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Homotopy Classification of Minimizers of the Ginzburg–Landau Energy and the Existence of Permanent Currents

Jacob Rubinstein[1,★], **Peter Sternberg**[2,★★]

[1] Department of Mathematics, Technion, Israel Institute of Technology, Haifa 32000, Israel
[2] Department of Mathematics, Indiana University, Bloomington, IN 47405, USA

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Abstract: Our objective is to explain the phenomenon of permanent currents within the context of the Ginzburg–Landau model for superconductors. Using variational techniques we make a connection between the formation of permanent currents and the topology of the superconducting sample.

1. Introduction

Superconductors are materials whose electrical resistivity is effectively zero. They are also known for their peculiar magnetic properties. For example, the magnetic fluxoids, defined precisely in (3.15) below, can only have discrete values. Another interesting property of superconductors is the existence of permanent currents. Such currents are created by submitting a superconducting ring to an external magnetic field. The currents are observed to persist even after the applied field is turned off. The main objective of this paper is to explain the phenomenon of permanent currents and their relation to fluxoids. In particular, we consider the connection between the formation of permanent currents and the topology of the superconducting sample.

We shall use the Ginzburg–Landau theory to model the superconductor. For this purpose we denote the superconducting electrons density by $u(x)$, and set A to be the magnetic vector potential. In the absence of an applied magnetic field, the energy is described by the functional (see e.g. [A, DGP])

$$E_\varepsilon(u,A) = \int_\Omega \frac{1}{2}|(\nabla - iA)u|^2 + \varepsilon^{-2}V(u)\,dx + \int_{\mathbf{R}^3} \frac{1}{2}|\nabla \times A|^2\,dx\,, \qquad (1.1)$$

where $V(u) = \frac{1}{4}(|u|^2 - 1)^2$, Ω is a bounded domain in $\mathbf{R}^3$, u is a complex-valued function defined on Ω, $A : \mathbf{R}^3 \to \mathbf{R}^3$ and ε^{-1} is the Ginzburg–Landau parameter. Note that (1.1) consists of two terms. The first is the energy associated with the superconducting electrons, which are confined to the domain Ω, occupied by the

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superconducting material. The second term is the energy of the magnetic field which is defined over the entire space. It is the absence of an applied field which leads one to consider $A : \mathbf{R}^3 \to \mathbf{R}^3$ rather than $A : \Omega \to \mathbf{R}^3$.

Assume now that Ω is a bounded domain in $\mathbf{R}^3$ which is topologically equivalent to a solid torus and which has Lipschitz continuous boundary. Our main result is a theorem stating that for each integer m, and for sufficiently small ε, there exists a nontrivial minimizer $(u_\varepsilon^m, A_\varepsilon^m)$ of (1.1) associated with the m^{th} element of the homotopy group π_1 of mappings from $S^1 \to S^1$. Our proof consists of two parts. We first introduce the functional

$$E_0(u,A) = \int_\Omega \frac{1}{2}|(\nabla - iA)u|^2\,dx + \int_{\mathbf{R}^3} \frac{1}{2}|\nabla \times A|^2\,dx\,. \tag{1.2}$$

Minimizers (u^m, A^m) (lying in appropriate Sobolev spaces) are constructed and classified according to their 1-homotopy type using the direct method of the calculus of variations. We then show that each (u^m, A^m) can be approximated (for small ε) by a local minimizer of (1.1).

We note that the existence of stable nontrivial solutions to the Ginzburg–Landau equations in non-simply connected domains was first established by Jimbo and Morita [JM] who considered the special case where the domain Ω is a solid of revolution having convex cross-section. This assumption allows them to seek solutions via separation of variables and so involves quite different techniques than those used here.

We further remark that we limit ourselves here to domains that are topologically equivalent to the solid torus for simplicity, and because this is the most relevant case in physics. Nevertheless, our method can be easily extended to establish the classification by homotopy type of local minimizers for the Ginzburg–Landau functional in smooth multiply connected domains with arbitrary topology.

