_z^\sigma \cdot [X, c^\sigma] : -2h^\vee \mathrm{tr}(A_z^\sigma X) , \tag{B.2}$$

$$T_{zz} = : \partial_z b_z^\sigma \cdot c^\sigma : - : b_z^\sigma \cdot [A_z^\sigma, c^\sigma] : + h^\vee \mathrm{tr}(A_z^\sigma A_z^\sigma) - \frac{c_{gh}}{12} S_{zz} , \tag{B.3}$$

where $\sigma X = \sum_{\mathrm{a}} \sigma_{\mathrm{a}} X^{\mathrm{a}}$, $c_{gh} = -2\dim H$ and $S_{zz} = \partial_z \omega_z - \frac{1}{2}\omega_z^2$.

The WZW Model. To the current, we associate an $\mathfrak{h}_{\mathbf{C}}^*$-valued holomorphic differential J_z^σ defined by

$$J_z \cdot \sigma X = J_z^\sigma \cdot X - k\,\mathrm{tr}(A_z^\sigma X)\,. \tag{B.4}$$

Define the regularized product $:J_z^\sigma(z)J_z^\sigma(w):$ by

$$J_z^\sigma(z)\cdot X\; J_z^\sigma(w)\cdot Y = \frac{k\,\mathrm{tr}(XY)}{(z-w)^2} + \frac{J_z^\sigma(w)\cdot[X,Y]}{z-w} + :J_z^\sigma(z)\cdot X\; J_z^\sigma(w)\cdot Y:\,. \tag{B.5}$$

Then, the Sugawara form of the energy momentum tensor is

$$T_{zz} = \frac{\eta^{\mathrm{ab}}}{2(k+h^\vee)} : J_z^\sigma \cdot e_{\mathrm{a}}\; J_z^\sigma \cdot e_{\mathrm{b}} : - J_z^\sigma \cdot A_z^\sigma + \frac{k}{2}\,\mathrm{tr}(A_z^\sigma A_z^\sigma) - \frac{c_{H,k}}{12} S_{zz}\,, \tag{B.6}$$

where $\eta^{\mathrm{ab}}\mathrm{tr}(e_{\mathrm{b}}e_{\mathrm{c}}) = \delta_{\mathrm{c}}^{\mathrm{a}}$ and $c_{H,k} = \frac{k}{k+h^\vee}\dim H$. This leads to differential equations of correlation functions [20, 42, 43].

Appendix C

This appendix gives an outline of the proof of the transformation rule (4.46) of γ_x. The γ_x-transform $(\mathscr{P}^\gamma, f^\gamma)$ of an $H_{\mathbf{C}}$-bundle $\mathscr{P}$ described by $\sigma(qz) = \sigma(z)h(q;z)$ with a flag $\sigma(1)h_f$ is defined by the relations (4.44) and (4.45) of an admissible section σ_0^γ around $z=1$ and a section σ' over $\mathbf{C}^* - q^{\mathbf{Z}}$.

We shall find an everywhere regular (but multivalued) section σ^γ. We put $\sigma^\gamma(z) = \sigma'(z)\tilde{\chi}(z)$ for $z \neq 1$ and require the relation $\sigma^\gamma(qz) = \sigma^\gamma(z)h^\gamma(q;z)$ to hold. The task is then to find such $\tilde{\chi}(z)$ that

$$\begin{cases} \tilde{\chi}(zq) = h(q;z)^{-1}\tilde{\chi}(z)h^\gamma(q;z) \\ \chi(z) = h_\gamma(z-1)^{-1}h_f^{-1}\tilde{\chi}(z) \quad \text{is regular as } z \to 1\,. \end{cases} \tag{C.1}$$

The latter condition arises from the requirement that $\sigma_0^\gamma(z) = \sigma^\gamma(z)\chi(z)^{-1}$ is an admissible section around $z=1$. The solution is exhibited below as $(\mathscr{P}, f) \to (\mathscr{P}^\gamma f^\gamma) : \tilde{\chi}(z)$.

