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Absolutely Continuous Spectrum of One-Dimensional Schrödinger Operators and Jacobi Matrices with Slowly Decreasing Potentials

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Abstract: We prove that for any one-dimensional Schrödinger operator with potential $V(x)$ satisfying decay condition $|V(x)| \leqq Cx^{-3/4-\varepsilon}$, the absolutely continuous spectrum fills the whole positive semi-axis. The description of the set in $\mathbb{R}^+$ on which the singular part of the spectral measure might be supported is also given. Analogous results hold for Jacobi matrices.

Introduction

Let $H_V = -\frac{d^2}{dx^2} + V(x)$ be the one-dimensional Schrödinger operator acting on $L^2(0,\infty)$. We assume $V(x)$ is a real-valued locally integrable function which goes to zero at infinity. It is a well-known fact that if we fix some self-adjoint boundary condition at zero, the expression H_V has unique self-adjoint realization in $L^2(0,\infty)$. The essential spectrum of the operator H_V, $\sigma_{\text{ess}}(H_V)$, coincides with the positive semi-axis since the potential vanishing at infinity constitutes a relatively compact perturbation of the free Hamiltonian.

In this paper, we explore the problem of dependence of the spectral properties of H_V for positive energies on the rate of decay of the potential V. In particular, the interesting question is to determine the critical rate of decay which can lead to the complete or partial destruction of the absolutely continuous spectrum on the positive half-axis, and, correspondingly, to find out which classes of potentials are not strong enough to seriously affect the absolutely continuous spectrum inherent for the free Hamiltonian. As is generally known, if $V(x)$ belongs to $L^1(0,\infty)$ then the spectrum on the positive semi-axis is purely absolutely continuous (see, e.g., [29]). The situation is not so clear for decreasing potentials which are not absolutely integrable. There are many results on the absolute continuity of the spectrum on the positive semi-axis (except perhaps for a finite number of resonances in some cases) for certain classes of decaying potentials, such as potentials of bounded variation [29] or specific oscillating potentials (see, e.g., [1, 11, 30, 16] for further references). But no general relations between the rate of decay and spectral properties, apart from the absolutely integrable class, seem to be known.

The results concerning the spectral properties of Schrödinger operators with random potentials, however, suggest that there may be a general relation between the rate of decay of the potential and the preservation of the absolutely continuous spectrum on $\mathbb{R}^+ = (0, \infty)$. Namely, Kotani–Ushiroya [15] show that when $q(x) = a(x)F(Y_x(\omega))$, where $a(x)$ is a smooth power decaying non-random factor, $Y_x(\omega)$ is a Brownian motion on a compact Riemannian manifold M with the volume element μ and $F : M \to \mathbb{R}$ is a non-flattening C^∞ function satisfying $\int_M F d\mu = 0$, then the question of whether the rate of decay of $a(x)$ is faster or slower than $x^{-1/2}$ is crucial for the spectral properties of the corresponding random Schrödinger operator. When $a(x) = (1 + |x|)^{-\alpha}$ with $0 < \alpha < \frac{1}{2}$, the spectrum on $\mathbb{R}^+$ is pure point with probability one; when $\alpha > \frac{1}{2}$, then the spectrum on the positive semi-axis is a.e. purely absolutely continuous.

The methods of [15] are probabilistic in nature and cannot provide information on what happens in general for potentials satisfying $|V(x)| \leqq C(1 + |x|)^{-\alpha}$, $\alpha > \frac{1}{2}$. Although the set of potentials leading to purely absolutely continuous spectrum is "big" in a certain sense [15], examples with eigenvalues on $\mathbb{R}^+$ show there may be exceptions. Moreover, if one could find at least one potential satisfying $|V(x)| \leqq C(1 + |x|)^{-\alpha}$, for certain $\alpha > \frac{1}{2}$ and C, which gives rise to purely singular spectrum on the positive semi-axis, then by general principles of the genericity of singular continuous spectrum [24], there would exist another "big" (in a topological sense) set of potentials obeying the same decay condition and yielding purely singular continuous spectrum on $\mathbb{R}^+$. Namely, this set would be a dense G_δ in the space of all potentials satisfying the power decay estimate $|V(x)| \leqq C(1 + |x|)^{\alpha}$, equipped with the L^∞ norm. An analogous situation is exactly the case for $\alpha < \frac{1}{2}$, when the spectrum on the positive semi-axis is dense pure point with probability one by [15], but, at the same time, by the recent result of Simon [24], there exists a dense G_δ set of potentials leading to purely singular continuous spectrum on the R^+.

To further illustrate the difficulty of the passage from random to deterministic results, we note that [15] implies that there exist "many" potentials with power decay (slower than $x^{-1/2}$) yielding dense pure point spectrum on $\mathbb{R}^+$. But, nevertheless, there are no deterministic examples of potentials with power decay even leading to just purely singular spectrum (to construct an explicit example of a potential having dense pure point spectrum should be much harder, since an arbitrarily small change in the boundary condition may change the spectrum to purely singular continuous [4, 9]). In fact, the only known explicit examples of decaying potentials yielding purely singular spectrum on $\mathbb{R}^+$ are due to Pearson [20] and these potentials exhibit slower than power-rate decay.

The main result we prove in this paper says that all potentials decaying faster than $Cx^{-3/4-\varepsilon}$, with no additional conditions, preserve absolutely continuous spectrum on the positive semi-axis, although of course embedded singular spectrum may appear. This result provides a new general class of decaying potentials preserving absolutely continuous spectrum of the free Hamiltonian. It also shows that there is indeed a deterministic analog of the random potential results, at least in the range of power decay $\alpha \in (\frac{3}{4}, 1]$. The main new idea we use in the proof is a combination of a certain ODE asymptotic technique, which has been commonly used for the treatment of oscillating potentials, with some results from harmonic analysis related to the almost everywhere convergence of Fourier integrals.

Another interesting aspect of the spectral behaviour of Schrödinger operators with decreasing potentials is a phenomena of positive eigenvalues. Eastham–Kalf [6] show that if $V(x) = o(1/x)$ as $x \to \infty$, then H_V does not have eigenvalues above zero. If $V(x) = O(1/x)$, there are no eigenvalues above a certain constant. On the other hand, Eastham–Leod [7], with further developments by Thurlow [28], show how to construct potentials $V(x)$ of the type $V(x) = \frac{C(x)}{x}$, with $C(x)$ converging to infinity as x tends to infinity, such that a prescribed countable set of isolated points represents embedded positive eigenvalues of H_V. These authors use the Gel'fand–Levitan approach. Later, Naboko [18] described a construction which allows for an arbitrary countable set T of rationally independent numbers in $(0,\infty)$ (and so possibly a dense set) to find a potential $V(x)$ satisfying $|V(x)| \leqq \frac{C(x)}{x}$ with $C(x) \xrightarrow{x\to\infty} \infty$ monotonously at an arbitrarily slow given rate, such that the corresponding Schrödinger operator has the set T among its eigenvalues. Recently, Simon [25] has found a different construction that does away with the rational independence assumption. The constructions of Naboko and Simon do not give information about other kinds of spectrum on $\mathbb{R}^+$ in such a situation. In particular, it was not clear whether there is any other spectrum but pure point in the case when the set T of prescribed eigenvalues is dense in $\mathbb{R}^+$. The present paper settles the questions arising from Naboko's and Simon's constructions. Moreover, together with these works, it provides explicit examples of potentials yielding an arbitrary dense (countable) set of eigenvalues embedded in the absolutely continuous spectrum.

We should also mention that the results for random decaying potentials for the discrete Schrödinger operators (Jacobi matrices) [5] raise parallel questions in the discrete case. There is also a discrete analog to the continuous case of Naboko's construction by Naboko and Yakovlev [19] which allows one to find a potential decaying arbitrarily slower than $\frac{1}{n}$ such that the corresponding discrete Schrödinger operator has eigenvalues dense in the essential spectrum $[-2,2]$.

The paper is organized as follows. In the first section we prove our main result for power decaying potentials. In the second section we consider an application of our method to certain more general classes of potentials, including some potentials of the bump type. In the third section we show similar results for Jacobi matrices.

1. Main Results for Power Decaying Potentials

Let us first set up some notation we will need. Suppose the function $f(x)$ belongs to $L^2(0,\infty)$. Then we denote by $\Phi(f)(k)$ the Fourier transform of the function f,

$$\Phi(f)(k) = L^2 - \lim_{N\to\infty} \int_{-N}^{N} \exp(ikt) f(t)\, dt\,.$$

We also use the notation $M^+(g)$ for the following function corresponding to the function $g \in L^p(R)$, $1 \leqq p \leqq \infty$:

$$M^+(g)(x) = \sup_{h>0} \frac{1}{h} \int_0^h |g(x+t) + g(x-t)|\, dt$$

and notation $\mathcal{M}^+(g)$ for the set

$$\mathcal{M}^+(g) = \{x \mid M^+(g)(x) < \infty\}\,.$$

The function $M^+(g)$ is "almost" a maximal function of the function g; in particular, $M^+(g)$ is finite whenever the maximal function of g is finite. By well-known properties of the maximal function (see, e.g. [23]) we have then that $M^+(g)$ is finite a.e. and therefore the complement of the set $\mathcal{M}^+(g)$ has measure zero.