2. Preliminaries and Notation: Sobolev Spaces and 1-Homotopy Type

Before proceeding with our variational approach, we must introduce appropriate Sobolev spaces for the arguments u and A of the functionals E_0 and E_ε defined by (1.2) and (1.1). We begin with spaces for $u : \Omega \to \mathbb{C}$. Throughout, we are assuming $\Omega \subset \mathbf{R}^3$ is topologically equivalent to a solid torus.

As is standard, we will denote by $H^1(\Omega)$ the space of real-valued functions lying in $L^2(\Omega)$ and having weak derivatives lying in $L^2(\Omega)$. We denote by $H^1(\Omega; \mathbb{C})$ the space of complex-valued functions having this property. Then $H^1(\Omega; S^1)$ represents all $H^1(\Omega; \mathbb{C})$ functions having modulus 1 at almost every $x \in \Omega$. It is this space which we wish to partition according to 1-homotopy type. Recall that the 1-homotopy type of a *continuous* function $u : \Omega \to \mathbb{C}$, is the 1-homotopy type of the restriction of u to any 1-dimensional skeleton of a triangulation of Ω. Equivalently, and more simply, the 1-homotopy type corresponds to the winding number of u when restricted to any closed curve in Ω which loops once around the hole of the torus. The extension of homotopy classification from smooth maps to functions lying in certain Sobolev spaces is nontrivial and was accomplished by White [W]. We will take a slightly different tack than that used in [W] by combining the following two theorems, which are specific applications of much more general results by White and Bethuel–Zheng, respectively.

Theorem 2.1 (cf. [W], Theorem 2.1). *For each $K > 0$ there is an $\varepsilon > 0$ such that if f_1 and f_2 are Lipschitz mappings from Ω into S^1 satisfying the conditions*

$$\|f_1 - f_2\|_{L^2(\Omega)} < \varepsilon, \qquad \|\nabla f_i\|_{L^2(\Omega)} \leqq K \quad \textit{for } i = 1,2\,,$$

then f_1 and f_2 have the same 1-homotopy type.

Theorem 2.2 (cf. [B], Theorem 1, also [BZ]). *Smooth maps from any compact Riemannian 3-manifold M to S^1 are dense in $H^1(M;S^1)$*

From Theorem 2.1 one sees, in particular, that any two smooth maps which are sufficiently close in H^1-norm must have the same 1-homotopy type. From Theorem 2.2 we can then use approximation by smooth maps to define the 1-homotopy type of any function in $H^1(\Omega;S^1)$ via density. Henceforth, for each integer m, we will denote by $H^1_m(\Omega;S^1)$ those $H^1(\Omega;S^1)$ functions that can be approached in the H^1-norm by smooth functions having 1-homotopy type m. Fixing any smooth element Ψ in $H^1_1(\Omega;S^1)$ note that any element u of $H^1_m(\Omega;S^1)$ can be expressed as

$$u = \Psi^m e^{i\zeta}\,, \tag{2.1}$$

where $\zeta \in H^1(\Omega)$. In light of Theorem 2.1, note also that 1-homotopy type is preserved under bounded weak H^1-convergence. That is, if $\{u_j\}$ is a sequence of functions in $H^1_m(\Omega;S^1)$ satisfying a uniform bound

$$\|u_j\|_{H^1(\Omega;S^1)} \leqq C\,,$$

then there exists a function $u \in H^1_m(\Omega;\,S^1)$ such that $u_{j_k} \to u$ weakly in $H^1(\Omega;S^1)$. This compactness property allows one to apply the direct method from the calculus of variations while working within a given space $H^1_m(\Omega;S^1)$.

We discuss now appropriate spaces for the magnetic vector potential A. Given the nature of the functionals E_0 and E_ε an obvious space to work in is $H^1(\mathbf{R}^3;\mathbf{R}^3)$, which one can take as the closure of C_0^∞ vector fields v under the norm $\int_{\mathbf{R}^3} |\nabla v|^2\,dx$. However, the functionals (1.1) and (1.2) are invariant under the gauge transformation

$$u \to ue^{i\psi}, \qquad A \to A + \nabla\psi\,, \tag{2.2}$$

where ψ is an arbitrary function in $H^2_{\text{loc}}(\mathbf{R}^3;\mathbf{R})$. To fix the gauge, we will frequently find it convenient to assume $\operatorname{div} A = 0$. Specifically, we denote by $\check{H}^1_{\text{div}}$ the closure of C_0^∞ vector fields v having divergence zero under the norm $\int_{\mathbf{R}^3} |\nabla v|^2\,dx$. We note for future reference that the quantities $\int_{\mathbf{R}^3} |\nabla A|^2\,dx$ and $\int_{\mathbf{R}^3} |\nabla \times A|^2\,dx$ are equivalent for $A \in \check{H}^1_{\text{div}}(\mathbf{R}^3;\mathbf{R}^3)$

