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Deformations of Compact Quantum Groups via Rieffel's Quantization

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Abstract: It is shown that compact quantum groups containing torus subgroups can be deformed into new compact quantum groups under Rieffel's quantization. This is applied to showing that the two classes of compact quantum groups K_q^u and K_q studied by Levendorkii and Soibelman are strict deformation quantization of each other, and that the quantum groups $A_u(m)$ have many deformations.

1. Introduction

This paper answers in the affirmative the following two questions of Rieffel's: (1) Are Drinfeld's algebraic twistings K_q^u of the quantum groups K_q, as studied in [20, 12, 13], strict deformation quantizations of K_q? (2) Can the quantum groups $A_u(m)$ constructed in [25, 26] be deformed? The key to answering these questions is a result, in the spirit of [17], on deformations of arbitrary compact quantum groups (instead of only compact groups as treated there). We believe this result is of interest in its own right.

We now describe the results of this paper in more detail. Let A be a Woronowicz Hopf C^*-algebra in the sense of [30, 2, 25, 26], whose coproduct is denoted by Φ. We will also call it a compact quantum group, referring to its dual object (cf. [26]). Suppose that the quantum group A has an abelian Lie subgroup T. This means that there is a surjective C^*-algebra homomorphism π from A to $C(T)$ preserving the coproducts (see [25, 26]). For any element h in T, denote by E_h the corresponding evaluation functional on $C(T)$. Assume that η is a continuous homomorphism from a vector space Lie group $\mathbb{R}^n$ to T, where n is allowed to be different from the dimension of T. Define an action α of $\mathbb{R}^d := \mathbb{R}^n \times \mathbb{R}^n$ on the C^*-algebra A as follows:

$$\alpha_{(s,u)} = \lambda_{\eta(s)}\rho_{\eta(u)}.$$

In the above,

$$\lambda_{\eta(s)} = (E_{\eta(-s)}\pi \otimes id)\Phi, \quad \rho_{\eta(u)} = (id \otimes E_{\eta(u)}\pi)\Phi,$$

where id is the identity map on A. For any skew-symmetric operator S on $\mathbb{R}^n$, one may apply Rieffel's quantization procedure [16] for the action α above to obtain a deformed C^*-algebra A_J, where $J = S \oplus (-S)$. The family $A_{\hbar J}$ ($\hbar \in \mathbb{R}$) is a strict deformation quantization of A (see Chapter 9 of [16]). Our main result for answering Rieffel's questions is the following theorem.

1.1. Theorem. (See 3.9) *The deformation A_J is also a compact quantum group (namely Woronowicz Hopf C^*-algebra) containing T as a (quantum) subgroup; A_J is a compact matrix quantum group if and only if A is.*

The construction of the action α above is a rather straightforward reformulation, in the general compact quantum group setting, of the construction of [17], where Rieffel deals with only compact groups. The theorem above is a generalization of the main theorem in the paper cited above. However, the proof of the above theorem is quite different from the proof of the corresponding theorem in [17]. In [17], Rieffel works with the algebra of smooth functions on the undeformed compact Lie group to obtain the coproduct on the deformed algebra, which is remarkable in that it gives the first example of a deformation quantization of the entire algebra of smooth functions on a smooth manifold. In the present setting, since it is not clear what it should mean by "the algebra of smooth functions on a compact (matrix) quantum group," we work with the Krein algebra $\mathscr{A}$ of A (in the sense of [25, 28]) to obtain the quantum group structure on A_J. We refer the reader to [18] for a generalization of the construction in [17] to the case of non-compact Lie groups, and to [10, 5] for deformations of non-compact Lie groups by using techniques that are different from those used by Rieffel (See also [11]).

Using the construction of the Drinfeld-Jimbo quantum groups [4, 8], Levendorskii and Soibelman introduced two families of compact quantum groups K_q and K_q^u for each simple compact Lie group K [20, 12, 13]. The maximal torus T of K is still a subgroup of both the quantum groups K_q and K_q^u. Applying the construction above (with an appropriate skew-symmetric operator S), we have the following result, which answers Rieffel's first question.

1.2. Theorem. (See 4.2) *The compact quantum groups K_q and K_q^u are strict deformation quantization of each other in the sense of Rieffel [16].*

Applying the above construction to the quantum groups $A_u(m)$ constructed in [26] (since they contain many abelian Lie subgroups) yields an answer to Rieffel's second question.

1.3. Theorem. (See 5.1) *The quantum groups $A_u(m)$ can be deformed under Rieffel's quantization. The deformed quantum groups $A_u(m)_J$ are quantum subgroups of $A_u(m)$. The quantum groups constructed in [17] are quantum subgroups of $A_u(m)_J$ for suitable m.*

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2. Preliminaries

For convenience of the reader, we recall in this section some basic definitions. For use in later sections of this paper, we also collect a few elementary results concerning actions of groups on Hopf C^*-algebras. For a positive integer d, $M_d = B(\mathbb{C}^d)$ denotes the C^*-algebra of $d \times d$ matrices over the complex numbers $\mathbb{C}$.

2.1. Definition. (cf. [30, 2, 25, 26]) *A* **Woronowicz Hopf C^*-algebra** *(or a* **compact quantum group***) is a unital C^*-algebra A together with a dense *-subalgebra $\mathcal{A}$ generated by u^p_{ij} (where $p \in \mathsf{N}$ and $i, j \in \{1, \ldots, d_p\}$, and N is an indexing set), a C^*-homomorphism $\Phi: \ A \longrightarrow A \otimes A$, and a linear algebra-antihomomorphism $\kappa: \mathcal{A} \longrightarrow \mathcal{A}$, such that,*

(1) The matrix $u^p = (u^p_{ij})$ is a unitary (or equivalently, an invertible) element of $M_{d_p} \otimes A$, for all $p \in \mathsf{N}$;

(2) For $p \in \mathsf{N}$, and $i, j \in \{1, \ldots, d_p\}$, $\Phi(u^p_{ij}) = \sum_{k=1}^{d_p} u^p_{ik} \otimes u^p_{kj}$;

(3) For $a \in \mathcal{A}$, and $p \in \mathsf{N}$, $\kappa(\kappa(a^*)^*) = a$, and $(id \otimes \kappa)(u^p) = (u^p)^{-1}$.

We denote the above Woronowicz Hopf C^*-algebra simply by A. For a Woronowicz Hopf C^*-algebra A, we will also call it a compact quantum group, referring to its dual object G, as in the case of a compact group G with Woronowicz Hopf C^*-algebra $A = C(G)$ (cf. [25, 26]). The algebra $\mathcal{A}$ is the generic example of Krein algebras [28, 25].