$$\begin{aligned} (\mathscr{P}_u^{(0)}, 1) \to (\mathscr{P}_F^{(1)}, y_u) &: \begin{pmatrix} r_{-u}R_u(z) & ie^{-2\pi i u}q^{-\frac{1}{4}}r_{-u}R_{u-\frac{\tau}{2}}(z) \\ -r_uR_{-u}(z) & -ie^{-2\pi i u}q^{-\frac{1}{4}}r_uR_{-u-\frac{\tau}{2}}(z) \end{pmatrix};\; u \neq 0\,, \\ (\mathscr{P}_{00}^{(0)}, \infty) \to (\mathscr{P}_F^{(1)}, y_0) &: \begin{pmatrix} R_0(z)F(z) & iq^{-\frac{1}{4}}R_{-\frac{\tau}{2}}(z)G(z) \\ R_0(z) & iq^{-\frac{1}{4}}R_{-\frac{\tau}{2}}(z) \end{pmatrix}, \end{aligned} \tag{C.2}$$

where

$$\begin{aligned} R_u(z) &= \frac{\vartheta(\zeta + 2u + \tau, 2\tau)}{(\vartheta(\zeta + \frac{\tau+1}{2}, \tau))^{\frac{1}{2}}}, \qquad r_u = c_\tau \cdot (z-1)^{\frac{1}{2}} R_u(z)|_{z=1}\,, \\ F(z) &= 2z\frac{\partial}{\partial z}\log\vartheta(\zeta+\tau, 2\tau) - 1, \qquad G(z) = 2z\frac{\partial}{\partial z}\log\vartheta(\zeta, 2\tau)\,, \end{aligned}$$

in which $z = \mathrm{e}^{-2\pi i\zeta}$, c_τ is a constant, and ϑ is the theta function.

Acknowledgement. I wish to thank Y. Kazama for advice and encouragement throughout my graduate course. I thank T. Eguchi, N. Hayashi, A. Kato, N. Kawamoto, T. Kohno, K. Mohri, K. Yano and T. Yoneya for valuable discussions and helpful suggestions. I thank H. Nakajima for explaining the Hecke correspondence and N. Iwase for instructions on some algebraic topology including the isomorphism (6.1). I thank S. Higuchi for calling attention to ref. [48].

References

1. Gawędzki, K., Kupiainen, A.: Nucl. Phys. **B320**, 625–668 (1989); Phys. Lett. **215B**, 119–123 (1988)
2. Bardakci, K., Rabinovici, E., Säring, B.: Nucl. Phys. **B299**, 151–182 (1988); Altschuler, D., Bardakci, K., Rabinovici, E.: Commun. Math. Phys. **118**, 241–261 (1989)
3. Kalabali, D., Park, Q.-H., Schnitzer, H.J., Yang, Z.: Phys. Lett. **216B**, 307–312 (1989); Schnitzer, H.J.: Nucl. Phys. **B324**, 412–426 (1989); Kalabali, D., Schnitzer, H.J.: Nucl. Phys. **B329**, 649–666 (1990)
4. Bowcock, P.: Nucl. Phys. **B316**, 80–100 (1989)
5. Bardacki, K., Crescimanno, M., Rabinovici, E.: Nucl. Phys. **B344**, 344–370 (1990)
6. Belavin, A.A., Polyakov, A.M., Zamolodchikov, A.B.: Nucl. Phys. **B241**, 333–380 (1984)
7. Friedan, D., Qiu, Z., Shenker, S.: Phys. Lett. **151B**, 37–43 (1985)
8. Zamolodchikov, A.B., Fateev, V.A.: Sov. Phys. JETP **62**, 215–225 (1985); Gepner, D., Qiu, Z.: Nucl. Phys. **B285**, 423–453 (1987); Gepner, D.: Nucl. Phys. **B290**, 10–24 (1987)
9. Goddard, P., Kent, A., Olive, D.: Commun. Math. Phys. **103**, 105–119 (1987)
10. Gepner, D.: Phys. Lett. **222B**, 207–212 (1989)
11. Moore, G., Seiberg, N.: Phys. Lett. **220B**, 422–430 (1989)
12. Lerche, W., Vafa, C., Warner, N.P.: Nucl. Phys. **B324**, 427–474 (1989)
13. Witten, E.: Commun. Math. Phys. **92**, 455–472 (1984)
14. Felder, G., Gawędzki, K., Kupiainen, A.: Commun. Math. Phys. **117**, 127–158 (1988)
15. Polyakov, A.M., Wiegmann, P.B.: Phys. Lett. **141B**, 223–228 (1984)
16. Pressley, A., Segal, G.: Loop Groups. Oxford: Clarendon Press, 1986
17. Kac, V.G., Peterson, D.H.: Adv. Math. **53**, 125–264 (1984)
18. Goddard, P., Olive, D.: Int. J. Mod. Phys. **A1**, 303–414 (1986)
19. Bernard, D.: Nucl. Phys. **B288**, 628–648 (1987)
20. Knizhnik, V.G., Zamolodchikov, A.B.: Nucl. Phys. **B247**, 83–103 (1984)
21. Bourbaki, N.: Groupes et Algèbres de Lie, Chapitres 4, 5 et 6. Paris: Masson, 1981
22. Alvarez Gaumé, L., Bost, J.-B., Moore, G., Nelson, P., Vafa, C.: Commun. Math. Phys. **112**, 503–552 (1987)
23. Gawędzki, K., Kupiainen, A.: Commun. Math. Phys. **135**, 531–546 (1994)
24. Falceto, F., Gawędzki, K: Commun. Math. Phys. **159**, 471–503 (1994)
25. Polyakov, A.M.: Phys. Lett. **108B**, 207–210 (1981)
26. Alvarez Gaumé, L., Gomez, C., Moore, G., Vafa, C.: Nucl. Phys. **B303**, 455–521 (1988)
27. Friedan, D., Martinec, E., Shenker, S.: Nucl. Phys. **B271**, 93–165 (1986)
28. Grothendieck, A.: Am. J. Math. **79**, 121–138 (1957)
29. Atiyah, M.F., Bott, R.: Philos. Trans. R. Soc. Lond. **308A**, 523–615 (1982)
30. Narasimhan, M.S., Seshadri, C.S.: Ann. Math. **82**, 540–567 (1965)
31. Ramanathan, A.: Math. Ann. **213**, 129–152 (1975)
32. Donaldson, S.K.: J. Differ. Geom. **18**, 269–277 (1983)
33. Serre, J.P.: Séminaire BOURBAKI **100**, 1–8 (1954)
34. Bott, R.: Ann. Math. **66**, 203–248 (1957)
35. Kostant, B.: Ann. Math. **74**, 329–382 (1961)
36. Bouwknegt, P., McCarthy, J., Pilch, K.: Semi-infinite cohomology in conformal field theory and 2d gravity. In: The Proceedings of the XXV Karpacz Winter School of Theoretical Physics, 1992
37. Mehta, V.B., Seshadri, C.S.: Math. Ann. **248**, 205–239 (1980)
38. Atiyah, M.F.: Proc. Lond. Math. Soc. **7**, 414–452 (1957)
39. Narasimhan, M.S., Ramanan, S.: Geometry of Hecke cycles-I. In: C.P. Ramanujam : A tribute. Berlin, Heidelberg, New York: Springer, 1978, pp. 291–345
40. Segal, G., Wilson, G.: Pub. Math. I.H.E.S. **61**, 5–65 (1985)

