On the Continuous Limit of Integrable Lattices
I. The Kac–Moerbeke System and KdV Theory

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Abstract: KdV theory is constructed systematically through the continuous limit of the Kac–Moerbeke system The infinitely many commuting vector fields, the conserved functionals, the Lax pairs and the biHamiltonian structure are recovered as the limits of suitably defined linear combinations of homologous objects for the Kac–Moerbeke system The combinatorial aspects of this recombination method are treated in detail

1. Introduction

In the literature about integrable lattices, one meets the statement that the KdV-type equations can be obtained as continuous limits of the evolution equations of suitably chosen nonlinear lattices After the pioneering work of Zabusky and Kruskal [ZK] on the continuous limit of the Fermi–Pasta–Ulam lattice, Toda pointed attention to the *integrable* lattice theories giving the KdV (or the Boussinesq) equation in the continuous limit The limit process for the Toda lattice (from now on indicated with the acronym TL) was first described by Toda and Wadati [TW], this was the beginning of an effort, aiming to get more insight on the relation between discrete integrable systems and KdV-type theories The second edition of Toda's book [Tod] quotes, for example, Saitoh's papers [Sai], other references will be given in the sequel

The discussion of the continuous limit has a counterpart in the realm of inverse scattering The papers of Case, Chiu and Kac [CCK] are a classical reference on this topic in the discrete version of inverse scattering presented here, the limit process is easily performed at any step of the construction, yielding to the standard theory for the Schrödinger spectral problem

Leaving aside the inverse scattering, and coming to the structural analysis of the integrable evolution equations, we cannot avoid quoting Kupershmidt's monograph [Kup] Here a general, purely algebraic setting is proposed for the integrable lattices, extending to discrete systems the techniques for integrable KdV-type field theories developed by the Russian school [Dic]; after constructing a formal variational

calculus and a theory of differential-difference operators, Kupershmidt discusses in this framework the limit case in which the lattice spacing goes to zero

In this paper we are interested, in particular, in the ordinary KdV theory as a continuous limit of the so-called *Kac–Moerbeke system* This is a *reduction* of the usual TL (from now on denoted with the acronym KM), if $a = (a_i)$ and $b = (b_i)$ are the standard Flaschka coordinates for TL (i being the index for lattice sites), then the phase space of the KM system is the submanifold defined by the constraint $b_i = 0$ (see Sect 2 for the necessary details) The system was considered in [KM], where the angle-action coordinates were constructed, it was also discussed by Zakharov, Musher, Rubenchik and Manakov [ZMR, Man] The KM Lax operator is the differential-difference operator already considered in the papers [CCK] on inverse scattering

The fact that the KM evolution equation gives the ordinary KdV equation in the continuous limit was understood early (as reported in [Sch], Kac obtained the KdV one-soliton solution from the limit process) The convergence of the KM solutions to KdV solutions, for arbitrary initial conditions, was proved rigorously by Schwarz in the quoted reference

In spite of the above mentioned relevant results on the correspondence between KM and KdV theory, we think that there is still an opportunity to say something new on this topic We list a number of problems which, in our opinion, are essentially open

i) Most of the literature aims to recover from the continuous limit of the KM system the KdV evolution equation

$$u_\tau = u_{xxx} + 6uu_x \tag{1 1}$$

and/or its Lax formulation

$$\frac{dL^{\mathrm{KdV}}}{d\tau} = [B_2^{\mathrm{KdV}}, L^{\mathrm{KdV}}] , \tag{1 2}$$

where $L^{\mathrm{KdV}}(u) = \partial_{xx} + u$ and $B_2^{\mathrm{KdV}}(u) = 4\partial_{xxx} + 6u\partial_x + 3u_x$. (Here, we denote by τ the evolution parameter, reserving the letter t for the KM system) On the other hand, it is known that the vector field (1 1) is a member of an infinite *hierarchy* of commuting vector fields, it would be interesting to obtain all these fields, and the related Lax pairs, through a systematic limit procedure on the KM hierarchy

ii) Of course, one would like to obtain in the same scheme the full sequence of KdV conserved functionals

iii) The KdV and Toda-type hierarchies are *biHamiltonian* each vector field has two different Hamiltonian descriptions, given in terms of two compatible Poisson tensors It would be interesting to recover the KdV biHamiltonian structure as a continuous limit In the quoted book, Kupershmidt discusses the limit of Poisson structures on a lattice, even though his treatment allows to obtain the first KdV Poisson tensor, it is not clear how to recover the second one Indeed, should one interpret too naively some statement of the author (see p 119), he could be led to the conclusion that the Poisson structures arising as continuous limits of a lattice theory contain only the first power of the operator ∂_x, this would exclude the possibility to recover the second KdV Poisson tensor, which is a third order differential operator in ∂_x

In this paper, we propose a solution for problems i), ii) and iii). Our derivation of KdV theory from the KM system is based on a systematic method, which can be as well applied to other systems, starting from a different lattice, we plan to recover the Boussinesq theory in a subsequent paper

The biHamiltonian formalism plays a central role in our construction, for this reason, our attitude is to *define* the KM and KdV systems in biHamiltonian terms, and to introduce subsequently the Lax formalism, we take advantage from this viewpoint in order to prove that the continuous limit can be performed for the whole KM hierarchy

In order to set up the limit process for the KM system, we interpolate the sequence (a_t) of phase space coordinates with a smooth function $a(x)$ of a continuous variable x, and introduce a rescaled field variable u related to a by

$$a(x) = 2 + \varepsilon^2 u(x),\qquad(1\,3)$$

ε being the lattice spacing This is, essentially, the variable change considered in [Kup] to obtain Eq (1 1) and the KdV Lax operator in the $\varepsilon \mapsto 0$ limit

The distinctive feature of our approach is a *recombination principle* which allows to construct *the full KdV* theory (the infinitely many vector fields and conserved functionals, the Poisson tensors, the Lax pairs) taking the limits of some suitable chosen *linear combinations* of homologous objects for the KM system. We anticipate here the main results arising through the application of this method, the proofs will be given in Sect 5 (Propositions 5 4–6)

Let $f_s^\varepsilon(u)$ ($s = 0, 1, 2,\quad$) be the sequence of conserved functionals for the KM system, given in terms of the u variable on account of the transformation (1 3), similarly, let $X_s^\varepsilon(u)$ ($s = 1, 2, 3,\quad$) be the KM vector fields (see Sects 3 and 4) For each s we form the linear combinations

$$h_s^\varepsilon(u) \;=\; f_s^\varepsilon(u) + \sum_{j=0}^{s-1} c_{sj} f_j^\varepsilon(u) - c_s,\qquad Z_s^\varepsilon(u) \;=\; X_s^\varepsilon(u) + \sum_{j=1}^{s-1} c_{sj} X_j^\varepsilon(u),\qquad(1\,4)$$

where c_{sj} and c_s are numerical coefficients determined by an explicit prescription (see Sect 5) Our result is that the zero-spacing limits of the h_s^ε and Z_s^ε are the conserved functionals and the vector fields of order s of the KdV hierarchy

$$\lim_{\varepsilon \to 0} \frac{h_s^\varepsilon(u)}{4^s \varepsilon^{2s+2}} = h_s^{\mathrm{KdV}}(u),\qquad \lim_{\varepsilon \to 0} \frac{Z_s^\varepsilon(u)}{4^{s+1}\varepsilon^{2s-1}} = Z_s^{\mathrm{KdV}}(u)\qquad(1\,5)$$

Even in the case of the KdV equation (1 1) (i.e, the vector field Z_2^{KdV} in our notations), the derivation based on Eq (1 5) turns out to be different from the derivation given in [Kup], this point will be discussed in the Remark at the end of Sect. 4

In our approach, the above results can be derived from the $\varepsilon \mapsto 0$ limit of the KM biHamiltonian recursion scheme, the limit of the biHamiltonian structure can also be treated by means of a recombination In fact, let Q_u^ε and P_u^ε be the Poisson tensors of the KM system, we will show that Q_u^ε and $S_u^\varepsilon = P_u^\varepsilon - 16 Q_u^\varepsilon$ give respectively, for $\varepsilon \mapsto 0$, the first and the second KdV Poisson tensors

$$\lim_{\varepsilon \to 0} \frac{1}{4}\varepsilon^3 Q_u^\varepsilon = Q_u^{\mathrm{KdV}},\qquad \lim_{\varepsilon \to 0} \frac{1}{16}\varepsilon S_u^\varepsilon = S_u^{\mathrm{KdV}}\qquad(1\,6)$$

The same limit technique can be applied to the Lax pairs For each s, let us consider the KM vector field X_s^ε and its Lax formulation $dL^\varepsilon/dt_s = [A_s^\varepsilon, L^\varepsilon]$, to recover from

here the KdV Lax pairs $dL^{\text{KdV}}/d\tau_s = [B_s^{\text{KdV}}, L^{\text{KdV}}]$ we will consider the operators

$$B_s^\varepsilon(u) := A_\varepsilon^s(u) + \sum_{j=1}^{s-1} c_{sj} A_j^\varepsilon(u),\qquad\qquad(1.7)$$

and show that

$$\lim_{\varepsilon\mapsto 0} \frac{L^\varepsilon(u) - 4}{2\varepsilon^2} = L^{\text{KdV}}(u),\qquad \lim_{\varepsilon\mapsto 0} \frac{B_s^\varepsilon(u)}{4^{s+1}\varepsilon^{2s-1}} = B_s^{\text{KdV}}(u)\qquad\qquad(1.8)$$

One could be tempted to formulate the above results in a more picturesque style, using the language of renormalization well known from field theory [GJ] We confine this temptation to the next few lines, and dare to say that the lattice spacing ε is a cut-off, while $a(x)$ and $u(x) = \frac{a(x)-2}{\varepsilon^2}$ are the "bare" and "renormalized" fields. The recombination scheme for the functionals f_s^ε somehow recalls the Wick product in our model one gets the wanted $\varepsilon \mapsto 0$ limit replacing f_s^ε with $h_s^\varepsilon = f_s^\varepsilon + $ (a linear combination of the f_j^ε's for $j = 0, 1, \quad, s - 1$) + a constant, whereas in constructive field theory one operates on the action functional and replaces the powers ϕ^s of the field variable ϕ with the Wick powers $.\,\phi^s$, each of them being expressible as $\phi^s + $ (a linear combination of the lowest order powers ϕ^j) + a constant