2.2. Let A and B be compact quantum groups. *A* **homomorphism** *from the quantum group B to the quantum group A is defined to be a unital C^*-algebra homomorphism from A to B that preserves the coproducts. The quantum group B is called an* **embedded quantum subgroup** *(or simply* **quantum subgroup***) of A if there exists a surjective C^*-algebra homomorphism from A to B that preserves the coproducts (cf [25, 26]).*

2.3. We define a **Hopf C^*-algebra** *(see e.g. [2]) to be a C^*-algebra A (unital or not) such that there is a nondegenerate (see [2]) C^*-algebra homomorphism Φ from A to the multiplier algebra $M(A \otimes A)$ satisfying the coassociativity condition: $(id \otimes \Phi)\Phi = (\Phi \otimes id)\Phi$.*

A **left invariant mean** *(resp.* **right invariant mean***) on a Hopf C^*-algebra A is defined to be a state ϕ on A that satisfies*

$$(\psi \otimes \phi)\Phi = \phi \quad (\textit{resp.} \quad (\phi \otimes \psi)\Phi = \phi)$$

for any state ψ on A. If ϕ is both a left and right invariant mean, we simply call it an **invariant mean**.

On every compact quantum group, there exists a unique invariant mean, which is called the Haar measure of the quantum group ([30, 25, 26, 23]).

2.4. Notation. In the following, let A be a compact quantum group such that the space $X(A)$ of non-zero $*$-homomorphisms from A to the algebra $\mathbb{C}$ of complex numbers is non-empty. Then $X(A)$ is a compact group, and it is a compact Lie group if A is a compact matrix quantum group in the sense of [30]. Consequently the counit of A is continuous. The group $X(A)$ is called the **maximal subgroup** of A (see the end of Sect. 2 of [26]).

Let $x, y \in X(A)$. Define endomorphisms λ_x, ρ_y of the C^*-algebra A by

$$\lambda_x = (E_{x^{-1}} \otimes id)\Phi, \quad \rho_y = (id \otimes E_y)\Phi,$$

where E_x is identified with the homomorphism x from A to $\mathbb{C}$ and x^{-1} is the inverse of x in the group $X(A)$. In the notation of [30] (see near the end of Sect. 1 there),

$$\lambda_x(a) = a * E_{x^{-1}}, \quad \rho_y(a) = E_y * a$$

for $a \in A$. Since A is a bimodule over the algebra A^* of continuous functionals of A (in the sense of Sect. 1.2.4 of [25]) and $X(A)$ is a group, we see that both λ_x and ρ_y are automorphisms of the C^*-algebra A commuting with each other, and that $x \longrightarrow \lambda_x$ and $y \longrightarrow \rho_y$ define strongly continuous actions of the compact group $X(A)$ on A. In other words, we have C^*-dynamical systems $(A, X(A), \lambda)$ and $(A, X(A), \rho)$. The actions λ and ρ clearly commute with each other.

2.5. Proposition. *Let $x \in X(A)$, and let (a_{kl}) be a finite dimensional representation of the quantum group A (not to be confused with the representation of the algebra A). Then $\lambda_x(a_{ij})$ (resp. $\rho_x(a_{ij})$) is a linear combination of the coefficients a_{kj} (resp. a_{il}), with k (resp. l) varying.*

Proof. The proof follows from unraveling the definitions of the actions λ and ρ, using $\Phi(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$. Q.E.D.

2.6. Proposition. *We have $(id \otimes \lambda_x)\Phi = (\rho_{-x} \otimes id)\Phi$ for any $x \in X(A)$.*

Proof. This follows immediately from the coassociativity of Φ. Q.E.D.

2.7. Proposition. *We have the equalities*

$$\Phi\lambda_x = (\lambda_x \otimes id)\Phi, \quad \Phi\rho_x = (id \otimes \rho_x)\Phi,$$

for any $x \in X(A)$.

Proof. Let (a_{ij}) be any finite dimensional representation of the quantum group A, so $\Phi(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$. Then

$$\begin{aligned}\Phi\lambda_x(a_{ij}) &= \sum_{kl}(E_{x^{-1}} \otimes id \otimes id)(a_{ik} \otimes a_{kl} \otimes a_{lj}) \\ &= (E_{x^{-1}} \otimes id \otimes id)(\Phi \otimes id)\Phi(a_{ij}) \\ &= ((E_{x^{-1}} \otimes id)\Phi \otimes id)\Phi(a_{ij}) = (\lambda_x \otimes id)\Phi(a_{ij}).\end{aligned}$$

From this we see that $\Phi\lambda_x = (\lambda_x \otimes id)\Phi$.

Similarly we have $\Phi\rho_x = (id \otimes \rho_x)\Phi$. Q.E.D.

2.8. Proposition. *The equalities*

$$\lambda_x\kappa = \kappa\rho_x, \quad \rho_x\kappa = \kappa\lambda_x$$

are valid as maps on $\mathscr{A}$ for any $x \in X(A)$.

Proof. Since $E_{x^{-1}}\kappa = E_x$, using $\Phi\kappa = \sigma(\kappa \otimes \kappa)\Phi$ (see [30, 25]), we have

$$\begin{aligned}\lambda_x\kappa &= (E_{x^{-1}} \otimes id)\Phi\kappa = (E_{x^{-1}} \otimes id)\sigma(\kappa \otimes \kappa)\Phi \\ &= (id \otimes E_{x^{-1}})(\kappa \otimes \kappa)\Phi = (\kappa \otimes E_{x^{-1}}\kappa)\Phi \\ &= \kappa(id \otimes E_x)\Phi = \kappa\rho_x,\end{aligned}$$

where σ is the flip map on $A \otimes A$ sending $a \otimes b$ to $b \otimes a$. Similarly, we have $\rho_x\kappa = \kappa\lambda_x$. Q.E.D.

2.9. Remark. We note that the actions λ and ρ preserve the Haar measure of the quantum group. This is an immediate consequence of the definition of these actions and invariance properties of the Haar measure.

2.10. If A is a (not necessarily unital) Hopf C^*-algebra such that the space $X(A)$ of non-zero $*$-homomorphisms from A to $\mathbb{C}$ is non-empty, then $X(A)$ is a locally compact semi-group, and the observations in 2.4, 2.6, 2.9 are still valid if we replace $X(A)$ by a subgroup of $X(A)$, and 2.7 and 2.8 are also valid for x in subgroups of $X(A)$ if A has a continuous antipodal map.

2.11. Deformation quantization by actions of $\mathbb{R}^d$. We briefly recall Rieffel's deformation quantization [16] from actions of the abelian Lie group $V = \mathbb{R}^d$ on (not necessarily commutative!) C^*-algebras. Let (A, V, α) be a C^*-dynamical system. Denote by A^∞ the algebra of smooth vectors of the action α endowed with the Frechet topology coming from the action of the Lie algebra of V on A^∞. Let J be any skew-symmetric operator on V. For $a, b \in A^\infty$, define $a \times_J b$ by the following oscillatory integral (for discussions on general oscillatory integrals, see Chapter 1 of [16]):

$$a \times_J b = \int_V \int_V \alpha_{Ju}(a)\alpha_v(b)e(u \cdot v),$$

where $u \cdot v$ is the inner product of u and v on V and e is the function $e(t) = \exp(2\pi i t)$ for a real number t. Then A^∞ is an associative *-algebra under the product $\times_J$ and the involution of A restricted to A^∞. Let $\mathscr{S}^A$ be the space of A-valued smooth functions on V such that the products of their derivatives with any complex valued polynomials on V are bounded under the evident super norm of $\mathscr{S}^A$. Then $\mathscr{S}^A$ is a pre-Hilbert (right) A-module with A-valued inner product defined by