The main result of this section is the following theorem:

Theorem 1.1. *Suppose that the potential* $V(x)$ *satisfies* $|V(x)| \leqq Cx^{-3/4-\varepsilon}$ *for* $x \in (a, \infty)$ *with some positive constants* ε, a, C. *Then the absolutely continuous spectrum of the operator* H_V *fills the whole positive semi-axis, in the sense that the absolutely continuous component* ρ_{ac} *of the spectral measure* ρ *satisfies* $\rho_{\mathrm{ac}}(T) > 0$ *for any measurable set* $T \subset (0, \infty)$ *with* $|T| > 0$ *(where* $|\cdot|$ = *Lebesgue measure). The singular spectrum on* $(0, \infty)$ *may be located only on the complement of the set*

$$S = \frac{1}{4}(\mathcal{M}^+(\Phi(V(x)x^{1/4})))^2$$

(i.e., quarters of squares of the points from $\mathcal{M}^+(\Phi(V(x)x^{1/4}))$*), so that* $\rho_{\mathrm{sing}}(S) = 0$. *Moreover, for every energy* $\lambda \neq 0$ *from the set S we have two linearly independent solutions* ϕ_λ, $\overline{\phi_\lambda}$ *(=complex conjugation of* ϕ_λ*) of the equation* $H_V\phi - \lambda\phi = 0$ *with the following asymptotics as x goes to infinity:*

$$\phi_\lambda(x) = \exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int_0^x V(s)\,ds\right)(1 + O(x^{-\varepsilon}\log x)) \qquad (1)$$

(which is exactly the WKB formula).

The main idea behind the proof is a combination of the following three ingredients:

(i) The recent studies on the connection between asymptotic behavior of solutions of the Schrödinger equation and spectral properties, which allow one to conclude the absolute continuity of the spectrum on a certain set from the boundedness of all solutions corresponding to the energies from this set;

(ii) The methods of studying the asymptotics of solutions, namely the "$I + Q$" transformation technique introduced by Harris and Lutz [10] and later used by many authors for treating Schrödinger operators with oscillating potentials;

(iii) The results from the theory of Fourier integrals; in particular, the question of a.e. convergence of the partial integral $\int_{-N}^{N} \exp(ikt)f(t)\,dt$ to the Fourier transform of f under certain conditions and an estimation of the rate of convergence.

As a preparation for the proof, we need several lemmas. The first lemma allows us to reduce the proof of Theorem 1.1 to the study of generalized eigenfunction asymptotics.

Lemma 1.2. *Suppose that for every* λ *from the set B, all solutions of the equation* $H_V\phi - \lambda\phi = 0$ *are bounded. Then on the set B, the spectral measure* ρ *of the operator* H_V *is purely absolutely continuous in the following sense:*

(i) $\rho_{\mathrm{ac}}(A) > 0$ for any $A \subseteq B$ with $|A| > 0$,

(ii) $\rho_{\mathrm{sing}}(B) = 0$.

Proof. For a large class of potentials, including those we consider here, this lemma follows from the Gilbert and Pearson subordinacy theory [8], as shown by Stolz [27]. Also, in a recent paper, Jitomirskaya and Last [12] obtained a rather transparent proof of more general results. For a direct simple proof of the lemma we refer to a paper of Simon [26]. □

The complement of the set S in the statement of Theorem 1.1 has Lebesgue measure zero (which of course follows from the fact that the complement of the set $\mathcal{M}^+(\Phi(x^{1/4}\ V(x)))$ has measure zero). Therefore, we see that (assuming Lemma 1.2) for the proof of Theorem 1.1, it suffices to prove the stated asymptotics of generalized eigenfunctions for the energies from the set S.

The second lemma we need deals with certain properties of the Fourier integral.

Lemma 1.3. *Consider the function $f(x) \in L^2(\mathbb{R})$. Then for every $k_0 \in \mathcal{M}^+(\Phi(f))$, we have*

$$\int_{-N}^{N} f(x)\exp(ik_0x)\,dx = O(\log N)\,.$$

Before giving the proof, let us point out the relation between the question we study and one of the subtle problems of harmonic analysis. The Fourier transform of the square integrable function $f(x)$ is usually defined as a limit in L^2-norm as $N \to \infty$ of the functions $\int_{-N}^{N} f(x)\exp(-ikx)\,dx$. The question of whether these integrals converge to the Fourier transform of f in an ordinary sense for almost all values of k is, roughly speaking, equivalent to Lusin's hypothesis that the Fourier series of square integrable function converge almost everywhere, resolved positively by Carleson [2] in 1966. All that our simple lemma says is that we have an estimate from above on the speed of divergence of partial integrals, but for a rather explicitly described set of values of the parameter k of full measure. In the next section, which treats certain non-power decaying potentials, we will need more refined results on the a.e. convergence of Fourier integral.

Proof of Lemma 1.3. The proof uses the Parseval equality:

$$\int_{-N}^{N} f(x)\exp(ik_0x)\,dx = \frac{1}{\pi}\int_{R} \Phi(f)(k)\frac{\sin N(k_0-k)}{k_0-k}\,dk$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin Nk}{k}\,(\Phi(f)(k_0-k)+\Phi(f)(k_0+k))\,dk\,.$$

We split the last integral into three parts and estimate them separately:

$$\left|\int_{1}^{\infty} \frac{\sin Nk}{k}\,(\Phi(f)(k_0-k)+\Phi(f)(k_0+k))\,dk\right| \leqq 4\pi \left\|\frac{\sin Nk}{k}\right\|_{L^2(1,\infty)} \|\, f\,\|_{L^2(-\infty,\infty)},$$

and so this part is bounded when $N \to \infty$;

$$\left|\int_{0}^{1} \frac{\sin Nk}{k}\,(\Phi(f)(k_0-k)+\Phi(f)(k_0+k))\,dk\right|$$

$$\leqq N\int_{0}^{1/N} |\Phi(f)(k_0+k)+\Phi(f)(k_0-k)|\,dk$$

$$+\int_{1/N}^{1} \frac{1}{k}\,|\Phi(f)(k_0+k)+\Phi(f)(k_0-k)|\,dk\,.$$

In the last expression the first summand is bounded by $M^+(\Phi(f))(k_0)$, while in the second we perform integration by parts:

$$\int_{1/N}^{1} \frac{1}{k} |\Phi(f)(k_0+k)+\Phi(f)(k_0-k)|\,dk$$

$$= \int_{1/N}^{1} |\Phi(f)(k_0+k)+\Phi(f)(k_0-k)|\,dk$$

$$+ \int_{1/N}^{1} \frac{1}{k^2} \int_{1/N}^{k} |\Phi(f)(k_0+t)+\Phi(f)(k_0-t)|\,dt\,dk$$

$$\leqq M^+(\Phi(f))(k_0) + \int_{1/N}^{1} \frac{1}{k} M^+(\Phi(f))(k_0)\,dk = O(\log N)\,. \qquad \square$$

To begin with the proof of the theorem, we rewrite equation $H_V\phi - \lambda\phi = 0$ as a system of first-order equations:

$$w'(x) = \begin{pmatrix} 0 & 1 \\ V(x)-\lambda & 0 \end{pmatrix} w(x)\,, \tag{2}$$

where w and ϕ are clearly related by $w(x) = \binom{\phi(x)}{\phi'(x)}$. We perform two transformations with the system (2), the first of which is the variation of the parameter formula,

$$w(x) = \begin{pmatrix} \psi_1(x) & \psi_2(x) \\ \psi_1'(x) & \psi_2'(x) \end{pmatrix} y(x)\,, \tag{3}$$

where it is convenient for our purpose to choose $\psi_1(x) = \exp(i\sqrt{\lambda}x)$, $\psi_2(x) = \exp(-i\sqrt{\lambda}x)$. Substituting (3) into (2), we get for $y(x)$:

$$y'(x) = \frac{i}{2\sqrt{\lambda}} \begin{pmatrix} -V(x) & -V(x)\exp(-2i\sqrt{\lambda}x) \\ V(x)\exp(2i\sqrt{\lambda}x) & V(x) \end{pmatrix} y(x)\,. \tag{4}$$

We can also write this system as

$$y' = (\mathscr{D} + \mathscr{W})y\,, \tag{5}$$

where $\mathscr{D}$ stays for the diagonal part of the system and $\mathscr{W}$ for the non-diagonal part which we would like to consider as a perturbation. The matrices $\mathscr{D}$ and $\mathscr{W}$ have the form

$$\mathscr{D} = \begin{pmatrix} D(x) & 0 \\ 0 & \overline{D}(x) \end{pmatrix}, \qquad \mathscr{W} = \begin{pmatrix} 0 & W(x) \\ \overline{W}(x) & 0 \end{pmatrix}$$

with $D(x) = -\frac{i}{2\sqrt{\lambda}}V(x)$ and $W(x) = -\frac{i}{2\sqrt{\lambda}}V(x)\exp(-2i\sqrt{\lambda}x)$ in our case.

The main approach to the study of the asymptotics of solutions for systems similar to (5) is to attempt to find some transformation which will reduce the off-diagonal terms so that they will become absolutely integrable and then try to apply Levinson's theorem [3] on the L^1-perturbations of the systems of linear differential equations. It was discovered by Harris and Lutz [10] that when $W(x)$ is a conditionally integrable function, the following simple transformation of the system (5) works in some cases. We let

$$y(x) = (I + \mathcal{Q})z(x)\,, \tag{6}$$

where I is an identity matrix, while $\mathcal{Q}$ satisfies $\mathcal{Q}' = \mathcal{W}$, that is,

$$\mathcal{Q}(x) = \begin{pmatrix} 0 & q(x) \\ \overline{q}(x) & 0 \end{pmatrix}$$

with $q(x) = -\int_x^\infty W(x)\,dx$. In this case $q(x) \overset{x\to\infty}{\longrightarrow} 0$, so that for large enough x the transformation (7) is non-singular and preserves the asymptotics of the solutions. For the new variable $z(x)$ we have:

$$z' = (I + \mathcal{Q})^{-1}(\mathcal{D} + \mathcal{D}\mathcal{Q} + \mathcal{W}\mathcal{Q})z\,,$$

which after calculation leads to

$$z' = \left(\begin{pmatrix} D & 0 \\ 0 & \overline{D} \end{pmatrix} + (1 - |q|^2)^{-1} \begin{pmatrix} \overline{W}q + 2|q|^2\overline{D} & 2\overline{q}\overline{D} - \overline{q}^2 W \\ 2qD - q^2\overline{W} & 2|q|^2 D + \overline{q}W \end{pmatrix} \right) z\,. \tag{7}$$

Since $q(x)$ decays at infinity, there is hope that $q(x)D(x)$ and $q(x)^2W(x)$ may be both absolutely integrable, even if initially $W(x)$ was not.

