Commun. Math. Phys. 178, 1 – 26 (1996)

Communications in
Mathematical
Physics

Yang–Mills–Higgs versus Connes–Lott

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Received: 3 April 1995

Abstract: By a suitable choice of variables we show that every Connes–Lott model is a Yang–Mills–Higgs model. The contrary is far from being true. Necessary conditions are given. Our analysis is pedestrian and illustrated by examples.

Despite its impressing success in describing particles and interactions, the Yang–Mills–Higgs (YMH) model building kit has conceptual shortcomings:

- its rules are essentially unmotivated,
- its complicated input comprising a Lie group and three representations is only justified by experiment,
- the model singled out by more and more precise experiments, namely the standard $SU(3) \times SU(2) \times U(1)$ model of electro-weak and strong interactions, is ugly and nobody really believes it to be the last word.

Concerning the first two points, the Connes–Lott (CL) model building kit [1] does better. Its rules have a precise motivation from non-commutative geometry and its input, comprising an involution algebra and two representations, is infinitely more restricted than the YMH input. Nevertheless, the standard model is also a CL model [1–4], a fact that by itself does not improve its beauty, but that perhaps allows unification with gravity. Indeed, the Einstein–Hilbert action as well may be formulated naturally in the setting of non-commutative geometry [5–7].

The purpose of this work is to show that the CL models represent a very small subset of the YMH models, where we restrict ourselves to "local" models, i.e. models defined on trivial bundles. Also we restrict ourselves to CL models defined by means of a finite dimensional algebra $\mathscr{A}$ tensorized with the algebra of functions on (a compact, Euclidean) "spacetime" of dimension 4. These particular models can be computed with elementary mathematics [8] and compare naturally to YMH models. Models whose algebras are not such tensor products, as the non-commutative torus [9], the fuzzy sphere [10] or a quantum space time [11] are

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much more involved mathematically and appear as natural candidates for the above mentioned unification.

1. Yang–Mills–Higgs Models

Let us first set up our notations of a YMH model. It is defined by the following input:

- a finite dimensional, real, compact Lie group G,
- a positive definite, bilinear invariant form on the Lie algebra $\mathfrak{g}$ of G, this choice being parameterized by a few positive number g_i, the coupling constants,
- a (unitary) representation ρ_L on a Hilbert space $\mathcal{H}_L$ accommodating the left-handed fermions ψ_L,
- a representation ρ_R on $\mathcal{H}_R$ for the right-handed fermions ψ_R,
- a representation ρ_S on $\mathcal{H}_S$ for the scalars φ,
- an invariant, positive polynomial $V(\varphi)$, $\varphi \in \mathcal{H}_S$, of order 4, the Higgs potential,
- one complex number or Yukawa coupling g_Y for every trilinear invariant, i.e. for every one dimensional invariant subspace, "singlet," in the decomposition of the representation associated to $(\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes \mathcal{H}_S) \oplus (\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes \mathcal{H}_S^*)$.

The standard model is defined by the following input:

$$G = SU(3) \times SU(2) \times U(1)$$

with three coupling constants g_3, g_2, g_1,

$$\begin{aligned}
\mathcal{H}_L &= \bigoplus_1^3 \left[(1,2,-1) \oplus \left(3,2,\frac{1}{3}\right)\right], \\
\mathcal{H}_R &= \bigoplus_1^3 \left[(1,1,-2) \oplus \left(3,1,\frac{4}{3}\right) \oplus \left(3,1,-\frac{2}{3}\right)\right], \\
\mathcal{H}_S &= \left(1,2,-\frac{1}{2}\right),
\end{aligned} \tag{1}$$

where (n_3, n_2, y) denotes the tensor product of an n_3 dimensional representation of $SU(3)$, an n_2 dimensional representation of $SU(2)$ and the one dimensional representation of $U(1)$ with hypercharge y:

$$\rho(e^{i\theta}) = e^{iy\theta}\,, \quad y \in \mathbb{Q},\ \theta \in [0, 2\pi)\,,$$

$$V(\varphi) = \lambda(\varphi^*\varphi)^2 - \frac{\mu^2}{2}\varphi^*\varphi\,, \quad \varphi \in \mathcal{H}_S,\ \lambda, \mu > 0\,.$$

There are 27 Yukawa couplings in the standard model.

The gauge symmetry is said to be spontaneously broken if every minimum $v \in \mathcal{H}_S$ of the Higgs potential is gauge variant, $\rho_S(g)v \neq v$ for some $g \in G$. Any such minimum v is called a vacuum. For simplicity let us assume that the vacuum is non-degenerate, i.e. all minima lie on the same orbit under G. Then, the little group G_l is by definition the isotropy group of the vacuum, $\rho(g_l)v = v$. For example, in the standard model, any doublet of length $\sqrt{\varphi^*\varphi} = \mu/(2\sqrt{\lambda})$ is a minimum and the little group is $G_l = SU(3) \times U(1)_{em}$. To do perturbation theory, we have to introduce a scalar variable h, that vanishes in the vacuum,

$$h(x) := \varphi(x) - v\,,$$

x a point in spacetime M. With this change of variables, the Klein–Gordon Lagrangian is $(\mathrm{D}\varphi)^* * \mathrm{D}\varphi$. The Hodge star $*\,\cdot$ should be distinguished from the Hilbert space dual $\cdot\,^*$, wedge symbols are suppressed. We denote by D the covariant exterior derivative, here for scalars $\mathrm{D}\varphi := \mathrm{d}\varphi + \tilde\rho_S(A)\varphi$, φ is now a multiplet of *fields*, i.e. a 0-form on spacetime with values in the scalar representation space, $\varphi \in \Omega^0(M, \mathcal{H}_S)$, while the vacuum v remains constant over spacetime so that it also minimizes the kinetic term $\mathrm{d}\varphi^* * \mathrm{d}\varphi$. The gauge fields are 1-forms with values in the Lie algebra of G: $A \in \Omega^1(M, \mathfrak{g})$, $\tilde\rho_S$ denotes the Lie algebra representation on $\mathcal{H}_S$. The Klein–Gordon Lagrangian produces the mass matrix for the gauge bosons A. This mass matrix is given by the (constant) symmetric, positive semi-definite form on the Lie algebra of G,

$$(\tilde\rho_S(A)v)^*\tilde\rho_S(A)v\,.$$

It contains the masses of the gauge bosons and vanishes on the generators of the little group. In the example of the standard model, the little group is generated by the gluons and the photon which remain massless.

In the following we are more concerned with the fermionic mass matrix $\mathcal{M}$, a linear map $\mathcal{M} : \mathcal{H}_R \to \mathcal{H}_L$. We want to produce it in the same way we produced the mass matrix for the gauge bosons, via the change of variables $h(x) := \varphi(x) - v$. For this purpose, we add by hand to the Dirac Lagrangian gauge invariant trilinears

$$\sum_{j=1}^{n} g_{Y_j}(\psi_L^*, \psi_R, \varphi)_j + \sum_{j=n+1}^{m} g_{Y_j}(\psi_L^*, \psi_R, \varphi^*)_j + \text{complex conjugate}\,, \tag{2}$$

n is the number of singlets in $(\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes \mathcal{H}_S)$, $m+n$ the number of singlets in $(\mathcal{H}_L^* \otimes \mathcal{H}_R \otimes \mathcal{H}_S^*)$. For $h = 0$ again, we obtain the fermionic mass matrix $\mathcal{M}$ as a function of the Yukawa couplings g_{Y_j} and the vacuum v,

$$\psi_L^*\mathcal{M}\psi_R := \sum_{j=1}^{n} g_{Y_j}(\psi_L^*, \psi_R, v)_j + \sum_{j=n+1}^{m} g_{Y_j}(\psi_L^*, \psi_R, v^*)_j\,.$$

As the gauge boson masses, the fermionic mass terms $\psi_L^*\mathcal{M}\psi_R$ are not gauge invariant in general. They are gauge invariant if $\rho_L(g^{-1})\mathcal{M}\rho_R(g) = \mathcal{M}$ for all $g \in G$ and in analogy with the little group, we define $G_\mathcal{M}$ to be the subgroup of G, that leaves $\mathcal{M}$ invariant,

$$\rho_L(g_\mathcal{M}^{-1})\mathcal{M}\rho_R(g_\mathcal{M}) = \mathcal{M} \quad \text{for all } g_\mathcal{M} \in G_\mathcal{M}\,.$$

In the standard model with its 27 Yukawa couplings, the mass matrix $\mathcal{M}$ can be any matrix yielding mass terms invariant under the little group.

