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Quantum $\mathcal{W}_N$ Algebras and Macdonald Polynomials

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Abstract: We derive a quantum deformation of the $\mathcal{W}_N$ algebra and its quantum Miura transformation, whose singular vectors realize the Macdonald polynomials.

1. Introduction

The excited states of the Calogero–Sutherland model [14] and its relativistic model (the trigonometric limit of the Ruijsenaars model) [11] are described by the Jack polynomials [13] and their q-analog (the Macdonald polynomials) [6], respectively. Since the Jack polynomials coincide with certain correlation functions of the $\mathcal{W}_N$ algebra [8, 1], it is natural to expect that the Macdonald polynomials are also realized by those of a deformation of $\mathcal{W}_N$ algebra.

In a previous paper [12], we derived a quantum Virasoro algebra whose singular vectors are some special kinds of Macdonald polynomials. On the other hand, E. Frenkel and N. Reshetikhin succeeded in constructing the Poisson $\mathcal{W}_N$ algebra and its quantum Miura transformation in the analysis of the $U_q(\widehat{sl_N})$ algebra at the critical level [4]. Like the classical case [3], these two works, q-Virasoro and q-Miura transformation, are essential to find and study a quantum $\mathcal{W}_N$ algebra. In this article, we present a q-$\mathcal{W}_N$ algebra[1] whose singular vectors realize the general Macdonald polynomials.

This paper is arranged as follows: In Sect. 2, we define a quantum deformation of $\mathcal{W}_N$ algebras and its quantum Miura transformation. The screening currents and a vertex operator are derived in Sects. 3 and 4. A relation with the Macdonald polynomials is obtained in Sect. 5. Section 6 is devoted to conclusion and discussion. Finally we recapitulate the q-Virasoro algebra and the integral formula for the Macdonald polynomials in the appendices.

★ JSPS fellow.

[1] After finishing of this work, we received the preprint "*Quantum $\mathcal{W}$-algebras and elliptic algebras*" by B Feigin and E Frenkel (q-alg/9508009) They discuss similar things as Sects 2 1, 2 3, 3 1 and Eq (8) of ours Although the algebra of screening currents is considered there, the normal ordering of q-$\mathcal{W}$ generators and the relation with the Macdonald polynomial are not given

2. Quantum Deformation of $\mathcal{W}_N$ Algebra

We start with defining a new quantum deformation of the $\mathcal{W}_N$ algebra by a quantum Miura transformation.

2.1. Quantum Miura Transformation. First we define fundamental bosons h_n^i and Q_h^i for $i = 1,2,\ldots,N$ and $n \in \mathbf{Z}$ such that[2]

$$
\begin{aligned}
[h_n^i, h_m^j] &= -\frac{1}{n}(1-q^n)(1-t^{-n})\frac{1-p^{(\delta_{ij}N-1)n}}{1-p^{Nn}}p^{Nn\theta(i<j)}\delta_{n+m,0}\,, \\
[h_0^i, Q_h^j] &= \delta_{ij} - \frac{1}{N}, \qquad \sum_{i=1}^{N} p^{in}h_n^i = 0, \qquad \sum_{i=1}^{N} Q_h^i = 0\,,
\end{aligned}
\tag{1}
$$

with $q,\ t \equiv q^\beta \in \mathbf{C}$ and $p \equiv q/t$. Here $\theta(P) \equiv 1$ or 0 if the proposition P is true or false, respectively. These bosons correspond to the weights of the vector representation h_i whose inner-product is $(h_i \cdot h_j) = (\delta_{ij}N - 1)/N$.

Let us define fundamental vertices $\Lambda_i(z)$ and q-$\mathcal{W}_N$ generators $W^i(z)$ for $i = 1,2,\ldots,N$ as follows:

$$
\begin{aligned}
\Lambda_i(z) &\equiv\ : \exp\left\{\sum_{n\neq 0} h_n^i z^{-n}\right\} : q^{\sqrt{\beta}h_0^i} p^{\frac{N+1}{2}-i}\,, \\
W^i(zp^{\frac{1-i}{2}}) &\equiv \sum_{1\leqq j_1 < \cdots < j_i \leqq N} :\Lambda_{j_1}(z)\Lambda_{j_2}(zp^{-1})\cdots\Lambda_{j_i}(zp^{1-i}):\,,
\end{aligned}
\tag{2}
$$

and $W^0(z) \equiv 1$. Here $:*:$ stands for the usual bosonic normal ordering such that the bosons h_n^i with non-negative mode $n \geqq 0$ are in the right. Note that

$$
W^N(zp^{\frac{1-N}{2}}) = \ :\Lambda_1(z)\Lambda_2(zp^{-1})\cdots\Lambda_N(zp^{1-N}): \ = 1\,. \tag{3}
$$

If we take the limit $t \to 1$ with q fixed, the above generators reduce to those of Ref. [4]. These generators are obtained by the following quantum Miura transformation:

$$
\begin{aligned}
&:(p^{D_z} - \Lambda_1(z))(p^{D_z} - \Lambda_2(zp^{-1}))\cdots(p^{D_z} - \Lambda_N(zp^{1-N})): \\
&= \sum_{i=0}^{N}(-1)^i W^i(zp^{\frac{1-i}{2}})p^{(N-i)D_z}\,,
\end{aligned}
\tag{4}
$$

with $D_z \equiv z\frac{\partial}{\partial z}$. We remark that p^{D_z} is the p-shift operator such that $p^{D_z}f(z) = f(pz)$.

2.2. Relations of q-$\mathcal{W}_N$ Generators. Next we give the algebra of the above q-$\mathcal{W}_N$ generators. Let $W^i(z) = \sum_{n\in\mathbf{Z}} W_n^i z^{-n}$. Let us define a new normal ordering $\circ\!\!\circ * \circ\!\!\circ$

[2] We found this commutation relation by comparing the Poisson bracket in Frenkel–Reshetikhin's work [4] and the commutator in ours [12]. The oscillator a_n used in [12] is given by $a_n = -nh_n^1 p^{-n/2}/(1-t^n)$ and $a_{-n} = nh_{-n}^1 p^{n/2}(1+p^n)/(1-t^{-n})$ for $n > 0$.