We now return to a particular case of the system (5) we consider. Our $W(x)$ is equal to $-\frac{i}{2\sqrt{\lambda}}V(x)\exp(-2i\sqrt{\lambda}x)$, depending not only on x but also on the energy λ, and we are seeking to define $q(x,\lambda) = \frac{i}{2\sqrt{\lambda}}\int_x^\infty V(s)\exp(-2i\sqrt{\lambda}s)$. The next technical lemma shows that under our assumption on the decay of the potential we can do it and, in fact, rather successfully for every energy λ which belongs to the set S in the statement of Theorem 1.1.

Lemma 1.4. *Suppose that $V(x) \in L^{1,\mathrm{loc}}$ satisfies $V(x) \leqq C|x|^{-3/4-\varepsilon}$ for $|x| > a$ with some positive constants C, a, ε. Then for every $k \in \mathcal{M}^+(\Phi(V(x)x^{1/4}))$, the integral $\int_x^\infty \exp(-iks)\,V(s)\,ds$ converges and moreover*

$$\int_x^\infty \exp(-iks)V(s)\,ds = O(x^{-1/4}\log x)$$

as $x \to \infty$.

Proof. Note that $V(x)x^{1/4}$ is square integrable and therefore by Lemma 1.3, for every $k \in \mathcal{M}^+(\Phi(V(x)x^{1/4}))$ we have as $x \to \infty$,

$$\int_0^x V(s)s^{1/4}\exp(-iks)\,ds = O(\log x)$$

(we change ik in Lemma 1.3 to $-ik$, but since V is real, it does not change the set S). Writing $V(s) = V(s)s^{-1/4}s^{1/4}$ and integrating by parts we get

$$\int_0^x V(s)\exp(-iks)\,ds = x^{-1/4}\int_0^x V(s)s^{1/4}\exp(-iks)\,ds$$
$$+ \frac{1}{4}\int_0^x t^{-5/4}\int_0^t V(s)s^{1/4}\exp(-iks)\,ds\,dt\,.$$

The first summand clearly behaves at infinity like $O(x^{-1/4}\log x)$, while the second is absolutely convergent, since

$$\left| t^{-5/4} \int_0^t V(s) s^{1/4} \exp(-iks)\, ds \right| \leqq C_1 t^{-5/4} \log t \tag{8}$$

by Lemma 1.3 for all $t > 2$ with some constant C_1. The integral over $(0, 2)$ is finite since $V(x) \in L^{1,\mathrm{loc}}$. Therefore, the integral $\int_0^x V(s)\exp(-iks)\,ds$ is conditionally convergent and its "tail" is equal to

$$\int_x^\infty V(s)\exp(-iks)\,ds = -x^{-1/4}\int_0^x V(s)s^{1/4}\exp(-iks)\,ds + \frac{1}{4}\int_x^\infty t^{-5/4}\int_0^t V(s)s^{1/4}\exp(-iks)\,ds\,dt\,,$$

which we can estimate for x large enough using Lemma 1.3 and (8):

$$\left|\int_x^\infty V(s)\exp(-iks)\,ds\right| \leqq C_1 x^{-1/4}\log x + C_1\int_x^\infty t^{-5/4}\log t\,dt = O(x^{-1/4}\log x)\,. \qquad \square$$

Now we note that the condition $\lambda \in S$ is equivalent to $2\sqrt{\lambda} \in \mathscr{M}^+(\Phi(V(x)x^{1/4}))$ by the definition of the set S. Therefore, for every $\lambda \in S$ there is a number a_λ such that for any $x > a_\lambda$ the function $q(x,\lambda)$ is less than $\frac{1}{2}$. Applying the "$I + \mathscr{Q}$" transformation for $x > a_\lambda$ for each $\lambda \in S$, we get a system (7) for $x > a_\lambda$. The shift on the finite distance from the origin certainly does not affect asymptotics since the evolution, corresponding to such a shift is just multiplication by some constant (for each λ) matrix. Lemma 1.4 allows us to see that the non-diagonal part and, in fact, the whole second summand of the matrix in the system (7) is now absolutely integrable. Indeed, every element of this matrix is equal to the product of some bounded function and the function $V(x)q(x,\lambda)$, the latter being absolutely integrable and moreover, by our assumption on V and Lemma 1.4, satisfying $|V(x)q(x,\lambda)| < C_2(\lambda)x^{-1-\varepsilon}\log x$ for every $\lambda \in S$ with the constant C_2 depending on λ. We could now apply Levinson's theorem, but in our situation we do not need the whole power of this result. Rewriting the system (7) for every $\lambda \in S$ as

$$z'(x) = \left(\frac{i}{2\sqrt{\lambda}}\begin{pmatrix} -V(x) & 0 \\ 0 & V(x)\end{pmatrix} + R(x,\lambda)\right) z(x)$$

with $V(x)$ real and $\|R(x,\lambda)\| \in L^1$ with $\int_x^\infty \|R(s,\lambda)\|\,ds = O(x^{-\varepsilon}\log x)$, we can in a standard way transfer this system into the system of integral equations, apply the Gronwall lemma and prove (see [22] for the details) that for each $\lambda \in S$ there exist solutions $z_\lambda(x)$, $\bar{z}_\lambda(x)$ with the asymptotics

$$z_\lambda(x) = \exp\left(-\frac{i}{2\sqrt{\lambda}}\int_0^x V(x)\,dx\right)(1 + O(x^{-\varepsilon}\log x))\,.$$

Applying now transformations (6) and (3) to obtain the solution asymptotics of the initial problem, we conclude the proof of Theorem 1.1.

Remarks. 1. One may apply the proven results to the study of the absolutely continuous spectrum of Schrödinger operators with spherically symmetric potentials in $\mathbb{R}^n$, satisfying $|V(r)| \leqq Cr^{-3/4-\varepsilon}$. In a standard way, one decomposes the Schrödinger operator H_V into a direct sum of one-dimensional operators $H_{V,l} = -\frac{d^2}{dr^2} + (f_n(l)r^{-2} + V(r))$ acting on different moment subspaces (see, e.g. [21]). It is easy to see that the set S_l of energies for which all solutions of the equation $H_{V,l}\phi - \lambda\phi$ are bounded will be in fact independent of l, since the term $f_n(l)r^{-2}$ decays fast at infinity. Correspondingly, the singular spectrum of H_V on $\mathbb{R}^+$ may only be supported on the complement of the set S.

2. With very little effort, the introduced method yields results for the whole-line problem for the Schrödinger operator H_V with potential $V \in L^{1,\mathrm{loc}}$ satisfying $|V(x)| \leqq C(1+|x|)^{-3/4-\varepsilon}$ for $|x|$ large enough. The substitution of Lemma 1.2 for the whole line can be easily recovered from the remark in [26] and says that on the set $S_+ \cup S_-$, where S_+ and S_- are the sets of energies for which all solutions are bounded as x approaches correspondingly plus or minus infinity, the spectrum is purely absolutely continuous of multiplicity two (in the sense of Lemma 1.2). Of course, Lemmas 1.3 and 1.4 can be used for studying the asymptotics of solutions at $-\infty$ as well as at $+\infty$. We get in this case that the whole positive half-axis is filled by the absolutely continuous spectrum of multiplicity two and the singular spectrum may only be supported on the complement of $S_+ \cup S_-$. Moreover, it is a known fact [13] that the multiplicity of the singular spectrum may only be one for the whole-line Sturm–Liouville operators.

3. In fact, Theorem 1.1 is more than a deterministic analog of the Kotani–Ushiroya theorem in the power range $\alpha \in (\frac{3}{4}, 1]$. Indeed, one can check that from the assumption $\int_M F\,d\mu = 0$ in their random model it follows that a.e. potential is conditionally integrable and satisfies

$$\int_x^\infty V(t,\omega)\,dt \leqq C(\omega)(1+|x|)^{-\beta}$$

for every $\beta < \alpha - \frac{1}{2}$ with probability one. Assuming conditional integrability of V and certain power-decay estimate on the "tail" of potential, we can extend our result about the presence of the absolutely continuous spectrum on potentials satisfying only $|V(x)| \leqq Cx^{-2/3-\varepsilon}$. We treat this case in the Appendix.

As a byproduct of the computations we performed, let us formulate the following proposition, which is in fact a slight variation of Theorem 2.1 from Harris and Lutz [10]:

Proposition 1.5. *Suppose that for given energy $\lambda > 0$, the function $V(x)\int_x^\infty \exp(-2i\sqrt{\lambda}t)\,V(t)\,dt$ is well-defined and belongs to $L^1(0,\infty)$. Then there exist two linearly independent solutions ϕ_λ, $\overline{\phi_\lambda}$ of the equation $H_V\phi - \lambda\phi = 0$ with the following asymptotics as $x \to \infty$:*

$$\phi_\lambda(x) = \exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int_0^x V(s)\,ds\right)$$

$$\times\left(1 + O\left(\int_x^\infty \left|V(s)\int_s^\infty V(t)\exp(-2i\sqrt{\lambda}t)\,dt\right|\,ds\right)\right).$$

In particular, all solutions are bounded.