In general however, the little group is only a subgroup of $G_\mathcal{M}$,

$$G_l \subset G_\mathcal{M}\,.$$

For example, if we modify the standard model by choosing the scalars in the adjoint representation of $SU(2)$, then the little group becomes $G_l = SU(3) \times U(1) \times U(1)$, there are no trilinear invariants, the mass matrix $\mathcal{M}$ vanishes, and $G_\mathcal{M} = SU(3) \times SU(2) \times U(1)$.

2. Connes–Lott Models

With the two specializations mentioned in the introduction, a Connes–Lott model is defined by the following choices:

- a finite dimensional, associative, algebra $\mathscr{A}$ over the field $\mathbb{R}$ or $\mathbb{C}$ with unit 1 and involution *,
- two *-representations of $\mathscr{A}$, ρ_L and ρ_R, on Hilbert spaces $\mathscr{H}_L$ and $\mathscr{H}_R$ over the field, such that $\rho := \rho_L \oplus \rho_R$ is faithful,
- a mass matrix $\mathscr{M}$, i.e. a linear map $\mathscr{M} : \mathscr{H}_R \to \mathscr{H}_L$,
- a certain number of coupling constants depending on the degree of reducibility of $\rho_L \oplus \rho_R$.

The data $(\mathscr{H}_L, \mathscr{H}_R, \mathscr{M})$ plays a fundamental role in non-commutative Riemannian geometry, where it is called K-cycle.

With this input and the rules of non-commutative geometry, Connes and Lott construct a YMH model. Their starting point is an auxiliary differential algebra $\Omega\mathscr{A}$, the so-called universal differential envelope of $\mathscr{A}$:

$$\Omega^0\mathscr{A} := \mathscr{A} \,,$$

$\Omega^1\mathscr{A}$ is generated by symbols δ_a, $a \in \mathscr{A}$ with relations

$$\delta 1 = 0, \qquad \delta(ab) = (\delta a)b + a\delta b \,.$$

Therefore $\Omega^1\mathscr{A}$ consists of finite sums of terms of the form $a_0\delta a_1$,

$$\Omega^1\mathscr{A} = \left\{ \sum_j a_0^j \delta a_1^j,\ a_0^j, a_1^j \in \mathscr{A} \right\} ,$$

and likewise for higher p,

$$\Omega^p\mathscr{A} = \left\{ \sum_j a_0^j \delta a_1^j \cdots \delta a_p^j,\ a_q^j \in \mathscr{A} \right\} .$$

The differential δ is defined by $\delta(a_0\delta a_1 \cdots \delta a_p) := \delta a_0 \delta a_1 \cdots \delta a_p$.

Two remarks: The universal differential envelope $\Omega\mathscr{A}$ of a commutative algebra $\mathscr{A}$ is not necessarily graded commutative. The universal differential envelope of any algebra has no cohomology. This means that every closed form ω of degree $p \geqq 1$, $\delta\omega = 0$, is exact, $\omega = \delta\kappa$ for some $(p-1)$ form κ.

The involution * is extended from the algebra $\mathscr{A}$ to $\Omega^1\mathscr{A}$ by putting

$$(\delta a)^* := \delta(a^*) =: \delta a^* \,.$$

Note that Connes defines $(\delta a)^* := -\delta(a^*)$ which amounts to replacing δ by $i\delta$. With the definition $(\omega\kappa)^* = \kappa^*\omega^*$, the involution is extended to the whole differential envelope.

The next step is to extend the representation $\rho := \rho_L \oplus \rho_R$ on $\mathscr{H} := \mathscr{H}_L \oplus \mathscr{H}_R$ from the algebra $\mathscr{A}$ to its universal differential envelope $\Omega\mathscr{A}$. This extension is the central piece of Connes' algorithm and deserves a new name:

$$\pi : \Omega\mathscr{A} \to \mathrm{End}(\mathscr{H}) \,,$$

$$\pi(a_0\delta a_1 \cdots \delta a_p) := (-i)^p \rho(a_0)[\mathscr{D}, \rho(a_1)] \cdots [\mathscr{D}, \rho(a_p)] \,, \tag{3}$$

where $\mathscr{D}$ is the linear map from $\mathscr{H}$ into itself

$$\mathscr{D} := \begin{pmatrix} 0 & \mathscr{M} \\ \mathscr{M}^* & 0 \end{pmatrix} .$$

In non-commutative geometry, $\mathscr{D}$ plays the role of the Dirac operator and we call it the internal Dirac operator. Note that in Connes' notations there is no factor $(-i)^p$ on the rhs of Eq. (3). A straightforward calculation shows that π is in fact a representation of $\Omega\mathscr{A}$ as an involution algebra, and we are tempted to define also a differential, again denoted by δ, on $\pi(\Omega\mathscr{A})$ by

$$\delta\pi(\omega) := \pi(\delta\omega) . \tag{4}$$

However, this definition does not make sense if there are forms $\omega \in \Omega\mathscr{A}$ with $\pi(\omega) = 0$ and $\pi(\delta\omega) \neq 0$. By dividing out these unpleasant forms, Connes constructs a new differential algebra $\Omega_{\mathscr{D}}\mathscr{A}$, the interesting object:

$$\Omega_{\mathscr{D}}\mathscr{A} := \frac{\pi(\Omega\mathscr{A})}{J}$$

with

$$J := \pi(\delta \ker \pi) =: \bigoplus_p J^p$$

(J for junk). On the quotient now, the differential (4) is well defined. Degree by degree we have:

$$\Omega^0_{\mathscr{D}}\mathscr{A} = \rho(\mathscr{A})$$

because $J^0 = 0$,

$$\Omega^1_{\mathscr{D}}\mathscr{A} = \pi(\Omega^1\mathscr{A})$$

because ρ is faithful, and in degree $p \geqq 2$,

$$\Omega^p_{\mathscr{D}}\mathscr{A} = \frac{\pi(\Omega^p\mathscr{A})}{\pi(\delta(\ker\pi)^{p-1})} .$$

While $\Omega\mathscr{A}$ has no cohomology, $\Omega_{\mathscr{D}}\mathscr{A}$ does in general. In fact, in infinite dimensions, if $\mathscr{F}$ is the algebra of complex functions on spacetime M and if the K-cycle is obtained from the Dirac operator, then $\Omega_{\not\partial}\mathscr{F}$ is de Rham's differential algebra of differential forms on M.

We come back to our finite dimensional case. Remember that the elements of the auxiliary differential algebra $\Omega\mathscr{A}$ that we introduced for book-keeping purposes only, are abstract entities defined in terms of symbols and relations. On the other hand, the elements of $\Omega_{\mathscr{D}}\mathscr{A}$, the "forms," are operators on the Hilbert space $\mathscr{H}$, i.e. concrete matrices of complex numbers. Therefore there is a natural scalar product defined by

$$\langle\hat{\omega},\hat{\kappa}\rangle := \mathrm{tr}(\hat{\omega}^*\hat{\kappa}), \qquad \hat{\omega},\hat{\kappa} \in \pi(\Omega^p\mathscr{A}) \tag{5}$$

for elements of equal degree and by zero for two elements of different degree. With this scalar product $\Omega_{\mathscr{D}}\mathscr{A}$ is a subspace of $\pi(\Omega\mathscr{A})$, by definition orthogonal to the junk. As a subspace, $\Omega_{\mathscr{D}}\mathscr{A}$ inherits a scalar product which deserves a special name (,). It is given by

$$(\omega,\kappa) = \langle\hat{\omega},P\hat{\kappa}\rangle, \qquad \omega,\kappa \in \Omega^p_{\mathscr{D}}\mathscr{A} ,$$

where P is the orthogonal projector in $\pi(\Omega\mathscr{A})$ onto the ortho-complement of J and $\hat{\omega}$ and $\hat{\kappa}$ are any representatives in the classes ω and κ. Again the scalar product vanishes for forms with different degree. For real algebras, all traces must be understood as the real part of the trace.