for the q-$\mathcal{W}_N$ generators as follows:

$$
\begin{aligned}
&{}^\circ_\circ W^i(rw)W^j(w){}^\circ_\circ \\
&\equiv \oint \frac{dz}{2\pi iz}\left\{\frac{1}{1-rw/z}f^{ij}\left(\frac{w}{z}\right)W^i(z)W^j(w)+\frac{z/rw}{1-z/rw}W^j(w)W^i(z)f^{ji}\left(\frac{z}{w}\right)\right\} \\
&=\sum_{n\in\mathbf{Z}}\sum_{m\geqq 0}\sum_{l=0}^{m} f_l^{ij}\{r^{m-l}\cdot W^i_{-m}W^j_{n+m}+r^{l-m-1}\cdot W^j_{n-m-1}W^i_{m+1}\}w^{-n}\,,
\end{aligned}
\tag{5}
$$

with

$$
\begin{aligned}
&f^{ij}(x)\equiv\exp\left\{\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})\frac{1-p^{in}}{1-p^n}\frac{1-p^{(N-j)n}}{1-p^{Nn}}p^{\frac{j-i}{2}n}x^n\right\}\,, \\
&f^{ji}(x)\equiv f^{ij}(x),\quad (i\leqq j)\,,
\end{aligned}
\tag{6}
$$

and $f^{ij}(x)\equiv\sum_{l\geqq 0}f_l^{ij}x^l$. Here $(1-x)^{-1}$ stands for $\sum_{n\geqq 0}x^n$. We remark that this normal ordering ${}^\circ_\circ * {}^\circ_\circ$ is a generalization of the following usual one $(*)$ used in conformal field theory:

$$
\begin{aligned}
(AB)(w)&\equiv\oint_w\frac{dz}{2\pi i}\frac{1}{z-w}A(z)B(w) \\
&\equiv\oint_0\frac{dz}{2\pi iz}\left\{\frac{1}{1-w/z}A(z)B(w)+\frac{z/w}{1-z/w}B(w)A(z)\right\}\,.
\end{aligned}
\tag{7}
$$

The relation of the q-$\mathcal{W}_N$ generators should be written in this normal ordering. Here we present some examples of them. The relation of $W^1(z)$ and $W^j(z)$ for $j\geqq 1$ is

$$
\begin{aligned}
&f^{1j}\left(\frac{w}{z}\right)W^1(z)W^j(w)-W^j(w)W^1(z)f^{j1}\left(\frac{z}{w}\right) \\
&=-\frac{(1-q)(1-t^{-1})}{1-p}\left\{\delta\left(p^{\frac{j+1}{2}}\frac{w}{z}\right)W^{j+1}(p^{\frac12}w)-\delta\left(p^{-\frac{j+1}{2}}\frac{w}{z}\right)W^{j+1}(p^{-\frac12}w)\right\}\,,
\end{aligned}
\tag{8}
$$

with $\delta(x)\equiv\sum_{n\in\mathbf{Z}}x^n$; and that of $W^2(z)$ and $W^j(z)$ for $j\geqq 2$ is

$$
\begin{aligned}
&f^{2j}\left(\frac{w}{z}\right)W^2(z)W^j(w)-W^j(w)W^2(z)f^{j2}\left(\frac{z}{w}\right) \\
&=-\frac{(1-q)(1-t^{-1})}{1-p}\frac{(1-qp)(1-t^{-1}p)}{(1-p)(1-p^2)} \\
&\quad\times\left\{\delta\left(p^{\frac{j}{2}+1}\frac{w}{z}\right)W^{j+2}(pw)-\delta\left(p^{-\frac{j}{2}-1}\frac{w}{z}\right)W^{j+2}(p^{-1}w)\right\} \\
&\quad-\frac{(1-q)(1-t^{-1})}{1-p}\left\{\delta\left(p^{\frac{j}{2}}\frac{w}{z}\right){}^\circ_\circ W^1(p^{-\frac12}z)W^{j+1}(p^{\frac12}w){}^\circ_\circ\right. \\
&\quad\left.-\delta\left(p^{-\frac{j}{2}}\frac{w}{z}\right){}^\circ_\circ W^1(p^{\frac12}z)W^{j+1}(p^{-\frac12}w){}^\circ_\circ\right\}
\end{aligned}
$$

$$+\frac{(1-q)^2(1-t^{-1})^2}{(1-p)^2}\left\{\delta\left(p^{\frac{j}{2}}\frac{w}{z}\right)\left(\frac{p^2}{1-p^2}W^{j+2}(pw)+\frac{1}{1-p^j}W^{j+2}(w)\right)\right.$$

$$\left.-\delta\left(p^{-\frac{j}{2}}\frac{w}{z}\right)\left(\frac{p^j}{1-p^j}W^{j+2}(w)+\frac{1}{1-p^2}W^{j+2}(p^{-1}w)\right)\right\}, \tag{9}$$

with $W^i(z)\equiv 0$ for $i>N$. The main terms of

$$f^{ij}\left(\frac{w}{z}\right)W^i(z)W^j(w)-W^j(w)W^i(z)f^{ji}\left(\frac{z}{w}\right)\quad (i\leqq j)$$

is

$$-\frac{(1-q)(1-t^{-1})}{1-p}\sum_{k=1}^{\min(i,N-j)}\prod_{l=1}^{k-1}\frac{(1-qp^l)(1-t^{-1}p^l)}{(1-p^l)(1-p^{l+1})}$$

$$\times\left\{\delta\left(p^{\frac{j-i}{2}+k}\frac{w}{z}\right)\,{}^{\circ}_{\circ}W^{i-k}(p^{-\frac{k}{2}}z)W^{j+k}(p^{\frac{k}{2}}w){}^{\circ}_{\circ}\right.$$

$$\left.-\delta\left(p^{\frac{i-j}{2}-k}\frac{w}{z}\right)\,{}^{\circ}_{\circ}W^{i-k}(p^{\frac{k}{2}}z)W^{j+k}(p^{-\frac{k}{2}}w){}^{\circ}_{\circ}\right\}.$$

To obtain the above relations, the fundamental formula is

$$f^{11}\left(\frac{w}{z}\right)\Lambda_i(z)\Lambda_i(w)-\Lambda_i(w)\Lambda_i(z)f^{11}\left(\frac{z}{w}\right)=0\,,$$

$$f^{11}\left(\frac{w}{z}\right)\Lambda_i(z)\Lambda_j(w)-\Lambda_j(w)\Lambda_i(z)f^{11}\left(\frac{z}{w}\right)$$

$$=\frac{(1-q)(1-t^{-1})}{1-p}\left(\delta\left(\frac{w}{z}\right)-\delta\left(p\frac{w}{z}\right)\right):\Lambda_i(z)\Lambda_j(w):\,,$$

for $i<j$; here we use[3]

$$\exp\left\{\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})x^n\right\}-\exp\left\{\sum_{n>0}\frac{1}{n}(1-q^{-n})(1-t^n)x^{-n}\right\}$$

$$=\frac{(1-q)(1-t^{-1})}{1-p}(\delta(x)-\delta(px))\,. \tag{10}$$

To calculate the general relations, the following formulae are useful:

$$\exp\left\{\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})(1+r^n)x^n\right\}$$

$$-\exp\left\{\sum_{n>0}\frac{1}{n}(1-q^{-n})(1-t^n)(1+r^{-n})x^{-n}\right\}$$

$$=\frac{(1-q)(1-t^{-1})}{(1-p)(1-r)}\left\{(1-qr)(1-t^{-1}r)\frac{\delta(x)-\delta(prx)}{1-pr}\right.$$

$$\left.-(r-q)(r-t^{-1})\frac{\delta(rx)-\delta(px)}{r-p}\right\}, \tag{11}$$

with $r\neq 0$; for $r=1$ or $p^{\pm1}$, the right-hand side of (11) should be understood as the limit $r\to 1$ or $p^{\pm1}$, respectively; and $f^{ij}(x)=\prod_{k=1}^{i}f^{1j}(p^{\frac{i+1}{2}-k}x)$ for $i\leqq j$.