   The paper is organized as follows In Sect 2, we review the basic facts on the KM system (also giving its second Poisson structure, which does not seem to be well known in the literature), in the same section, we recall how to obtain the KM system as a reduction of TL, and we fix our notational standards about KdV theory In Sect 3, we discuss the interpolation of the discrete KM system in terms of continuous field variables In Sect 4, we introduce the recombination method and the zero-spacing limit procedure In Sect 5, we give the general proofs of the statements (1 5), (1 6) and (1.8) As an exercise, in Sect 6 we show that the one-soliton solutions for the evolution equations of the first three recombined vector fields Z_s^ε converge for $\varepsilon \mapsto 0$ to the one-soliton solutions of the first three vector fields Z_s^{KdV}, here, we also include some remarks on the general problem of the $\varepsilon \mapsto 0$ limit for the flows of the recombined hierarchy Some of the computations presented in Sects 2,4 have been performed using the MATHEMATICA symbolic manipulation system Sects 2–5 of the paper could well be written in the merely algebraic language of formal variational calculus, but we prefer to adopt a less abstract presentation, a formulation of our results in the very pure style of [Kup] will be given elsewhere

2. Preliminaries: Kac–Moerbeke Versus KdV

In this section, we work with spaces of real sequences $p = (p_i)_{i \in \mathscr{I}}$, where $\mathscr{I}$ is the set of integers $\mathbb{Z}$ (infinite case) or $\mathscr{I} = \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, with N any integer (periodic case) When necessary, suitable boundary conditions will be assumed in the infinite case concerning the behaviour of sequences for $i \to \pm\infty$ If p is a sequence and $r \in \mathscr{I}$, $p_{(r)}$ is the r-shifted sequence of elements $p_{(r)i} = p_{i+r}$ [1] The shift operator sending p into $p_{(1)}$ is denoted by Δ, obviously, the power Δ^r sends p into $p_{(r)}$

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[1] On the contrary, we denote by p^r the rth power of the sequence p, with elements $(p^r)_i := (p_i)^r$

If we consider the real sequences over $\mathscr{L}$ as an algebra with the point-wise product $(pq)_i := p_i q_i$, then $\varDelta$ turns out to be an algebra automorphism. For any fixed sequence g and each integer r, composing the shift and the multiplication operator by g we can form the operators $g\varDelta^r$ and $\varDelta^r g$, these act on any sequence $\psi = (\psi_i)_{i \in \mathscr{L}}$ according to the rules $(g\varDelta^r)(\psi) = g\psi_{(r)}$, $(\varDelta^r g)(\psi) = g_{(r)}\psi_{(r)}$, and we have $\varDelta^r g = g_{(r)}\varDelta^r$. If a second sequence h is fixed, by composition we can construct the operator $g\varDelta^r h$, and so on.

Following [Kup], one can also introduce the associative algebra $\mathfrak{g}$ of formal pseudo differential-difference operators

$$G = \sum_{r=-\infty}^{r_{\max}} g_r \varDelta^r \ , \tag{2.1}$$

where each coefficient g_r is a sequence over $\mathscr{L}$, $r_{\max}$ is an arbitrary integer (depending on G). This algebra carries a trace: if the coefficient g_0 in Eq. (2.1) is a sequence with elements $g_{0,i}$, then $\operatorname{Tr} G := \sum_{i \in \mathscr{L}} g_{0,i}$.

2.1 The Kac–Moerbeke (KM) system. The phase space of the system is a conveniently chosen set $\mathscr{A}$ of real sequences $a = (a_i)_{i \in \mathscr{L}}$. We denote by $T_a \mathscr{A}$ and $T_a^* \mathscr{A}$ the tangent and cotangent spaces at any point a, their elements are represented as real sequences, written as $\dot{a} = (\dot{a}_i)$ and $\delta a = (\delta a_i)$ respectively. We have the pairing $\langle \delta a, \dot{a} \rangle = \sum_{i \in \mathscr{L}} \delta a_i \dot{a}_i$.

The manifold $\mathscr{A}$ carries a biHamiltonian structure, i.e., a pair of Poisson tensors Q, P which are mutually compatible in the sense of Magri [Mag]. At each point a, we have $Q_a : T_a^* \mathscr{A} \to T_a \mathscr{A}$, $\delta a \mapsto \dot{a} = \frac{1}{2} a(a_{(1)} \delta a_{(1)} - a_{(-1)} \delta a_{(-1)})$, or, in operator form,

$$\dot{a} = \frac{1}{2}(a\varDelta a - a\varDelta^{-1}a)(\delta a) \tag{2.2}$$

The second tensor $P_a : T_a^* \mathscr{A} \to T_a \mathscr{A}$ sends a covector δa into

$$\dot{a} = \frac{1}{2}(a^3 \varDelta a - a\varDelta^{-1}a^3 + a\varDelta a^3 - a^3 \varDelta^{-1}a + a\varDelta a^2 \varDelta a - a\varDelta^{-1}a^2 \varDelta^{-1}a)(\delta a) \tag{2.3}$$

An infinite hierarchy of commuting vector fields X_s $(s = 1, 2, 3, .\ )$ and conserved functionals f_s $(s = 0, 1, 2, \ )$ can be defined recursively on $\mathscr{A}$ using the biHamiltonian structure Q, P. The starting point is the function

$$f_0(a) := \sum_{i \in \mathscr{L}} \log a_i \ , \tag{2.4}$$

which is a Casimir of Q: $Q df_0 = 0$. The subsequent functionals and the vector fields of the hierarchy are defined by means of the recursion relations

$$X_{s+1} = Q df_{s+1} = P df_s \quad (s = 0, 1, 2, \ ) \tag{2.5}$$

If f_s is known, the above equation gives X_{s+1} and fixes f_{s+1} up to the addition of a Casimir of Q (i.e., f_0 or an arbitrary function of f_0), from the functional form of f_0, it is evident that the ambiguity in f_{s+1} can be removed with the following prescription: f_{s+1} must be the sum over i of a *polynomial* expression in a_i, $a_{i \pm 1}$, $a_{i \pm 2}$, *including no constant term*. Using this algorithm,

one finds

$$f_1(a) = \sum_{i \in \mathscr{I}} a_i^2, \qquad f_2(a) = \sum_{i \in \mathscr{I}} \left(\frac{1}{2} a_i^4 + a_i^2 a_{i+1}^2 \right), \qquad (2\ 6)$$

$$X_1(a) = a(a_{(1)}^2 - a_{(-1)}^2),$$

$$X_2(a) = a(a_{(1)}^4 - a_{(-1)}^4 + a^2 a_{(1)}^2 - a^2 a_{(-1)}^2 + a_{(1)}^2 a_{(2)}^2 - a_{(-1)}^2 a_{(-2)}^2),$$

$$X_3(a) = a(a_{(1)}^6 - a_{(-1)}^6 + a^4 a_{(1)}^2 - a^4 a_{(-1)}^2 + 2a^2 a_{(1)}^4 - 2a^2 a_{(-1)}^4$$

$$+ a_{(1)}^2 a_{(2)}^4 - a_{(-1)}^2 a_{(-2)}^4 + 2a_{(1)}^4 a_{(2)}^2 - 2a_{(-1)}^4 a_{(-2)}^2 + a^2 a_{(1)}^2 a_{(2)}^2$$

$$- a^2 a_{(-1)}^2 a_{(-2)}^2 + a_{(1)}^2 a_{(2)}^2 a_{(3)}^2 - a_{(-1)}^2 a_{(-2)}^2 a_{(-3)}^2), \qquad (2\ 7)$$

and so on

Equation (2 5) tells us that each vector field X_{s+1} of the KM hierarchy is Hamiltonian with respect to both Q and P, the Hamiltonian functions being f_{s+1} and f_s respectively As it is known from the standard theory of biHamiltonian systems, this implies that the functionals f_s are conserved quantities for all vector fields of the hierarchy, and they are in involution for both the Q and P Poisson brackets, moreover, the vector fields X_s mutually commute

One can also produce a Lax formulation of the KM hierarchy, resting on the Lax operator

$$L(a) = \Lambda^{-1} a + a\Lambda \qquad (2\ 8)$$

If d/dt_s denotes the Lie derivative along the vector field X_s, we have

$$\frac{dL}{dt_s} = [A_s, L], \qquad A_s = (L^{2s})_{\text{skew}}, \qquad (2\ 9)$$

where, for each G as in Eq (2 1), G_{skew} denotes the skew-symmetric part of G, given by $G_{\text{skew}} = \sum_{l=1}^{l_{\max}} (g_l \Lambda^l - \Lambda^{-l} g_l)$ Equation (2 9) can be as well employed in place of the biHamiltonian recursion (2 5) to *define* the vector fields X_s For $s = 1, 2, 3, \quad$, the Hamiltonians f_s can also be characterized in terms of the Lax operator, since

$$f_s(a) = \frac{1}{2s} \operatorname{Tr} L^{2s}(a) \qquad (2\ 10)$$

2 2 The KM system as a reduction of TL The relation between the two systems is apparent in the Lax approach, and a bit more subtle from the biHamiltonian viewpoint To clarify this point, it is convenient to review some facts about TL

We systematically use the Flaschka variables [Fla]. The phase space $\mathscr{M}$ of TL is a set of pairs $m = (a, b)$, where a and b are $\mathscr{I}$-indexed real sequences, at each point m, $T_m \mathscr{M}$ and $T_m^* \mathscr{M}$ denote respectively the tangent and the cotangent spaces, whose elements are represented as pairs $\dot{m} = (\dot{a}, \dot{b})$ and $\delta m = (\delta a, \delta b)$, with the pairing $\langle \delta m, \dot{m} \rangle = \sum_{i \in \mathscr{I}} (\delta b_i \dot{b}_i + \delta a_i \dot{a}_i)$ Two compatible Poisson tensors P_1 and P_2 are defined on $\mathscr{M}$, at each point m we have $P_{1m} \quad T_m^* \mathscr{M} \to T_m \mathscr{M}$, $\delta m \mapsto \dot{m} = P_{1m} \delta m$, with $\dot{a} = a(\delta b_{(1)} - \delta b)$ and $b = a\delta a - a_{(-1)}\delta a_{(-1)}$, or, in matrix form,

$$\begin{pmatrix} \dot{a} \\ b \end{pmatrix} = \begin{pmatrix} 0 & a(\Lambda - 1) \\ (1 - \Lambda^{-1})a & 0 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta b \end{pmatrix} \qquad (2\ 11)$$

Similarly, P_{2m} sends a covector $(\delta a, \delta b)$ into the tangent vector

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}a(\Delta - \Delta^{-1})a & a(\Delta - 1)b \\ b(1 - \Delta^{-1})a & 2a^2\Delta - 2\Delta^{-1}a^2 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta b \end{pmatrix}. \qquad (2\ 12)$$

An infinite hierarchy of commuting vector fields $\mathcal{X}_s$ ($s = 1, 2, 3, \ \ $) and Hamiltonians φ_s ($s = 0, 1, 2, \ \ $) can be constructed recursively on $\mathcal{M}$ starting from

$$\varphi_0(m) := \sum_{i \in \mathcal{I}} \log a_i, \qquad \varphi_1(m) = \sum_{i \in \mathcal{I}} b_i ; \qquad (2\ 13)$$