In Yang–Mills models coupling constants appear as parametrization of the most general gauge invariant scalar product. In the same spirit, we want the most general scalar product on $\pi(\Omega\mathscr{A})$ compatible with the underlying algebraic structure. It is given by

$$\langle\hat{\omega},\hat{\kappa}\rangle_z := \mathrm{tr}(\hat{\omega}^*\hat{\kappa}z), \qquad \hat{\omega},\hat{\kappa}\in\pi(\Omega^p\mathscr{A}), \tag{6}$$

where z is a positive operator on $\mathscr{H}$, that commutes with $\rho(\mathscr{A})$ and with the Dirac operator $\mathscr{D}$ and that leaves $\mathscr{H}_L$ and $\mathscr{H}_R$ invariant. A natural subclass of these scalar products is constructed with operators z in the image under ρ of the center of $\mathscr{A}$.

Since π is a homomorphism of involution algebras the product in $\Omega_{\mathscr{D}}\mathscr{A}$ is given by matrix multiplication followed by the projection P. The involution is given by transposition and complex conjugation, i.e. the dual with respect to the scalar product of the Hilbert space $\mathscr{H}$. Note that this scalar product admits no generalization. W. Kalau et al. [12] discuss the computation of the junk and of the differential for matrix algebras.

At this stage there is a first contact with gauge theories. Consider the vector space of anti-Hermitian 1-forms $\{H\in\Omega^1_{\mathscr{D}}\mathscr{A},\ H^*=-H\}$. A general element H is of the form

$$H = i\begin{pmatrix}0 & h\\ h^* & 0\end{pmatrix}$$

with h a finite sum of terms $\rho_L(a_0)[\rho_L(a_1)\mathscr{M}-\mathscr{M}\rho_R(a_1)] : \mathscr{H}_R\to\mathscr{H}_L,\ a_0,a_1\in\mathscr{A}$. These elements are called Higgses or gauge potentials. In fact the space of gauge potentials carries an affine representation of the group of unitaries

$$G := \{g\in\mathscr{A},\ gg^* = g^*g = 1\},$$

defined by

$$\begin{aligned} H^g &:= \rho(g)H\rho(g^{-1}) + \rho(g)\delta(\rho(g^{-1})) = \rho(g)H\rho(g^{-1}) + (-i)\rho(g)[\mathscr{D},\rho(g^{-1})]\\ &= \rho(g)[H-i\mathscr{D}]\rho(g^{-1}) + i\mathscr{D} = i\begin{pmatrix}0 & h^g\\ (h^g)^* & 0\end{pmatrix}\end{aligned}$$

with $h^g - \mathscr{M} := \rho_L(g)[h-\mathscr{M}]\rho_R(g^{-1})$. H^g is the "gauge transformed of H." As usual every gauge potential H defines a covariant derivative $\delta + H$, covariant under the left action of G on $\Omega_{\mathscr{D}}\mathscr{A}$:

$${}^g\omega := \rho(g)\omega, \quad \omega\in\Omega_{\mathscr{D}}\mathscr{A},$$

which means

$$(\delta + H^g)\,{}^g\omega = {}^g[(\delta+H)\omega].$$

Also we define the curvature C of H by

$$C := \delta H + H^2 \in \Omega^2_{\mathscr{D}}\mathscr{A}.$$

Note that here and later, H^2 is considered as element of $\Omega^2_{\mathscr{D}}\mathscr{A}$ which means it is the projection P applied to $H^2\in\pi(\Omega^2\mathscr{A})$. The curvature C is a Hermitian 2-form with *homogeneous* gauge transformations

$$C^g := \delta(H^g) + (H^g)^2 = \rho(g)C\rho(g^{-1})\,.$$

Finally, we define the preliminary Higgs potential $V_0(H)$, a functional on the space of gauge potentials, by

$$V_0(H) := (C, C) = \mathrm{tr}[(\delta H + H^2)P(\delta H + H^2)]\,.$$

It is a polynomial of degree 4 in H with real, non-negative values. Furthermore it is gauge invariant, $V_0(H^g) = V_0(H)$, because of the homogeneous transformation property of the curvature C and because the orthogonal projector P commutes with all gauge transformations, $\rho(g)P = P\rho(g)$. The most remarkable property of the preliminary Higgs potential is that, in most cases, its vacuum spontaneously breaks the group G. To see this, define

$$\mathscr{D}_G := -i\int_G \pi(g^{-1}\delta g)\,\mathrm{d}g\,,$$

where $\mathrm{d}g$ is the Haar measure of the compact Lie group G. Thus $\mathscr{D}_G$ is in $\Omega^1_{\mathscr{D}}\mathscr{A}$, unlike the internal Dirac operator $\mathscr{D}$ which is not necessarily in $\Omega^1_{\mathscr{D}}\mathscr{A}$, see the next example. Moreover

$$\mathscr{D}_G = \mathscr{D} - \int_G \rho(g^{-1})\mathscr{D}\rho(g)\,\mathrm{d}g = \begin{pmatrix} 0 & \mathscr{M}_G \\ \mathscr{M}_G^* & 0 \end{pmatrix}\,,$$

where

$$\mathscr{M}_G := \mathscr{M} - \int_G \rho_L(g^{-1})\mathscr{M}\rho_R(g)\,\mathrm{d}g\,.$$

Note that $\mathscr{M} - \mathscr{M}_G$ leads to gauge invariant mass terms and $G_{\mathscr{M}_G} = G_{\mathscr{M}}$. We now introduce the change of variables

$$\Phi := H - i\mathscr{D}_G =: i\begin{pmatrix} 0 & \varphi \\ \varphi^* & 0 \end{pmatrix} \in \Omega^1_{\mathscr{D}}\mathscr{A} \qquad (7)$$

with $\varphi = h - \mathscr{M}_G$. Then, assuming of course a gauge invariant internal Dirac operator, $\mathscr{D}^g = \mathscr{D}$, Φ is homogeneously transformed into

$$\begin{aligned}\Phi^g = H^g - i\mathscr{D}^g_G &= \rho(g)[H - i\mathscr{D}]\rho(g^{-1}) + i\mathscr{D} - i\mathscr{D} + i\int_G \rho(g'^{-1})\mathscr{D}\rho(g')\mathrm{d}g' \\ &= \rho(g)\left[H - \left(i\mathscr{D} - i\int_G \rho(g'^{-1})\mathscr{D}\rho(g')\mathrm{d}g'\right)\right]\rho(g^{-1}) = \rho(g)\Phi\rho(g^{-1})\,, \qquad (8)\end{aligned}$$

and

$$\varphi^g = \rho_L(g)\varphi\rho_R(g^{-1})\,.$$

Now $h = 0$, or equivalently $\varphi = -\mathscr{M}_G$, is certainly a minimum of the preliminary Higgs potential and this minimum spontaneously breaks G if it is gauge variant and non-degenerate.

Consider two extreme classes of examples, vector-like and left-right models.

A *vector-like model* is defined by an arbitrary internal algebra $\mathscr{A}$ with identical left and right representations, $\rho_L = \rho_R$, and with a mass matrix proportional to the identity in each irreducible component. As we shall see, every vector-like model produces a Yang–Mills model with unbroken parity and unbroken gauge symmetry, $G_{\mathscr{M}} = G_l = G$, as electromagnetism or chromodynamics. Since $\mathscr{D}$ and ρ commute, the internal differential algebra is trivial, $\Omega^p_{\mathscr{D}}\mathscr{A} = 0$ for $p \geqq 1$, and the space of Higgses is zero, $H = 0$. The new variable Φ vanishes as well, because $\mathscr{D}_G$ vanishes:

$$\int_G \rho_L(g^{-1})\mathscr{M}\rho_R(g)\mathrm{d}g = \int_G \rho_L(g^{-1})\mathscr{M}\rho_L(g)\mathrm{d}g$$

$$= \int_G \rho_L(g^{-1})\rho_L(g)\mathscr{M}\mathrm{d}g = \int_G \mathscr{M}\mathrm{d}g = \mathscr{M}\,.$$

The preliminary Higgs potential vanishes identically, but its minimum is non-degenerate. In this example, the simpler variable $\Phi = H - i\mathscr{D}$ would not be in a vector space, because $\mathscr{D} \notin \Omega^1_{\mathscr{D}}\mathscr{A}$.