[3] In these kinds of formulae we use $\exp\{-\sum_{n>0}x^n/n\}=1-x=-x\exp\{-\sum_{n>0}x^{-n}/n\}$.

2.3. Example of q-$\mathscr{W}_3$. $N=2$ case is $\mathscr{V}ir_{q,t}$ studied in Ref. [12] (see Appendix A). Here we give an example when $N=3$. The generators are

$$\begin{aligned} W^1(z) &= \Lambda_1(z)+\Lambda_2(z)+\Lambda_3(z)\,, \\ W^2(z) &= \Lambda_1(zp^{\frac12})\Lambda_2(zp^{-\frac12})+\Lambda_1(zp^{\frac12})\Lambda_3(zp^{-\frac12})+\Lambda_2(zp^{\frac12})\Lambda_3(zp^{-\frac12})\,. \end{aligned} \tag{12}$$

The relation of these generators is

$$\begin{aligned} & f^{11}\left(\frac{w}{z}\right)W^1(z)W^1(w)-W^1(w)W^1(z)f^{11}\left(\frac{z}{w}\right) \\ &= -\frac{(1-q)(1-t^{-1})}{1-p}\left\{\delta\left(\frac{w}{z}p\right)W^2(wp^{\frac12})-\delta\left(\frac{w}{z}p^{-1}\right)W^2(wp^{-\frac12})\right\}, \\ & f^{12}\left(\frac{w}{z}\right)W^1(z)W^2(w)-W^2(w)W^1(z)f^{21}\left(\frac{z}{w}\right) \\ &= -\frac{(1-q)(1-t^{-1})}{1-p}\left\{\delta\left(\frac{w}{z}p^{\frac32}\right)-\delta\left(\frac{w}{z}p^{-\frac32}\right)\right\}, \\ & f^{22}\left(\frac{w}{z}\right)W^2(z)W^2(w)-W^2(w)W^2(z)f^{22}\left(\frac{z}{w}\right) \\ &= -\frac{(1-q)(1-t^{-1})}{1-p}\left\{\delta\left(\frac{w}{z}p\right)W^1(zp^{-\frac12})-\delta\left(\frac{w}{z}p^{-1}\right)W^1(zp^{\frac12})\right\}, \end{aligned}$$

with

$$f^{11}(x)=\exp\left\{\sum\frac{1}{n}(1-q^n)(1-t^{-n})\frac{1-p^{2n}}{1-p^{3n}}x^n\right\}=f^{22}(x)\,,$$

$$f^{12}(x)=\exp\left\{\sum\frac{1}{n}(1-q^n)(1-t^{-n})\frac{1-p^{n}}{1-p^{3n}}p^{\frac{n}{2}}x^n\right\}=f^{21}(x)\,.$$

Note that there is no difference between W^1 and W^2 in algebraically.

2.4. Highest Weight Module of q-$\mathscr{W}_N$ Algebra. Here we refer to the representation of the q-$\mathscr{W}_N$ algebra. Let $|\lambda\rangle$ be the highest weight vector of the q-$\mathscr{W}_N$ algebra which satisfies $W_n^i|\lambda\rangle=0$ for $n>0$ and $i=1,2,\ldots,N-1$ and $W_0^i|\lambda\rangle=\lambda^i|\lambda\rangle$ with $\lambda^i\in\mathbf{C}$. Let M_λ be the Verma module over the q-$\mathscr{W}_N$ algebra generated by $|\lambda\rangle$. The dual module M_λ^* is generated by $\langle\lambda|$ such that $\langle\lambda|W_n^i=0$ for $n<0$ and $\langle\lambda|W_0^i=\lambda^i\langle\lambda|$. The bilinear form $M_\lambda^*\otimes M_\lambda\to\mathbf{C}$ is uniquely defined by $\langle\lambda|\lambda\rangle=1$.

A singular vector $|\chi\rangle\in M_\lambda$ is defined by $W_n^i|\chi\rangle=0$ for $n>0$ and $W_0^i|\chi\rangle=(\lambda^i+N^i)|\chi\rangle$ with $N^i\in\mathbf{C}$.

3. Screening Currents and Singular Vectors

Next we turn to the screening currents, a commutant of the q-$\mathscr{W}_N$ algebra, which construct the singular vectors.

3.1. Screening Currents. Let us introduce root bosons $\alpha_n^i\equiv h_n^i-h_n^{i+1}$ and $Q_\alpha^i\equiv Q_h^i-Q_h^{i+1}$ for $i=1,2,\ldots,N-1$. Then they satisfy

$$
\begin{aligned}
[\alpha^i_n, \alpha^j_m] &= -\frac{1}{n}(1-q^n)(1-t^{-n})\left\{(1+p^{-n})\delta_{i,j} - \delta_{i+1,j} - p^{-n}\delta_{i-1,j}\right\}\delta_{n+m,0}\,, \\
[\alpha^i_0, Q^j_\alpha] &= 2\delta_{i,j} - \delta_{i+1,j} - \delta_{i-1,j}\,,
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
[h^i_n, \alpha^j_m] &= \frac{1}{n}(1-q^{-n})(1-t^{-n})\left\{q^n\delta_{i,j} - t^n\delta_{i,j+1}\right\}\delta_{n+m,0}\,, \\
[h^i_0, Q^j_\alpha] &= \delta_{i,j} - \delta_{i,j+1}, \qquad [\alpha^i_0, Q^j_h] = \delta_{i,j} - \delta_{i+1,j}.
\end{aligned} \tag{14}
$$

Note that $[h^i_n + p^n h^{i+1}_n, \alpha^i_m] = 0$.