Based on the technique introduced in the proof of our main theorem, we now prove the result showing that certain conditions on the Fourier transform of potentials decaying faster than $x^{-3/4-\varepsilon}$ are sufficient to ensure the absence of the singular component of the spectrum on the positive semi-axis.

Theorem 1.6. *Suppose potential $V(x)$ satisfies $|V(x)| < Cx^{-3/4-\varepsilon}$ for all $x > a$ and the Fourier transform $\Phi(x^{1/4}V(x))(k)$ belongs to $L^{p,\mathrm{loc}}$ for some $p > 1/\varepsilon$. Then the spectrum of the operator H_V on the positive semi-axis is purely absolutely continuous, and for every energy $\lambda \in (0,\infty)$ there exist two solutions ϕ_λ, $\overline{\phi}_\lambda$ with the asymptotics as $x \to \infty$,*

$$\phi_\lambda(x) = \exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int_0^x V(s)\,ds\right)(1 + O(x^{-\varepsilon+1/p}))\,.$$

It is clear that we can concentrate on proving the stated asymptotics for every λ in the positive half-axis. A slight modification of Lemma 1.3 is needed:

Lemma 1.7. *Suppose that the Fourier transform $\Phi(f)(k)$ of the function $f(x) \in L^2$ belongs to $L^{p,\mathrm{loc}}$, $p > 2$. Then for every value of k,*

$$\int_{-N}^{N} f(x)\exp(ikx)\,dx = O(N^{1/p})\,.$$

Proof. As in the proof of Lemma 1.3 making use of the Parseval equality, we get

$$\int_{-N}^{N} f(x)\exp(ikx)\,dx = \int_0^\infty \frac{\sin Nt}{t}(\Phi(f)(k-t) + \Phi(f)(k+t))\,dt\,.$$

Again, the integral from 1 to ∞ is bounded uniformly in N by the product of L^2-norms of the functions under the integral. The remaining part we split into two integrals and estimate them using Hölder's inequality:

$$\left|\int_{1/N}^{1} \frac{\sin Nt}{t}(\Phi(f)(k-t) + \Phi(f)(k+t))\,dt\right|$$

$$\leqq \left(\int_{1/N}^{1} (1/t)^{p'}\,dt\right)^{1/p'} \left(\int_{k-1}^{k+1} |\Phi(f)(t)|^p\,dt\right)^{1/p} = O(N^{1/p})\,,$$

where p' is a conjugate exponent for $p : p' = \frac{p}{p-1}$. The second integral is estimated in a similar way:

$$\left|\int_0^{1/N} \frac{\sin Nt}{t}(\Phi(f)(k-t) + \Phi(f)(k+t))\,dt\right|$$

$$\leqq \left(\int_0^{1/N} N^{p'}\,dt\right)^{1/p'} \left(\int_{k-1/N}^{k+1/N} |\Phi(f)(t)|^p\,dt\right)^{1/p} = O(N^{1/p})\,. \qquad \square$$

Proof of Theorem 1.6. The same calculation which we performed proving Lemma 1.4 (integration by parts) shows that under the conditions of Theorem 1.6 for every positive λ, we have

$$q(x,\lambda) = \int_x^\infty V(x)\exp(-i\sqrt{\lambda}x)\,dx = O(x^{-1/4+1/p})$$

as $x \to \infty$. This implies that for all energies the function $V(x)q(x,\lambda)$ is absolutely integrable and moreover satisfies the estimate for large enough x,

$$|V(x)q(x,\lambda)| < C(\lambda)x^{-1-\varepsilon+1/p}\,.$$

By Proposition 1.5, the proof is complete. □

Remark. It is easy to modify the proof of Lemma 1.7 and Theorem 1.6 to obtain a local criteria for the absence of singular spectrum. That is, if V satisfies the conditions of Theorem 1.6 and $\Phi(x^{1/4}V(x))(k)$ belongs to $L^p(a,b)$, $b > a > 0$, then the spectrum of the operator H_V is purely absolutely continuous in the energy interval $(\frac{a^2}{4}, \frac{b^2}{4})$.

We note that the conditions stated in the theorem are rather precise. For example, in the celebrated Wigner–von Neumann example (historically the first example of the decaying potential having positive eigenvalue embedded in the absolutely continuous spectrum), the asymptotic behavior of the potential at infinity is $V(x) = -8(\sin 2x)/x + O(x^{-2})$ (see, e.g. [22]) so that $\varepsilon = \frac{1}{4}$, while the singularity of the Fourier transform of $x^{1/4}V(x)$ is easily seen to be of the order $(k-2)^{-1/4}$ which belongs to $L^{p,\mathrm{loc}}$ with $p < 4$. It is an open question whether one can replace condition $\Phi(x^{1/4}V(x)) \in L^{p,\mathrm{loc}}$, $p > 1/\varepsilon$ with the simpler one $\Phi(x^{1/4}V(x)) \in L^{1/\varepsilon,\mathrm{loc}}$ so that the last theorem still remains true.

2. Non-Power Decreasing Potentials

In this section we apply the method described in the preceding section of the paper to a wider class of potentials. This class will include, in particular, certain potentials of the bump type, which are "mostly" zero but have bumps decaying at infinity.

Let us introduce the class of potentials we will treat.

Definition. *We say that the potential* $\tilde{V}(x) \in L^\infty(0,\infty)$ *belongs to the class* $\mathscr{P}_{-\alpha}(0,\infty)$ *if there exists a potential* $V(x) \in L^\infty(0,\infty)$ *satisfying* $|V(x)| \leqq Cx^{-\alpha}$ *for x large enough and a countable collection of disjoint intervals in* $(0,\infty)$ $\{(a_j,b_j)\}_{j=1}^\infty$, $b_j \leqq a_{j+1}$ $\forall j$, *such that*

$$\tilde{V}(x) = \begin{cases} 0, & x \in (a_n,b_n) \\ V(x - \sum_{j=1}^n (b_j - a_j)), & x \in (b_n, a_{n+1}) \end{cases}.$$

Roughly, the potential $\tilde{V}(x)$ is obtained from $V(x)$ by inserting a countable number of intervals on which $\tilde{V}(x)$ vanishes; while on the rest of the axis, it is $V(x)$ shifted on the distance which is equal to the sum of the lengths of the intervals inserted so far. Of course, $\tilde{V}(x) \in \mathscr{P}_{-\alpha}$ need not decay faster than any power at infinity. However, if we "compress" $\tilde{V}(x)$ by collapsing all intervals on which it vanishes, we get a potential which is bounded by $Cx^{-\alpha}$ for large x.

The following theorem holds for potentials from the class $\mathscr{P}_{-3/4-\varepsilon}$:

Theorem 2.1. *Suppose* $\tilde{V}(x) \in \mathscr{P}_{-3/4-\varepsilon}$, $\varepsilon > 0$. *Then the absolutely continuous part of the spectral measure fills the whole positive semi-axis, in the sense that* $\rho_{ac}(T) > 0$ *for any measurable set* $T \subset (0,\infty)$ *with* $|T| > 0$. *For almost every energy* $\lambda \in (0,\infty)$, *there exist two solutions* ϕ_λ, $\overline{\phi_\lambda}$ *with the asymptotics as* $x \to \infty$,

$$\phi_\lambda(x) = \exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int_0^x \tilde{V}(s)\,ds\right)(1+o(1)).$$

Proposition 1.5 implies that to prove the stated result, we need only to show that the function $R(t) = \tilde{V}(x)\int_x^\infty \tilde{V}(t)\exp(-2i\sqrt{\lambda}t)\,dt$ is well defined and belongs to $L^1(0,\infty)$ for a.e. $\lambda \in \mathbb{R}^+$. To proceed with the proof, we need some further facts from the theory of Fourier integral.

The following result is due to Zygmund [31].

Theorem (Zygmund). *If* $f \in L^p(-\infty,\infty)$, *where* $1 \leqq p < 2$, *then the integral*

$$F(f)(k,N) = \frac{1}{\sqrt{2\pi}}\int_{-N}^{N} f(x)\exp(-ikx)\,dx$$

converges as $N \to \infty$, *in an ordinary sense for almost every value of* k.

This will serve us as an analog of Lemma 1.3. However, it is a much more sophisticated result by itself. One of the consequences is that we do not have a description of the exceptional set on which convergence fails (and correspondingly, where the singular spectrum may be supported). For future reference, let us denote by $A(f)$ the set of full measure for which the integral $F(f)(k,N)$ does converge.

The main idea now is the same as before: to perform in some "clever" way integration by parts to get estimates on the tail

$$\tilde{q}(x,\lambda) = \int_x^\infty \tilde{V}(t)\exp(-2i\sqrt{\lambda}t)\,dt$$

for a.e. λ. Of course, there is no hope anymore that $q(x,\lambda)$ will, in general, decay even as some power for potentials we now consider. However, the special structure of the potentials allows us to overcome this problem.