We define a *left-hand model* by an internal algebra consisting of a sum of a "left-handed" and a "right-handed" algebra, $\mathscr{A} = \mathscr{A}_L \oplus \mathscr{A}_R$ with the left-handed algebra acting only on left-handed fermions and similarly for right-handed,

$$\rho_L(a_L, a_R) = \rho_L(a_L, 0)\ , \quad \rho_R(a_L, a_R) = \rho_R(0, a_R)\ , \qquad a_L \in \mathscr{A}_L, \quad a_R \in \mathscr{A}_R\,.$$

Now, any non-vanishing fermion mass matrix breaks the gauge invariance, $G_{\mathscr{M}} \neq G$, $\mathscr{M} \neq 0$. At the same time, the internal Dirac operator is a 1-form, $\mathscr{D} = \mathscr{D}_G \in \Omega^1_{\mathscr{D}}\mathscr{A}$, because

$$\int_G \rho_L(g^{-1})\mathscr{M}\rho_R(g)\mathrm{d}g = \int_{G_L \times G_R} \rho_L(g_L^{-1}, 1)\mathscr{M}\rho_R(1, g_R)\mathrm{d}g_L\mathrm{d}g_R$$

$$= \left(\int_{G_L} \rho_L(g_L^{-1}, 1)\mathrm{d}g_L\right)\mathscr{M}\left(\int_{G_R} \rho_R(1, g_R)\mathrm{d}g_R\right) = 0\,.$$

In left-right models, we have $\Phi = H - i\mathscr{D}$. A more interesting, intermediate example will be discussed in Sect. 3.1 below.

In the next step, the vectors ψ_L, ψ_R, and H are promoted to genuine fields, i.e. rendered spacetime dependent. As already known in classical quantum mechanics, this is achieved by tensorizing with functions. Let us denote by $\mathscr{F}$ the algebra of (smooth, real or complex valued) functions over spacetime M. Consider the algebra $\mathscr{A}_t := \mathscr{F} \otimes \mathscr{A}$. The group of unitaries of the tensor algebra $\mathscr{A}_t$ is the gauged version of the group of unitaries of the internal algebra $\mathscr{A}$, i.e. the group of functions from spacetime into the group G. Consider the representation $\rho_t := \underline{\cdot} \otimes \rho$ of the tensor algebra on the tensor product $\mathscr{H}_t := \mathscr{S} \otimes \mathscr{H}$, where $\mathscr{S}$ is the Hilbert space of square integrable spinors on which functions act by multiplication: $(\underline{f\psi})(x) := f(x)\psi(x)$, $f \in \mathscr{F}$, $\psi \in \mathscr{S}$. The definition of the tensor product of Dirac operators.

$$\mathscr{D}_t := \not{\partial} \otimes 1 + \gamma_5 \otimes \mathscr{D}$$

comes from non-commutative geometry. We now repeat the above construction for the infinite dimensional algebra $\mathscr{A}_t$ and its K-cycle. As already stated, for $\mathscr{A} = \mathbb{C}$, $\mathscr{H} = \mathbb{C}$, $\mathscr{M} = 0$ the differential algebra $\Omega_{\mathscr{D}_t}\mathscr{A}_t$ is isomorphic to the de Rham algebra of differential forms $\Omega(M, \mathbb{C})$. For general $\mathscr{A}$, using the notations of [8], an anti-Hermitian 1-form $H_t \in \Omega^1_{\mathscr{D}_t}\mathscr{A}_t$,

$$H_t = A + H\,,$$

contains two pieces, an anti-Hermitian Higgs *field* $H \in \Omega^0(M, \Omega^1_{\mathscr{D}}\mathscr{A})$ and a genuine gauge field $A \in \Omega^1(M, \rho(\mathrm{g}))$ with values in the Lie algebra of the group of unitaries, $\mathrm{g} := \{X \in \mathscr{A},\ X^* = -X\}$, represented on $\mathscr{H}$. The curvature of H_t,

$$C_t := \delta_t H_t + H_t^2 \in \Omega^2_{\mathscr{D}}\mathscr{A}_t\ ,$$

contains three pieces,

$$C_t = C + F - \mathrm{D}\Phi\gamma_5\ ,$$

the ordinary, now x-dependent, curvature $C = \delta H + H^2$, the field strength

$$F := \mathrm{d}A + \frac{1}{2}[A, A] \in \Omega^2(M, \rho(\mathrm{g}))\ ,$$

and the covariant derivative of Φ,

$$\mathrm{D}\Phi = \mathrm{d}\Phi + [A, \Phi] \in \Omega^1(M, \Omega^1_{\mathscr{D}}\mathscr{A})\ .$$

Note that the covariant derivative may be applied to Φ thanks to its homogeneous transformation law, Eq. (8).

The definition of the Higgs potential in the infinite dimensional space

$$V_t(H_t) := (C_t, C_t)$$

requires a suitable regularisation of the sum of eigenvalues over the space of spinors $\mathscr{S}$. Here, we have to suppose spacetime to be compact and Euclidean. Then the regularisation is achieved by the Dixmier trace which allows an explicit computation of V_t. One of the miracles in CL models is that V_t alone reproduces the complete bosonic action of a YMH model. Indeed it contains of three pieces, the Yang–Mills action, the covariant Klein–Gordon action and an integrated Higgs potential,

$$V_t(A + H) = \int_M \mathrm{tr}(F * F z) + \int_M \mathrm{tr}(\mathrm{D}\Phi^* * \mathrm{D}\Phi z) + \int_M * V(H)\ . \tag{9}$$

As the preliminary Higgs potential V_0, the (final) Higgs potential V is calculated as a function of the fermion masses,

$$V := V_0 - \mathrm{tr}[\alpha C^* \alpha C z] = \mathrm{tr}[(C - \alpha C)^*(C - \alpha C) z]\ .$$

The linear map $\alpha : \Omega^2_{\mathscr{D}}\mathscr{A} \to \rho(\mathscr{A}) + \pi(\delta(\ker \pi)^1)$ is determined by the two equations

$$\mathrm{tr}[R^*(C - \alpha C) z] = 0 \quad \text{for all } R \in \rho(\mathscr{A})\ ,$$

$$\mathrm{tr}[K^* \alpha C z] = 0 \quad \text{for all } K \in \pi(\delta(\ker \pi)^1)\ . \tag{10}$$

All remaining traces are over the finite dimensional Hilbert space $\mathscr{H}$. Note the "wrong" signs of the first and third terms in Eq. (9). The signs are in fact correct for Euclidean spacetime.

Another miracle happens in the fermionic sector, where the completely covariant action $\psi^*(\mathscr{D}_t + iH_t)\psi$ reproduces the complete fermionic action of a YMH model. We denote by

$$\psi = \psi_L + \psi_R \in \mathscr{H}_t = \mathscr{S} \otimes (\mathscr{H}_L \oplus \mathscr{H}_R)$$

the multiplets of spinors and by ψ^* the dual of ψ with respect to the scalar product of the concerned Hilbert space. Then

$$\begin{aligned}\psi^*(\mathscr{D}_t + iH_t)\psi &= \int_M * \psi^*(\not{\partial} + i\gamma(A))\psi - \int_M * (\psi_L^* h\gamma_5\psi_R + \psi_R^* h^*\gamma_5\psi_L)\\ &\quad + \int_M * (\psi_L^* \mathscr{M}\gamma_5\psi_R + \psi_R^* \mathscr{M}^*\gamma_5\psi_L)\\ &= \int_M * \psi^*(\not{\partial} + i\gamma(A))\psi - \int_M * (\psi_L^* \varphi\gamma_5\psi_R + \psi_R^* \varphi^*\gamma_5\psi_L)\\ &\quad + \int_M * (\psi_L^*(\mathscr{M} - \mathscr{M}_G)\gamma_5\psi_R + \psi_R^*(\mathscr{M} - \mathscr{M}_G)^*\gamma_5\psi_L) \qquad (11)\end{aligned}$$