φ_0 is a Casimir of both Poisson tensors P_1 and P_2, while φ_1 is a Casimir of P_1 The higher order Hamiltonians and the vector fields of the hierarchy are defined stipulating the recursion relation

$$\mathcal{X}_s = P_2 d\varphi_s = P_1 d\varphi_{s+1} \quad (s = 1, 2, 3, \ \ ). \qquad (2\ 14)$$

For each s, the above equation determines φ_{s+1} from φ_s up to the addition of a Casimir of P_1 The nature of the Casimirs allows to remove this ambiguity with the following prescription φ_{s+1} must be (the sum over i of) a *polynomial* in a_i, b_i and their shifts of any order $a_{i+i}, b_{i+i'}$, *including neither linear terms in b nor constant terms* In particular, the first vector field

$$\mathcal{X}_1(m) = \begin{pmatrix} a(b_{(1)} - b) \\ 2(a^2 - a_{(-1)}^2) \end{pmatrix} \qquad (2\ 15)$$

gives, in the Flaschka variables, the evolution equations for the chain of exponential springs first considered by Toda

All vector fields of the TL hierarchy admit a Lax formulation in terms of the Lax operator

$$\Lambda(m) = \Delta^{-1}a + b + a\Delta \qquad (2\ 16)$$

If d/dt_s denotes the derivative along $\mathcal{X}_s$, we have

$$\frac{d\Lambda}{dt_s} = [(\Lambda^s)_{\mathrm{skew}}, \Lambda] , \qquad (2\ 17)$$

where $_{\mathrm{skew}}$ is the projection already considered in Subsect 2.1 For $s = 1, 2, 3, \ \ $, the Hamiltonians φ_s considered above can be expressed as

$$\varphi_s(m) = \frac{1}{s} \operatorname{Tr} \Lambda^s(m) \qquad (2\ 18)$$

Now, let us observe that the TL Lax operator $\Lambda(m)$ becomes the KM operator $L(a)$ if one sets $b = 0$ in Eq (2 16) From a geometrical viewpoint, this amounts to consider in the phase space $\mathcal{M}$ the submanifold formed by pairs $m = (a, 0)$, this submanifold can be identified with the phase space $\mathcal{A}$ of the KM system, and it can be shown that the geometrical structures considered in Subsect 2 1 are restrictions to $\mathcal{A}$ of homologous structures living on $\mathcal{M}$

Let us start from the biHamiltonian formalism The TL Poisson tensor P_1 cannot be restricted[2] to $\mathcal{A}$, conversely, P_2 restricts into the Poisson tensor Q of Eq (2 2)

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[2] For a review on the restriction techniques of Poisson tensors, see for example [MMR]

In order to recover the second tensor P of the KM system, we must consider, besides P_1 and P_2, the higher order TL Poisson structures These can be introduced using the recursion operator $N_m = P_{2m} \circ P_{1m}^{-1}$ (where the inverse of P_1 is understood in the framework of formal pseudo differential-difference operators) At each point $m = (a, b)$, we have

$$N_m = \begin{pmatrix} (ab - a\Delta b)(a - a\Delta)^{-1} & \frac{1}{2}(a + a\Delta) \\ 2(\Delta^{-1}a^2 - a^2\Delta)(a - a\Delta)^{-1} & b \end{pmatrix} \qquad (2\ 19)$$

By means of N, one can define the tensors $P_s = N^{s-2}P_2$ ($s = 3, 4, 5, \quad$), which form (together with P_1 and P_2) a sequence of mutually compatible Poisson structures[3] It turns out that N^2 restricts from $\mathcal{M}$ to $\mathcal{A}$, this implies restrictability of $P_4 = N^2 P_2$, which goes into the KM Poisson tensor P of Eq (2 3)

Using the reducibility of N^2, one can show that all even order vector fields $\mathcal{X}_{2s}$ of the TL hierarchy are tangent to the submanifold $\mathcal{A}$, for $s = 1, 2, 3, \quad$, the restrictions of the $\mathcal{X}_{2s}$ are just the KM vector fields X_s Similarly, the restrictions to $\mathcal{A}$ of the Hamiltonians φ_{2s} are the KM Hamiltonians f_s (the odd order Hamiltonians φ_{2s+1} are identically zero on $\mathcal{A}$)

2 3 A glossary of KdV theory In order to fix the notations, we add a few lines about KdV theory Here the phase space is a set $\mathcal{U}$ of real smooth functions u of a variable x, ranging over the real line $\mathbb{R}$ or the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ Elements of the tangent space $T_u\mathcal{U}$ and the cotangent space $T_u^*\mathcal{U}$ are smooth functions of x, denoted typically by $\dot{u}$ and δu, we have the pairing $\langle \delta u, \dot{u} \rangle = \int dx\, \delta u(x)\, \dot{u}(x)$ The geometrical and algebraic structures attached to the phase space $\mathcal{U}$ will be marked with the letters KdV, in order to avoid confusions in the rest of the paper We have two Poisson tensors Q^{KdV}, S^{KdV}, at each point u, Q_u^{KdV} . $T_u^*\mathcal{U} \mapsto T_u\mathcal{U}$ sends a covector δu into the tangent vector

$$\dot{u} = \delta u_x , \qquad (2\ 20)$$

while S_u^{KdV} sends δu into

$$\dot{u} = (\partial_{xxx} + 4u\partial_x + 2u_x)(\delta u) \qquad (2\ 21)$$

We denote with Z_s^{KdV} ($s = 1, 2, 3, \quad .$) and h_s^{KdV} ($s = 0, 1, 2, \quad$) the vector fields and the Hamiltonians of the familiar KdV hierarchy, these can be defined recursively starting from

$$h_0^{\mathrm{KdV}}(u) = \frac{1}{2}\int dx\, u(x) \qquad (2\ 22)$$

(which is a Casimir of Q) and stipulating that

$$Z_{s+1}^{\mathrm{KdV}} = Q^{\mathrm{KdV}} dh_{s+1}^{\mathrm{KdV}} = S^{\mathrm{KdV}} dh_s^{\mathrm{KdV}} \quad (s = 0, 1, 2, \quad .), \qquad (2.23)$$

with the supplementary requirement that h_{s+1}^{KdV} be (the integral over x of) a polynomial in u and its derivatives, *containing neither linear terms in u nor constant terms*. For example $Z_1^{\mathrm{KdV}}(u) = u_x$, $Z_2^{\mathrm{KdV}}(u)$ is the r h s of Eq (1 1), $h_1^{\mathrm{KdV}}(u) = \frac{1}{2}\int dx\, u^2$, $h_2^{\mathrm{KdV}}(u) = \int dx\,(u^3 - \frac{1}{2}u_x^2)$, etc

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[3] This sequence could also be constructed via the so-called "mastersymmetry approach," see e g [Dam]

Of course, this hierarchy can be described in terms of the Lax operator

$$L^{\mathrm{KdV}}(u) = \hat{\partial}_x + u \tag{2 24}$$

If $d/d\tau_s$ is the derivative along Z_s^{KdV}, we have

$$\frac{dL^{\mathrm{KdV}}}{d\tau_s} = [B_s^{\mathrm{KdV}}, L^{\mathrm{KdV}}], \qquad B_s^{\mathrm{KdV}} = 4^{s-1}((L^{\mathrm{KdV}})^{s-\frac{1}{2}})_+ , \tag{2 25}$$

where $+$ denotes the projection on the nonnegative powers of $\hat{\partial}_x$ in the algebra of formal pseudo differential operators, furthermore,

$$h_s^{\mathrm{KdV}}(u) = \frac{4^s}{2s+1} \operatorname{Tr}(L^{\mathrm{KdV}})^{s+\frac{1}{2}}(u) , \tag{2 26}$$

where Tr is the well known Adler traceform carried by this algebra [Dic]

3. Interpolating the KM System

The constructions of Sect. 2 arise from an abstract algebraic structure, which can be described as follows we have a commutative algebra (the sequences over $\mathscr{Z}$, with the pointwise product), carrying a distinguished automorphism (the shift operator Δ) and a linear functional (the sum $\sum_{i\in\mathscr{A}}$) invariant under this automorphism There is an obvious, alternative realization of the same abstract structure, where the sequences are replaced by real smooth functions of a continuous variable x, ranging over the real line $\mathbb{R}$ or the torus $\mathbb{T}$. In the present section we consider this alternative realization, the system arising in this way can be easily interpreted as an interpolation of the discrete KM lattice of Sect 2 (see the final Remark) and provides the starting point to recover KdV theory through an appropriate limit process

Let us consider the algebra of real smooth functions a on $\mathbb{R}$ or $\mathbb{T}$ (with appropriate boundary conditions at infinity in the former case) We fix a fundamental spacing $\varepsilon \in \mathbb{R}$, the shift operator on smooth functions is the map Δ_ε $a \mapsto a_{(\varepsilon)}$, where $a_{(\varepsilon)}(x) = a(x+\varepsilon)$ For each integer r, the rth power of Δ_ε is the operator sending a into $a_{(r\varepsilon)}$, where $a_{(r\varepsilon)}(x) = a(x+r\varepsilon)$, this operator will be denoted by $\Delta_{r\varepsilon}$ The linear functional $(a_i) \mapsto \sum_{i\in\mathscr{A}} a_i$ on sequences is replaced in the present framework by the functional $a \mapsto \int dx\, a(x)$ (integration over the real line or the torus), which is clearly shift invariant [4]

If g and h are fixed smooth functions, we can form operators such as $g\Delta_{r\varepsilon}$, $\Delta_{r\varepsilon}g$, $g\Delta_{r\varepsilon}h$, for example, the third one acts on a function ψ with the rule $(g\Delta_{r\varepsilon}h)(\psi)(x) = g(x)h(x+r\varepsilon)\psi(x+r\varepsilon)$ Also, we can introduce an algebra of formal pseudo differential-difference operators $G_\varepsilon = \sum_{i=-\infty}^{i_{max}} g_i \Delta_{i\varepsilon}$, where the coefficients g_i are smooth functions of x This is clearly the counterpart of the algebra $\mathfrak{g}$ in Sect 2, it carries a traceform Tr, where $\operatorname{Tr} G_\varepsilon = \int dx\, g_0(x)$

---

[4] In the case where x ranges over the line $\mathbb{R}$, it is necessary for our purposes to define the integral $\int dx\, a(x)$ for each function of the form $a(x) = c + v(x)$, with c a constant and v vanishing rapidly at infinity If a is of this form, we put $\int dx\, a(x) := \int dx\, v(x)$ ("subtraction of the vacuum infinities"); according to this prescription, for each constant c it is $\int dx\, c = 0$ An analogous definition could well be given for $\sum_i a_i$, when i ranges over the set $\mathbb{Z}$ of integers and (a_i) is the sum of a constant and a rapidly vanishing sequence