41. Gepner, D., Witten, E.: Nucl. Phys. **B278**, 493–549 (1986)
42. Bernard, D.: Nucl. Phys. **B303**, 77–93 (1988)
43. Eguchi, T., Ooguri, H.: Nucl. Phys. **B282**, 308–328 (1987)
44. Kac, V.G.: Infinite Dimensional Lie Algebra. Cambridge: Cambridge University Press, 1990
45. Ray, D.B., Singer, I.M.: Ann. Math. **98**, 154–177 (1973)
46. Verlinde, E.: Nucl. Phys. **B300**, 360–376 (1988)
47. Schellekens, A.N., Yankielowicz, S.: Nucl. Phys. **B334**, 67–102 (1990); Int. J. Mod. Phys. **A5**, 2903–2952 (1990)
48. Dijkgraaf, R., Witten, E.: Commun. Math. Phys. **129**, 393–429 (1990)
49. Axelrod, S.E.: Geometric Quantization Of Chern–Simons Gauge Theory. Ph.D. Thesis, Princeton University (1991)
50. Witten, E.: Commun. Math. Phys. **144**, 189–212 (1992)
51. Cappelli, A.: Phys. Lett. **B185**, 82–88 (1987)
52. Kazama, Y., Suzuki, H.: Nucl. Phys. **B321**, 232–268 (1989)
53. Hosono, S., Tsuchiya, A.: Commun. Math. Phys. **136**, 451–486 (1991)
54. Gepner, D.: Commun. Math. Phys. **142**, 433–491 (1991)
55. Schellekens, A.N.: Nucl. Phys. **B366**, 27–73 (1991)
56. Nakatsu, T., Sugawara, Y.: Nucl. Phys. **B405**, 695–743 (1993)
57. Fuchs, J., Schweigert, C.: Nucl. Phys. **B411**, 181–222 (1994)

Communicated by R.H. Dijkgraaf