Proof of Theorem 2.1. Let us factorize $\tilde{V}(x) = \tilde{V}_1(x)\tilde{V}_2(x)$ in the following way: if $\tilde{V}(x) = V(x, \{(a_j,b_j\}_{j=1}^{j=\infty})$, then

$$\tilde{V}_1(x) = \begin{cases} 0, & x \in (a_n,b_n) \\ (x - \sum_{j=1}^n (b_j - a_j))^{1/4} V(x - \sum_{j=1}^n (b_j - a_j)), & x \in (b_n, a_{n+1}) \end{cases}$$

and

$$\tilde{V}_2(x) = \begin{cases} (a_n - \sum_{j=1}^{n-1}(b_j - a_j))^{-1/4}, & x \in (a_n,b_n) \\ (x - \sum_{j=1}^n (b_j - a_j))^{-1/4}, & x \in (b_n,a_{n+1}) \end{cases}.$$

Therefore, $\tilde{V}_1(x)$ is obtained from the function $x^{1/4}V(x)$ in the same way as $\tilde{V}(x)$ is obtained from $V(x)$, while the quotient $\frac{\tilde{V}(x)}{\tilde{V}_1(x)} = \tilde{V}_2(x)$ is a continuous piecewise differentiable non-increasing function. Since $|V(x)| < Cx^{-3/4-\varepsilon}$ for large enough x, we have that $x^{1/4}V(x)$, and therefore also $\tilde{V}_1(x)$ belong to $L^{2-\varepsilon}(0,\infty)$. Then by Zygmund's theorem, for all λ from the set $\frac{1}{4}(A(\tilde{V}_1(x))^2$ (quarters of the squares of the points from the set $A(F(\tilde{V}_1(x))))$ of full measure in the positive semi-axis the limit $\lim_{N\to\infty}\int_0^N \tilde{V}_1(t)\exp(-2i\sqrt{\lambda}t)\,dt$ exists, so that we can consider for these λ the conditionally convergent integral $\int_x^\infty \tilde{V}_1(t)\exp(-2i\sqrt{\lambda}t)\,dt$. Let us integrate by parts the expression

$$\begin{aligned}\tilde{q}(x,\lambda) &= \int_x^\infty \tilde{V}_1(t)\tilde{V}_2(t)\exp(-2i\sqrt{\lambda}t)\,dt\\ &= \tilde{V}_2(x)\int_x^\infty \tilde{V}_1(t)\exp(-2i\sqrt{\lambda}t)\,dt + \int_x^\infty \tilde{V}_2'(t)\int_t^\infty \tilde{V}_1(s)\exp(-2i\sqrt{\lambda}s)\,ds\,dt\,.\end{aligned}$$

For the values of λ which we consider, the absolute value of the integral $\int_x^\infty \tilde{V}_1(t)$ $\exp(2i\sqrt{\lambda}t)\,dt$ goes to zero at infinity and therefore is bounded by some constant C (depending on λ) for all values of x. Hence, we can estimate the right-hand side in the last equation by

$$C\left(\tilde{V}_2(x) + \int_x^\infty |\tilde{V}_2'(t)|\,dt\right) \leqq 2C\tilde{V}_2(x)\,,$$

since $\tilde{V}_2(x)$ is a non-increasing positive continuous piecewise differentiable function. Thus, we get that for a.e. λ

$$\left|\tilde{V}(x)\int_x^\infty \tilde{V}(t)\exp(2i\sqrt{\lambda}t)\,dt\right| \leqq C(\lambda)|\tilde{V}(x)\tilde{V}_2(x)|\,.$$

To conclude the proof, we notice that the function $\tilde{V}(x)\tilde{V}_2(x)$ is absolutely integrable by the way we constructed the functions $\tilde{V}(x)$ and $\tilde{V}_2(x)$; the L^1-norm of their product is equal to the L^1-norm of the function $x^{-1/4}V(x)$. On the intervals (a_n,b_n), where $\tilde{V}_2(x)$ is defined to be constant, $\tilde{V}(x)$ vanishes, and on the intervals where $\tilde{V}(x)$ is equal to shifted $V(x)$, $\tilde{V}_2(x)$ is just shifted $x^{-1/4}$. □

One of the situations to which Theorem 2.1 applies is when we have a sequence of repeating bumps of the same shape but with decreasing magnitude. Fix $U(x) \in L^\infty(0,a)$ and let

$$\tilde{V}(x) = \sum_{n=1}^\infty g_n U(x-a_n), \quad a_n - a_{n-1} > a\,.$$

For potentials of this type, Pearson [20] has shown that if one chooses the distances between bumps to be big enough, then if $\sum_{n=1}^\infty g_n^2 = \infty$, the corresponding Schrödinger operator has purely singular continuous spectrum on $\mathbb{R}^+$ and if $\sum_{n=1}^\infty g_n^2 < \infty$, the spectrum on the positive semi-axis is purely absolutely continuous. Otherwise, there was essentially nothing known about possible spectral behavior for Schrödinger operators with bump potentials which are not absolutely integrable and not power decaying. From the last theorem it follows that if $|g_n| < Cn^{-3/4-\varepsilon}$, the absolutely continuous spectrum remains on the positive semi-axis, no matter how $U(x)$ looks and which distances between bumps we take.

3. Jacobi Matrices

Now we prove similar results for Jacobi matrices. We consider the self-adjoint operator h_v on $l^2(\mathbb{Z}_+)$ (with $\mathbb{Z}_+ = \{1,2,\ldots\}$) given by

$$\begin{aligned} h_v u(n) &= u(n+1) + u(n-1) + v(n)u(n)\,, \\ u(0) &= 0\,, \end{aligned} \tag{9}$$

where $v(n)$ is real valued, tending to zero at infinity sequence. All the theorems we have proven for Schrödinger operators in the first two sections have their analogs for Jacobi matrices. Of course, we need to replace the positive semi-axis by the segment $(-2,2)$, the interior of the essential spectrum of the free discrete Schrödinger operator. Since we consider only decaying potentials, the essential spectrum is the same for h_v. The way the argument goes in the Jacobi matrices case is very close to the continuous analog and hence sometimes we will omit the proofs. We will still use the notation $\Phi(f)(k)$, but now for the Fourier transform of the l^2-sequence $f(n)$:

$$\Phi(f)(k) = l^2 - \lim_{N\to\infty} \sum_{l=-N}^{N} \exp(ikl)f(l)\,.$$

All other notations introduced in the preceding sections of the paper also remain valid. Let us begin by stating our main theorem for Jacobi matrices:

Theorem 3.1. *Suppose that $v(n)$ satisfies $|v(n)| < Cn^{-3/4-\varepsilon}$ for some positive constants C, ε. Then the absolutely continuous component ρ_{ac} of the spectral measure ρ of the operator h_v fills the whole segment $(-2,2)$, in the sense that $\rho_{ac}(T) > 0$ for any measurable set $T \subset (-2,2)$ with positive Lebesgue measure. The singular component of the spectral measure may be supported only on the complement of the set $S = 2\cos(\frac{1}{2}\mathcal{M}^+(\Phi(n^{1/4}v(n))) \cap (-2,2)$ (values of energy such that* 2 arccos *of half their value belongs to the set $\mathcal{M}^+(\Phi(n^{1/4}V(n))$; we fix the range of the* arccos *to be $[0,\pi]$. Moreover, for every $\lambda \in S$ there exist two linearly independent solutions $\psi_\lambda(n)$, $\overline{\psi_\lambda}(n)$ with the following asymptotics as $n \to \infty$:*

$$\psi_\lambda(n) = \exp\left(ikn + \frac{i}{2\sin k}\sum_{l=1}^{n} V_l\right)(1 + O(n^{-1/4}\log n))\,,$$

where $k =$ arccos $\frac{1}{2}\lambda$.

The strategy of the proof is the same as in the Schrödinger operators case. The analogs of the three lemmas we used heavily are as follows:

Lemma 3.2. *Assume that for every λ from the set B, all solutions of the equation $h_v\phi - \lambda\phi$ are bounded. Then on the set B, the spectral measure ρ of the operator h_v is purely absolutely continuous in the following sense*:

(i) $\rho_{ac}(A) > 0$ *for any $A \subseteq B$ with* $|A| > 0$,
(ii) $\rho_{sing}(B) = 0$.

Proof. This lemma follows from the subordinacy theory for infinite matrices, developed by Khan and Pearson [14]. Recently, Jitomirskaya and Last proved more general results for Jacobi matrices [12]. The reference for a simple direct proof of the lemma is the paper of Simon [26]. □

Lemma 3.3. *Consider the function $f(n) \in l^2(\mathbb{Z})$. Then for every $k_0 \in \mathcal{M}^+(\Phi(f))$, we have*

$$\sum_{l=-N}^{N} f(x)\exp(ik_0 l) = O(\log N)\,.$$

Proof. The Parseval equality in this case yields

$$\sum_{l=-N}^{N} f(l)\exp(ik_0 l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(N+1/2)(k_0-k)}{\sin\frac{1}{2}(k_0-k)}\Phi(v)(k)\,dk$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} \frac{\sin(N+1/2)(k)}{\sin\frac{1}{2}k}(\Phi(v)(k_0+k)+\Phi(v)(k_0-k))\,dk\,.$$

The final expression may be estimated exactly as in the proof of Lemma 1.3. □

Lemma 3.4. *Suppose that sequence $v(n)$ satisfies $|v(n)| < Cn^{-3/4-\varepsilon}$ with some positive constants C, ε. Then for every $k \in \mathcal{M}^+(v(n)n^{1/4})$, the sum $\sum_{l=n}^{\infty}\exp(-ikl)v(l)$ converges and moreover,*

$$\sum_{l=n}^{\infty}\exp(-ikl)v(l) = O(n^{-1/4}\log n)$$

as $n \to \infty$.