containing the ordinary Dirac action and the Yukawa couplings. If the minimum $\varphi = v$ is non-degenerate, we retrieve the input fermionic mass matrix $\mathscr{M}$ on the output side by setting the perturbative variables h to zero in the first equation in (11). The rhs of the second equation in (11) is the fermionic action written with the homogeneous scalar variables φ. The second term yields the trilinear invariants (2) with Yukawa couplings fixed such that $\mathscr{M}$ is the fermionic mass matrix. As already pointed out, the third term is an invariant mass term and therefore admissible in a YMH Lagrangian. Consequently every CL model with non-degenerate vacuum is a YMH model with $\mathscr{H}_S = \{H \in \Omega^1_{\mathscr{D}}\mathscr{A},\ H^* = -H\}$. Note that $\mathscr{H}_S$ carries a group representation, that is not necessarily an algebra representation and we have the following inclusion of group representations $\mathscr{H}_S \subset (\mathscr{H}_L^* \otimes \mathscr{H}_R) \oplus (\mathscr{H}_R^* \otimes \mathscr{H}_L)$. Furthermore $G_l = G_{\mathscr{M}} = G_{\mathscr{M}_G}$. We have nothing to say about degenerate vacua, i.e. minima of the Higgs potential, that lie on distinct gauge orbits. In fact, whether these are allowed in YMH models is a question of taste for some, a question of quantum corrections for others. We shall indicate a few examples. A final remark concerns the unusual appearance of γ_5 in the fermionic action (11). Just as the "wrong" signs in the bosonic action (9), these γ_5 are proper to the Euclidean signature and disappear in the Minkowski signature.

3. Examples with Degenerate Vacua

3.1. Discrete Degeneracy. Our first example is in between vector-like models, $\mathscr{M}_G = 0$, and left-right models, $\mathscr{M}_G = \mathscr{M}$, in the sense that here $\mathscr{M}_G \neq 0$ and $\neq \mathscr{M}$. Choose as internal algebra $\mathscr{A} = M_2(\mathbb{C})$, the algebra of complex 2×2 matrices. Both left- and right-handed fermions come in N generations of doublets, $\mathscr{H}_L = \mathscr{H}_R = \mathbb{C}^2 \otimes \mathbb{C}^N$. These Hilbert spaces carry identical left and right representations

$$\rho_L(a) = \rho_R(a) := a \otimes 1_N, \quad a \in \mathscr{A}\,.$$

The fermion mass matrix is chosen block diagonal to ensure conservation of the electric charge, $G_{\mathscr{M}} = U(1)$:

$$\mathscr{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix},$$

m_1 and m_2 are complex $N \times N$ matrices which should be thought of as mass matrices of the quarks of electric charge 2/3 and $-1/3$, and we suppose them different, $m_1 \neq m_2$. Then

$$\begin{aligned}\mathscr{M}_G &= \mathscr{M} - \int_{U(2)} (g^{-1} \otimes 1_N)\mathscr{M}(g \otimes 1_N)\,\mathrm{d}g \\ &= 1_2 \otimes \frac{1}{2}(m_1 + m_2) + \sigma_3 \otimes \frac{1}{2}(m_1 - m_2) - \int_{U(2)} g^{-1} 1_2 g\,\mathrm{d}g \otimes \frac{1}{2}(m_1 + m_2) \\ &\quad - \int_{U(2)} g^{-1}\sigma_3 g\,\mathrm{d}g \otimes \frac{1}{2}(m_1 - m_2) = \frac{1}{2}\sigma_3 \otimes \mu\,,\end{aligned}$$

where we have put $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mu := m_1 - m_2$, and we have used the identity

$$\int_{U(2)} g^{-1} A g\,\mathrm{d}g = \frac{1}{2}(\mathrm{tr}\,A)\,1_2, \quad A \in M_2(\mathbb{C})\,.$$

A general component of the Higgs field takes the form $h = h_1 \otimes \mu$, h_1 being an arbitrary Hermitian 2×2-matrix. Likewise $\varphi = \varphi_1 \otimes \mu$ and $\varphi_1 = h_1 - 1/2\sigma_3$. In these variables, the Higgs potential can be computed to be [13]

$$V(H) = 2\left(\mathrm{tr}((\mu\mu^*)^2) - \frac{(\mathrm{tr}\,\mu\mu^*)^2}{N}\right)\mathrm{tr}[(\varphi_1 + 1_2/2)^2(\varphi_1 - 1_2/2)^2]\,.$$

z is necessarily a positive scalar and we have put $z = 1_{4N}$. For $N = 1$ generation, the Higgs potential vanishes identically, and any point in $\mathscr{H}_S$ is minimum. The situation is more exciting in the presence of two or more generations. Then, the minima lie on three disconnected pieces, the orbit of $\varphi = -\mathscr{M}_G$ with a little group $G_l = G_{\mathscr{M}} = U(1)$, are two isolated points $\varphi_1 = \pm 1_2/2$ with a little group $G_l = U(2)$. We may wonder if quantum corrections [14] do lift this degeneracy and if so, in favor of which vacuum.

Since $\mathscr{D}_G$ is a 1-form, one can compute the curvature of the Higgs $i\mathscr{D}_G$:

$$\delta(i\mathscr{D}_G) + (i\mathscr{D}_G)^2 = \frac{1}{2}\left(\mathrm{tr}[\mathscr{M}\mathscr{M}^*] - \frac{1}{2}|\mathrm{tr}\,\mathscr{M}|^2\right) 1_4 \otimes 1_N, \quad \text{for all } N\,.$$

We remark that, in the similar looking model by M. Dubois-Violette, R. Kerner and J. Madore [15], this curvature vanishes.

3.2. Continuous Degeneracy. In the last example we had a finite, discrete degeneracy: the vacuum consisted of three disconnected orbits. Now we would like to present a left-right model with continuous degeneracy, the orbits of the minimum will lie on a horizontal gutter. Consider the complex algebra $\mathscr{A} = M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ with representations on $\mathscr{H}_L = \mathbb{C}^2$, $\mathscr{H}_R = \mathbb{C}^2$ given by

$$\rho_L(a,b,b') = a, \qquad \rho_R(a,b,b') = \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix} =: B, \quad a \in M_2(\mathbb{C}), \quad (b,b') \in \mathbb{C} \oplus \mathbb{C}\,.$$

Let the mass matrix be as in the last example with one generation,

$$\mathscr{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad m_1, m_2 \in \mathbb{C}, \ |m_1| \neq |m_2|\,.$$

Recall that for any left-right model we have $\mathscr{M}_G = \mathscr{M}$ and $\mathscr{D}_G = \mathscr{D}$. A general element of $\Omega^1_{\mathscr{D}}\mathscr{A}$ is of the form

$$\pi((a_0, b_0, b_0')\delta(a_1, b_1, b_1')) = i \begin{pmatrix} 0 & a_0(a_1 - B_1)\mathscr{M} \\ -\mathscr{M}^* B_0(a_1 - B_1) & 0 \end{pmatrix}$$

$$= H = i \begin{pmatrix} 0 & h \\ \tilde{h}^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & h_1\mathscr{M} \\ \mathscr{M}^* \tilde{h}_1^* & 0 \end{pmatrix}, \quad h_1, \tilde{h}_1 \in M_2(\mathbb{C}) .$$

As an element of $\pi(\Omega^2\mathscr{A})$, δH is

$$\delta H = \pi(\delta(a_0, b_0, b_0')\delta(a_1, b_1, b_1'))$$

$$= \begin{pmatrix} (a_0 - B_0)\mathscr{M}\mathscr{M}^*(a_1 - B_1) & 0 \\ 0 & \mathscr{M}^*(a_0 - B_0)(a_1 - B_1)\mathscr{M} \end{pmatrix}$$

$$= \begin{pmatrix} \Sigma(a_0 - B_0)(a_1 - B_1) + \Delta(a_0 - B_0)\sigma_3(a_1 - B_1) & 0 \\ 0 & \mathscr{M}^*(a_0 - B_0)(a_1 - B_1)\mathscr{M} \end{pmatrix},$$

where we have used the decomposition

$$\mathscr{M}\mathscr{M}^* = \begin{pmatrix} |m_1|^2 & 0 \\ 0 & |m_2|^2 \end{pmatrix} = \Sigma 1_2 + \Delta\sigma_3$$

with

$$\Sigma := \frac{1}{2}(|m_1|^2 + |m_2|^2) , \qquad \Delta := \frac{1}{2}(|m_1|^2 - |m_2|^2) .$$