By using these root bosons, we define screening currents as follows:

$$
\begin{aligned}
S^i_+(z) &\equiv :\exp\left\{\sum_{n\neq 0}\frac{\alpha^i_n}{1-q^n}z^{-n}\right\}: e^{\sqrt{\beta}Q^i_\alpha} z^{\sqrt{\beta}\alpha^i_0}\,, \\
S^i_-(z) &\equiv :\exp\left\{-\sum_{n\neq 0}\frac{\alpha^i_n}{1-t^n}z^{-n}\right\}: e^{-\frac{1}{\sqrt{\beta}}Q^i_\alpha} z^{-\frac{1}{\sqrt{\beta}}\alpha^i_0}\,.
\end{aligned} \tag{15}
$$

Then we have

Proposition. *The screening currents satisfy*

$$
\begin{aligned}
&[:(p^{D_z} - \Lambda_1(z))(p^{D_z} - \Lambda_2(zp^{-1}))\cdots(p^{D_z} - \Lambda_N(zp^{1-N})):, S^i_\pm(w)] \\
&\quad = (1-q^{\pm 1})(1-t^{\mp 1})\frac{d}{d_{\frac{q}{t}}w} :(p^{D_z} - \Lambda_1(z))\cdots(p^{D_z} - \Lambda_{i-1}(zp^{2-i})) \\
&\qquad \times w\delta\left(\frac{w}{z}p^{i-1}\right)A^i_\pm(w)p^{D_z}(p^{D_z} - \Lambda_{i+2}(zp^{-1-i}))\cdots(p^{D_z} - \Lambda_N(zp^{1-N})):\,,
\end{aligned}
$$

with

$$
\begin{aligned}
A^i_+(w) &= :\exp\left\{\sum_{n\neq 0}\frac{h^i_n - q^n h^{i+1}_n}{1-q^n}w^{-n}\right\}: e^{\sqrt{\beta}Q^i_\alpha} w^{\sqrt{\beta}\alpha^i_0} q^{\sqrt{\beta}h^{i+1}_0} p^{\frac{N+1}{2}-i-1}\,, \\
A^i_-(w) &= :\exp\left\{-\sum_{n\neq 0}\frac{t^n h^i_n - h^{i+1}_n}{1-t^n}w^{-n}\right\}: e^{-\frac{1}{\sqrt{\beta}}Q^i_\alpha} w^{-\frac{1}{\sqrt{\beta}}\alpha^i_0} q^{\sqrt{\beta}h^i_0} p^{\frac{N+1}{2}-i}\,.
\end{aligned}
$$

Here $\frac{d}{d_\xi w}f(w) \equiv (f(w) - f(\xi w))/((1-\xi)w)$.

Proof. First, we have

$$
\begin{aligned}
[\Lambda_i(z), S^j_+(w)] &= (t-1)\delta_{i,j}\delta\left(\frac{w}{z}q\right) :\Lambda_j(z)S^j_+(w): \\
&\quad + (t^{-1}-1)\delta_{i,j+1}\delta\left(\frac{w}{z}\right) :\Lambda_{j+1}(z)S^j_+(w):\,, \\
[\Lambda_i(z), S^j_-(w)] &= (q^{-1}-1)\delta_{i,j}\delta\left(\frac{w}{z}\right) :\Lambda_j(z)S^j_-(w): \\
&\quad + (q-1)\delta_{i,j+1}\delta\left(\frac{w}{z}t\right) :\Lambda_{j+1}(z)S^j_-(w):\,.
\end{aligned} \tag{16}
$$

Here we use the following formula:

$$q^{\mp 1}\exp\left\{\pm\sum_{n>0}\frac{1}{n}(1-q^n)x^n\right\}-\exp\left\{\pm\sum_{n>0}\frac{1}{n}(1-q^{-n})x^{-n}\right\}$$
$$=(q^{\mp 1}-1)\delta(xq^{\frac{1\mp 1}{2}}). \tag{17}$$

The operator parts are

$$:\Lambda_j(wq)S_+^j(w):=A_+^j(wq)p, \qquad :\Lambda_{j+1}(w)S_+^j(w):=A_+^j(w)\,,$$
$$:\Lambda_j(w)S_-^j(w):=A_-^j(w), \qquad :\Lambda_{j+1}(wt)S_-^j(w):=A_-^j(wt)p^{-1}\,. \tag{18}$$

Next,

$$[\Lambda_i(z)+\Lambda_{i+1}(z),S_\pm^i(w)]=-(1-q^{\pm 1})(1-t^{\mp 1})\frac{d}{d_{\frac{q}{t}}w}\left\{w\delta\left(\frac{w}{z}\right)A_\pm^i(w)\right\}\,,$$
$$[:\Lambda_i(z)\Lambda_{i+1}(zp^{-1}):,S_\pm^i(w)]=0\,. \tag{19}$$

Hence,

$$[:(p^{D_z}-\Lambda_i(z))(p^{D_z}-\Lambda_{i+1}(zp^{-1})):,S_\pm^i(w)]$$
$$=(1-q^{\pm 1})(1-t^{\mp 1})\frac{d}{d_{\frac{q}{t}}w}\left\{w\delta\left(\frac{w}{z}\right)A_\pm^i(w)\right\}p^{D_z}. \tag{20}$$

This gives us the proposition. □

Therefore, the screening currents $S_\pm^i(z)$ commute with any q-$\mathscr{W}_N$ generators up to total difference. Thus we obtain

Theorem. *Screening charges $\oint dzS_\pm^i(z)$ commute with any q-$\mathscr{W}_N$ generators.*

3.2. Singular Vectors. Let $\mathscr{F}_\alpha$ be the boson Fock space generated by the highest weight state $|\alpha\rangle$ such that $\alpha_n^i|0\rangle=0$ for $n\geqq 0$ and $|\alpha\rangle\equiv\exp\{\sum_{i=1}^{N-1}\alpha^iQ_\Lambda^i\}|0\rangle$ with $Q_\Lambda^i\equiv\sum_{j=1}^iQ_h^j$. Note that $\alpha_0^i|\alpha\rangle=\alpha^i|\alpha\rangle$. And this state $|\alpha\rangle$ is also the highest weight state of the q-$\mathscr{W}_N$ algebra.