Proof. Summation by parts gives

$$\sum_{l=1}^{n}\exp(-ikl)v(l) = n^{-1/4}\sum_{l=1}^{n}\exp(-ikl)(v(l)l^{1/4})$$

$$+\sum_{l=1}^{n-1}((l^{-1/4}-(l+1)^{-1/4})\sum_{j=1}^{l}\exp(-ikj)v(j)j^{1/4})$$

and applying Lemma 3.3, we obtain that for the values of $k \in \mathcal{M}^+(v(n)n^{1/4})$ the sum converges as $n \to \infty$. For the speed of convergence we have an estimate:

$$\left|\sum_{l=n}^{\infty}\exp(-ikl)f(l)\right| \leqq \left|n^{-1/4}\sum_{l=1}^{n}\exp(-ikl)(v(l)l^{1/4})\right|$$

$$+\left|\sum_{l=1}^{n-1}((l^{-1/4}-(l+1)^{-1/4})\sum_{j=1}^{l}\exp(-ikj)v(j)j^{1/4})\right|$$

$$\leqq C\left(n^{-1/4}\log n + \sum_{l=n}^{\infty} l^{-5/4}\log l\right) = O(n^{-1/4}\log n)\,. \qquad \square$$

In the discrete case, the solution ψ of the formal equation $h_v\psi = \lambda\psi$ satisfies the recursion relation

$$\begin{pmatrix}\psi(n+1)\\ \psi(n)\end{pmatrix} = \begin{pmatrix}\lambda - v(n) & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi(n)\\ \psi(n-1)\end{pmatrix}\,. \tag{10}$$

Let $k = \arccos \frac{1}{2}\lambda$ for $\lambda \in (-2, 2)$. Applying to the system (10) a discrete analog of the variation of the parameters formula

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = \begin{pmatrix} \exp(ik(n+1)) & \exp(-ik(n+1)) \\ \exp(ikn) & \exp(-ikn) \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \tag{11}$$

we get for new variables the finite difference system

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{iv(n)}{2\sin k} \begin{pmatrix} 1 & \exp(-2ikn) \\ -\exp(2ikn) & -1 \end{pmatrix} \right) \begin{pmatrix} A_n \\ B_n \end{pmatrix}. \tag{12}$$

Now we are in a position to apply the discrete analog of the Harris–Lutz technique to study the asymptotics of the solutions of the system (12). For every $\lambda \in (-2, 2)$ such that $2k = 2\arccos \frac{1}{2}\lambda$ belongs to $\mathscr{M}^+(\Phi(n^{1/4}v(n)))$, by Lemma 3.4 we can define

$$q(n,k) = -\frac{i}{2\sin k} \sum_{l=n}^{\infty} v(l) \exp(-2ikl),$$

and moreover, $q(n,k)$ behaves as $O(n^{-1/4}\log n)$ as n goes to infinity. The "$I + Q$" transformation will be

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} 1 & q(n,k) \\ \overline{q}(n,k) & 1 \end{pmatrix} \begin{pmatrix} C_n \\ D_n \end{pmatrix}. \tag{13}$$

This transformation is non-singular as far as n is large enough and so we can reconstruct the asymptotics of our generalized eigenfunctions from the asymptotics of the variables C_n, D_n. Substitution of (13) into the system (12) yields

$$\begin{pmatrix} C_{n+1} \\ D_{n+1} \end{pmatrix} = \left(\begin{pmatrix} 1 + \frac{i}{2\sin k}v(n) & 0 \\ 0 & 1 - \frac{i}{2\sin k}v(n) \end{pmatrix} + R(n,k) \right) \begin{pmatrix} C_n \\ D_n \end{pmatrix}. \tag{14}$$

Direct computation shows that every element of the matrix $R(n,k)$ is a product of numbers, uniformly bounded in n for each $k \in \frac{1}{2}\mathscr{M}^+(\Phi(n^{1/4}v(n)))$ and $q(n,k)v(n)$ or $q(n+1,k)v(n)$. Hence by Lemma 3.4 and our assumptions on potential v, we have $\|R(n,k(\lambda))\| = O(n^{-1-\varepsilon}\log n)$ at infinity for every λ from the set S in the statement of Theorem 3.1. We can further simplify (14) by applying the transformation

$$\begin{pmatrix} C_n \\ D_n \end{pmatrix} = \begin{pmatrix} \exp(\frac{i}{2\sin k}\sum_{l=1}^{n-1} v(l)) & 0 \\ 0 & \exp(-\frac{i}{2\sin k}\sum_{l=1}^{n-1} v(l)) \end{pmatrix} \begin{pmatrix} E_n \\ F_n \end{pmatrix}. \tag{15}$$

For E, F variables we have

$$\begin{pmatrix} E_{n+1} \\ F_{n+1} \end{pmatrix} = (I + \tilde{R}(n,k)) \begin{pmatrix} E_n \\ F_n \end{pmatrix}, \tag{16}$$

where I is an identity matrix and $\tilde{R}(n,k)$ satisfies the same norm decaying conditions as $R(n,k)$. One can also directly check by looking at the transformations we performed with the initial system (10) that the determinant of the matrix

$\tilde{R}(n,k)$ is equal to $\frac{1-|q(n,k)|^2}{1-|q(n+1,k)|^2}$. A simple argument, carried out in Lemma 3.5 immediately below, shows that there exists a solution of (16) with the asymptotics at infinity

$$\begin{pmatrix} E_n \\ F_n \end{pmatrix} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(n^{-1/4} \log n) \right).$$

The application of transformations (15), (13), (11) allows us to compute the asymptotics of the generalized eigenfunction ψ_λ and therefore concludes the proof. □

Lemma 3.5. *Suppose we have a recursive relation*

$$\begin{pmatrix} E_{n+1} \\ F_{n+1} \end{pmatrix} = (I + \tilde{R}(n)) \begin{pmatrix} E_n \\ F_n \end{pmatrix} \tag{17}$$

and the matrix $\tilde{R}(n)$ *satisfies* $\{\|\tilde{R}(n)\|\}_{n=1}^{\infty} \in l^1(\mathbb{Z}_+)$. *Moreover, suppose that determinants of the matrices* $\prod_{l=1}^{n}(I + R(l))$ *are bounded away from zero. Then there exists a solution* H_n *of* (17) *such that*

$$\left\| H_n - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = O\left(\sum_{l=n}^{\infty} \|\tilde{R}(n)\| \right)$$

as $n \to \infty$.

Proof. A standard argument shows that the product $\prod_{l=1}^{n}(I + R(l))$ converges as n goes to infinity under the conditions of the lemma to a matrix we will denote $R_\infty = \prod_{l=1}^{\infty}(I + R(l))$. The condition on the determinants of finite products ensures that R_∞ is invertible. Pick the vector $H_1 = R_\infty^{-1}\binom{1}{0}$. Then for n large enough so that $\sum_{l=n}^{\infty} \|R(l)\| < 1$, we have

$$\begin{aligned} \left\| H_n - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| &\leqq \left\| \left(I - \prod_{l=n}^{\infty} (I + R(l)) \right) \left(\prod_{l=1}^{n} (I + R(l)) H_1 \right) \right\| \\ &\leqq \frac{\sum_{l=n}^{\infty} \|R(l)\|}{1 - \sum_{l=n}^{\infty} \|R(l)\|} \exp \sum_{l=1}^{n} \|R(l)\| = O\left(\sum_{l=n}^{\infty} \|R(l)\| \right). \end{aligned}$$ □

Let us specifically stress one consequence of the calculations we performed and formulate

Proposition 3.6. *For discrete Schrödinger operators, similar to the continuous ones, in order to prove that for a certain energy* $\lambda \in (-2,2)$, *all solutions of the equation* $h_v\psi - \lambda\psi = 0$ *are bounded, it is enough to show that the sequences* $q(n,k)v(n)$ *and* $q(n+1,k)v(n)$ (*where* $k = \arccos \frac{\lambda}{2}$) *belong to* $l^1(\mathbb{Z}_+)$.

This observation leads to the following theorem, which provides conditions under which the singular component of the spectral measure of the operator h_v on $(-2,2)$ is void. It is an analog of Theorem 1.6:

Theorem 3.7. *Suppose that* $|v(n)| < Cn^{-3/4-\varepsilon}$ *and the Fourier transform* $\Phi(n^{1/4}v(n))(k)$ *belongs to* $L^p(0,2\pi)$ *with* $p > 1/\varepsilon$. *Then the spectrum of the operator* h_v *on the segment* $(-2,2)$ *is purely absolutely continuous. Moreover, for every value* *of* $\lambda \in (-2,2)$ *there exist two solutions* ψ_λ *and* $\overline{\psi_\lambda}$ *of the equation* $h_v\psi - \lambda\psi = 0$ *with the following asymptotics as* $n \to \infty$:

$$\psi_\lambda = \exp\left(ikn + \frac{i}{2\sin k}\sum_{l=1}^{n} v(l)\right)(1 + O(n^{-\varepsilon+1/p})),$$

where $k = \arccos \frac{1}{2}\lambda$.

Proof. The proof is a complete analogy of the proof of Theorem 1.5. One only needs to replace integration by parts with Abel's transformation (summation by parts). □

Finally, we discuss Jacobi matrices with non-power decaying potentials. The class of potentials we treat is again potentials which are "mostly" zero and become power decaying after "compression."