$$< f, g >_A = \int_V f(v)^* g(v) dv$$

for $f, g \in \mathscr{S}^A$. For each $a \in A$, define an operator $L_{\tilde{a}}$ on $\mathscr{S}^A$ by

$$L_{\tilde{a}}(f)(x) = \int_V \int_V \tilde{a}(x + Ju) f(x + v) e(u \cdot v)$$

for $f \in \mathscr{S}^A$, where $\tilde{a}(x) = \alpha_x(a)$ for $x \in V$. Then L_a is a bounded operator having an adjoint on the pre-Hilbert A-module $\mathscr{S}^A$, and $a \mapsto L_{\tilde{a}}$ is a $*$-representation of $(A^\infty, \times_J)$ into the C^*-algebra of bounded operators on $\mathscr{S}^A$. Now define

$$\|a\|_J = \|L_{\tilde{a}}\|.$$

Then $\| \ \|_J$ is a pre-C^*-norm on the algebra A^∞ endowed with the new product $\times_J$. The completion of this pre-C^*-algebra is denoted by A_J. To summarize: for every quadruple (A, V, α, J), Rieffel associates a deformed C^*-algebra A_J; the family $A_{\hbar J}$ ($\hbar \in \mathbb{R}$) is a strict deformation quantization of A.

3. Deformations of Compact Quantum Groups

The goal of this section is to prove Theorem 1.1. We first fix the set-up of the section.

3.1. Assumptions and Notation. Our standing assumptions throughout this section are as follows: A is any compact quantum group (with the notation as in 2.1), T is a subgroup of the quantum group A, where T is a compact abelian Lie group. Namely, there is a surjective homomorphism π of C^*-algebras from A to $C(T)$ preserving the coproducts. For any element h in T, denote by E_h the corresponding evaluation functional on $C(T)$. Let η be a continuous homomorphism from a vector space Lie group $\mathbb{R}^n$ to T, and S a skew-symmetric operator on $\mathbb{R}^n$. The letters $m, I, id, \Phi, \epsilon, \kappa$ will denote, respectively, the product map from $\mathcal{A} \otimes \mathcal{A}$ to $\mathcal{A}$, the unit of A, the identity map on A, the coproduct of A, the counit of A, the coinverse of A.

We remark that the above assumptions on A are equivalent to the the following assumptions: the space $X(A)$ of non-zero $*$-homomorphism from A to the algebra of complex numbers is a compact group, and there is a continuous injective group homomorphism from T to $X(A)$. In particular the counit of A is continuous. These assumptions are fulfilled by all the nontrivial compact quantum groups constructed so far (see e.g. Sects. 4 and 5 in the following).

Put

$$\lambda_{\eta(s)} = (E_{\eta(-s)}\pi \otimes id)\Phi, \quad \rho_{\eta(u)} = (id \otimes E_{\eta(u)}\pi)\Phi.$$

Note that the meanings of λ and ρ here are slightly different from those in Sect. 2, but this should not cause confusion. The action α of $\mathbb{R}^d := \mathbb{R}^n \times \mathbb{R}^n$ (so $d = 2n$) on the C^*-algebra A is defined by

$$\alpha_{(s,u)} = \lambda_{\eta(s)}\rho_{\eta(u)},$$

where $s, u \in \mathbb{R}^n$ (the u here is not to be confused with the u^p's in 2.1). Let $J = S \oplus (-S)$. Applying Rieffel's strict deformation quantization [16] to the quadruple $(A, \mathbb{R}^d, \alpha, J)$, one obtains a deformed C^*-algebra A_J.

Our goal in this section is to show that A_J is also a compact quantum group. First we show that the space $\mathcal{A}$ is an algebra under the product $\times_J$ and is dense in the C^*-algebra A_J. This is the main reason why we are able to work with the Krein algebra $\mathcal{A}$, instead of a larger algebra, as is the case in [17].

3.2. Proposition. *The space $\mathcal{A}$ is contained in the space A^∞ of smooth vectors of the action α. Under the product and involution of A_J, the space $\mathcal{A}$ is an involutive algebra. The space $\mathcal{A}$ is dense in A^∞ under the Frechet topology of A^∞ and is therefore dense in the C^*-algebra A_J under the C^*-norm of A_J.*

Proof. Let $a_{ij} \in \mathcal{A}$ be a coefficient of a finite dimensional representation (a_{kl}) of the quantum group A. Then by Proposition 2.5, $\alpha_{(s,u)}(a_{ij})$ is a linear combination of the a_{kl}'s. The coefficients in this linear combination are smooth complex valued functions on $\mathbb{R}^d$, because the map π sends the Krein algebra $\mathcal{A}$ of A to that of $C(T)$ (see [25, 26]) and the Krein algebra of $C(T)$ is contained in the algebra of smooth functions on T (note that η is automatically smooth). This proves the first statement of the proposition.

Let (b_{ij}) be another finite dimensional representation of the quantum group A. By definition, the formula for $a_{ij} \times_J b_{kl}$ is

$$
\begin{aligned}
a_{ij} \times_J b_{kl} &= \int \alpha_{J(s,u)}(a_{ij})\alpha_{(t,v)}(b_{kl})e(st + uv) \\
&= \int \lambda_{\eta(Ss)}\rho_{\eta(-Su)}(a_{ij})\lambda_{\eta(t)}\rho_{\eta(v)}(b_{kl})e(st + uv),
\end{aligned}
$$

which, by Proposition 2.5, is easily seen to be a linear combination of the products $a_{i'j'}b_{k'l'}$'s. Hence $a_{ij} \times_J b_{kl}$ is in the subspace $\mathscr{A}$ of A^∞, though the above integral is defined in the Frechet space A^∞ and requires the completeness of A^∞ (see [16] for the precise definition of the integral). Since elements of $\mathscr{A}$ are linear combinations of the coefficients of finite dimensional representations of the quantum group A by virtue of the Peter-Weyl theorem for compact quantum groups (cf. [30, 25, 26]), we see that $\mathscr{A}$ is indeed an algebra under the product $\times_J$. Since $\mathscr{A}$ is an involutive algebra under the product of A and the action α preserves its involution, we also see from the above that $\mathscr{A}$ is an involutive algebra.

Let $\iota : A^\infty \longrightarrow A$ be the inclusion map from the Frechet space A^∞ to the Frechet space A, where the Frechet topology on A is given by the C^*-norm thereon. Then ι is equivariant and continuous, where A^∞ and A are endowed with the actions $\beta := \alpha|_{A^\infty}$ and α, respectively. Hence we can invoke Proposition 1.1 of [17], with (A^∞, β) and (A, α) here being the (B, β) and (A, α) there — the conclusion of that proposition is still true if we replace $B^\infty = (A^\infty)^\infty$ there by $\mathscr{A}$, because $(A^\infty)^\infty = A^\infty$ and for $b \in \mathscr{A}$ the element $\beta_\phi(b)$ defined in the proof there by

$$
\beta_\phi(b) = \int_V \int_V \phi(s,u)\alpha_{(s,u)}(b) = \int_V \int_V \phi(s,u)\lambda_{\eta(s)}\rho_{\eta(u)}(b)
$$

is in $\mathscr{A}$ because of Proposition 2.5. This proves the last statement of the proposition. Q.E.D.