We now come to the *interpolated* KM system, its construction rephrases Subsect 2 2, so it is hardly the case to mention the essential points Concerning the notations, our style will be to maintain the same letters for the homologous objects in the discrete and continuous cases, adding the mark ε to any structure which depends on the spacing

So, the phase space of the interpolated system will be denoted by $\mathscr{A}$, its elements are real smooth functions a on $\mathbb{R}$ or $\mathbb{T}$ The pairing between cotangent vectors δa and tangent vectors $\dot{a}$ is given by $\langle \delta a, \dot{a} \rangle = \int dx\, \delta a(x) \dot{a}(x)$. We have a pair of Poisson tensors Q^{ε}, P^{ε}, defined by formulas analogous to (2.2), (2 3) (provided that we systematically replace the symbols $a_{(i)}$ and Δ' with $a_{(i\varepsilon)}$ and $\Delta_{i\varepsilon}$), the recursion relation

$$X_{s+1}^{\varepsilon} = Q^{\varepsilon} df_{s+1}^{\varepsilon} = P^{\varepsilon} df_{s}^{\varepsilon}, \quad (s = 0,1,2, \quad) \tag{3.1}$$

gives rise to a hierarchy of vector fields X_{s}^{ε} and Hamiltonians f_{s}^{ε}, starting from

$$f_{0}^{\varepsilon}(a) = \int dx \log a(x) \tag{3 2}$$

(in fact, ε independent) For $s = 1,2,3, \quad$, we stipulate that f_{s}^{ε} be the integral over x of a polynomial in $a(x)$ and the shifts $a_{(i\varepsilon)}(x)$, *including no constant term*, with this supplementary requirement, the recursion formula (3 1) uniquely determines f_{s+1}^{ε}, if f_{s}^{ε} is known The first Hamiltonians after f_{0}^{ε} are

$$f_{1}^{\varepsilon}(a) = \int dx\, a^{2}(x), \qquad f_{2}^{\varepsilon}(a) = \int dx \left(\frac{1}{2} a^{4}(x) + a^{2}(x) a^{2}(x + \varepsilon) \right) \tag{3.3}$$

The explicit expressions for the vector fields X_{1}^{ε}, X_{2}^{ε} and X_{3}^{ε} are analogous to formulas (2 7), with the obvious prescription of replacing $a_{(i)}$ with $a_{(i\varepsilon)}$

The hierarchy has a Lax operator $L^{\varepsilon}(a) = a\Delta_{\varepsilon} + \Delta_{\varepsilon}^{-1} a$; Eqs (2 9) and (2 10) are still satisfied, i e ,

$$\frac{dL^{\varepsilon}}{dt_{s}} = [A_{s}^{\varepsilon}, L^{\varepsilon}], \qquad A_{s}^{\varepsilon} = ((L^{\varepsilon})^{2s})_{\text{skew}} \tag{3 4}$$

and

$$f_{s}^{\varepsilon}(a) = \frac{1}{2s} \operatorname{Tr}(L^{\varepsilon})^{2s}(a) \,, \tag{3 5}$$

intending the trace as explained above

Remark   Let us add some comments on the relation between the interpolated and the discrete KM systems Of course, the discrete index i of Sect 2, ranging over $\mathscr{L}$, can be viewed as the label for a network of equally spaced sites on $\mathbb{R}$ or $\mathbb{T}$, of coordinates $x_{i} = i\varepsilon$ If $\mathscr{L} = \mathbb{Z}$, the points x_{i} lie on the line and ε is arbitrary, if $\mathscr{L} = \mathbb{Z}_{N}$, the N points x_{i} are placed on the torus, with $\varepsilon = \pm 1/N$

Given a smooth function a of x and a sequence (a_{i}), assume that $a(x_{i}) = a_{i}$ for each $i \in \mathscr{L}$, in this case, we say that *a interpolates* (a_{i}), or, equivalently, that (a_{i}) is the *discretization* of the function a

Interpolation can be described geometrically, in terms of a projection More precisely, let Π^{ε} be the map sending the function a of x into the sequence $a_{i} = a(x_{i})$, Π^{ε} is a surjective application between the phase spaces of the interpolated and the discrete systems The vector fields X_{s} of Subsect 2 2 and their continuous

homologues X_s^ε are related by

$$X_s((a_t)) = (T_a \Pi^\varepsilon)(X_s^\varepsilon(a)) , \tag{3.6}$$

where $T\Pi^\varepsilon$ denotes the tangent map and a is any function such that $\Pi^\varepsilon(a) = (a_t)$
The geometrical property expressed by Eq (3 6) corresponds to the general notion
of projectability for vector fields

4. The Zero Spacing Limit and KdV Theory

4 1 Field rescaling Our purpose in this section is to recover KdV theory from
the interpolated KM system through the $\varepsilon \mapsto 0$ limit As a preliminary step, we
introduce (for ε finite and arbitrary) a rescaled field variable u, related to a by
Eq (1 3) For any fixed $\varepsilon \neq 0$, we can describe the rescaling by saying that there
is a diffeomorphism

$$\Phi^\varepsilon \quad \mathscr{A} \to \mathscr{U}, \qquad a \mapsto u = \frac{a - 2}{\varepsilon^2} \tag{4 1}$$

(with $\mathscr{U}$ an appropriate set of functions[5]) At any point $a \in \mathscr{A}$, the tangent map of
Φ^ε and its adjoint are

$$T_a \Phi^\varepsilon \quad T_a \mathscr{A} \to T_u \mathscr{U}, \qquad \dot{a} \mapsto \dot{u} = \frac{\dot{a}}{\varepsilon^2} , \tag{4 2}$$

$$T_a^* \Phi^\varepsilon \quad T_u^* \mathscr{U} \to T_a^* \mathscr{A}, \qquad \delta u \mapsto \delta a = \frac{1}{\varepsilon^2} \, \delta u \tag{4 3}$$

We can employ the diffeomorphism Φ^ε to transport on $\mathscr{U}$ the whole structure of the
KM system The correspondents of the objects considered in the previous section
will be denoted for simplicity with the same letters So, $\mathscr{U}$ carries a hierarchy of
vector fields

$$X_s^\varepsilon(u) := (T_a \Phi^\varepsilon)(X_s^\varepsilon(a))\big|_{a=(\Phi^\varepsilon)^{-1}(u)} = \frac{1}{\varepsilon^2} X_s^\varepsilon(a)\bigg|_{a=2+\varepsilon^2 u} \tag{4 4}$$

and Hamiltonians

$$f_s^\varepsilon(u) = f_s^\varepsilon(a)\big|_{a=(\Phi^\varepsilon)^{-1}(u)} = f_s^\varepsilon(a)\big|_{a=2+\varepsilon^2 u} . \tag{4 5}$$

The first Poisson tensor on $\mathscr{U}$ is

$$Q_u^\varepsilon = T_a \Phi^\varepsilon \circ Q_a^\varepsilon \circ T_a^* \Phi^\varepsilon\big|_{a=(\Phi^\varepsilon)^{-1}(u)} = \frac{1}{\varepsilon^4} Q_a^\varepsilon\bigg|_{a=2+\varepsilon^2 u} , \tag{4.6}$$

i e , from Eq (2 2), $Q_u^\varepsilon = \frac{1}{2\varepsilon^4}((2 + \varepsilon^2 u)\Delta_\varepsilon(2 + \varepsilon^2 u) - (2 + \varepsilon^2 u)\Delta_{-\varepsilon}(2 + \varepsilon^2 u))$ The
second Poisson tensor P_u^ε is constructed similarly
   The Lax operator for the KM hierarchy on $\mathscr{U}$ is

$$L^\varepsilon(u) := L^\varepsilon(a)\big|_{a=(\Phi^\varepsilon)^{-1}(u)} = (2 + \varepsilon^2 u)\Delta_\varepsilon + \Delta_{-\varepsilon}(2 + \varepsilon^2 u) \tag{4 7}$$

---

[5] If λ ranges over the line $\mathbb{R}$, it is convenient to assume that, for all functions $a \in \mathscr{A}$ and $u \in \mathscr{U}$,
$a - 2$ and u be rapidly vanishing at infinity

4.2 The $\varepsilon \mapsto 0$ limit. The Taylor formula $\psi(x + r\varepsilon) = \sum_{j=0}^{+\infty}(1/j!)(r\varepsilon)^j \frac{d^j \psi}{dx^j}(x)$, for the shift of a function ψ, means that $\Delta_{r\varepsilon} = \exp(r\varepsilon\partial_x)$. So, if we take the Lax operator $L^\varepsilon(u)$ in Eq. (4.7) and expand it in powers of ε up to order 2, we find (similarly to [Kup])

$$L^\varepsilon(u) = 4 + 2\varepsilon^2(\partial_{xx} + u) + O(\varepsilon^3) \qquad (4.8)$$

The appearance of the KdV Lax operator is a hint that we are following the right path towards our goal. Now, let us pass to the vector fields X_s^ε. The first one is given by

$$X_1^\varepsilon(u) = \frac{1}{\varepsilon^2}(2 + \varepsilon^2 u)\left[(2 + \varepsilon^2 u_{(\varepsilon)})^2 - (2 + \varepsilon^2 u_{(-\varepsilon)})^2\right]. \qquad (4.9)$$

From the Taylor expansion of $u(x \pm \varepsilon)$, we have $u_{(\pm\varepsilon)} = u \pm \varepsilon u_x + \frac{1}{2}\varepsilon^2 u_{xx} + \ \cdots \ + (\varepsilon^6/720)u_{xxxxxx} + O(\varepsilon^7)$, which implies

$$X_1^\varepsilon(u) = 16\varepsilon u_x + \frac{8}{3}\varepsilon^3(u_{xxx} + 6uu_x)$$

$$+ \frac{2}{15}\varepsilon^5(u_{xxxxx} + 20uu_{xxx} + 30u_x u_{xx} + 30u^2 u_x) + O(\varepsilon^7) \qquad (4.10)$$

Similarly, we find

$$X_2^\varepsilon(u) = 384\varepsilon u_x + 128\varepsilon^3(u_{xxx} + 6uu_x)$$

$$+ \frac{96}{5}\varepsilon^5\left(u_{xxxxx} + \frac{40}{3}uu_{xxx} + \frac{70}{3}u_x u_{xx} + 30u^2 u_x\right) + O(\varepsilon^7), \qquad (4.11)$$

$$X_3^\varepsilon(u) = 7680\varepsilon u_x + 3840\varepsilon^3(u_{xxx} + 6uu_x)$$

$$+ 960\varepsilon^5(u_{xxxxx} + 12uu_{xxx} + 22u_x u_{xx} + 30u^2 u_x) + O(\varepsilon^7) \qquad (4.12)$$