3. Existence of Minimizers

In this section we carry out the process of obtaining local minimizers of E_ε in a neighborhood of local minimizers of E_0. We begin by establishing existence and uniqueness of local minimizers for the functional E_0.

Theorem 3.1.

(i) *For each integer m there exists a minimizing pair* (u^m, A^m) *to the problem*

$$\inf E_0(u,A)$$

among all $u \in H^1_m(\Omega; S^1)$ *and all* $A \in \check{H}^1_{\mathrm{div}}(\mathbf{R}^3; \mathbf{R}^3)$.

(ii) *For each integer m and each pair* (u^m, A^m) *minimizing* E_0 *in* $H^1_m(\Omega; S^1) \times \check{H}^1_{\mathrm{div}}(\mathbf{R}^3; \mathbf{R}^3)$ *there exists a positive constant* γ_m *such that if* $(v, A) \in H^1(\Omega; S^1) \times \check{H}^1_{\mathrm{div}}(\mathbf{R}^3; \mathbf{R}^3)$ *satisfy*

$$\|v - u^m\|_{H^1(\Omega; S^1)} \leqq \gamma_m, \qquad E_0(v,A) \leqq E_0(u^m, A^m), \tag{3.1}$$

then $v = e^{i\alpha} u^m$ *for some constant* α *and* $A = A^m$.

Proof of (i). Existence follows easily from the direct method. A minimizing sequence $(u_j, A_j) \in H^1_m(\Omega; S^1) \times \check{H}^1_{\mathrm{div}}(\mathbf{R}^3; \mathbf{R}^3)$ will be bounded and so (weakly) compact using Theorem 2.1. Using that $|u_j| = 1$, we then write $E_0(u_j, A_j)$ as

$$E_0(u_j, A_j) = \int_\Omega \frac{1}{2}|\nabla u_j|^2 + \frac{1}{2}|A_j|^2 - 2\,\mathrm{Im}[\langle A_j, \nabla u_j\rangle \overline{u_j}]\,dx + \frac{1}{2}\int_{\mathbf{R}^3} |\nabla \times A_j|^2\,dx\,,$$

to most readily see that the integrals will be lower-semicontinuous under weak convergence in $H^1_m(\Omega; S^1) \times \check{H}^1_{\mathrm{div}}(\mathbf{R}^3; \mathbf{R}^3)$.

Proof of (ii). Let (u^m_1, A^m_1) be any pair minimizing E_0 in $H^1_m(\Omega; S^1) \times \check{H}^1_{\mathrm{div}}(\mathbf{R}^3; \mathbf{R}^3)$. We will first argue that if (u^m_2, A^m_2) is any other pair in this space satisfying $E_0(u^m_1, A^m_1) = E_0(u^m_2, A^m_2)$, then

$$u^m_2 = u^m_1 e^{i\alpha} \tag{3.2}$$

for some constant α and

$$A^m_1 = A^m_2\,. \tag{3.3}$$

In view of (2.1), we will now fix a smooth function $\Psi \in H^1_1(\Omega; S^1)$. For convenience, we will take as our Ψ the function which minimizes the Dirichlet integral among $H^1_1(\Omega; S^1)$ competitors (i.e. Ψ is the harmonic map in this homotopy class). The existence of such a minimizer follows from Theorem 2.1 using the direct method. Henceforth we write any element v of the space $H^1_m(\Omega; S^1)$ as $v = \Psi^m e^{i\zeta}$ for some $\zeta \in H^1(\Omega)$. We will denote by $\theta : \Omega \to \mathbf{R}$ the multi-valued function $\arg \Psi$. Note that the function $\nabla\theta$ $(= -i\Psi^{-1}\nabla\Psi)$ is a smooth function on Ω and that $\Delta\theta = 0$. Thus consideration of the functional E_0 restricted to $H^1_m(\Omega; S^1) \times H^1(\mathbf{R}^3; \mathbf{R}^3)$ is seen to be equivalent to consideration of the functional E^m_0 depending on arguments $(\zeta, A) \in H^1(\Omega) \times H^1(\mathbf{R}^3; \mathbf{R}^3)$ given by