A general element in $(\ker \pi)^1$ is a finite sum of the form $\sum_j (a_0^j, b_0^j, b_0'^j)\delta(a_1^j, b_1^j, b_1'^j)$ with the two conditions

$$\left[\sum_j a_0^j(a_1^j - B_1^j)\right] \mathscr{M} = 0 , \qquad \mathscr{M}^* \left[\sum_j B_0^j(a_1^j - B_1^j)\right] = 0 .$$

Therefore the corresponding general element in $\pi(\delta(\ker \pi)^1)$ is

$$\delta H = \begin{pmatrix} \Sigma \sum_j (a_0^j - B_0^j)(a_1^j - B_1^j) + \Delta \sum_j (a_0^j - B_0^j)\sigma_3(a_1^j - B_1^j) & 0 \\ 0 & 0 \end{pmatrix}$$

still subject to the two conditions. Recall that $\Delta \neq 0$ by assumption and we have the following inclusion:

$$\pi(\delta(\ker \pi)^1) \supset \left\{ \begin{pmatrix} \Delta \sum_j a_0^j \sigma_3 a_1^j & 0 \\ 0 & 0 \end{pmatrix}, \sum_j a_0^j a_1^j = 0 \right\} = \left\{ \begin{pmatrix} \Delta k & 0 \\ 0 & 0 \end{pmatrix}, k \in M_2(\mathbb{C}) \right\} .$$

To prove the last equality, we note that the subspace is a two-sided ideal in rhs and non-zero. The algebra $M_2(\mathbb{C})$ being simple, the subspace is the whole algebra. Consequently the junk is

$$J^2 = \pi(\delta(\ker \pi)^1) = \left\{ \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, k \in M_2(\mathbb{C}) \right\} .$$

Now we compute the quotient $\Omega^2_{\mathscr{D}}\mathscr{A} = \pi(\Omega^2\mathscr{A})/J^2$ as an orthogonal complement of the junk is $\pi(\Omega^2\mathscr{A})$ with respect to the scalar product (5) with $z = 1_4$,

$$\Omega^2_{\mathscr{D}}\mathscr{A} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \mathscr{M}^* c_1 \mathscr{M} \end{pmatrix}, \; c_1 \in M_2(\mathbb{C}) \right\} .$$

Let us recapitulate:

$$\Omega^0_{\mathscr{D}}\mathscr{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}, \; a \in M_2(\mathbb{C}), \; B = \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix} \right\} ,$$

$$\Omega^1_{\mathscr{D}}\mathscr{A} = \left\{ i \begin{pmatrix} 0 & h_1\mathscr{M} \\ \mathscr{M}^* \tilde{h}_1^* & 0 \end{pmatrix}, \; h_1, \tilde{h}_1 \in M_2(\mathbb{C}) \right\} ,$$

$$\Omega^2_{\mathscr{D}}\mathscr{A} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \mathscr{M}^* c \mathscr{M} \end{pmatrix}, \; c \in M_2(\mathbb{C}) \right\} .$$

Since π is a *-homomorphism, the product in $\Omega_{\mathscr{D}}\mathscr{A}$ is given by matrix multiplication followed by the projection

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1_2 \end{pmatrix} ,$$

and the involution is given by transposition and complex conjugation. In order to calculate the differential δ, we went back to the differential envelope:

$$\delta : \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} \in \Omega^0_{\mathscr{D}}\mathscr{A} \mapsto i \begin{pmatrix} 0 & (a-B)\mathscr{M} \\ -\mathscr{M}^*(a-B) & 0 \end{pmatrix} \in \Omega^1_{\mathscr{D}}\mathscr{A} ,$$

$$\delta : i \begin{pmatrix} 0 & h_1\mathscr{M} \\ \mathscr{M}^* \tilde{h}_1^* & 0 \end{pmatrix} \in \Omega^1_{\mathscr{D}}\mathscr{A} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & \mathscr{M}^*(h_1 + \tilde{h}_1^*)\mathscr{M} \end{pmatrix} \in \Omega^2_{\mathscr{D}}\mathscr{A} .$$

Let now

$$H = i \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix} = i \begin{pmatrix} 0 & h_1\mathscr{M} \\ \mathscr{M}^* h_1^* & 0 \end{pmatrix}, \quad h_1 \in M_2(\mathbb{C}) ,$$

be a Higgs. Its homogeneous variable is

$$\Phi := H - i\mathscr{D}_G = H - i\mathscr{D} = i \begin{pmatrix} 0 & \varphi \\ \varphi^* & 0 \end{pmatrix} = i \begin{pmatrix} 0 & \varphi_1\mathscr{M} \\ \mathscr{M}^* \varphi_1^* & 0 \end{pmatrix} .$$

In other words, $\varphi_1 = h_1 - 1_2$ is an arbitrary, complex 2×2 matrix. Under the group of unitaries $G = U(2) \times U(1) \times U(1)$, is still decomposes into two irreducible pieces, its two column vectors, $\varphi_1 =: (\varphi_{11}, \varphi_{12})$. In terms of these variables, the curvature reads

$$C := \delta H + H^2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathscr{M}^* c \mathscr{M} \end{pmatrix} \in \Omega^2_{\mathscr{D}}\mathscr{A}$$

with $c = h_1 + h_1^* - h_1^* h_1 = 1_2 - \varphi_1^* \varphi_1$. The preliminary Higgs potential is

$$\begin{aligned} V_0(H) &= \mathrm{tr}[C^2] = \mathrm{tr}[(\mathscr{M}^*(1_2 - \varphi_1^*\varphi_1)\mathscr{M})^2] \\ &= |m_1|^4 + |m_2|^4 + |m_1|^4(\varphi_{11}^*\varphi_{11})^2 + |m_2|^4(\varphi_{12}^*\varphi_{12})^2 \\ &\quad - 2|m_1|^4\varphi_{11}^*\varphi_{11} - 2|m_2|^4\varphi_{12}^*\varphi_{12} + 2|m_1|^2|m_2|^2(\varphi_{11}^*\varphi_{12})(\varphi_{12}^*\varphi_{11}) . \end{aligned}$$

Its minimum is non-degenerate and spontaneously breaks the gauge symmetry. However with

$$\alpha C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & |m_1|^2 c_{11} & 0 \\ 0 & 0 & 0 & |m_2|^2 c_{22} \end{pmatrix},$$

the Higgs potential

$$V = \mathrm{tr}[(C - \alpha C)^2] = 2|m_1|^2 |m_2|^2 |\varphi_{11}^* \varphi_{12}|^2$$

has continuously degenerate vacua which also include the gauge invariant point, $\varphi_{11} = \varphi_{12} = 0$. Indeed, the Higgs potential vanishes if and only if the two complex doublets φ_{11} and φ_{12} are orthogonal, irrespective of their lengths. Finally, we remark that the Higgs potential has only symmetry breaking minima for two and more generations.