The inspection of the above expansions suggests that the first three vector fields of the KdV hierarchy could be recovered in the $\varepsilon \mapsto 0$ limit by suitable recombinations, so that the lowest order term in the expansion of each recombined field is a KdV vector field. Indeed, this is obtained introducing, for each finite ε, the vector fields

$$Z_1^\varepsilon := X_1^\varepsilon, \qquad Z_2^\varepsilon := X_2^\varepsilon - 24X_1^\varepsilon, \qquad Z_3^\varepsilon := X_3^\varepsilon - 40X_2^\varepsilon + 480X_1^\varepsilon \qquad (4.13)$$

(that we can consider to be defined on both phase spaces $\mathscr{A}$ and $\mathscr{U}$). The choice of the coefficients in Eq. (4.13) implies that

$$Z_1^\varepsilon(u) = 16\varepsilon u_x + O(\varepsilon^3) = 16\varepsilon Z_1^{\mathrm{KdV}}(u) + O(\varepsilon^3),$$

$$Z_2^\varepsilon(u) = 64\varepsilon^3(u_{xxx} + 6uu_x) + O(\varepsilon^5) = 64\varepsilon^3 Z_2^{\mathrm{KdV}}(u) + O(\varepsilon^5),$$

$$Z_3^\varepsilon(u) = 256\varepsilon^5(u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x) + O(\varepsilon^7)$$

$$= 256\varepsilon^5 Z_3^{\mathrm{KdV}}(u) + O(\varepsilon^7) \qquad (4.14)$$

These experimental facts lead us to conjecture that we could generate all vector fields of the KdV hierarchy with the same recombination method. This result will be presented in the next section; for the moment, we maintain an experimental attitude and test the $\varepsilon \mapsto 0$ behaviour of the first KM Hamiltonians. We have[6]

$$f_0^\varepsilon(u) = \int dx\, \log(2 + \varepsilon^2 u)$$

$$= \log 2 \int dx + \frac{\varepsilon^2}{2} \int dx\, u - \frac{\varepsilon^4}{8} \int dx\, u^2 + \frac{\varepsilon^6}{24} \int dx\, u^3 + O(\varepsilon^8) ,$$

$$f_1^\varepsilon(u) = \int dx\, (2 + \varepsilon^2 u)^2 = 4 \int dx + 4\varepsilon^2 \int dx\, u + \varepsilon^4 \int dx\, u^2 ,$$

$$f_2^\varepsilon(u) = \int dx\, \left(\frac{1}{2}(2 + \varepsilon^2 u)^4 + (2 + \varepsilon^2 u_{(\varepsilon)})^2(2 + \varepsilon^2 u)^2 \right)$$

$$= 24 \int dx + 48\varepsilon^2 \int dx\, u + 36\varepsilon^4 \int dx\, u^2 + 4\varepsilon^6 \int dx\, (3u^3 - 2u_x^2) + O(\varepsilon^8)$$

$$(4.15)$$

So, setting

$$h_0^\varepsilon = f_0^\varepsilon - \log 2 \int dx, \qquad h_1^\varepsilon = f_1^\varepsilon - 8 f_0^\varepsilon - (4 - 8\log 2) \int dx ,$$

$$h_2^\varepsilon = f_2^\varepsilon - 24 f_1^\varepsilon + 96 f_0^\varepsilon - (96 \log 2 - 72) \int dx , \qquad (4.16)$$

we infer

$$h_0^\varepsilon(u) = \frac{1}{2}\varepsilon^2 \int dx\, u + O(\varepsilon^4) = \varepsilon^2\, h_0^{\mathrm{KdV}}(u) + O(\varepsilon^4) ,$$

$$h_1^\varepsilon(u) = 2\varepsilon^4 \int dx\, u^2 + O(\varepsilon^6) = 4\varepsilon^4\, h_1^{\mathrm{KdV}}(u) + O(\varepsilon^6) ,$$

$$h_2^\varepsilon(u) = 8\varepsilon^6 \int dx\, (2u^3 - u_x^2) + O(\varepsilon^8) = 16\varepsilon^6\, h_2^{\mathrm{KdV}}(u) + O(\varepsilon^8) \qquad (4.17)$$

Finally, let us consider the Poisson tensors Q^ε, P^ε (see Eq. (4.6)). We have

$$Q_u^\varepsilon(\delta u) = \frac{4}{\varepsilon^3} \delta u_x + \frac{2}{3\varepsilon} (\delta u_{xxx} + 6u\delta u_x + 3u_x\delta u) + O(\varepsilon) , \qquad (4.18)$$

$$P_u^\varepsilon(\delta u) = \frac{64}{\varepsilon^3} \delta u_x + \frac{80}{3\varepsilon} \left(\delta u_{xxx} + \frac{24}{5} u\delta u_x + \frac{12}{5} u_x\delta u \right) + O(\varepsilon) , \qquad (4.19)$$

this implies

$$Q_u^\varepsilon = \frac{4}{\varepsilon^3} \partial_x + O\left(\frac{1}{\varepsilon}\right), \qquad S_u^\varepsilon = \frac{16}{\varepsilon} (\partial_{xxx} + 4u\partial_x + 2u_x) + O(\varepsilon) , \qquad (4.20)$$

---

[6] In the periodic case, $\int dx = 1$ (the total measure of the torus $\mathbb{T}$). We recall that we have defined $\int dx := 0$ if x ranges over the line $\mathbb{R}$: see the footnote in Sect. 3. In computing the expansions of the Hamiltonians, we take into account that, for any polynomial φ in u and its derivatives, it is $\int dx\, \varphi_x = 0$; this is granted when x ranges over $\mathbb{T}$, or, alternatively, when x ranges over $\mathbb{R}$ and u is rapidly vanishing at infinity.

where we have put

$$S^{\varepsilon} := P^{\varepsilon} - 16 Q^{\varepsilon} \qquad (4\,21)$$

In Eq (4 20), one recognizes the KdV Poisson tensors, this fact will be basic in the next section We observe that, for any finite ε, Q^{ε} and S^{ε} form a pair of compatible Poisson tensors on the phase space $\mathcal{U}$ Also, the definition (4 21) of S^{ε} is consistent with the recombination rules found for the first vector fields and Hamiltonian functions, in the sense that we still have a recursion scheme, in fact, using Eqs (4 13), (4 16), (4 21), and Eq (3 1), one can check that, for the lowest values of s,

$$Z^{\varepsilon}_{s+1} = Q^{\varepsilon} dh^{\varepsilon}_{s+1} = S^{\varepsilon} dh^{\varepsilon}_{s} \qquad (4.22)$$

We are now going to develop systematically the approach suggested here, so as to recover not only the first vector fields and Hamiltonians, but the whole KdV hierarchy

Remark. From our viewpoint, the role of the first KM vector field X^{ε}_1 is simply to give, in the $\varepsilon \mapsto 0$ limit, the vector field $Z^{\mathrm{KdV}}_1(u) = u_x$, i.e , the infinitesimal generator of space translations on $\mathcal{U}$, to obtain the ordinary KdV equation (1 1), i e , the vector field Z^{KdV}_2, we take the limit of the combination $X^{\varepsilon}_2 - 24 X^{\varepsilon}_1$ For completeness, we compare this construction with Kupershmidt's derivation of Eq (1 1) Kupershmidt employs only the first KM vector field X^{ε}_1, according to Eq (4 10), the evolution equation $du/dt_1 = X^{\varepsilon}_1(u)$ becomes, at the lowest orders in ε,

$$u_{t_1} = 16 \varepsilon u_x + \frac{8}{3} \varepsilon^3 (u_{xxx} + 6 u u_x) \,. \qquad (4\,23)$$

The manipulation performed by Kupershmidt on this equation is described in terms of his abstract language of evolutionary derivations, but amounts essentially to consider a space-time coordinate change

$$\xi = x + 16 \varepsilon t_1, \qquad \tau_1 = \frac{8}{3} \varepsilon^3 t_1 \,, \qquad (4\,24)$$

under which Eq (4 23) becomes $u_{\tau_1} = u_{\xi\xi\xi} + 6 u u_{\xi}$ The τ_1 variable is clearly a rescaled time, while ξ can be interpreted as the space coordinate for an observer translating with speed 16ε with respect to the lattice[7], the introduction of this observer, which is a standard trick in the literature, is essentially motivated by the aim to reabsorb the term in εu_x from Eq (4 23) No attempt is made in [Kup] to recover the subsequent vector fields of the KdV hierarchy

When we subtract from X^{ε}_2 the vector field X^{ε}_1 to get the KdV, we essentially perform the transformation (4 24) on the evolution equation $du/dt_2 = X^{\varepsilon}_2(u)$, because $X^{\varepsilon}_1(u)$ behaves for $\varepsilon \mapsto 0$ as $16\varepsilon u_x$, which is the generator of the one-parameter group of transformations $u(x) \mapsto u(x + 16\varepsilon t_1)$ No interpretation of this sort, in terms of a space-time variable change, is possible for the higher order recombinations such as the vector field $X^{\varepsilon}_3 - 40 X^{\varepsilon}_2 + 480 X^{\varepsilon}_1$

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[7] One should not be astonished that ε can be both a lattice spacing and a speed, since all the space-time and field variables employed in the paper are dimensionless

5. The General Recombination Scheme

The general recombination formulas for the infinitely many vector fields X_s^ε and Hamiltonians f_s^ε will be given in terms of a set of coefficients (c_{sj})

5.1. Definition. $(c_{sj})_{\substack{s=0\,1\,2 \\ j=0\,1}}$ is the family of real coefficients defined recursively in the following way

$$c_{00} := 1 ,\qquad (5 1)$$

$$c_{s+1,j} = \begin{cases} -4^{s+1}\binom{2s+2}{s+1} - \sum_{i=1}^{s} 4^i \binom{2i}{i}(c_{s,i-1} - 16c_{si}) & \text{for } j = 0 \\ c_{s,j-1} - 16c_{sj} & \text{for } j = 1, 2, \quad , s \\ 1 & \text{for } j = s+1 \end{cases} \qquad (5 2)$$

(here $\binom{r}{k}$ denotes as usual the binomial coefficient, for each pair r, k) □

One can arrange the above coefficients into an infinite lower triangular matrix

$$\begin{pmatrix} c_{00} & 0 & 0 & 0 \\ c_{10} & c_{11} & 0 & 0 & \ldots \\ c_{20} & c_{21} & c_{22} & 0 \\ & & & & \end{pmatrix} \qquad (5 3)$$

Knowledge of any row in this matrix allows to compute the subsequent one, according to Eq (5 2), it should be noted that all elements c_{ss} on the main diagonal are equal to 1, and that

$$c_{s0} = -4^s \binom{2s}{s} - \sum_{i=1}^{s-1} 4^i \binom{2i}{i} c_{si} \quad (s = 1, 2, 3, \quad) \qquad (5 4)$$