$$E^m_0(\zeta, A) = \int_\Omega \frac{1}{2}|m\nabla\theta + \nabla\zeta|^2 + \frac{1}{2}|A|^2 - \langle A, m\nabla\theta + \nabla\zeta\rangle + \frac{1}{2}\int_{\mathbf{R}^3} |\nabla \times A|^2\,dx\,.$$

Note that in this formulation, gauge invariance (2.2) can be expressed as

$$E^m_0(\zeta, A) = E^m_0(\zeta + \psi, A + \nabla\psi) \quad \text{for any } \psi \in H^2_{\mathrm{loc}}(\mathbf{R}^3)\,. \tag{3.4}$$

To establish (3.2) and (3.3) we write u_1^m and u_2^m as $u_i^m = e^{i(m\theta+\phi_i^m)}$, $i = 1,2$. Since (ϕ_i^m, A_i^m) minimize E_0^m among competitors in $H^1(\Omega) \times \check{H}^1_{\mathrm{div}}(\mathbf{R}^3;\mathbf{R}^3)$ for $i = 1,2$, the gauge invariance (3.4) implies that these pairs minimize E_0^m among competitors in $H^1(\Omega) \times H^1(\mathbf{R}^3;\mathbf{R}^3)$ as well. Taking variations of E_0^m with respect to its first and second arguments separately then implies

$$\int_\Omega \langle \nabla\phi_i^m, \nabla\beta\rangle\, dx = 0 \quad i = 1,2 \tag{3.5}$$

for all $\beta \in H^1(\Omega)$ and

$$\int_\Omega \langle A_i^m - (m\nabla\theta + \nabla\phi_i^m), B\rangle\, dx + \int_{\mathbf{R}^3} \langle \nabla \times A_i^m, \nabla \times B\rangle = 0 \quad i = 1,2\,, \tag{3.6}$$

for all $B \in H^1(\mathbf{R}^3;\mathbf{R}^3)$. In (3.5) we have used the harmonicity of θ and that $\operatorname{div} A_i^m = 0$. As (3.5) implies that the functions ϕ_i^m are themselves harmonic (and in particular smooth), we may introduce functions $\tilde{\phi}_i^m$, $i = 1,2$ as any $H^2_{\mathrm{loc}}(\mathbf{R}^3)$ extensions of ϕ_i^m for $i = 1,2$ respectively. Then for $i = 1,2$, let $B_i = A_i^m - \nabla\tilde{\phi}_i^m$ and consider the four relations derived by setting $i = 1$ in (3.6) with $B = B_1$, $i = 2$ with $B = B_1$, $i = 1$ with $B = B_2$ and $i = 2$ with $B = B_2$. Adding the first and fourth and then subtracting the second and third of these equations yields

$$\int_\Omega |(A_1^m - A_2^m) - \nabla(\phi_1^m - \phi_2^m)|^2\, dx + \int_{\mathbf{R}^3} |\nabla \times (A_1^m - A_2^m)|^2\, dx = 0\,. \tag{3.7}$$

Since $\operatorname{div} A_i^m = 0$, we conclude from this that

$$A_1^m - A_2^m = \nabla\psi \quad \text{for some } \psi \text{ such that } \nabla\psi = \nabla\phi_1^m - \nabla\phi_2^m \text{ in } \Omega\,, \tag{3.8}$$

and such that $\Delta\psi = 0$ in $\mathbf{R}^3$ with $\nabla\psi \in \check{H}^1(\mathbf{R}^3;\mathbf{R}^3)$. In particular, by Sobolev imbedding, $\nabla\psi$ is a harmonic function in $L^6(\mathbf{R}^3;\mathbf{R}^3)$; hence ψ is a constant and (3.2) and (3.3) are established.