We will denote the algebra $(\mathscr{A}, \times_J)$ simply by $\mathscr{A}_J$. Now we define the coproduct and the coinverse on A_J and verify the axioms of Definition 2.1.

To proceed, as in [17], let

$$
C = \{F \in A \otimes A \mid (\rho_h \otimes id)F = (id \otimes \lambda_{h^{-1}})F, \quad \text{for all} \quad h \in T\}.
$$

Then by 2.6, $\Phi(A)$ is contained in C. Note that C is a C^*-algebra. Let Ψ denote Φ viewed as a homomorphism from A to C. Define an action β of $\mathbb{R}^d$ on C by

$$
\beta_{(s,u)} = \lambda_{\eta(s)} \otimes \rho_{\eta(u)}.
$$

Note that β is initially defined on $A \otimes A$, but since λ and ρ commute with each other, we see that it can be restricted to an action of V on C. By 2.7, we see that the map Ψ is equivariant for the actions α on A and β on C.

Similarly, define an action γ of $\mathbb{R}^d \times \mathbb{R}^d$ on $A \otimes A$ by $\gamma = \alpha \otimes \alpha$. Then γ restricts to an action of $\mathbb{R}^{2d}$ on the subalgebra C of $A \otimes A$, because T is abelian and λ and ρ commute with each other. Let

$$
L = J \oplus J = S \oplus (-S) \oplus S \oplus (-S).
$$

As shown in [17], we have $C_L^\gamma = C_J^\beta$. Let

$$
\Phi_J = \varrho \Psi_J,
$$

where ϱ is the inclusion from C_L^γ to $(A \otimes A)_L^\gamma$. Since $(A \otimes A)_L^\gamma = A_J^\alpha \otimes A_J^\alpha$ by 2.2 of [17] and both Ψ_J and ϱ are unital C^*-algebra homomorphisms, thus we obtain a unital C^*-algebra homomorphism Φ_J from A_J to $A_J \otimes A_J$.

3.3. Proposition. *On the dense subalgebra $\mathscr{A}_J$ of A_J, we have for each $p \in \mathbb{N}$,*

$$\Phi_J(u_{ij}^p) = \sum_{k=1}^{d_p} u_{ik}^p \otimes u_{kj}^p.$$

Therefore axiom (2) of Definition 2.1 is satisfied.

Proof. Recall that on A^∞, we have that $\Psi_J = \Psi$ (see [16]), and that Ψ is simply Φ on A. From Proposition 3.2, $\mathscr{A}$ is contained in A^∞. The rest is now clear. Q.E.D.

We remark that the coassociativity $(id_J \otimes \Phi_J)\Phi_J = (\Phi_J \otimes id_J)\Phi_J$ of Φ_J is a consequence of the axiom (2) of Definition 2.1, where id_J is the identity map on A_J. So the above proposition generalizes Theorem 2.3 of [17].

As in [17], we have a surjective homomorphism π_J from A_J to $C(T)_J = C(T)$.

3.4. Proposition. *The map π_J preserves the coproducts. Namely, we have*

$$(\pi_J \otimes \pi_J)\Phi_J = \Phi_T \pi_J,$$

where Φ_T is the coproduct on $C(T)$.

Proof. Since by definition [16], $\pi_J|_{A^\infty} = \pi|_{A^\infty}$, and on the subspace $\mathscr{A}_J$ of A^∞,

$$(\pi_J \otimes \pi_J)\Phi_J = (\pi \otimes \pi)\Phi = \Phi_T \pi = \Phi_T \pi_J,$$

by the density of $\mathscr{A}_J$ in A_J and the continuity of π_J, we see that π_J preserves the coproducts. Q.E.D.

Let ϵ_T be the counit of $C(T)$. Define $\epsilon_J := \epsilon_T \pi_J$.

3.5. Proposition. *The map ϵ_J has the property*

$$\epsilon_J(u_{ij}^p) = \delta_{ij}$$

for any u_{ij}^p.

Proof. This follows immediately from the definition of ϵ_J, because $\epsilon_T \pi = \epsilon$ and $\pi_J|_{\mathscr{A}} = \pi|_{\mathscr{A}}$. Q.E.D.

We remark that as in [17], ϵ_J satisfies the counit property

$$(id_J \otimes \epsilon_J)\Phi_J = id_J = (\epsilon_J \otimes id_J)\Phi_J$$

on A_J (not just on $\mathscr{A}_J$): this follows from Propositions 3.3 and 3.5.

Define the map κ_J on $\mathscr{A}_J$ to be the same as κ. We can do this because as a vector space, $\mathscr{A}_J$ is the same as $\mathscr{A}$.

3.6. Proposition. *The map κ_J is an anti-algebra homomorphism for the product $\times_J$*

of $\mathcal{A}_J$, *and for* $a \in \mathcal{A}_J$, $\kappa_J(\kappa_J(a^*)^*) = a$.

Proof. Let $a, b \in \mathcal{A}$. First we note that

$$\kappa_J(a \times_J b) = \int\int \kappa(\alpha_{J(s,u)}(a)\alpha_{(t,v)}(b))e(s \cdot t + u \cdot v)$$

even though κ is not assumed to be continuous because by Proposition 2.5, for fixed a and b, there is a finite number of finite dimensional representations of the quantum group A such that the expression

$$\alpha_{J(s,u)}(a)\alpha_{(t,v)}(b)e(s \cdot t + u \cdot v)$$

is a linear combination of the products of the matrix coefficients of these representations. Then by Proposition 2.8 we have

$$\begin{aligned}
\kappa_J(a \times_J b) &= \int\int \kappa(\alpha_{J(s,u)}(a)\alpha_{(t,v)}(b))e(s \cdot t + u \cdot v) \\
&= \int\int \kappa(\lambda_{\eta(Ss)}\rho_{\eta(-Su)}(a)\lambda_{\eta(t)}\rho_{\eta(v)}(b))e(s \cdot t + u \cdot v) \\
&= \int\int \kappa(\lambda_{\eta(t)}\rho_{\eta(v)}(b))\kappa(\lambda_{\eta(Ss)}\rho_{\eta(-Su)}(a))e(s \cdot t + u \cdot v) \\
&= \int\int \rho_{\eta(t)}\lambda_{\eta(v)}\kappa(b)\rho_{\eta(Ss)}\lambda_{\eta(-Su)}\kappa(a)e(s \cdot t + u \cdot v),
\end{aligned}$$

which, by Proposition 1.13 of [16],

$$\begin{aligned}
&= \int\int \rho_{\eta(-St)}\lambda_{\eta(Sv)}\kappa(b)\rho_{\eta(s)}\lambda_{\eta(u)}\kappa(a)e(s \cdot t + u \cdot v) \\
&= \kappa_J(b) \times_J \kappa_J(a).
\end{aligned}$$

Now the identity $\kappa_J(\kappa_J(a^*)^*) = a$ is immediate because the involution is not deformed. Q.E.D.