5.2. Definition. For each ε, Z_s^ε $(s = 1, 2, 3, \quad)$ and h_s^ε $(s = 0, 1, 2, \quad)$ are the vector fields and the Hamiltonians defined by

$$Z_s^\varepsilon := \sum_{j=1}^{s} c_{sj} X_j^\varepsilon , \qquad (5 5)$$

$$h_s^\varepsilon = \sum_{j=0}^{s} c_{sj} f_j^\varepsilon - c_s , \qquad (5 6)$$

where the constants c_s are given by

$$c_s = \left(c_{s0} \log 2 + \sum_{j=1}^{s} \frac{2^{2j-1}}{j} \binom{2j}{j} c_{sj} \right) \int dx . \quad \square \qquad (5 7)$$

For the lowest values of s, Eqs (5 1–7) give the vector fields and the Hamiltonians considered in the previous section

5.3. Proposition. Let Q^ε and $S^\varepsilon = P^\varepsilon - 16Q^\varepsilon$ be the Poisson tensors considered in Sect 4 Then $Q^\varepsilon dh_0^\varepsilon = 0$ and

$$Z_{s+1}^\varepsilon = Q^\varepsilon dh_{s+1}^\varepsilon = S^\varepsilon dh_s^\varepsilon \quad (s = 0, 1, 2, \quad) \qquad (5 8)$$

Proof   We already know that $h_0^\varepsilon = f_0^\varepsilon - c_0$ is a Casimir of Q^ε From Eqs (5 6)
(3 1) and (5 5) it follows that

$$Q^\varepsilon dh_{s+1}^\varepsilon = \sum_{j=0}^{s+1} c_{s+1,j} Q^\varepsilon df_j^\varepsilon = \sum_{j=1}^{s+1} c_{s+1,j} X_j^\varepsilon = Z_{s+1}^\varepsilon \qquad (5\ 9)$$

Also taking into account the definition of S^ε and the recursion rule (5 2), we obtain

$$S^\varepsilon dh_s^\varepsilon = \sum_{j=0}^{s} c_{sj}(P^\varepsilon - 16Q^\varepsilon)df_j^\varepsilon = X_{s+1}^\varepsilon + \sum_{j=1}^{s} (c_{s,j-1} - 16c_{sj})X_j^\varepsilon = Z_{s+1}^\varepsilon \qquad (5\ 10)$$

It should be noted that our argument entails only a part of the recurrence law,
namely the formula for $c_{s+1,j}$ for j running from 1 to $s+1$; for the moment, the
rule for $c_{s+1,0}$ is immaterial, as well as the explicit expression stipulated for the
constants c_s in Eq (5 7). On the contrary, these prescriptions will be essential in
the sequel, to characterize the behaviour of the Hamiltonians h_s^ε at the lowest orders
in ε □

For convenience, we now restate a result essentially known from the previous
section

5.4. Proposition.   *We have*

$$Q_u^\varepsilon = \frac{4}{\varepsilon^3} Q_u^{\text{KdV}} + O\left(\frac{1}{\varepsilon}\right), \qquad S_u^\varepsilon = \frac{16}{\varepsilon} S_u^{\text{KdV}} + O(\varepsilon) \qquad (5\ 11)$$

These are the leading terms of two expansions of the form

$$Q_u^\varepsilon = \sum_{m=0}^{+\infty} \varepsilon^{2m-3} Q_{2m-3,u}, \qquad S_u^\varepsilon = \sum_{m=0}^{+\infty} \varepsilon^{2m-1} S_{2m-1,u} \qquad (5.12)$$

Proof   The method of Sect 4 ensures that such expansions in ε are possible, that
the leading terms are in $1/\varepsilon^3$ for Q_u^ε and $1/\varepsilon$ for S_u^ε and have explicit expressions as
in Eq (5 11) (compare with Eq. (4 20)). It remains to show that, in both cases, only
odd powers of ε appear in the developments. Indeed, it is evident from Eqs (4 6),
(2 2) and (2 3) that $Q_u^\varepsilon = -Q_u^{-\varepsilon}$ and $P_u^\varepsilon = -P_u^{-\varepsilon}$, the tensor S_u^ε is also an odd
function of ε, since it is a linear combination of Q_u^ε and P_u^ε □

From the previous two propositions, we finally infer

5.5. Proposition.   *One has*

$$h_s^\varepsilon(u) = 4^s \varepsilon^{2s+2} h_s^{\text{KdV}}(u) + O(\varepsilon^{2s+4}) \quad (s = 0, 1, 2,\ ), \qquad (5\ 13)$$

$$Z_s^\varepsilon(u) = 4^{s+1} \varepsilon^{2s-1} Z_s^{\text{KdV}}(u) + O(\varepsilon^{2s+1}) \quad (s = 1, 2, 3,\ ) \qquad (5\ 14)$$

Proof   The essential point consists in showing Eq. (5 13), our arguments will be
divided in three steps

Step 1 We start by analyzing the expansions in ε of the Hamiltonians f_s^ε
Let us employ the representation $f_s^\varepsilon(u) = \frac{1}{2s} \text{Tr} L_\varepsilon^{2s}(a)|_{a=2+\varepsilon^2 u}$, holding for $s = 1, 2, 3,\ $, and note that we can write $L^\varepsilon(a) = a\Lambda_\varepsilon + \Lambda_{-\varepsilon}a = a\Lambda_\varepsilon + a_{(-\varepsilon)}\Lambda_{-\varepsilon} =$

$\sum_{j=\pm 1} a_{(\frac{1}{2}j\varepsilon - \frac{1}{2}\varepsilon)} \varLambda_{j\varepsilon}$ This implies

$$L_\varepsilon^{2s}(a) = \sum_{r_1, \ldots, r_{2s}=\pm 1} a_{(\frac{r_1}{2}\varepsilon - \frac{1}{2}\varepsilon)} a_{(r_1\varepsilon + \frac{r_2}{2}\varepsilon - \frac{1}{2}\varepsilon)} \cdots a_{(r_1\varepsilon + r_2\varepsilon + \cdots + r_{2s-1}\varepsilon + \frac{r_{2s}}{2}\varepsilon - \frac{1}{2}\varepsilon)} \varLambda_{(r_1 + \cdots + r_{2s})\varepsilon} ,$$

$$(5\ 15)$$

so, taking the trace, we find

$$f_s^\varepsilon(u)$$

$$= \frac{1}{2s} \int dx \sum_{\substack{r_1, \ldots, r_{2s}=\pm 1 \\ r_1 + \cdots + r_{2s} = 0}} a_{(\frac{r_1}{2}\varepsilon - \frac{1}{2}\varepsilon)} a_{(r_1\varepsilon + \frac{r_2}{2}\varepsilon - \frac{1}{2}\varepsilon)} \cdots a_{(r_1\varepsilon + r_2\varepsilon + \cdots + r_{2s-1}\varepsilon + \frac{r_{2s}}{2}\varepsilon - \frac{1}{2}\varepsilon)} , \quad (5.16)$$

with $a = 2 + \varepsilon^2 u$ We wish to make explicit the lowest order terms in the expansion in ε of (5 16) To this purpose we note that, for each $r_1, \ldots, r_k$, it is

$$a_{(r_1\varepsilon + r_2\varepsilon + \cdots + r_{k-1}\varepsilon + \frac{r_k}{2}\varepsilon - \frac{1}{2}\varepsilon)} = 2 + \varepsilon^2 u + \varepsilon^3 \left(r_1 + r_2 + \cdots + r_{k-1} + \frac{r_k}{2} - \frac{1}{2} \right) u_x + O(\varepsilon^4)$$

$$(5\ 17)$$

So, for each $r_1, \ldots, r_{2s}$, we have

$$a_{(\frac{r_1}{2}\varepsilon - \frac{1}{2}\varepsilon)} a_{(r_1\varepsilon + \frac{r_2}{2}\varepsilon - \frac{1}{2}\varepsilon)} \cdots a_{(r_1\varepsilon + r_2\varepsilon + \cdots + r_{2s-1}\varepsilon + \frac{r_{2s}}{2}\varepsilon - \frac{1}{2}\varepsilon)}$$

$$= 2^{2s} + 2^{2s} s\varepsilon^2 u + 2^{2s-1} \varepsilon^3 \left(\sum_{k=1}^{2s} \left(2s - k + \frac{1}{2} \right) r_k - s \right) u_x + O(\varepsilon^4) \quad (5\ 18)$$

If we integrate over x, the term in ε^3 in the above formula disappears, because $\int dx\, u_x = 0$, if we also sum over $r_1, \ldots, r_{2s}$, we conclude that

$$f_s^\varepsilon(u) = 2^{2s-1} \#(s) \left(\frac{1}{s} \int dx + \varepsilon^2 \int dx\, u \right) + O(\varepsilon^4) , \quad (5\ 19)$$

where $\#(s)$ is the number of finite sequences $(r_1, \ldots, r_{2s})$ with values in ± 1 and zero sum Elementary combinatorics gives $\#(s) = \binom{2s}{s}$

We note that developing $f_s^\varepsilon(u)$ up to $O(\varepsilon^4)$, we find no terms of order ε and ε^3 In general, no odd powers of ε appear in the development· this follows from the identity $f_s^{-\varepsilon}(u) = f_s^\varepsilon(u)$, which can be derived directly, without expanding, from the representation (5 16) So, we can state that

$$f_s^\varepsilon(u) = \sum_{l=0}^{+\infty} \varepsilon^{2l} f_{s,2l}(u) , \quad (5\ 20)$$

in particular, the first two terms are as we found above, i e ,

$$f_{s0}(u) = \frac{2^{2s-1}}{s} \binom{2s}{s} \int dx, \qquad f_{s2}(u) = 2^{2s-1} \binom{2s}{s} \int dx\, u \quad (5\ 21)$$

The above equations hold for each $s \geq 1$ For $s = 0$, the corresponding Hamiltonian f_0^ε can also be expanded as in Eq (5 20), the first two terms being

$$f_{00}(u) = \log 2 \int dx, \qquad f_{02}(u) = \frac{1}{2} \int dx\, u \quad (5\ 22)$$

For our purposes, it is not necessary to make explicit the terms of order ≥ 4 in the expansions of the Hamiltonians, it suffices to remark that, for $2l \geq 4$, each functional $f_{s,2l}(u)$ is (the integral of) a polynomial in u and its derivatives, containing no costant term and no linear term in u

Step 2 Let us come to the Hamiltonians h_s^ε ($s = 0, 1, 2, .$) Each of them has an expansion

$$h_s^\varepsilon(u) = \sum_{l=0}^{+\infty} \varepsilon^{2l} h_{s,2l}(u) \tag{5 23}$$

Our previous considerations enable us to state that

$$h_{00}(u) = 0, \qquad h_{s0}(u) = h_{s2}(u) = 0 \quad (s = 1, 2, 3, .) \tag{5 24}$$