3.3. Complete Symmetry Breakdown. We have seen that, in CL models with non-degenerate vacuum, the little group coincides with $G_{\mathcal{M}}$. The latter is controlled immediately by the input. We take advantage of this to construct a model with complete, spontaneous symmetry breakdown, i.e. a finite little group. Consider a left-right model with *real* internal algebra $\mathcal{A} = \mathbb{H} \oplus \mathbb{C}$, $\mathcal{A}_L = \mathbb{H}$ being the quaternions, and two generations of fermions

$$\mathcal{H}_L = \mathbb{C}^2 \otimes \mathbb{C}^2, \qquad \rho_L(a,b) = a \otimes 1_2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad a \in \mathbb{H},$$

$$\mathcal{H}_R = (\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^2, \qquad \rho_R(a,b) = B \otimes 1_2 = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}, \quad b \in \mathbb{C},\ B := \begin{pmatrix} b & 0 \\ 0 & b^* \end{pmatrix}.$$

We choose the mass matrix

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{pmatrix}, \qquad \mathcal{M}_1 := \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \qquad \mathcal{M}_2 := \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix},$$

with $m_1, m_2, m \in \mathbb{R}$, $m_1 \neq m_2$, $m \neq 0$. Therefore $G_{\mathcal{M}} = \mathbb{Z}_2$. A general 1-form is a finite sum of terms

$$H = -i\pi((a_0, b_0)\delta(a_1, b_1)) = i \begin{pmatrix} 0 & 0 & h_1 \mathcal{M}_1 & 0 \\ 0 & 0 & 0 & h_2 \mathcal{M}_2 \\ \mathcal{M}_1 \tilde{h}_1 & 0 & 0 & 0 \\ 0 & \mathcal{M}_2 \tilde{h}_2 & 0 & 0 \end{pmatrix}$$

with

$$h_1 := a_0(a_1 - B_1), \qquad h_2 := a_0(a_1 - B_1^*),$$

$$\tilde{h}_1 := -B_0(a_1 - B_1), \qquad \tilde{h}_2 := -B_0^*(a_1 - B_1^*).$$

After the finite summation, the four quaternions $h_1, h_2, \tilde{h}_1, \tilde{h}_2$ are independent in general. The junk in degree two is

$$\pi(\delta(\ker \pi)^1) = \left\{ \begin{pmatrix} i(m_1^2 - m_2^2)k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ k \in \mathbb{H} \right\}$$

and

$$\delta H = \begin{pmatrix} \frac{1}{2}(m_1^2+m_2^2)(h_1+\tilde{h}_1) & 0 & 0 & 0 \\ 0 & m^2(h_2+\tilde{h}_2) & 0 & 0 \\ 0 & 0 & \mathcal{M}_1(h_1+\tilde{h}_1)\mathcal{M}_1 & 0 \\ 0 & 0 & 0 & \mathcal{M}_2(h_2+\tilde{h_2})\mathcal{M}_2 \end{pmatrix} .$$

A Higgs, an anti-Hermitian 1-form, is characterized by two independent quaternions, h_1 and h_2,

$$h = \begin{pmatrix} h_1\mathcal{M}_1 & 0 \\ 0 & h_2\mathcal{M}_2 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} \varphi_1\mathcal{M}_1 & 0 \\ 0 & \varphi_2\mathcal{M}_2 \end{pmatrix},$$

with $\varphi_j = h_j - 1_2,\ j = 1,2$. Let us decompose each quaternion

$$\varphi_j = \begin{pmatrix} x_j & -y_j^* \\ y_j & x_j^* \end{pmatrix}, \quad x_j, y_j \in \mathbb{C}$$

into its two column vectors

$$\varphi_j = \left(\varphi_{j1}, -i\sigma_2\varphi_{j1}^{*T}\right), \quad \varphi_{j1} = \begin{pmatrix} x_j \\ y_j \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

They define the irreducible pieces of the Higgs under a unitary transformation $g = (g_2, g_1) \in SU(2) \times U(1)$:

$$\varphi_{11}^g = g_2\varphi_{11}g_1^{-1}, \qquad \varphi_{21}^g = g_2\varphi_{21}g_1 .$$

In other words the Higgs consists of two complex $SU(2)$-doublets with opposite $U(1)$-charges. Note that if $\mathcal{M}_2$ was also diagonal, we would only have one complex Higgs doublet. Now the computation of the Higgs potential is lengthy, but straightforward. In terms of the two doublets, the result for $z = 1_8$ is

$$\begin{aligned} V(H) = (m_1^4 + m_2^4)[1 - \varphi_{11}^*\varphi_{11}]^2 + 2m^4[1 - \varphi_{21}^*\varphi_{21}]^2 \\ - (m_1^2 + m_2^2)m^2[1 - \varphi_{11}^*\varphi_{11}][1 - \varphi_{21}^*\varphi_{21}] . \end{aligned}$$

The Higgs potential is zero if and only if both complex doublets φ_{11} and φ_{21} have length one. Since their relative orientation is arbitrary and gauge invariant, the vacua are continuously degenerate. However, in every vacuum, all four gauge bosons are massive and the four masses are independent of the relative orientation. Furthermore, the little groups of all vacua are equal, $G_l = \{-1, +1\}$, as expected.

4. Necessary Conditions

One may very well do general relativity using only Euclidean geometry. Still, we agree that Riemannian geometry is the natural setting of general relativity. A main argument in favor of this attitude is that there are infinitely more gravitational theories in Euclidean geometry than in Riemannian geometry. The same is true for the standard model. Its natural setting, to our taste, is non-commutative geometry. The fact that today's Yang–Mills–Higgs model of electro-weak and strong interactions falls in the infinitely smaller class of Connes–Lott models is remarkable. The purpose of this section is to show to what extent it is remarkable. We give a list of constraints on the input of a YMH model. They are necessary conditions for the existence of a corresponding CL model.

4.1. The Group. The compact Lie group G defining a Yang–Mills model must be chosen such that its Lie algebra $\mathfrak{g}$ admits an invariant scalar product. Therefore $\mathfrak{g}$ is a direct sum of simple and abelian algebras. After complexification, the simple Lie algebras are classified according to E. Cartan, into four infinite series, $su(n+1)$, $n \geqq 1$, $o(2n+1)$, $n \geqq 2$, $sp(n)$, $n \geqq 3$, $o(2n)$, $n \geqq 4$ and five exceptional algebras G_2, F_4, E_6, E_7, E_8. To define a CL model, we need a real or complex involution algebra $\mathscr{A}$ admitting a finite dimensional, faithful representation. Their classification is easy. In the complex case, such an algebra is a direct sum of matrix algebras $M_n(\mathbb{C})$, $n \geqq 1$. In the real case, we have direct sums of matrix algebras with real, complex or quaternionic coefficients, $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, $M_n(\mathbb{H})$, $n \geqq 1$. The corresponding groups of unitaries are $O(n,\mathbb{R}), U(n), USp(n)$. Note the two isomorphisms, $USp(2) \cong SU(2)$ and $USp(4)/\mathbb{Z}_2 \cong SO(5,\mathbb{R})$.

Let us outline the proof of the classification. Since $\mathscr{A}$ has a faithful representation on a Hilbert space it is semi-simple [16]. Then $\mathscr{A}$ is a finite sum of $n \times n$ matrices over finite dimensional division algebras [17]. There are only three finite, real division algebras, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ [18].

The groups accessible in a CL model therefore belong to the second, third, and forth Cartan series. Furthermore we have $u(n) \cong su(n) \oplus u(1)$. Up to the $u(1)$ factor, this is the first series. At the group level, this factor is disposed of by a condition on the determinant. In the algebraic setting there is a similar condition, that reduces the group of unitaries to a subgroup, here $SU(n)$. This condition is called unimodularity and is discussed in the next section. To sum up, all classical Lie groups are accessible in a CL model but the exceptional ones.

4.2. The Fermion Representation. In a YMH model, the left- and right-handed fermions come in unitary representations of the chosen group G. Every G has an infinite number of irreducible, unitary representations. They are classified by their maximal weight. On the other hand, the above involution algebras $\mathscr{A}$ admit only one or two irreducible representations. The reason is that an algebra representation has to respect the multiplication and the linear structure, while a group representation has to respect only the multiplication. In particular, the tensor product of two group representations is a group representation, while the tensor product of two algebra representations is not an algebra representation, in general.

The only irreducible representation of $M_n(\mathbb{C})$ as a complex algebra is the fundamental one on $\mathscr{H} = \mathbb{C}^n$. Also $M_n(\mathbb{R})$ and $M_n(\mathbb{H})$ have only the fundamental representations on $\mathscr{H} = \mathbb{R}^n$ and $\mathscr{H} = \mathbb{C}^n \otimes \mathbb{C}^2$, while $M_n(\mathbb{C})$ considered as a real algebra has two inequivalent, irreducible representations, the fundamental one: $\mathscr{H} = \mathbb{C}^n$, $\rho_1(a) = a$, $a \in M_n(\mathbb{C})$, and its conjugate: $\mathscr{H} = \mathbb{C}^n$, $\rho_2(a) = \bar{a}$.