We denote the negative mode part of $S_+^i(z)$ as $(S_+^i(z))_-\equiv\exp\{\sum_{n<0}\frac{\alpha_n^i}{1-q^n}z^{-n}\}$. Then we have

Proposition. *For a set of non-negative integers s_a and $r_a\geqq r_{a+1}\geqq 0$, $(a=1,\dots,N-1)$, let*

$$\alpha_{r,s}^a=\sqrt{\beta}(1+r_a-r_{a-1})-\frac{1}{\sqrt{\beta}}(1+s_a),\quad r_0=0\,,$$
$$\widetilde{\alpha}_{r,s}^a=\sqrt{\beta}(1-r_a+r_{a+1})-\frac{1}{\sqrt{\beta}}(1+s_a),\quad r_N=0\,. \tag{21}$$

Then the singular vectors $|\chi_{rs}^+\rangle \in \mathscr{F}_{\alpha_{rs}^+}$ *are realized by the screening currents as follows*:

$$
\begin{aligned}
|\chi_{r,s}\rangle &= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} dx_j^a \cdot S_+^1(x_1^1) \cdots S_+^1(x_{r_1}^1) \cdots S_+^{N-1}(x_1^{N-1}) \cdots S_+^{N-1}(x_{r_{N-1}}^{N-1}) |\widetilde{\alpha}_{r,s}\rangle \\
&= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dx_j^a}{x_j^a} \cdot \prod_{a=1}^{N-1} \Pi(\overline{x^a}, px^{a+1}) \Delta(x^a) C(x^a) \prod_{j=1}^{r_a} (x_j^a)^{-s_a} (S_+^a(x_j^a))_- \cdot |\alpha_{r,s}\rangle
\end{aligned} \tag{22}
$$

with $x^N = 0$, $\bar{x} = 1/x$ *and*

$$
\Pi(x,y) = \prod_{ij} \exp\left\{ \sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} x_i^n y_j^n \right\}, \qquad \Delta(x) = \prod_{i \neq j}^{r} \exp\left\{ -\sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} \frac{x_j^n}{x_i^n} \right\},
$$

$$
C(x) = \prod_{i<j}^{r} \exp\left\{ \sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} \left(\frac{x_i^n}{x_j^n} - p^n \frac{x_j^n}{x_i^n} \right) \right\} \prod_{i=1}^{r} x_i^{(r+1-2i)\beta}. \tag{23}
$$

Proof. The operator product expansion of the screening currents is

$$
S_+^a(x) S_+^a(y) = \exp\left\{ -\sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} (1+p^n) \frac{y^n}{x^n} \right\} x^{2\beta} : S_+^a(x) S_+^a(y) : ,
$$

$$
S_+^a(x) S_+^{a\pm1}(y) = \exp\left\{ \sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} p^{\frac{1\pm1}{2}n} \frac{y^n}{x^n} \right\} x^{-\beta} : S_+^a(x) S_+^{a\pm1}(y) : . \tag{24}
$$

Since

$$
\begin{aligned}
S_+^a(x_1) \cdots S_+^a(x_r) &= \prod_{i<j} \exp\left\{ -\sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} (1+p^n) \frac{x_j^n}{x_i^n} \right\} \prod_{i=1}^{r} x_a^{2\beta(r-i)} : \prod_{i=1}^{r} S_+^a(x_i) : \\
&= \Delta(x) C(x) \prod_{i=1}^{r} x_i^{(r-1)\beta} : \prod_{i=1}^{r} S_+^a(x_i) : ,
\end{aligned} \tag{25}
$$

and

$$
: \prod_{a=1}^{N-1} \prod_{i=1}^{r_a} S_+^a(x_i) : |\widetilde{\alpha}_{r,s}\rangle = \prod_{a=1}^{N-1} \prod_{i=1}^{r_a} (x_i^a)^{(1-r_a+r_{a+1})\beta-(1+s_a)} (S_+^a(x_i))_- \cdot |\alpha_{r,s}\rangle , \tag{26}
$$

we obtain the proposition. □

Note that $C(x)$ is a pseudo-constant under the q-shift, i.e., $q^{D_{x_i}} C(x) = C(x)$. The expression in (21) is the same as that of $q = 1$ case [1].

We remark that the singular vectors are also realized by using the other screening currents $S_-^i(x)$ by replacing t with q^{-1} and $\sqrt{\beta}$ with $-1/\sqrt{\beta}$ in (22), that is to say:

$$|\chi_{r,s}^{-}\rangle = \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a} dx_j^a \cdot S_-^1(x_1^1)\cdots S_-^1(x_{r_1}^1)\cdots S_-^{N-1}(x_1^{N-1})\cdots S_-^{N-1}(x_{r_{N-1}}^{N-1})|\widetilde{\alpha}_{r,s}^{-}\rangle$$

$$= \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a} \frac{dx_j^a}{x_j^a}\cdot \prod_{a=1}^{N-1} \Pi_-(\overline{x^a},x^{a+1})\Delta_-(x^a)C_-(x^a)$$

$$\times \prod_{j=1}^{r_a}(x_j^a)^{-s_a}(S_-^a(x_j^a))_-\cdot|\alpha_{r,s}^-\rangle\,, \tag{27}$$

where $\widetilde{\alpha}_{r,s}^{-}$, $\alpha_{r,s}^{-}$, Π_-, Δ_- and C_- are obtained from those without $-$ suffix by replacing t with q and $\sqrt{\beta}$ with $-1/\sqrt{\beta}$. And $(S_-^a(z))_-$ is the negative mode part of $S_-^a(z)$.

4. Vertex Operator of Fundamental Representation

Now we introduce a vertex operator. Let $V(z)$ be the vertex operator defined as

$$V(z) \equiv :\exp\left\{-\sum_{n\neq 0}\frac{h_n^1}{1-q^n}p^{-\frac{n}{2}}z^{-n}\right\}:\, e^{-\sqrt{\beta}Q_h^1}z^{-\sqrt{\beta}h_0^1}. \tag{28}$$

When $q=1$, this $V(z)$ coincides with the vertex operator of fundamental representation. Note that the fundamental vertex $\Lambda_1(z)$ can be realized by $V(z)$ as

$$\Lambda_1(zp^{\frac{1}{2}}) = :V(zq^{-1})V^{-1}(z):\; p^{\frac{N-1}{2}}\,. \tag{29}$$

Hence, this vertex operator $V(z)$ can be considered as one of the building blocks of the q-$\mathscr{W}_N$ generators. We have

Proposition. *The vertex operator $V(w)$ enjoys the following Miura-like relation*:

$$:\left(p^{D_z} - g^L\left(\frac{w}{z}\right)\Lambda_1(z)\right)\cdots\left(p^{D_z}-g^L\left(\frac{w}{zp^{1-N}}\right)\Lambda_N(zp^{1-N})\right): V(w)$$

$$-V(w):\left(p^{D_z}-\Lambda_1(z)g^R\left(\frac{z}{w}\right)\right)\cdots\left(p^{D_z}-\Lambda_N(zp^{1-N})g^R\left(\frac{zp^{1-N}}{w}\right)\right):$$

$$= p^{\frac{N-1}{2}}(1-t^{-1})\delta\left(\frac{w}{z}p^{\frac{1}{2}}\right)$$

$$\times :V(wq^{-1})(p^{D_z}-\Lambda_2(zp^{-1}))\cdots(p^{D_z}-\Lambda_N(zp^{1-N})):\,,$$

and

$$g^L(x) = \exp\left\{\sum_{n>0}\frac{1}{n}(1-t^n)\frac{1-p^n}{1-p^{Nn}}p^{\frac{n}{2}}x^n\right\}t^{-\frac{1}{N}}\,,$$

$$g^R(x) = \exp\left\{\sum_{n>0}\frac{1}{n}(1-t^{-n})\frac{1-p^{-n}}{1-p^{-Nn}}p^{-\frac{n}{2}}x^n\right\}. \tag{30}$$