To show that A_J is a compact quantum group, it remains to show that the u^p's are unitary and that $(id_J \otimes \kappa_J)u^p = (u^p)^{-1}$. For this we need to first show that κ_J satisfies the antipodal property on $\mathcal{A}_J$. Let m_J be the product map from $\mathcal{A}_J \otimes \mathcal{A}_J$ to $\mathcal{A}_J$.

3.7. Proposition. *On the algebra* $\mathcal{A}_J$, *we have*

$$m_J(id_J \otimes \kappa_J)\Phi_J = I_J\epsilon_J = m_J(\kappa_J \otimes id_J)\Phi_J,$$

where I_J *is the unit of the algebra* A_J.

Proof. Let (a_{ij}) be a finite dimensional unitary representation of the quantum group A, so $\kappa(a_{ij}) = a_{ji}^*$. Then by 2.8 and 2.6 we have

$$m_J(id_J \otimes \kappa_J)\Phi_J(a_{ij}) = m_J(id_J \otimes \kappa_J)\Phi(a_{ij}) = \sum_k a_{ik} \times_J \kappa(a_{kj})$$

$$= \sum_k \int\int m(\alpha_{J(s,u)} \otimes \alpha_{(t,v)}\kappa)(a_{ik} \otimes a_{kj})e(s \cdot t + u \cdot v)$$

$$= \sum_k \int\int m(id \otimes \kappa)(\lambda_{\eta(Ss)}\rho_{\eta(-Su)} \otimes \lambda_{\eta(v)}\rho_{\eta(t)})(a_{ik} \otimes a_{kj})e(s \cdot t + u \cdot v)$$

$$= \sum_k \int\int m(id \otimes \kappa)(\lambda_{\eta(Ss)}\rho_{\eta(-Su-v)} \otimes \rho_{\eta(t)})(a_{ik} \otimes a_{kj})e(s \cdot t + u \cdot v),$$

which, by 3.1 of [17] and 1.11 of [16], and then 2.8,

$$= \sum_k \int\int m(id \otimes \kappa)(\lambda_{\eta(Ss)}\rho_{\eta(-v)} \otimes \rho_{\eta(t)})(a_{ik} \otimes a_{kj})e(s \cdot t + u \cdot v)$$

$$= \sum_k \int\int m(id \otimes \kappa)(\lambda_{\eta(Ss)} \otimes \rho_{\eta(t)})(a_{ik} \otimes a_{kj})e(s \cdot t)$$

$$= \sum_{lkr} \int\int E_{\eta(-Ss)}\pi(a_{il})a_{lk}a_{rk}^* E_{\eta(t)}\pi(a_{rj})e(s \cdot t)$$

$$= \sum_r \int\int E_{\eta(-Ss)}\pi(a_{ir})E_{\eta(t)}\pi(a_{rj})e(s \cdot t)$$

$$= \int\int (E_{\eta(-Ss)} \otimes E_{\eta(t)})\Phi_T\pi(a_{ij})e(s \cdot t)$$

$$= \int\int E_{\eta(-Ss+t)}\pi(a_{ij})e(s \cdot t) = \int\int E_{\eta(t)}\pi(a_{ij})e(s \cdot t)$$

$$= E_{\eta(0)}\pi(a_{ij}) = \epsilon_T\pi(a_{ij}) = \epsilon(a_{ij}) = \epsilon_J(a_{ij}).$$

That is on $\mathscr{A}_J$, $m_J(id_J \otimes \kappa_J)\Phi_J = I_J\epsilon_J$.
Similarly $m_J(\kappa_J \otimes id_J)\Phi_J = I_J\epsilon_J$ on $\mathscr{A}_J$. Q.E.D.

3.8. Proposition. *For every p, the matrix u^p is an unitary element of the *-algebra $M_{d_p} \otimes \mathscr{A}_J$, and*

$$(id_p \otimes \kappa_J) = (u^p)^{-1},$$

where the inverse $(u^p)^{-1}$ takes place in $M_{d_p} \otimes \mathscr{A}_J$.

Proof. The argument is contained in the proof of 3.2 of [30]. For reader's convenience we present the detailed proof anyway. The equalities

$$\Phi_J(u_{ij}^p) = \sum_k u_{ik}^p \otimes u_{kj}^p$$

can be written as the equality

$$\sum_{ij} e_{ij}^p \otimes \Phi_J(u_{ij}^p) = \sum_{ijk} e_{ij}^p \otimes u_{ik}^p \otimes u_{kj}^p,$$

where e_{ij}^p is the standard matrix units of M_{d_p}. Applying the map

$$id_p \otimes m_J(\kappa_J \otimes id_J)$$

to both sides of this equality, and using 3.7, 3.5 and 2.1.(3), where id_p is the identity map on M_{d_p}, we obtain

$$I_p \otimes I_J = \sum_{ij} e_{ij}^p \otimes \kappa(u_{ik}^p) \times_J u_{kj}^p = \sum_{ij} e_{ij}^p \otimes u_{ki}^{p\,*} \times_J u_{kj}^p.$$

Similarly, applying the map $id_p \otimes m_J(id_J \otimes \kappa_J)$ to both sides of the same equality, we obtain,

$$I_p \otimes I_J = \sum_{ij} e_{ij}^p \otimes u_{ik}^p \times_J \kappa(u_{kj}^p) = \sum_{ij} e_{ij}^p \otimes u_{ik}^p \times_J u_{jk}^{p\,*}.$$

This proves the proposition. Q.E.D.

Summarizing 3.2, 3.3, 3.4, 3.6 and 3.8, we have proved part (1) of the following theorem.

3.9. Theorem. *(1) With the dense subalgebra $\mathscr{A}_J$, its generating elements u_{ij}^p, the coproduct Φ_J, and the coinverse κ_J, as defined above, A_J is a compact quantum group. The compact abelian Lie group T is still a subgroup of the quantum group A_J.*

(2) A_J is a compact matrix quantum group if and only if A is.

Proof. The "if" part of (2) is clear from the proof of (1). Because T is again a subgroup of A_J, A_J satisfies the assumptions set forth at the beginning of this section. Hence we can apply the deformation process developed in this section to A_J. Note that $(A_J)_{-J} = A$ (see 7.5 of [16]), so the "only if" part follows from the "if" part. Q.E.D.

Notation. Let $A = C(G)$ be a Woronowicz Hopf C^*-algebra so that the construction A_J in the theorem above can be applied. We will also denote A_J by $C(G_J)$.

3.10. Remarks. (1) We remark that for $A = C(G)$ with G a compact Lie group, the above theorem becomes precisely the main theorem in [17].

(2) As in [17], the Haar measure h_J of the quantum group A_J is still the same as the Haar measure h of A, namely on the common subspace $\mathscr{A}$ of both A and A_J, we have

$$h_J(a) = h(a), \quad \text{for} \quad a \in \mathscr{A}.$$

From this we see that if A is a compact matrix quantum group of Kac type (one on which the Haar measure is a trace, see [2]), then A_J is also one such. (See also 5.2 in the following.) Therefore all of the compact quantum groups constructed in [17] are of Kac type.