This follows immediately from the definition (5 6) of h_s^ε as a linear combination of the Hamiltonians f_j^ε, from Eqs (5 21), (5 22) and from Eqs (5 4), (5 7) Now, we state that it is also

$$h_{s4}(u) = h_{s6}(u) = \quad = h_{s,2s}(u) = 0 \tag{5 25}$$

for each $s \geq 2$ If $s = 2$, this is already known (see Eq (4 17)) The general proof can be obtained by induction so, we assume that Eq (5 25) holds for a given s and show that it also holds for $s + 1$, on the grounds of the biHamiltonian recursion relations In fact, we have $Q_u^\varepsilon d_u h_{s+1}^\varepsilon = S_u^\varepsilon d_u h_s^\varepsilon = O(1/\varepsilon)O(\varepsilon^{s+2})$, i e,

$$Q_u^\varepsilon d_u h_{s+1}^\varepsilon = O(\varepsilon^{s+1}) \tag{5 26}$$

The expansion of $Q_u^\varepsilon d_u h_{s+1}^\varepsilon$ can be computed in terms of the developments (5 12) for Q_u^ε and (5 23–24) for h_{s+1}^ε If we set to zero the terms in ε^k for $k \leq s$, so as to satisfy Eq (5 26), we obtain a triangular system of $s - 1$ equations

$$Q_{-3,u} d_u h_{s+1 4} = 0, \qquad Q_{-3,u} d_u h_{s+1,6} + Q_{-1,u} d_u h_{s+1 4} = 0,$$

$$Q_{-3,u} d_u h_{s+1,2s+2} + Q_{-1,u} d_u h_{s+1,2s} + \quad + Q_{2s-5,u} d_u h_{s4} = 0 \tag{5 27}$$

Let us analyze this system, recalling that $Q_{-3,u} = 4\partial_x$ and that the functionals $h_{s+1 2l}$ are integrals of polynomial densities in u and its derivatives, containing neither constants nor linear terms in u Due to these features, the first equation $(d_u h_{s+1 4})_x = 0$ in the system (5 27) implies $h_{s+1 4} = 0$, substituting in the second equation we find that $(d_u h_{s+1 6})_x = 0$, and so $h_{s+1 6} = 0$, iterating this argument, we infer $d_u h_{s+1,2l} = 0$ for each $l \leq s + 1$

In conclusion, we have proved Eq (5 25), which implies, together with Eq (5 24),

$$h_s^\varepsilon(u) = \varepsilon^{2s+2} h_{s 2s+2}(u) + O(\varepsilon^{2s+4}) \quad (s = 0, 1, 2,) . \tag{5 28}$$

Step 3 We substitute Eq (5 28) in the biHamiltonian recursion formula (5 8), using again Eqs (5 11–12), we obtain that the sequence of functionals $h_{s,2s+2}$ satisfies the recursion scheme

$$Q_u^{\mathrm{KdV}} d_u h_{s+1,2s+4} = 4 S_u^{\mathrm{KdV}} d_u h_{s 2s+2} \quad (s = 0, 1, 2, .) , \tag{5 29}$$

which is very similar to Eq (2 23) The basic features of the functionals $h_{s,2s+2}$ with $s \geq 1$ (differential polynomials in u, with no constants and no linear terms in u) have

been already emphasized, on the other hand, the recursion scheme (2 23) admits no more than one solution with such features, for each choice of the starting point. The starting point of the sequence $h_{s,2s+2}$ is $h_{02}(u) = \frac{1}{2} \int dx\, u = h_0^{\mathrm{KdV}}(u)$, from here, and from Eq (2 23), one immediately gets

$$h_{s,2s+2}(u) = 4^s h_s^{\mathrm{KdV}}(u) \tag{5 30}$$

So, Eq (5 13) is proved.

The remaining Eq (5 14) follows straightforwardly, expanding to the lowest order in ε the identity $Z_s^\varepsilon = Q_s^\varepsilon dh_s^\varepsilon$ □

The connection between the KM system and KdV theory can be rephrased in terms of the Lax formalism By definition, the Z_s^ε vector fields are linear combinations of the X_s^ε's, which admit the Lax formulations (3 4), so, if $d/d0_s$ denotes the derivative along Z_s^ε, we have

$$\frac{dL^\varepsilon}{d0_s} = [B_s^\varepsilon, L^\varepsilon], \qquad B_s^\varepsilon(u) = \sum_{j=1}^s c_{sj} A_j^\varepsilon(u) \tag{5 31}$$

We already know the lower terms in development of $L^\varepsilon(u)$, we complete our analysis considering the behaviour of $B_s^\varepsilon(u)$ and summarize the results in the following

5.6. Proposition.   *One has*

$$L^\varepsilon(u) - 4 = 2\varepsilon^2 L^{\mathrm{KdV}}(u) + O(\varepsilon^3) , \tag{5 32}$$

$$B_s^\varepsilon(u) = 4^{s+1}\varepsilon^{2s-1} B_s^{\mathrm{KdV}}(u) + O(\varepsilon^{2s}) \quad (s = 1, 2, 3, \quad) \tag{5 33}$$

Proof   The result concerning the Lax operator is already known (see Eq. (4 8)) To derive Eq (5 33), we proceed in three steps, the integer s is fixed throughout the proof

Step 1 For subsequent use, it is useful to emphasize a feature of the operator $B_s^\varepsilon(u)$ at the point $u = 0$. On account of Eq (4 7), we have $L^\varepsilon(0) = 2\Delta_\varepsilon + 2\Delta_{-\varepsilon}$, so, from the expression (3 4) of A_j^ε and Eq (5 31) it follows that $B_s^\varepsilon(0)$ is a linear combination with constant coefficients of differences of the form $\Delta_{rc} - \Delta_{-rc}$ with r an integer between 0 and $2s$ Expanding in ε, we find only odd powers of ε and $\hat{c}_s$

$$B_s^\varepsilon(0) = \sum_{m=0}^{+\infty} b_{s,2m+1} \varepsilon^{2m+1} \hat{c}_s^{2m+1} , \tag{5 34}$$

where $b_{s1}, b_{s3},$ are real constants whose explicit expression is not relevant for our purposes At a generic point u, we have the expansion

$$B_s^\varepsilon(u) = \sum_{p=0}^{+\infty} \varepsilon^p B_{sp}(u) , \tag{5 35}$$

for each p, $B_{sp}(u)$ is a differential operator in $\hat{c}_s$, whose coefficients are polynomials in u and its derivatives In particular, at the point $u = 0$ we find, on account of Eq (5 34),

$$B_{sp}(0) = 0 \quad \text{for } p \text{ even}, \qquad B_{sp}(0) = b_{sp} \hat{c}_s^p \quad \text{for } p \text{ odd} \tag{5 36}$$

Step 2 The $\varepsilon \mapsto 0$ limits of the Z_s^ε vector fields are the KdV vector fields Z_s^{KdV}, according to Eq (5 14), so, by comparing the Lax formulations (5 31) and (2.25) we infer

$$[B_s^\varepsilon(u), L^\varepsilon(u)] = 2^{2s+3}\varepsilon^{2s+1}[B_s^{\text{KdV}}(u), L^{\text{KdV}}(u)] + O(\varepsilon^{2s+2}) \qquad (5\ 37)$$

Our aim is to obtain from this equation the maximum amount of information about the operator B_s^ε To this purpose, we use the expansion (5 35), and write $L^\varepsilon(u) = 4 + 2\varepsilon^2 L^{\text{KdV}}(u) + \sum_{q=3}^{+\infty} \varepsilon^q L_q(u)$, for each q, $L_q(u)$ is a differential operator Substituting these developments in Eq (5.37), and considering the terms up to ε^{2s}, we obtain a triangular system of $2s - 1$ equations

$$[B_{s0}(u), L^{\text{KdV}}(u)] = 0, \qquad 2[B_{s1}(u), L^{\text{KdV}}(u)] + [B_{s0}(u), L_3(u)] = 0, \qquad ,$$

$$2[B_{s,2s-2}(u), L^{\text{KdV}}(u)] + \sum_{q=3}^{2s}[B_{s,2s-q}(u), L_q(u)] = 0 \qquad (5\ 38)$$

Furthermore, extracting from (5.37) the terms in ε^{2s+1}, we find

$$2[B_{s,2s-1}(u), L^{\text{KdV}}(u)] + \sum_{q=3}^{2s+1}[B_{s,2s+1-q}(u), L_q(u)] = 2^{2s+3}[B_s^{\text{KdV}}(u), L^{\text{KdV}}(u)] \qquad (5\ 39)$$

Step 3 By discussing the triangular system (5 38), we will show that $B_{sp}(u) = 0$ for $p = 0, 1, 2, \quad , 2s - 2$, substituting this result in Eq (5 39), we will be able to determine the operator $B_{s,2s-1}(u)$, and this will conclude the proof

Let us start from the first equation in the system (5 38), which asserts that the differential operator $B_{s0}(u)$ is in the centralizer of $L^{\text{KdV}}(u)$ The well known characterization of the centralizer (see for example [DS]) implies in this case that $B_{s0}(u)$ is a linear combination of nonnegative integer powers of $L^{\text{KdV}}(u)$

$$B_{s0}(u) = \sum_j b_{s0j}(L^{\text{KdV}})^j(u), \qquad (5\ 40)$$

where the b_{s0j} are constant coefficients, independent of u On the other hand, at the point $u = 0$ the l h s of Eq. (5 40) vanishes on account of Eq (5 36), while $(L^{\text{KdV}})^j(0) = \partial_s^{2j}$, so, we conclude that $b_{s0j} = 0$ for each j, i e , $B_{s0}(u) = 0$ at each point u

Inserting this result in the second equation of the system (5 38), we obtain that also $B_{s1}(u)$ belongs to the centralizer of $L^{\text{KdV}}(u)$, so,

$$B_{s1}(u) = \sum_j b_{s1j}(L^{\text{KdV}})^j(u), \qquad (5\ 41)$$

where j ranges over a set of nonnegative integers and the b_{s1j} are real constants Again we fix our attention on the point $u = 0$, here the l h s of Eq (5 41) is $b_{s1}\partial_s$ due to Eq (5 36), while the r h s is $\sum_j b_{s1j}\partial_s^{2j}$ By comparison, we conclude that the constants b_{s1} and b_{s1j} all vanish, i e , that $B_{s1}(u) = 0$

The above arguments can be iterated on the remaining equations of the system (5 38), one finally obtains that, at each point u,

$$B_{sp}(u) = 0 \quad (p = 0, 1, \quad , 2s - 2) \qquad (5\ 42)$$

Keeping in mind this result, we turn our attention to Eq (5.39), from which we infer that $B_{s,2s-1}(u) - 4^{s+1}B_s^{\mathrm{KdV}}(u)$ is in the centralizer of $L^{\mathrm{KdV}}(u)$. So,

$$B_{s,2s-1}(u) = 4^{s+1}B_s^{\mathrm{KdV}}(u) + \sum_j b_{s,2s-1,j}(L^{\mathrm{KdV}})^j(u) \qquad (5\ 43)$$