Proof. The fundamental relation is

$$g^L\left(\frac{w}{z}\right)\Lambda_i(z)V(w)-V(w)\Lambda_i(z)g^R\left(\frac{z}{w}\right)=p^{\frac{N-1}{2}}(t^{-1}-1)\delta_{i,1}\delta\left(\frac{w}{z}p^{\frac{1}{2}}\right)V(wq^{-1}),\tag{31}$$

i.e.,

$$\left(p^{D_z}-g^L\left(\frac{w}{z}\right)\Lambda_i(z)\right)V(w)=V(w)\left(p^{D_z}-\Lambda_i(z)g^R\left(\frac{z}{w}\right)\right)$$
$$+p^{\frac{N-1}{2}}(1-t^{-1})\delta_{i,1}\delta\left(\frac{w}{z}p^{\frac{1}{2}}\right)V(wq^{-1}),\tag{32}$$

here we use $:\Lambda_1(wp^{\frac{1}{2}})V(w):=V(wq^{-1})p^{\frac{N-1}{2}}$. By using this relation (32) and $V(w)\Lambda_i(z)g^R(z/w)=:V(w)\Lambda_i(z):$, we obtain the proposition. □

For example, when $N=3$, the relation between the vertex operator $V(w)$ and the q-$\mathcal{W}_N$ generators is

$$g^L\left(\frac{w}{z}\right)W^1(z)V(w)-V(w)W^1(z)g^R\left(\frac{z}{w}\right)=p(t^{-1}-1)\delta\left(\frac{w}{z}p^{\frac{1}{2}}\right)V(wq^{-1}),$$

$$g^L\left(\frac{w}{z}\right)g^L\left(\frac{w}{z}p\right)W^2(zp^{-\frac{1}{2}})V(w)-V(w)W^2(zp^{-\frac{1}{2}})g^R\left(\frac{z}{w}\right)g^R\left(\frac{z}{w}p^{-1}\right)$$
$$=p(t^{-1}-1)\delta\left(\frac{w}{z}p^{\frac{1}{2}}\right)(:V(wq^{-1})\Lambda_2(wp^{-\frac{1}{2}}):+:V(wq^{-1})\Lambda_3(wp^{-\frac{1}{2}}):).\tag{33}$$

5. Macdonald Polynomials

Finally we present a relation with the Macdonald polynomials. The excited states of the trigonometric Ruijsenaars model are called Macdonald symmetric functions $P_\lambda(z)$ and they are defined as follows:

$$HP_\lambda(z_1,\dots,z_M)=\varepsilon_\lambda P_\lambda(z_1,\dots,z_M),$$
$$H=\sum_{i=1}^{M}\prod_{j\neq i}\frac{tz_i-z_j}{z_i-z_j}\cdot q^{D_{z_i}},\qquad \varepsilon_\lambda=\sum_{i=1}^{M}t^{M-i}q^{\lambda_i},\tag{34}$$

where the $\lambda=(\lambda_1\geqq\lambda_2\geqq\cdots\lambda_M\geqq 0)$ is a partition.

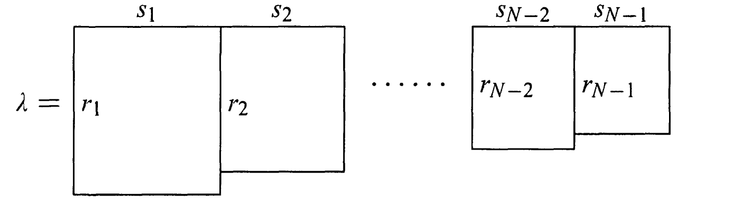
The Macdonald polynomials with general Young diagram λ are realized as some kind of correlation functions of the screening currents and vertex operators of the q-$\mathcal{W}_N$ algebra as follows:

Theorem. *The Macdonald polynomial $P_\lambda(z)$ with the Young diagram $\lambda=\sum_{i=1}^{N-1}(s_i^{r_i})$, $r_i\geqq r_{i+1}$ is written as*

$$P_\lambda(z_1,\dots,z_M)\propto\langle\alpha_{r,s}|\exp\left\{-\sum_{n>0}\frac{h_n^1}{1-q^n}\sum_{i=1}^{M}z_i^n\right\}|\chi_{r,s}\rangle.\tag{35}$$

Here $|\chi_{r,s}\rangle$ is a singular vector in (22).

Note that the operator part of the above equation is the positive mode part of the product of the vertex operators (28). The Young diagram is as follows:



Proof. First we have

$$\exp\left\{-\sum_{n>0}\frac{h_n^1}{1-q^n}\sum_{i=1}^{M}z_i^n\right\}S_+^a(w)=\Pi(z,px^1)^{\delta_{a,1}}S_+^a(w)\exp\left\{-\sum_{n>0}\frac{h_n^1}{1-q^n}\sum_{i=1}^{M}z_i^n\right\}. \tag{36}$$

By (22), the right-hand side of the equation of this theorem is

$$\oint\prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dx_j^a}{x_j^a}\cdot\Pi(z,px^1)\prod_{a=1}^{N-1}\Pi(\overline{x^a},px^{a+1})\Delta(x^a)C(x^a)\prod_{j=1}^{r_a}(x_j^a)^{-s_a}. \tag{37}$$

If we replace x^a with $(p^a x^a)^{-1}$ in (37), then the integrand coincides with that of the integral formula for Macdonald polynomials in Ref. [2] except for the $C(x)$ parts. For the integral representation of the Macdonald polynomial, we need only the property with respect to a q-shift. Since this $C(x)$ is a pseudo-constant under it, i.e., $q^{D_{x_i}}C(x)=C(x)$, they are integral representations of the Macdonald polynomial (see Appendix B). □

Remark that the Macdonald polynomials with the dual Young diagram $\lambda'=(r_1^{s_1},r_2^{s_2},\dots,r_{N-1}^{s_{N-1}})$ are realized by using the other screening currents $S_-^i(x)$ with $|\chi_{r,s}^-\rangle$ in (27) as

$$P_{\lambda'}(-z)\propto\langle\alpha_{r,s}^-|\exp\left\{-\sum_{n>0}\frac{h_n^1}{1-q^n}\sum_{i=1}^{M}z_i^n\right\}|\chi_{r,s}^-\rangle. \tag{38}$$

6. Conclusion and Discussion

We have derived a quantum $\mathcal{W}_N$ algebra for which some kind of correlation functions are the Macdonald polynomials.