We have now shown the uniqueness, up to a constant rotation, of any minimizing pair (u^m, A^m) in $H^1_m(\Omega;S^1) \times \check{H}^1_{\mathrm{div}}(\mathbf{R}^3;\mathbf{R}^3)$. However, by Theorem 2.1, there exists a constant $\gamma_m > 0$, such that any $v \in H^1(\Omega;S^1)$ satisfying

$$\|v - u^m\|_{H^1(\Omega;S^1)} \leqq \gamma_m \tag{3.9}$$

must lie in the space $H^1_m(\Omega;S^1)$. Hence, if (3.1) holds, then necessarily $E_0(v,A) = E_0(u^m, A^m)$ and we may apply the previous argument to find

$$v = u^m e^{i\alpha} \quad \text{and} \quad A = A^m\,. \qquad \square \tag{3.10}$$

Having established existence and uniqueness of minimizers of E_0 in each 1-homotopy class, we now establish the existence of corresponding local minimizers to E_ε.

Theorem 3.2. *For each integer m there exists a positive number* ε_0 *such that for all positive* $\varepsilon < \varepsilon_0$ *the functional* E_ε *possesses a local minimizer* $(u_\varepsilon^m, A_\varepsilon^m)$. *Furthermore,* $(u_\varepsilon^m, A_\varepsilon^m)$ *converges to* (u^m, A^m) *in* $H^1(\Omega;\mathbf{C}) \times \check{H}^1_{\mathrm{div}}(\mathbf{R}^3;\mathbf{R}^3)$ *as* $\varepsilon \to 0$, *where* (u^m, A^m) *minimizes* E_0 *in* $H^1_m(\Omega;S^1) \times \check{H}^1_{\mathrm{div}}(\mathbf{R}^3;\mathbf{R}^3)$.

Remark. In particular, the local minimizer $(u_\varepsilon^m, A_\varepsilon^m)$ will satisfy the Ginzburg–Landau system of equations that arises as the Euler–Lagrange equations for the functional E_ε:

$$(\nabla - iA)^2 u + \frac{1}{\varepsilon^2}(1 - |u_\varepsilon^m|^2)u_\varepsilon^m = 0 \quad \text{in } \Omega \tag{3.11}$$

and

$$\nabla \times \nabla \times A_\varepsilon^m + (A_\varepsilon^m |u_\varepsilon^m|^2 - \operatorname{Im}[\overline{u_\varepsilon^m} \nabla u_\varepsilon^m])\chi_\Omega = 0 \quad \text{in } \mathbf{R}^3 , \tag{3.12}$$

where χ_Ω denotes the characteristic function equaling 1 in Ω and 0 elsewhere.

Proof. Fix any integer m and consider the problem

$$\inf E_\varepsilon(u, A)$$

minimized over the set

$$\{(u, A) \in H^1(\Omega; \mathbf{C}) \times \check{H}^1_{\text{div}}(\mathbf{R}^3; \mathbf{R}^3) : \|u - u^m\|_{H^1(\Omega;\, \mathbf{C})} \leqq \gamma_m\} ,$$

where (u^m, A^m) minimizes E_0 in $H^1_m(\Omega; S^1) \times \check{H}^1_{\text{div}}(\mathbf{R}^3; \mathbf{R}^3)$ and the constant γ_m is taken from Theorem 3.1. Again the direct method yields a solution $(u_\varepsilon^m, A_\varepsilon^m)$ to this problem. We will show that $u_\varepsilon^m \to u^m$ in $H^1(\Omega; \mathbf{C})$ as $\varepsilon \to 0$, thus proving the theorem. Since

$$E_0(u_\varepsilon^m, A_\varepsilon^m) \leqq E_\varepsilon(u_\varepsilon^m, A_\varepsilon^m) \leqq E_\varepsilon(u^m, A^m) = E_0(u^m, A^m) , \tag{3.13}$$

we find immediately that

$$\int_\Omega V(u_\varepsilon^m) \leqq E_0(u^m, A^m)\varepsilon^2 ,$$

and therefore $|u_\varepsilon^m| \to 1$ pointwise a.e. as $\varepsilon \to 0$. Also the condition $\|u_\varepsilon^m - u^m\|_{H^1(\Omega;\mathbf{C})} \leqq \gamma_m$ implies that for a subsequence $\{\varepsilon_j\} \to 0$ one has