(3) Examining the proof of the above theorem, one can check that if we use either of the following two actions:

$$\alpha_{(s,u)} = \lambda_{\eta(-s)}\rho_{\eta(u)},$$

$$\alpha_{(s,u)} = \lambda_{\eta(s)}\rho_{\eta(-u)},$$

the theorem is still valid.

We conclude this section with a generalization of Example 4.3 of [27] (see also [25]). Recall [25, 27] that a Woronowicz Hopf C^*-dynamical system is defined to be a C^*-dynamical system with the additional assumption that the automorphism group preserves the coproduct of the Woronowicz Hopf C^*-algebra.

3.11. Proposition. *Let D be the subgroup of $\mathbb{R}^d$ consisting of vectors of the form (s, s) with s in $\mathbb{R}^n$. Put $\beta_s = \alpha_{(s,s)}$. Then (A, D, β) is a Woronowicz C^*-dynamical system equivariant for the action α, its deformation (A_J, D, β_J) is still a Woronowicz C^*-dynamical system.*

Proof. Same as the proof of 4.3 of [27]. Q.E.D.

4. Deformations of Quantum Groups K_q and K_q^u

Based on the earlier work [21], Soibelman [20] studied the representation theory of the function algebra of a general compact simple quantum group K_q, which are "compact real forms" of the Drinfeld-Jimbo quantum groups [4, 8]. In [12], Levendorskii introduced a deformation K_q^u of the quantum group K_q by a purely algebraic method (the so-called twisting construction of Drinfeld), and studied the representation theory of its function algebra. See also [13] for a summary of [20, 13]. For related work, see e.g. [31, 22, 19, 3, 9] for the analytical case, and [6, 7] for the algebraic case, as well as the literature cited in these papers. In [17], Rieffel raised the question as to whether the general quantum groups K_q can be deformed under *strict deformation quantization* into quantum groups K_q^u. This section is devoted to giving an affirmative answer to this question using the deformation developed in Sect. 3.

To establish the notation for this section, we first recall the notation of [20, 12, 13]. Let G be a simple complex Lie group with Lie algebra $\mathfrak{g}$. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, together with the corresponding decomposition $\Delta = \Delta_+ \cup \Delta_-$ of the root system and a fixed basis $\{\alpha_i\}_{i=1}^n$ for Δ_+. For each linear functional λ on $\mathfrak{h}$, H_λ denotes the element in $\mathfrak{h}$ corresponding to λ under the isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ determined by the Killing form (,) on $\mathfrak{g}$. Note that if the reader keeps the context in mind, the symbols α and λ used in this context should not cause confusion with the same symbols used in the previous sections in the definition of the action α. Let $\{X_\alpha\}_{\alpha\in\Delta} \cup \{H_i\}_{i=1}^n$ be a Weyl basis of $\mathfrak{g}$, where $H_i = H_{\alpha_i}$. This determines a Cartan involution ω_0 on $\mathfrak{g}$ with $\omega_0(X_\alpha) = -X_{-\alpha}$, $\omega_0(H_i) = -H_i$. Let $\mathfrak{k}$ be the compact real form of $\mathfrak{g}$ defined as the fixed points of ω_0 and K the associated compact real form of G. Put $\mathfrak{h}_{\mathbb{R}} = \oplus_{i=1}^n \mathbb{R} H_i$, and $T = \exp(i\mathfrak{h}_{\mathbb{R}})$, the later being the associated maximal torus of K.

Let $q > 1$. For $n, k \in \mathbb{N}$, $n \geq k$, define

$$\begin{aligned} [n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}}, \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q [n-1]_q \ldots [n-k+1]_q}{[k]_q [k-1]_q \ldots [1]_q}. \end{aligned}$$

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ [4, 8] is the complex associative algebra with generators $X_i^\pm$, $K_i^{\pm 1}$ ($i = 1, \cdots, n$) and defining relations:

$$K_iK_i^{-1} = 1 = K_i^{-1}K_i, \quad K_iK_j = K_jK_i,$$
$$K_iX_j^{\pm}K_i^{-1} = q^{\pm(\alpha_i,\alpha_j)}X_j^{\pm},$$
$$[X_i^+, X_j^-] = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^2 - q^{-2}},$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k} = 0 \ , \ i \neq j,$$

where $q_i = q^{(\alpha_i,\alpha_i)}$.

On $U_q(\mathfrak{g})$ there is a Hopf algebra structure with coproduct

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(X_i^{\pm}) = X_i^{\pm} \otimes K_i + K_i^{-1} \otimes X_i^{\pm},$$

and counit and antipode respectively

$$\varepsilon(X_i^{\pm}) = 0, \quad \varepsilon(K_i^{\pm 1}) = 1, \quad S(X_i^{\pm}) = -q_i^{\pm 1}X_i^{\pm}, \quad S(K_i^{\pm 1}) = K_i^{\mp 1}.$$

Under the *-structure defined by

$$(X_i^{\pm})^* = X_i^{\mp}, \quad K_i^* = K_i,$$

$U_q(\mathfrak{g})$ is a Hopf *-algebra.

The algebra $U_h(\mathfrak{g})$ is the $\mathbb{C}[[h]]$ algebra generated by $X_i^{\pm}$ and H_i with the defining relations

$$[H_i, H_j] = 0, \qquad [H_i, X_j^{\pm}] = \pm(\alpha_i, \alpha_j)X_j^{\pm},$$
$$[X_i^+, X_j^-] = \delta_{ij}\frac{sh(\frac{hH_j}{2})}{sh(\frac{h}{2})},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k} = 0 \ , \ i \neq j.$$

The Hopf *-algebra structure on $U_h(\mathfrak{g})$ is defined as above,

$$(X_i^{\pm})^* = X_i^{\mp}, \quad H_i^* = H_i,$$

where the involution of $\mathbb{C}[[h]]$ is given by $(ch)^* = \bar{c}h$ for $c \in \mathbb{C}$.

Let $u = \sum_{k,l} c_{kl}H_k \otimes H_l \in \wedge^2\mathfrak{h}_{\mathbb{R}}$. Then one can define a new coproduct on $U_h(\mathfrak{g})$ by

$$\Delta_u(\xi) = \exp(-ihu/2)\Delta(\xi)\exp(ihu/2),$$

where $\xi \in U_h(\mathfrak{g})$ and Δ is the original coproduct on $U_h(\mathfrak{g})$. The new Hopf *-algebra so obtained is denoted by $U_{h,u}(\mathfrak{g})$.

The function algebra $\mathbb{C}[K_q]$ of the compact quantum group K_q is defined to be a certain dual of the Hopf *-algebra $U_q(\mathfrak{g})$. It consists of matrix elements of finite dimensional representations ρ of $U_q(\mathfrak{g})$ such that eigenvalues of the endomorphisms $\rho(K_i)$ are positive. The function algebra $\mathbb{C}[K_q^u]$ of the compact quantum group K_q^u is defined to have the same elements as $\mathbb{C}[K_q]$ and the same *-structure as $\mathbb{C}[K_q]$, while the product of its elements is defined using Δ_u instead of Δ. The completions of the algebras $\mathbb{C}[K_q]$ and $\mathbb{C}[K_q^u]$ under their universal C^*-norms are denoted by

$C(K_q)$ and $C(K_q^u)$ respectively. The algebras $\mathbb{C}[K_q]$ and $\mathbb{C}[K_q^u]$ are examples of Krein algebras $\mathscr{A}$ in the sense of [28, 25].