Jack polynomials are realized in the following two ways (see also [5]): one is some kind of correlation function of $\mathcal{W}_N$ algebra [8, 1], the other is suitable combinations of correlation functions of $\widehat{sl}_N$ algebra [7]. The relations between Macdonald polynomials, the q-$\mathcal{W}_N$ algebra and the $U_q(\widehat{sl}_N)$ algebra are interesting.

In the classical limit $\hbar\to 0$ with $q\equiv e^{\hbar}$, the q-Miura transformation (4) reduces to the classical one. Since the right-hand side of it is order $\hbar^N$, the left-hand side must be the same order. To do so, the $\hbar$ expansion of the q-$\mathcal{W}_N$ generators must be nontrivial. Moreover, the classical generators are obtained as a linear combination of the q-$\mathcal{W}_N$ generators.

Appendix A: Quantum Virasoro Algebra

In this appendix, we give an example when $N=2$, i.e., $\mathscr{V}ir_{q,t}$ in [12]. The fundamental bosons h^1_n and Q^1_h satisfy

$$[h^1_n, h^1_m] = -\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+p^n}\delta_{n+m,0}, \qquad [h^1_0, Q^1_h] = \frac{1}{2}\,. \tag{39}$$

The root bosons are $\alpha^1_n = (1+p^{-n})h^1_n$ and $Q^1_\alpha = 2Q^1_h$.

The q-Virasoro generator $W^1(z)$, the screening currents $S^1_\pm(z)$ and the vertex operator $V(z)$ are now[4]

$$W^1(z) = :\exp\left\{\sum_{n\neq 0} h^1_n z^{-n}\right\}: q^{\sqrt{\beta}h^1_0}p^{\frac{1}{2}} + :\exp\left\{-\sum_{n\neq 0} h^1_n p^{-n} z^{-n}\right\}: q^{-\sqrt{\beta}h^1_0}p^{-\frac{1}{2}}\,,$$

$$S^1_\pm(z) = :\exp\left\{\pm\sum_{n\neq 0}\frac{1+p^{-n}}{1-r^n_\pm}h^1_n z^{-n}\right\}: e^{\pm 2\sqrt{\beta}^{\pm 1}Q^1_h} z^{\pm 2\sqrt{\beta}^{\pm 1}h^1_0}, \quad r_+ = q,\ r_- = t\,,$$

$$V(z) = :\exp\left\{-\sum_{n\neq 0}\frac{h^1_n}{1-q^n}p^{-\frac{n}{2}}z^{-n}\right\}:\ e^{-\sqrt{\beta}Q^1_h} z^{-\sqrt{\beta}h^1_0}\,. \tag{40}$$

The relations of them are

$$f^{11}\left(\frac{w}{z}\right)W^1(z)W^1(w) - W^1(w)W^1(z)f^{11}\left(\frac{z}{w}\right)$$

$$= -\frac{(1-q)(1-t^{-1})}{1-p}\left\{\delta\left(\frac{w}{z}p\right) - \delta\left(\frac{w}{z}p^{-1}\right)\right\}, \tag{41}$$

$$f^{11}(x) = \exp\left\{\sum_{n>0}\frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+p^n}x^n\right\},$$

$$[W^1(z), S^1_\pm(w)] = -(1-q^{\pm 1})(1-t^{\mp 1})\frac{d}{d_{r_\pm}w}\left\{w\delta\left(\frac{w}{z}\right)A^1_\pm(w)\right\},$$

$$A^1_\pm(w) = :\exp\left\{\sum_{n\neq 0}\frac{1+r^{\pm n}_\mp}{1-r^{\pm n}_\pm}h^1_n w^{-n}\right\}:\ e^{\pm 2\sqrt{\beta}^{\pm 1}Q^1_h} w^{\pm 2\sqrt{\beta}^{\pm 1}h^1_0} q^{\mp\sqrt{\beta}h^1_0} p^{\mp\frac{1}{2}}\,,$$

$$g^L\left(\frac{w}{z}\right)W^1(z)V(w) - V(w)W^1(z)g^R\left(\frac{z}{w}\right) = p^{\frac{1}{2}}(t^{-1}-1)\delta\left(\frac{w}{z}p^{\frac{1}{2}}\right)V(wq^{-1})\,,$$

$$g^{\substack{L\\R}}(x) = \exp\left\{\sum_{n>0}\frac{1}{n}\frac{1-t^{\pm n}}{1+p^{\pm n}}p^{\pm\frac{n}{2}}x^n\right\}t^{-\frac{1\pm 1}{4}}\,. \tag{42}$$

---

[4] The same operator with $S^1_+(z)$ was considered in [10].

For non-negative integers s and $r \geqq 0$, the singular vectors $|\chi_{rs}\rangle \in \mathcal{F}_{\alpha_{rs}}$ are

$$
\begin{aligned}
|\chi_{r,s}\rangle &= \oint \prod_{j=1}^{r} dx_j \cdot S_+^1(x_1)\cdots S_+^1(x_r)|\alpha_{-r,s}\rangle \\
&= \oint \prod_{j=1}^{r} \frac{dx_j}{x_j} \cdot \Delta(x)C(x)\prod_{j=1}^{r}(x_j)^{-s}(S_+(x_j))_- \cdot |\alpha_{r,s}\rangle\,, \qquad (43)
\end{aligned}
$$

with $\alpha_{r,s}^1 = \sqrt{\beta}(1+r) - \frac{1}{\sqrt{\beta}}(1+s)$. $\Delta(x)$ and $C(x)$ are the same as (23).

Appendix B: Integral Formula for the Macdonald Polynomials

Finally, we recapitulate the integral representation of the Macdonald polynomials [2] ([9, 1] in the $q = 1$ case). Let us denote the Macdonald polynomial defined by (34) as $P_\lambda(z;q,t)$ or $P_\lambda(z_1,\dots,z_M;q,t)$.