Namely, we say that a potential $\tilde{v}(n)$ belongs to the class $\mathscr{D}_\alpha$ if there exists a potential $v(n)$ such that $|v(n)| \leqq Cn^{-\alpha}$ and two sequences of positive integers $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$ satisfying $b_{i-1} < a_i < b_i$ for all i, such that

$$\tilde{v}(n) = \begin{cases} 0, & a_l \leqq n < b_l \\ v(n - \sum_{j=1}^{l}(b_j - a_j)), & b_l \leqq n < a_{l+1} \end{cases}.$$

We have the following theorem:

Theorem 3.8. *Let potential* $v(n)$ *belong to* $\mathscr{D}_{-3/4-\varepsilon}$. *Then the absolutely continuous spectrum of the operator* h_v *fills the whole segment* $[-2,2]$, *in the sense that for any measurable set* $T \subseteq [-2,2]$ *with positive Lebesgue measure we have* $\rho_{\mathrm{ac}}(T) > 0$. *Moreover, for a.e.* $\lambda \in (-2,2)$ *there exist two linearly independent solutions* ψ_λ, $\overline{\psi_\lambda}$ *with the following asymptotics as* $n \to \infty$:

$$\psi_\lambda = \exp\left(ikn + \frac{i}{2\sin k}\sum_{l=1}^{n} v(l)\right)(1 + O(n^{-\varepsilon+1/p})),$$

where $k = \arccos \frac{1}{2}\lambda$.

For the proof of this theorem we need an analog of Zygmund's result for the case of Fourier series instead of the Fourier integral. We refer to the work of Menchoff [17] for the following result:

Theorem (Menchoff). *Suppose* $\{\phi_n(x)\}_{n=1}^\infty$ *is an orthonormal system of functions on the interval* (a,b) *and the sequence* $\{c_n\}_{n=1}^\infty$ *belongs to* $l^p(\mathbb{Z})$ $0 < p < 2$. *Then the series*

$$\sum_{l=1}^{N} c_n\phi_n(x)$$

converges, in the ordinary sense, for almost every $x \in (a,b)$.

In particular, taking $\phi_n(x) = \exp(inx)$ and $(a,b) = (0,2\pi)$, we obtain an analog of Zygmund's theorem.

Proof of Theorem 3.8. Given Menchoff's theorem, the proof essentially repeats the argument we gave to prove Theorem 2.1 in the second section. □

Appendix

In this section we prove

Theorem A.1. *Suppose that potential V satisfies $|V(x)| \leqq C_1 x^{-\frac{2}{3}-\varepsilon}$ and is conditionally integrable with $|\int_x^\infty V(t)\,dt| \leqq C_2 x^{-\delta}$ for some positive δ. Then the absolutely continuous component of the spectral measure of the operator H_V fills the whole $\mathbb{R}^+$.*

For the proof of the theorem, we need to introduce some notation. Suppose that potential $V(x)$ satisfies $|V(x)| \leqq C_1 x^{-\alpha-\varepsilon}$, with some $\alpha > \frac{1}{2}$ and $\varepsilon > 0$. Then we denote by $S(V)$ the set of energies

$$S(V) = \left\{ \lambda \,\middle|\, |q(x,\lambda)| = \left| \int_x^\infty V(t)\exp(-2i\sqrt{\lambda}t)\,dt \right| \leqq C(\lambda)x^{-\alpha+\frac{1}{2}}\log x \right\}.$$

Following the proof of Lemma 1.4, it is easy to see that $S(V)$ contains the set $\mathscr{M}^+(\Phi(x^{\alpha-\frac{1}{2}}V(x)))$, and hence is a set of full measure.

Proof. To make the argument simpler, it is convenient to modify slightly the $\mathscr{I}+\mathscr{Q}$ transformation we applied to the system (4):

$$y'(x) = \frac{i}{2\sqrt{\lambda}}\begin{pmatrix} -V(x) & -V(x)\exp(-2i\sqrt{\lambda}x) \\ V(x)\exp(2i\sqrt{\lambda}x) & V(x) \end{pmatrix} y(x).$$

Now we let

$$y(x) = (1-|q|^2)^{-1/2}(\mathscr{I}+\mathscr{Q})z(x),$$

where $\mathscr{Q}$ and $q = q(x,\lambda)$ are the same as before. As we did earlier in Sect. 1, we will always assume that since we are interested in the asymptotics, we perform the $\mathscr{I}+\mathscr{Q}$ transformation "far enough" so that $|q| < 1$ for the x we consider. A calculation leads us to the following system for $z(x)$:

$$z' = \left(\begin{pmatrix} D & 0 \\ 0 & \overline{D} \end{pmatrix} + (1-|q|^2)^{-1} \times \begin{pmatrix} \frac{1}{2}(\overline{W}q - W\overline{q}) + 2|q|^2\overline{D} & 2\overline{q}\overline{D} - \overline{q}^2 W \\ 2qD - q^2\overline{W} & -\frac{1}{2}(\overline{W}q - W\overline{q}) + 2|q|^2 D \end{pmatrix} \right) z .$$

Here, as in the first section, D stands for $-\frac{i}{2\sqrt{\lambda}}V(x)$ and W for $-\frac{i}{2\sqrt{\lambda}}V(x)\exp(-2i\sqrt{\lambda}x)$. As we already mentioned above, on the set $S(V)$ of the full measure we have $|q(x,\lambda)| \leqq C(\lambda)x^{-1/6}\log x$. Hence, for all energies $\lambda \in S(V)$, the function $q^2(x,\lambda)V(x)$ is absolutely integrable and $|\int_x^\infty q^2(x,\lambda)V(x)\,dx| \leqq C(\lambda)x^{-1-\varepsilon}\log x$. Therefore, we can rewrite the system in the following way:

$$z' = \left(\begin{pmatrix} D + \frac{1}{2}(\overline{W}q - W\overline{q}) & 2\overline{q}\overline{D} \\ 2qD & \overline{D} - \frac{1}{2}(\overline{W}q - W\overline{q}) \end{pmatrix} + \mathscr{R}(x) \right) z , \tag{18}$$

where all entries of the matrix $\mathscr{R}$ are from L^1. The only dangerous terms are the off-diagonal terms in the matrix, since the diagonal terms are purely imaginary and alone would not lead to unbounded solutions. The main idea now is to iterate the $\mathscr{I}+\mathscr{Q}$ transformation, improving the rate of decay of the off-diagonal terms. To apply this procedure, we need first of all to ensure that $qD = \frac{1}{4\lambda}V(x)\int_x^\infty \exp(-2i\sqrt{\lambda}s)\,ds$ is an a.e. λ integrable function. For any $\lambda \in S(V)$, we have:

$$\begin{aligned}\int_0^x V(t)\int_t^\infty V(s)\exp(-2i\sqrt{\lambda}s)\,ds &= -\left(\int_0^\infty V(t)\,dt\right)\left(\int_0^\infty V(s)\exp(-2i\sqrt{\lambda}s)\,ds\right) \\ &+ \left(\int_x^\infty V(t)\,dt\right)\left(\int_x^\infty V(s)\exp(-2i\sqrt{\lambda}s)\,ds\right) \\ &- \int_0^x \left(V(t)\int_t^\infty V(s)\,ds\right)\exp(-2i\sqrt{\lambda}t)\,dt\,.\end{aligned}$$

Hence, it is easy to see that for the energies λ which lie in both $S(V(x))$ and $S(V(x)\int_x^\infty V(t)\,dt)$ we have (recall that by our assumption $|\int_x^\infty V(t)\,dt| \leqq C_1x^{-\delta}$)

$$\left|\int_x^\infty V(t)\int_t^\infty V(s)\exp(-2i\sqrt{\lambda}s)\,ds\,dt\right| \leqq C(\lambda)x^{-\frac{1}{6}-\delta}\log x\,.$$

Applying the modified $\mathscr{I}+\mathscr{Q}_1$-transformation

$$z = (\mathscr{I}+\mathscr{Q}_1)z_1 \quad \text{with } \mathscr{Q}_1 = \begin{pmatrix} 0 & q_1 \\ \overline{q}_1 & 0 \end{pmatrix},$$

where $q_1 = \frac{1}{4\lambda}\int_x^\infty V(t)\int_t^\infty V(s)\exp(2iks)\,ds\,dt$, we get after a computation similar to the one leading from the system (4) to (18),

$$z_1' = \left(\begin{pmatrix} D+\frac{1}{2}(\overline{W}q - W\overline{q}) & 2\overline{q}_1\overline{D} \\ 2q_1D & \overline{D}-\frac{1}{2}(\overline{W}q - W\overline{q}) \end{pmatrix} + \mathscr{R}_1(x)\right)z_1\,. \tag{19}$$

Here $\mathscr{R}_1$ is a matrix with entries from L^1. The off-diagonal terms in the system (19) have a rate of decay $|q_1(x,\lambda)V(x)| \leqq C(\lambda)x^{-\frac{5}{6}-\delta}\log x$ for a.e. λ.

To complete the proof, we need to apply the $\mathscr{I}+\mathscr{Q}$ transformation several times. The following lemma shows that under the assumptions of the theorem we can do this and it also determines the number of necessary iterations and the set of full measure for which we can derive the asymptotics of solutions.