For each dominant weight Λ of $\mathfrak{g}$, Soibelman introduces [20, 13] the matrix elements $C^{\Lambda}_{\nu,\mu,\Omega}$ of the highest weight $U_q(\mathfrak{g})$ module $(L(\Lambda), \rho_\Lambda)$ as follows. Let $\{v^{(i)}_\nu\}$ be an orthonormal weight basis for the unitary $U_q(\mathfrak{g})$ module $L(\Lambda)$, and let $\{l^{(i)}_\nu\}$ be the corresponding dual weight basis in $L^*(\Lambda)$. Put $\Omega = (i, j)$. Then $C^{\Lambda}_{\nu,\mu,\Omega}$ is defined by

$$C^{\Lambda}_{\nu,\mu,\Omega}(\xi) = l^{(i)}_\nu(\rho_\Lambda(\xi)v^{(j)}_\mu),$$

where $\xi \in U_q(\mathfrak{g})$.

To avoid confusion with the Killing form, we now use $s \oplus v$, instead of (s, v) (as used in the previous section), to denote an element of $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^n$. In the present setting, the space $\mathbb{R}^n$ is $\mathfrak{h}_{\mathbb{R}}$, with inner product $< \, , \, >= -(\, , \,)$, where $(\, , \,)$ is the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}_{\mathbb{R}}$. We will also use $< \, , \, >$ to denote the inner product on $\mathfrak{h}_{\mathbb{R}} \oplus \mathfrak{h}_{\mathbb{R}}$. Noting that the compact abelian group T is also a subgroup of both K_q and K_q^u (see [20, 12]), we can define as in Sect. 3 an action of $\mathbb{R}^d$ on $C(K_q)$ by

$$\alpha_{s\oplus v} = \lambda_{\exp(-2\pi i s)}\rho_{\exp(2\pi i v)}.$$

Thus the map η there in this case is defined by $\eta(s) = \exp(2\pi i s)$. This action may be viewed as an action of $H = T \times T$ in the sense of [16]. For each ν in the weight lattice P of $\mathfrak{g}$, the element H_ν is in $\mathfrak{h}_{\mathbb{R}}$. Keeping the notation of [16] for the spectral subspaces of the action α (see 2.22 there), we have the following lemma.

4.1. Lemma. *The matrix elements $C^{\Lambda}_{\nu,\mu,\Omega}$ are in the spectral subspaces $A_{-(H_\nu \oplus H_\mu)}$.*

Proof. In the notation of [16], we must show that

$$\alpha_{s\oplus v}(C^{\Lambda}_{\nu,\mu,\Omega}) = e^{-2\pi i < H_\nu \oplus H_\mu, s\oplus v>} C^{\Lambda}_{\nu,\mu,\Omega},$$

Let ξ be an element in $U_q(\mathfrak{g})$. We compute:

$$\begin{aligned}
\alpha_{s\oplus v}(C^{\Lambda}_{\nu,\mu,\Omega})(\xi) &= l^{(i)}_\nu(\rho_\Lambda(\exp(-2\pi i s))\rho_\Lambda(\xi)\rho_\Lambda(\exp(2\pi i v))v^{(j)}_\mu) \\
&= < \rho_\Lambda(\xi)\rho_\Lambda(\exp(2\pi i v))v^{(j)}_\mu, \rho_\Lambda(\exp(2\pi i s))v^{(i)}_{-\nu} > \\
&= < \rho_\Lambda(\xi)e^{2\pi i \mu(v)}v^{(j)}_\mu, e^{-2\pi i \nu(s)}v^{(i)}_{-\nu} > \\
&= e^{2\pi i(\nu(s)+\mu(v))}C^{\Lambda}_{\nu,\mu,\Omega}(\xi) \\
&= e^{-2\pi i < H_\nu \oplus H_\mu, s\oplus v>} C^{\Lambda}_{\nu,\mu,\Omega}(\xi).
\end{aligned}$$

This proves the lemma. Q.E.D.

In [12, 13], the following identity is obtained:

$$\begin{aligned}
&C^{\Lambda_1}_{\nu_1,\mu_1,\Omega_1} \circ C^{\Lambda_2}_{\nu_2,\mu_2,\Omega_2} \\
&= \exp(\frac{ih}{2}((\nu_1, \check{u}\nu_2) - (\mu_1, \check{u}\mu_2)))C^{\Lambda_1}_{\nu_1,\mu_1,\Omega_1}C^{\Lambda_2}_{\nu_2,\mu_2,\Omega_2},
\end{aligned}$$

where the left-hand side is the multiplication in $C(K_q^u)$ and the right-hand side is the multiplication in $C(K_q)$, and $\check{u}$ is the map on $\mathfrak{h}^*$ determined by u via the Killing form $(\, , \,)$ on $\mathfrak{g}$. Letting

$$p = -(H_{\nu_1} \oplus H_{\mu_1}), \quad q = -(H_{\nu_2} \oplus H_{\mu_2}),$$
$$J = \frac{h}{4\pi}(S_u \oplus (-S_u)),$$

where S_u is the skew-symmetric operator on $\mathfrak{h}_{\mathbb{R}}$ defined by

$$S_u(H_\nu) = \sum_{k,l} c_{kl}\nu(H_k)H_l,$$

the previous identity becomes

$$C^{\Lambda_1}_{\nu_1,\mu_1,\Omega_1} \circ C^{\Lambda_2}_{\nu_2,\mu_2,\Omega_2} = \exp(-2\pi i < p, Jq >) C^{\Lambda_1}_{\nu_1,\mu_1,\Omega_1} C^{\Lambda_2}_{\nu_2,\mu_2,\Omega_2}.$$

On the other hand from 2.22 of [16], we have by virtue of Lemma 4.1 that

$$C^{\Lambda_1}_{\nu_1,\mu_1,\Omega_1} \times_J C^{\Lambda_2}_{\nu_2,\mu_2,\Omega_2} = \exp(-2\pi i < p, Jq >) C^{\Lambda_1}_{\nu_1,\mu_1,\Omega_1} C^{\Lambda_2}_{\nu_2,\mu_2,\Omega_2}.$$

Note that the elements $C^{\Lambda}_{\nu,\mu,\Omega}$ generate $\mathbb{C}[K^u_q]$ and $(\mathbb{C}[K_q], \times_J)$, the later being the algebra with the product $\times_J$ on $\mathbb{C}[K_q]$. As a matter of fact, the elements $C^{\Lambda}_{\nu,\mu,\Omega}$, with Λ running over the fundamental weights, already generate these algebras. Summarizing these, we have reached a proof of the following result.