Proposition. *The Macdonald polynomials with the Young diagram* $\lambda = \sum_{i=1}^{N-1}(s_i^{r_i})$ *or with its dual* $\lambda' = (r_1^{s_1}, r_2^{s_2}, \dots, r_{N-1}^{s_{N-1}})$ *are realized as follows*:

$$
P_\lambda(z;q,t) \propto \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a} \frac{dx_j^a}{x_j^a} \cdot \Pi(z,\overline{x^1}) \prod_{a=1}^{N-1} \Pi(x^a,\overline{x^{a+1}})\Delta(x^a)C(x^a)\prod_{j=1}^{r_a}(x_j^a)^{s_a}\,,
$$

$$
P_{\lambda'}(z;t,q) \propto \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a} \frac{dx_j^a}{x_j^a} \cdot \widetilde{\Pi}(z,\overline{x^1}) \prod_{a=1}^{N-1} \Pi(x^a,\overline{x^{a+1}})\Delta(x^a)C(x^a)\prod_{j=1}^{r_a}(x_j^a)^{s_a}\,,
$$

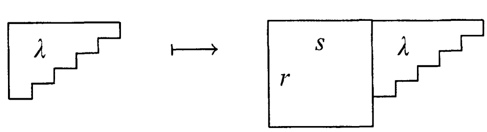
with an arbitrary pseudo-constant $C(x)$ *such that* $q^{D_{x_i}}C(x) = C(x)$. *Here* $\widetilde{\Pi}(x,y) \equiv \prod_{ij}(1 + x_i y_j)$. Π *and* Δ *are in* (23).

Proof. This proposition is proved by using two transformations in the following lemmas iteratively. The first transformation adds a rectangle to the Young diagram and the second one increases the number of variables. □

Lemma 1. *Galilean transformation.* (*Eq.* (VI.4.17) *in* [6])

$$
P_{\lambda+(s^r)}(x_1,\dots,x_r) = P_\lambda(x_1,\dots,x_r)\prod_{i=1}^{r} x_i^s\,. \qquad (44)
$$

This transformation adds a rectangle Young diagram to the original one:



.

Lemma 2. *The particle number changing transformation*:

$$
P_\lambda(x_1,\dots,x_N;q,t) \propto \oint \prod_{j=1}^{M} \frac{dy_j}{y_j}\Pi(x,\overline{y})\Delta(y)C(y)P_\lambda(y_1,\dots,y_M;q,t)\,,
$$

$$
P_{\lambda'}(x_1,\dots,x_N;t,q) \propto \oint \prod_{j=1}^{M} \frac{dy_j}{y_j}\widetilde{\Pi}(x,\overline{y})\Delta(y)C(y)P_\lambda(y_1,\dots,y_M;q,t)\,,
$$

here $C(y)$ is an arbitrary pseudo-constant $q^{D_{y_i}}C(y) = C(y)$ and λ' is a dual Young diagram of λ.

Proof. Let us define scalar products $\langle *, * \rangle$ and another one $\langle *, * \rangle'_N$ as follows:

$$\begin{aligned}\langle f, g\rangle &\equiv \oint \prod_{n>0} \frac{dp_n}{2\pi i p_n} f(\overline{p})\, g(p)\,,\\ \langle f, g\rangle'_N &\equiv \frac{1}{N!} \oint \prod_{j=1}^{N} \frac{dx_j}{2\pi i x_j} \Delta(x)\, f(\overline{x})\, g(x)\,,\end{aligned} \tag{45}$$

for the symmetric functions f and g with $p_n \equiv \sum_{i=1}^{N} x_i^N$, $\overline{p_n} \equiv n\frac{1-q^n}{1-t^n}\frac{\partial}{\partial p_n}$ and $\overline{x_j} \equiv 1/x_j$. Here we must treat the power-sums p_n as formally independent variables, i.e., $\frac{\partial}{\partial p_n} p_m = \delta_{n,m}$ for all $n, m > 0$. Then (Eq. (VI.4.13) and (VI.5.4) in [6])

$$\begin{aligned}\Pi(x, y) &= \sum_{\lambda} P_\lambda(x; q, t) P_\lambda(y; q, t) \langle P_\lambda, P_\lambda\rangle^{-1}\,,\\ \widetilde{\Pi}(x, y) &= \sum_{\lambda} P_\lambda(x; q, t) P_{\lambda'}(y; t, q)\,.\end{aligned} \tag{46}$$

Since the Macdonald operator is self-adjoint for another scalar product $\langle *, * \rangle'_N$, that is to say $\langle H f, g\rangle'_N = \langle f, H g\rangle'_N$ (Eq. (VI.9.4) in [6]), the Macdonald polynomials are orthogonal for this product $\langle P_\lambda, C P_\mu\rangle'_N \propto \delta_{\lambda,\mu}$ with an arbitrary pseudo-constant C. The proposition follows from the completeness (46) and the orthogonality of P_λ's. □

Remark that the above Lemma 2 is also proved directly by using the power-sum representation of the Macdonald operator [1]. Since that is also important to analyze the algebraic properties of the Macdonald polynomials, we review it here.

Proposition. *The Macdonald operator $H(x_1, \ldots, x_N)$ are written by the power sums $p_n \equiv \sum_{i=1}^{N} x_i^n$ as follows:*

$$H = \frac{t^N}{t-1} \oint \frac{d\xi}{2\pi i \xi} \exp\left\{\sum_{n>0} \frac{1-t^{-n}}{n} p_n \xi^n\right\} \exp\left\{\sum_{n>0} (q^n - 1) \frac{\partial}{\partial p_n} \xi^{-n}\right\} - \frac{1}{t-1}\,. \tag{47}$$

Proof. Since $q^{D_{x_i}} p_n = ((q^n - 1)x_i^n + p_n) q^{D_{x_i}}$, we have

$$q^{D_{x_i}} = : \exp\left\{\sum_{n>0} (q^n - 1) x_i^n \frac{\partial}{\partial p_n}\right\} : = \oint \frac{d\xi}{2\pi i \xi} \sum_{n\geqq 0} x_i^n \xi^n \cdot \exp\left\{\sum_{n>0} (q^n - 1) \frac{\partial}{\partial p_n} \xi^{-n}\right\}, \tag{48}$$

here $: * :$ stands for the normal ordering such that the differential operators $\frac{\partial}{\partial p_n}$ are in the right. It follows from Eq. (III.2.9) and (III.2.10) in [6] that

$$\sum_i \prod_{j\neq i} \frac{t x_i - x_j}{x_i - x_j} \sum_{n\geqq 0} x_i^n \xi^n = \frac{t^N}{t-1} \exp\left\{\sum_{n>0} \frac{1-t^{-n}}{n} p_n \xi^n\right\} - \frac{1}{t-1}\,. \tag{49}$$

This gives us the proposition. □

Let $\widetilde{H}_N(x_1,\ldots,x_N) \equiv t^{-N}((t-1)H(x_1,\ldots,x_N)+1)$, then

$$\widetilde{H}_N(x_1,\ldots,x_N)\Pi(x,y) = \widetilde{H}_M(y_1,\ldots,y_M)\Pi(x,y)\,. \tag{50}$$

With the self-adjointness of H for the another scalar product, we obtain Lemma 2 again.

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