Lemma A.1. *Under the assumptions of Theorem* A.1, *the function*

$$f_n(t_1,\lambda) = V(t_1)\int_{t_1}^\infty V(t_2)\int_{t_2}^\infty V(t_3)\cdots\int_{t_n}^\infty V(t_{n+1})\exp(-2i\sqrt{\lambda}t_{n+1})\,dt_{n+1}$$

is integrable for every $\lambda \in \tilde{S}_n = \bigcap_{j=0}^n S_j$, *where*

$$S_j = S\left(V(t_1)\left(\int_{t_1}^\infty V(t_2)\,dt_2\right)^j\right)$$

and moreover,

$$\left|\int_x^\infty f_n(t_1,\lambda)\,dt_1\right| \leqq Cx^{-\frac{1}{6}-n\delta}\log x\,.$$

Proof. The proof is by induction. We have already checked that for $n = 1$ the statement is true. For the sake of simplicity, we assume integrability and give an apriori estimate for the tail integral. Of course, one can easily prove integrability by essentially the same (but a longer) computation. Now, integrating by parts, we find that

$$\int_x^\infty f_n(t_1, \lambda)\, dt_1$$
$$= \left(\int_x^\infty V(t_1)\, dt_1\right)\left(\int_x^\infty f_{n-1}(t_1, \lambda)\, dt_1\right) - \int_x^\infty f_{n-1}(t_1, \lambda)\left(\int_{t_1}^\infty V(t_2)\, dt_2\right) dt_1 .$$

According to the induction hypothesis and our assumption on V, the first summand on the right-hand side is bounded by $C(\lambda)x^{-\frac{1}{6}-n\delta}\log x$ for every $\lambda \in \tilde{S}_{n-1}$. In the second summand we perform integration by parts, integrating $V(t_1)\int_{t_1}^\infty V(t_2)\, dt_2$. As a result we get:

$$-\int_x^\infty f_{n-1}(t_1, \lambda)\int_{t_1}^\infty V(t_2)\, dt_1 dt_2 = -\frac{1}{2}\left(\int_x^\infty V(t_1)\, dt_1\right)^2 \int_x^\infty f_{n-2}(t_1, \lambda)\, dt_1$$
$$+\frac{1}{2}\int_x^\infty \left(\int_{t_1}^\infty V(t_2)\, dt_2\right)^2 f_{n-2}(t_1, \lambda)\, dt_1 .$$

As before, the first term decays as $Cx^{-\frac{1}{6}-n\delta}\log x$ for every $\lambda \in \tilde{S}_{n-2}$. We continue to integrate by parts the second term, integrating $V(t_1)(\int_{t_1}^\infty V(t_2)\, dt_2)^2$; we again get a sum of two terms the first of which (off-integral) is well-behaved while the second is again integrated by parts. We perform such a procedure n times and in the end, summarizing the result of the whole calculation we find that

$$\int_x^\infty f_n(t_1, \lambda)\, dt_1 = g(x, \lambda) + \frac{(-1)^n}{n!}\int_x^\infty V(t_1)\left(\int_{t_1}^\infty V(t_2)\, dt_2\right)^n \exp(2i\sqrt{\lambda}t_1)\, dt_1 ,$$

where $g(x, \lambda)$ satisfies the decay condition

$$|g(x, \lambda)| \leqq C(\lambda)x^{-\frac{1}{6}-n\delta}\log x$$

for any $\lambda \in \tilde{S}_{n-1}$. The last term obviously satisfies the same estimate for every $\lambda \in S_n$. Hence, as claimed, $\int_x^\infty f_n(t, \lambda) \leqq C(\lambda)x^{-\frac{1}{6}-n\delta}\log x$ for every $\lambda \in \tilde{S}_n$. □

The proven lemma justifies the iteration of the $\mathscr{I} + \mathscr{Q}$ transformation, since on the n^{th} iteration, to obtain q_n, we need to integrate $q_{n-1}D$ which, up to irrelevant energy dependent constants, is exactly f_n from the statement of the lemma. After the n^{th} iteration we arrive at a system

$$z_n' = \left(\begin{pmatrix} D + \frac{1}{2}(\overline{W}q - W\overline{q}) & 2\overline{q}_n\overline{D} \\ 2q_nD & \overline{D} - \frac{1}{2}(\overline{W}q - W\overline{q}) \end{pmatrix} + \mathscr{R}_n(x)\right) z_1 ,$$

where the matrix $\mathscr{R}_n$ has absolutely integrable entries. Also by the second statement of the lemma, $|q_n(x, \lambda)V(x)| \leqq C(\lambda)x^{-\frac{5}{6}-n\delta}\log x$ for every $\lambda \in \tilde{S}_n$ and is therefore absolutely integrable as soon as $n > \frac{1}{6\delta}$. Therefore, for the energies from the set $\tilde{S}_m$ of full measure, $m = [\frac{1}{6\delta}] + 1$ iterations are enough to bring the system to the form where we can apply Levinson's theorem (or, as was noticed in Sect. 1, just use the integral equation technique, bearing in mind that our unperturbed eigenfunctons are bounded). We also note that for every $\lambda \in \tilde{S}_m$, transforming back, we get solutions $\phi_\lambda(x)$ and $\overline{\phi_\lambda(x)}$ with the asymptotics

$$\phi_\lambda(x) = \Bigg(\exp(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int_0^x V(t)\,dt + \frac{i}{4\lambda}\int_0^x V(t)\int_t^\infty \sin(2\sqrt{\lambda}(t-s))V(s)\,ds\,dt\Bigg) \times (1 + O(x^{-\rho}\log x)),$$

where $\rho = \min(\varepsilon, m\delta - \frac{1}{6})$. The solutions $\phi_\lambda(x)$ and $\overline{\phi_\lambda(x)}$ are bounded and clearly linearly independent. This completes the proof of the theorem. □

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References

1. Behncke, H.: Absolute continuity of Hamiltonians with von Neumann–Wigner potentials Proc Am. Math. Soc **111**, 373–384 (1991)
2. Carleson, L.: On convergence and growth of partial sums of Fourier series. Acta Math. **116**, 135–157 (1966)
3. Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations New York: McGraw-Hill, 1955, p. 92
4 del Rio, R , Makarov, N., Simon, B.: Operators with singular continuous spectrum, II. Rank one operators Commun. Math. Phys. **165**, 59–67 (1994)
5. Delyon, F., Simon, B., Souillard, B.: From power pure point to continuous spectrum in disordered systems. Ann Inst. H. Poincaré **42**, 283–309 (1985)
6. Eastham, M.S P , Kalf, H.: Schrödinger-type Operators with Continuous Spectra. Research Notes in Mathematics **65**, London: Pitman Books Ltd., 1982
7. Eastham, M.S P., Mc Leod, J.B : The existence of eigenvalues imbedded in the continuous spectrum of ordinary differential operators Proc Roy. Soc. Edinburgh, **79A**, 25–34 (1977)
8. Gilbert, D.J., Pearson, D.B.: On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators. J Math. Anal. Appl. **128**, 30–56 (1987)
9. Gordon, A.: Pure point spectrum under 1-parameter perturbations and instability of Anderson localization. Commun. Math. Phys. **164**, 489–505 (1994)
10. Harris, W.A., Lutz, D A : Asymptotic integration of adiabatic oscillator. J Math. Anal. Appl. **51**, 76–93 (1975)
11. Hinton, D.B., Shaw, J.K.: Absolutely continuous spectra of second-order differential operators with short and long range potentials. SIAM J Math. Anal. **17**, 182–196 (1986)
12. Jitomirskaya, S., Last, Y.: In preparation
13 Kac, I.S.: Multiplicity of the spectrum of differential operator and eigenfunction expansion Izv. Akad. Nauk USSR **27**, 1081–1112 (1963) (Russian)
14. Khan, S., Pearson, D.B.: Subordinacy and spectral theory for infinite matrices. Helv. Phys. Acta **65**, 505–527 (1992)
15. Kotani, S., Ushiroya, N.: One-dimensional Schrödinger operators with random decaying potentials. Commun. Math. Phys **115**, 247–266 (1988)
16. Matveev, V.B.: Wave operators and positive eigenvalues for a Schrödinger equation with oscillating potential Theoret. and Math. Phys. **15**, 574–583 (1973)
17. Menchoff, D.: Sur les séries de fonctions orthogonales. Fund Math. **10**, 375–420 (1927)

18 Naboko, S N.: Dense point spectra of Schrödinger and Dirac operators Theor Math. **68**, 18–28 (1986)
19 Naboko, S N , Yakovlev, S I : On the point spectrum of discrete Schrödinger operators Funct Anal. **26**(2), 145–147 (1992)
20 Pearson, D.B : Singular continuous measures in scattering theory. Commun. Math. Phys **60**, 13–36 (1978)
21 Reed, M., Simon, B.: Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness New York: Academic Press, 1975
22. Reed, M , Simon, B : Methods of Modern Mathematical Physics, III Scattering Theory New York: Academic Press, 1979
23 Rudin, W : Real and Complex Analysis New York: McGraw-Hill, 1987
24 Simon, B : Operators with singular continuous spectrum, I General operators Ann Math **141**, 131–145 (1995)
25 Simon, B.: Some Schrödinger operators with dense point spectrum Proc Am Math Soc (to appear)
26 Simon, B : Bounded eigenfunctions and absolutely continuous spectra for one-dimensional Schrödinger operators Proc. Am Math Soc (to appear)
27 Stolz, G : Bounded solutions and absolute continuity of Sturm–Liouville Operators J Math Anal Appl **169**, 210–228 (1992)
28 Thurlow, C R : The point-continuous spectrum of second-order, ordinary differential operators Proc. Roy Soc Edinburgh **84A**, 197–211 (1979)
29 Weidman, J : Spectral Theory of Ordinary Differential Operators Lecture Notes in Mathematics **1258**, Berlin: Springer-Verlag, 1987
30. White, D A W.: Schrödinger operators with rapidly oscillating central potentials. Trans. Am. Math. Soc **275**, 641–677 (1983)
31 Zygmund, A.: A remark on Fourier transforms Proc Camb. Phil. Soc **32**, 321–327 (1936)

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