$$u_{\varepsilon_j}^m \to U \quad \text{as } \varepsilon_j \to 0$$

weakly in $H^1(\Omega; \mathbf{C})$ and strongly in $L^2(\Omega; \mathbf{C})$ for some $H^1(\Omega; S^1)$ function U. Applying (3.13) to the sequence $\{A_\varepsilon^m\}$, we find that

$$A_{\varepsilon_j}^m \to A \quad \text{as } \varepsilon_j \to 0$$

weakly in $\check{H}^1_{\text{div}}(\mathbf{R}^3; \mathbf{R}^3)$ and strongly in $L^2(\Omega; \mathbf{R}^3)$ for some $\check{H}^1_{\text{div}}(\mathbf{R}^3; \mathbf{R}^3)$ function A. Furthermore, by lower-semicontinuity of the H^1-norm and of the expression $\int_{\mathbf{R}^3} |\nabla \times A|^2\, dx$ we have

$$\begin{aligned} \liminf_{\varepsilon_j \to 0} \int_\Omega |\nabla u_{\varepsilon_j}^m|^2\, dx &\geqq \int_\Omega |\nabla U|^2\, dx, \\ \liminf_{\varepsilon_j \to 0} \int_{\mathbf{R}^3} |\nabla \times A_{\varepsilon_j}^m|^2\, dx &\geqq \int_{\mathbf{R}^3} |\nabla \times A|^2\, dx , \end{aligned} \tag{3.14}$$

and therefore,

$$E_0(U, A) \leqq \liminf_{\varepsilon_j \to 0} E_0(u_{\varepsilon_j}^m, A_{\varepsilon_j}^m) . \tag{3.15}$$

Taking a limsup in the inequalities (3.13) and combining this with (3.15) we find that $U \in H^1(\Omega; S^1)$ satisfies

$$E_0(U, A) \leqq E_0(u^m, A^m) \quad \text{and} \quad \|U - u^m\|_{H^1(\Omega;\mathbf{C})} \leqq \gamma_m .$$

Hence, by Theorem 3.1, $U = e^{i\alpha}u^m$ for some α and $A = A^m$. Relabeling u_ε^m by $e^{-i\alpha}u_\varepsilon^m$ we find that $u_\varepsilon^m \to u^m$ weakly in $H^1(\Omega;\mathbb{C})$ and $A_\varepsilon^m \to A^m$ weakly in $\check{H}^1_{\text{div}}(\mathbf{R}^3;\mathbf{R}^3)$. However, (3.13) and (3.15) also imply that

$$\lim_{\varepsilon_j \to 0} E_0(u_{\varepsilon_j}^m, A_{\varepsilon_j}^m) = E_0(u^m, A^m) .$$

Together with (3.14), this implies that the L^2-norms of ∇u_ε^m and $\nabla \times A_\varepsilon^m$ converge to those of ∇u^m and $\nabla \times A^m$ on Ω and $\mathbf{R}^3$ respectively. Hence, the convergence is in the strong sense and the theorem is proved. □

Fluxoid quantization. The homotopy classification we have introduced is equivalent to the physical phenomena of *fluxoid quantization* (see [KZ]). To explain, let σ be a closed curve in Ω looping once around its hole. Then let Σ be any surface bounded by σ. Writing $u = \rho e^{i\Phi}$, we denote by $H = \nabla \times A$ the magnetic field and by $J = \rho^2(\nabla\Phi - A)$ the superconducting current. Then the fluxoid is defined by

$$FL = \int_\Sigma H \, d\Sigma + \int_\sigma \frac{J}{\rho^2} \, d\sigma . \tag{3.16}$$

Using Stokes theorem, we get

$$FL = \int_\sigma \nabla\Phi \, d\sigma .$$

For the solutions obtained in this section, $\Phi = \Phi_\varepsilon^m$ takes the form $\Phi = m\theta + \phi_\varepsilon^m$, where ϕ_ε^m is smooth. It then follows that the fluxoid value of an m-type minimizer is $2\pi m$. Hence, the m-type minimizers can be classified according to the values of their fluxoids.

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