To determine the constants $b_{s,2s-1,j}$, we use again the trick of specializing Eq (5 43) to the point $u = 0$, here the l h s is $b_{s,2s-1}\hat{\partial}_x^{2s-1}$ due to Eq (5 36), while the r h.s is $4^{2s}\hat{\partial}_x^{2s-1} + \sum_j b_{s,2s-1,j}\hat{\partial}_x^{2j}$ By comparison, we infer $b_{s,2s-1} = 4^{2s}$ and $b_{s,2s-1,j} = 0$ for each j In conclusion, at any point u we have

$$B_{s,2s-1}(u) = 4^{s+1}B_s^{\mathrm{KdV}}(u) ; \qquad (5\ 44)$$

the thesis (5 33) follows from this and from Eq (5 42)   □

   Propositions 5 4, 5 5 and 5 6 contain the main results of the paper, of course, their statements can be written as in Eqs (1.6), (1.5) and (1 8) of the Introduction.
   Let us note that Prop. 5 6 on the Lax pairs rests on the previous result in Prop 5 5, which has been proved using systematically the biHamiltonian recursion relations, so, in our approach, the continuous limit of the biHamiltonian scheme also gives the basic information about the limit of the Lax formalism

6. Final Remarks

In this paper, we have presented a zero-spacing limit procedure for the KM system, drawing attention to its algebraic/geometric structures (hierarchical organization of the conservation laws, biHamiltonian structure and Lax formalism) The convergence of the limit procedure for the *solutions* of the associated evolution equations deserve a separate discussion, it is outside our present purposes to give a systematic treatment of this topic, so we will limit ourselves to an example and some remarks
   A first exercise for going inside this matter is the study of the limit process for the soliton solutions; here, we will analyze the one-soliton case for the first three vector fields of the hierarchy It will turn out that the recombination principle employed throughout this paper plays a basic role also in the present discussion the coefficients c_{sj} of Sects 4–5 also control the recombination of the frequencies in the soliton solutions, and the prescriptions given for such coefficients allow to obtain the KdV one-soliton solutions through the limit process
   Let us consider the discrete KM system, in the infinite case (the index i labelling the sites runs over $\mathbb{Z}$) Let X_1 be the vector field given by Eq (2.7), it is known that the evolution equation $da/dt_1 = X_1(a)$ admits the one-soliton solution

$$a_i(t_1) = 2\sqrt{\frac{\cosh(3K) + \cosh 2[K(i - 1/2) + v_1(K)t_1 + A]}{\cosh(K) + \cosh 2[K(i - 1/2) + v_1(K)t_1 + A]}} , \qquad (6\ 1)$$

depending on the arbitrary real parameters K and A, with $v_1(K) = 8\sinh(2K)$ Equation (6 1) is equivalent to an equation appearing in [Man], clearly, the ratio $v_1(K)/K$ gives the speed of the soliton center, whose position at time $t_1 = 0$ is controlled by A Similarly, the vector fields X_2 and X_3 in Eq (2 7) yield to one-soliton solutions of the same form as in Eq (6 1), with $v_1(K)$ replaced, respectively, by

$v_2(K) = 32\sinh(4K) + 128\sinh(2K)$ and $v_3(K) = 128\sinh(6K) + 768\sinh(4K) + 1920\sinh(2K)$

We pass to the interpolated KM system (in the sense of Sect 3), and consider the vector fields X_s^ε ($s = 1, 2, 3$) For each choice of the lattice spacing ε, we can construct one-soliton solutions for these vector fields, replacing the index i in Eq. (6.1) with the ratio x/ε, the parameters K and A in the above equation can be chosen arbitrarily for any given ε If we make the choices $K = \varepsilon k$ (with k a real constant) and keep A independent of ε, we are led to the solutions

$$a^\varepsilon(x, t_s) = 2\sqrt{\frac{\cosh(3\varepsilon k) + \cosh 2[k(x - \varepsilon/2) + v_s^\varepsilon(k)t_s + A]}{\cosh(\varepsilon k) + \cosh 2[k(x - \varepsilon/2) + v_s^\varepsilon(k)t_s + A]}} \qquad (6\,2)$$

Here, k and A are arbitrary, while

$$v_1^\varepsilon(k) = 8\sinh(2\varepsilon k), \qquad v_2^\varepsilon(k) = 32\sinh(4\varepsilon k) + 128\sinh(2\varepsilon k),$$

$$v_3^\varepsilon(k) = 128\sinh(6\varepsilon k) + 768\sinh(4\varepsilon k) + 1920\sinh(2\varepsilon k) \qquad (6\,3)$$

In the previous two sections we have shown how to construct from the vector fields X_s^ε the recombinations Z_s^ε, in such a way that the ratio $Z_s^\varepsilon/\varepsilon^{2s-1}$ gives, in the $\varepsilon \mapsto 0$ limit, the vector field $4^{s+1}Z_s^{\mathrm{KdV}}$, in particular, the first three recombinations are as in Eq (4 13) Let us use again the field variable $a(x)$ rather than the rescaled field $u(x)$ of Eq (4 1), which will appear at the end of this computation, for $s = 1, 2, 3$, we consider the evolution equations $da/d\tau_s = Z_s^\varepsilon(a)/\varepsilon^{2s-1}$, which admit solutions of the same form as in Eq (6.2), with v_s^ε replaced by μ_s^ε, where

$$\mu_1^\varepsilon(k) = \frac{1}{\varepsilon}v_1^\varepsilon(k), \qquad \mu_2^\varepsilon(k) = \frac{1}{\varepsilon^3}(v_2^\varepsilon(k) - 24v_1^\varepsilon(k)),$$

$$\mu_3^\varepsilon(k) = \frac{1}{\varepsilon^5}(v_3^\varepsilon(k) - 40v_2^\varepsilon(k) + 480v_1^\varepsilon(k)) \qquad (6\,4)$$

Clearly, the preservation of the form (6 2) for the solutions, modulo a recombination of v_1^ε, v_2^ε and v_3^ε with the same coefficients as for the vector fields, is a consequence of the commutativity of the flows under examination.

The final step in our considerations is the $\varepsilon \mapsto 0$ limit of the solutions

$$a^\varepsilon(x, \tau_s) = 2\sqrt{\frac{\cosh(3\varepsilon k) + \cosh 2[k(x - \varepsilon/2) + \mu_s^\varepsilon(k)\tau_s + A]}{\cosh(\varepsilon k) + \cosh 2[k(x - \varepsilon/2) + \mu_s^\varepsilon(k)\tau_s + A]}} \qquad (6\,5)$$

It is easily checked that

$$\mu_1^\varepsilon(k) = 16k + \frac{32}{3}\varepsilon^2 k^3 + O(\varepsilon^3), \qquad \mu_2^\varepsilon(k) = 256k^3 + 256\varepsilon^2 k^5 + O(\varepsilon^3),$$

$$\mu_3^\varepsilon(k) = 4096k^5 + \frac{16384}{3}\varepsilon^2 k^7 + O(\varepsilon^3) \qquad (6\,6)$$

Inserting these expansions in Eq. (6 5) and developing again up to $O(\varepsilon^3)$, we finally obtain (for $s = 1, 2, 3$)

$$a^\varepsilon(x, \tau_s) = 2 + \varepsilon^2 \frac{2k^2}{\cosh^2(kx + 16^s k^{2s-1}\tau_s + A)} + O(\varepsilon^3) \qquad (6\,7)$$

Passing to the rescaled field variable u, as in Eq. (4.1), we see that the function $u^\varepsilon(x, \tau_s) = (1/\varepsilon^2)(a^\iota(x, \tau_s) - 2)$ converges, for $\varepsilon \mapsto 0$, to the familiar one-soliton solution for the vector field $4^{s+1}Z_s^{KdV}$. A similar analysis of the one-soliton solutions and their zero-spacing limit could be performed for the higher order vector fields, with $s > 3$.

After this example, we return to the general problem outlined in this section. Setting up general convergence results for the solutions (with arbitrary initial data) is a problem that should be attacked with appropriate tools. One should prove (for all values of s) that any solution $u(x, \tau_s)$ of the evolution equation $du/d\tau_s = Z_s^{KdV}(u)$ is the $\varepsilon \mapsto 0$ limit of a solution $u^\varepsilon(x, t_s)$, corresponding to the vector field $Z_s^\iota/\varepsilon^{2s-1}$.

In connection with this topic, we mention the rigorous results obtained in [Sch] on the continuous limit of the KM vector field X_1. In this paper, Schwarz discusses the KdV equation (1 1) as a limit of X_1, using a space-time coordinate change of the form (4 24) (incidentally, he also reports a computation of Kac on the limit of the one-soliton solutions for X_1, based on the above transformation). The beautiful convergence results of Schwarz are stated in the framework of Sobolev spaces, and concern the solutions of the Cauchy problem with arbitrary initial data.

Apart from the specification of the analytical setting, there are some technical differences between our approach and the one of Schwarz. Concerning the derivations of Eq (1 1), we repeat the comment following Eq (4 24) from our viewpoint, the KdV evolution equation (1 1) does not come from X_1 through a space-time transformation, but from a recombination of X_2 and X_1. Another difference is that Schwarz does not explicitly introduce an analogue of our "interpolating KM system," with a projection Π^ι onto the discrete system as explained in the Remark at the end of Sect 3, rather, he gives a constructive prescription to interpolate a sequence (a_i) with a function $a(x)$ (which amounts, in geometrical terms, to selecting a particular right inverse of our projection Π^ι).

In spite of the above facts, we think that Schwarz' method could be implemented into our scheme, modulo the necessary embedding of our setting into the rigorous framework of the Sobolev chains. In fact, the essential point in the convergence theorems of [Sch] is the possibility to obtain a priori bounds (*uniformly in* ε) on some conveniently defined differential-difference analogues of the Sobolev norms of $u^\varepsilon(\cdot, \tau)$. These bounds on the solutions are derived using the functionals h_s, which are constants of motion for the vector field X_1 under consideration. On the other hand, the h_s's or their recombinations are conserved quantities for the full KM hierarchy, so the technique proposed for X_1 can probably be extended to treat the higher order vector fields.

Schwarz employs the estimates on X_1 to infer a priori estimates for the solutions of the KdV equation, in the ordinary Sobolev norms, such a priori bounds mean that the solutions of the KdV equation do not develop shocks. As a final remark, we note that there are cases, known from the literature, in which the limiting system of an integrable lattice develops shocks, depending on the procedure that defines the limit. In particular, Bloch and Brockett [BB] have illustrated a limit process for the Toda lattice, yelding to a system of hydrodynamic type in which a shock phenomenon occurs.

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Communicated by M Jimbo