4.2. Theorem. *The Hopf *-algebras* $\mathbb{C}[K^u_q]$ *and* $(\mathbb{C}[K_q], \times_J)$ *are isomorphic.*

From the viewpoints in [28], the compact quantum groups K^u_q and $(K_q)_J$ are isomorphic, since the above theorem says that they have the same Krein algebras.

From the universality of the C^*-algebra $C(K^u_q)$, there is a map of Woronowicz Hopf C^*-algebras from $C(K^u_q)$ to $C(K_q)_J$ sending each of the elements $C^{\Lambda}_{\nu,\mu,\Omega}$ to itself (viewed as elements in different algebras). We believe this is an isomorphism. One possible way to prove this is to try to adapt the method of 10.2 in [16]: Show that there are no α-invariant ideals in $C(K^u_q)$. To achieve this, one would need to know in detail the ideal structure of the C^*-algebra $C(K^u_q)$, which is presumably more complicated than the ideal structure of the corresponding *-algebra $\mathbb{C}[K^u_q]$. If the map mentioned above is not an isomorphism of C^*-algebras, this is a kind of pathology analogous to the one that involves $C^*(F_2)$ and $C^*_r(F_2)$-the C^*-algebra $C^*_r(F_2)$ is a quotient of $C^*(F_2)$ by a pathological ideal, where F_2 is the free group on two generators. In any case, as noted in the previous paragraph (see also [28]), whether this map is an isomorphism or not is a question in C^*-algebras, not one in quantum groups.

5. Deformations of Quantum Groups $A_u(m)$

We now apply the construction in Sect. 3 to the quantum groups $A_u(m)$ (see [25, 26]) to answer Rieffel's second question mentioned at the beginning of this paper. Recall that for any positive integer $m \geq 2$, $A_u(m)$ is the universal C^*-algebra generated by m^2 elements a_{ij} such that both (a_{ij}) and (a^*_{ij}) are unitary elements of $M_m \otimes A_u(m)$, where (a^*_{ij}) is the matrix obtained from (a_{ij}) by applying the involution * to each of its entries. More explicitly, $A_u(m)$ is the universal C^*-algebra generated by a_{ij} subject to the relations

$$\sum_{k=1}^{m} a_{ik}a_{jk}^* = \delta_{ij}, \quad \sum_{k=1}^{m} a_{ki}^* a_{kj} = \delta_{ij},$$
$$\sum_{k=1}^{m} a_{ki}a_{kj}^* = \delta_{ij}, \quad \sum_{k=1}^{m} a_{ik}^* a_{jk} = \delta_{ij}$$

for $i, j = 1, \cdots, m$.

The unitary groups $U(k)$ are subgroups of the quantum groups $A_u(m)$ for all $k \leq m$. This can be seen as follows. Let u_{ij} be the coordinate functions on the group $U(k)$. Define

$$a'_{ij} = u_{ij} \text{ if } i, j \leq k, \quad \text{and} \quad a'_{ij} = \delta_{ij} \text{ if either } k < i \leq m \text{ or } k < j \leq m.$$

Then the map π_0 defined by $\pi_0(a_{ij}) = a'_{ij}$ defines an embedding of the unitary group $U(k)$ in the compact quantum group $A_u(m)$

Let T be a torus subgroup of $U(k)$, say, of dimension n. Then T can be viewed as a subgroup of the compact quantum group $A_u(m)$ with the surjective morphism π from $A_u(m)$ to $C(T)$ given by the composition of π_0 defined above and the restriction map from $C(U(k))$ to $C(T)$. The Lie algebra of T can be identified with the vector space $\mathbb{R}^n$ with the zero bracket. Let η be the exponential map from $\mathbb{R}^n$ to T, and let S be any skew-symmetric operator on $\mathbb{R}^n$. With the ingredients $A_u(m)$, T, π, η and S, we are in the setting to apply the construction in Sect. 3 to obtain quantum groups $A_u(m)_J$.

5.1. Theorem. *(1) The quantum groups $A_u(m)_J$ are quantum subgroups of the quantum groups $A_u(m)$.*

(2) For each compact quantum group $C(G)_J$ constructed in [17], there is an m such that one may define $A_u(m)_J$, and $C(G)_J$ are quantum subgroups of both $A_u(m)_J$ and $A_u(m)$.

Proof. Since $u = (a_{ij})$ and $\bar{u} = (a_{ij}^*)$ are both unitary representation of the quantum group $A_u(m)$, we see from the proof of 3.8 that u and $\bar{u}$ are still unitary representations of the quantum group $A_u(m)_J$. A moment's reflection shows that $A_u(m)_J$ is also generated by the a_{ij}'s as a C^*-algebra. Thus by the universal property of the C^*-algebra $A_u(m)$, there is a unital C^*-algebra homomorphism π from $A_u(m)$ to $A_u(m)_J$ sending the generators a_{ij} of $A_u(m)$ to the generators a_{ij} of $A_u(m)_J$ for each (i, j). It is clear that π preserves the coproducts. This proves (1).

Since G is a compact Lie group, it has faithful finite dimensional unitary representations. Let (u_{ij}) be one such, say, of dimension m. Hence G is a subgroup of the quantum group $A_u(m)$. Let π be the surjective Hopf C^*-algebra morphism from $A_u(m)$ to $C(G)$ sending a_{ij} to u_{ij} for each pair (i, j). Then under the obvious action of $\mathbb{R}^d$ on $A_u(m)$ coming from the action of $\mathbb{R}^d$ on $C(G)$, π is equivariant, so it can be deformed into a map π_J, which is easily seen to be still a surjective map of Woronowicz Hopf C^*-algebras. Hence $C(G)_J$ is a quantum subgroup of $A_u(m)_J$. Since $A_u(m)_J$ is a quantum subgroup of the quantum group $A_u(m)$, we see that $C(G)_J$ is a quantum subgroup of $A_u(m)$. This completes the proof of (2). Q.E.D.

5.2. Remarks. (1) From Sect. 5 of [30], one can see that the quantum groups $A_u(m)$ are universal in the following sense (see [24]): Any compact matrix quantum group of Kac type (see Remark 3.10.(2)) is a quantum subgroup of $A_u(m)$ for some m. It is easy to see that 5.1.(2) (and its proof) can be generalized to the following setting. Let A be a compact quantum group of Kac type with a torus subgroup T. Let π be a surjection of Woronowicz Hopf C^*-algebras from $A_u(m)$ to A. We can apply the construction in Sect. 3 to obtain A_J and $A_u(m)_J$. Then the deformed map π_J is again a surjection of Woronowicz C^*-algebras.

(2) For most invertible matrices Q, the quantum groups $A_u(Q)$ and $A_o(Q)$ constructed in [24] have many torus subgroups. Hence, as described in this paper, they are also subject to the Rieffel quantization.

(3) By the nature of Rieffel's quantization [16] (see Theorem 7.5 there), the quantum groups $A_u(m)_J$ defined above can also be deformed into $A_u(m)$ using $-J$ (Compare the proof of 3.9.(2)). It is natural to expect that $A_u(m)_J$ is isomorphic to $A_u(m)$. We believe that this is unlikely because of 2.22 of [16]. Therefore it is of interest to solve the following problem.

Problem: For each m, classify the quantum groups $A_u(m)_J$.

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