The proof of this classification relies on the facts that the centers of the above algebras $\mathscr{A} = M_n(\mathbb{R})$, $M_n(\mathbb{C})$, $M_n(\mathbb{H})$ are the corresponding division algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}$, and that the representations of $\mathscr{A}$ are classified (up to equivalence) by the automorphisms of their centers (as real algebras). Thus $M_n(\mathbb{C})$ is the only case with two inequivalent representations (Skolem–Noether theorem [19]).

Let us summarize. The only possible representations for fermions in a CL model are

- for $G = O(n, \mathbb{R})$, N generations of the fundamental representation on $\mathcal{H} = \mathbb{R}^n \otimes \mathbb{R}^N$,
- for $G = U(n)$ (or $SU(n)$), N generations of the fundamental representation on $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^N$ and $\bar{N}$ generations of its conjugate on $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^{\bar{N}}$,
- for $G = USp(n)$, N generations of the fundamental representation on $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$.

In a YMH model with $G = SU(2)$, the fermions can be put in any irreducible representations of dimension $1, 2, 3, \ldots$, while in the corresponding CL model with $\mathcal{A} = \mathbb{H}$, there is only one irreducible representation accessible for the fermions, the fundamental, two dimensional one. Similarly, in a YMH model with $G = U(1)$ the fermions can have any (electric) charge from $\mathbb{Z}$ or even from $\mathbb{R}$ if we allow "spinor" representations. In the corresponding CL model with $\mathcal{A} = \mathbb{C}$, fermions can only have charges plus and minus one. In any case, if we want more fermions in a CL model, we are forced to introduce families of fermions.

4.3. The Gauge Coupling Constants. In a YMH model, the gauge coupling constants parameterize the most general gauge invariant scalar product on the Lie algebra $\mathfrak{g}$ of G. In a CL model, see the rhs of Eq. (9), this scalar product is not general but comes from the trace over the fermion representation space $\mathcal{H}$, Eq. (6). The scalar product involves the positive operator z, that commutes with the internal Dirac operator and with the fermion representation $\rho(\mathcal{A})$ and that leaves $\mathcal{H}_L$ and $\mathcal{H}_R$ invariant. Depending on the details of the mass matrix and of the left- and right-handed representations ρ_L and ρ_R, the gauge coupling constants may be constraint or not. The examples of the last section will illustrate this point.

4.4. The Higgs Sector. As explained in Sect. 2, the scalar representation ρ_S on $\mathcal{H}_S$ in a CL model is a representation of the *group* of unitaries only. This representation is not chosen but it is calculated as a function of the left- and right-handed fermion representations and of the mass matrix. As illustrated by the examples of Sect. 3, the dependence of the scalar representation on this input is involved and we can make only one general statement:

$$\mathcal{H}_S \subset (\mathcal{H}_L^* \otimes \mathcal{H}_R) \oplus (\mathcal{H}_R^* \otimes \mathcal{H}_L) .$$

Nevertheless, this inclusion is sufficient to rule out the possibility of spontaneous parity breaking in left-right symmetric models à la Connes–Lott [13].

The Higgs potential as well, is on the output side of a CL model. Its calculation involves the positive operator z from the input and is by far, the most complicated calculation in this scheme. We know that $\varphi = -\mathcal{M}_G$ is an absolute minimum of the Higgs potential. If it is non-degenerate, the gauge and scalar boson masses are determined by the fermion masses and the entries of z. See the last section for examples.

Our last necessary condition concerns the Yukawa couplings. In a CL model, they are determined such that $\mathcal{M}$ is the fermionic mass matrix after spontaneous symmetry breaking. Up to the z dependent scalar normalization in the bosonic action (9), the Yukawa couplings are all one. Normalization details are relegated to the appendix.

5. The Unimodularity Condition

The purpose of the unimodularity condition is to reduce the group of unitaries $U(n)$ to its subgroup $SU(n)$. At the group level, this is easily achieved by the condition $\det g = 1$. However the determinant being a non-linear function is not available at the algebra level. We are lead to use the trace instead, together with the formula

$$\det e^{2\pi i X} = e^{2\pi i \,\mathrm{tr}\, X} .$$

Even in the infinite dimensional case, the connected component G^0 of the unit in the group of unitaries G is generated by elements $g = e^{2\pi i X}$, $X = X^* \in \mathcal{A}$. The desired reduction is achieved by using the phase, defined by [20],

$$\mathrm{phase}_\tau(g) := \frac{1}{2\pi i} \int_0^1 \tau\left(g(t) \frac{\mathrm{d}}{\mathrm{d}t} g(t)^{-1} \right) \mathrm{d}t ,$$

where τ is a linear form on $\mathcal{A}$ satisfying

$$\tau(1) \in \mathbb{Z}, \quad \tau(a^*) = \tau(a), \quad \tau(a) = \tau(g^* a g), \qquad g \in G,\ a \in \mathcal{A}^+ := \{bb^*, b \in \mathcal{A}\} ,$$

and where $g(t)$ is a curve in G^0 connecting the unit to g. We obtain the finite dimensional case above by putting $\tau(a) = \mathrm{tr}\rho(a)$ and $g(t) = e^{2\pi i X t}$. The definition of the phase involves two choices, that are easily controlled in finite dimensions: the most general linear form τ can be written as $\tau(a) = \mathrm{tr}\rho(ap)$, $a \in \mathcal{A}$, $p \in$ center $\mathcal{A}$, and the ambiguity in the choice of the curve $g(t)$ is controlled by the first fundamental group $\pi^1(G^0)$ which is contained in $\mathbb{Z}$, see the table below. Therefore the unimodularity condition

$$e^{2\pi i \,\mathrm{phase}_\tau(g)} = 1$$

is well defined and defines a subgroup

$$G_p := \left\{ g \in G^0,\ e^{2\pi i \,\mathrm{phase}_{\mathrm{tr}\rho(\,\cdot\, p)}(g)} = 1 \right\}$$

of G^0. For $\mathcal{A} = M_n(\mathbb{C})$, $n \geqq 2$, the center is spanned by 1_n and $G_1 = SU(n)$. The center of $\mathcal{A} = M_n(\mathbb{C}) \oplus M_m(\mathbb{C})$, $n, m \geqq 2$, is spanned by two elements, p_n and p_m, the projectors on $M_n(\mathbb{C})$ and on $M_m(\mathbb{C})$. We have

$$G_{p_n} = SU(n) \times U(m) ,$$

$$G_{p_m} = U(n) \times SU(m) ,$$

$$G_{p_n + p_m} = S(U(n) \times U(m)) .$$

We close this section with a remark: the described reduction of the group of unitaries G to a subgroup G_p is compatible with the model building kit of Sect. 2. In particular

$$\mathcal{D}_{G_p} = \mathcal{D}_{G^0} = \mathcal{D}_G \tag{12}$$

and the change of variables, Eq. (7), is untouched. The proof of Eqs. (12) is done case by case and is summarized in the following table.

G	G^0	G/G^0	G_1	G^0/G^1	$\pi^1(G^0)$
$O(n,\mathbb{R})$	$SO(n,\mathbb{R})$	$\{\mathrm{diag}(-1,1,\ldots,1),1_n\}$	$SO(n,\mathbb{R})$	$\{1_n\}$	$\mathbb{Z}_2$
$U(n)$	$U(n)$	$\{1_n\}$	$SU(n)$	$\left\{e^{2\pi i/n}1_n, e^{4\pi i/n}1_n,\ldots,e^{n2\pi i/n}1_n\right\}$	$\{1\}$
$USp(n)$	$USp(n)$	$\{1_{2n}\}$	$USp(n)$	$\{1_{2n}\}$	$\{1\}$

All elements of G/G^0 and G^0/G_1 are multiples of the identity except for $O(n,\mathbb{R})/SO(n,\mathbb{R})$. However, integrating $\rho(g^{-1})\mathscr{D}\rho(g)$ first over the normal subgroup $SO(n,\mathbb{R})$ yields a matrix whose blocks are already diagonal matrices.

6. The Standard Model

We would like to conclude by locating the standard model within the CL scheme. *The* pedagogical example to illustrate the YMH model building kit is the Georgi–Glashow $SO(3)$ model [21]. Miraculously enough, *the* pedagogical example in the CL subkit is almost the Glashow–Salam–Weinberg model. Indeed, this example is the electro-weak algebra $\mathscr{A}=\mathbb{H}\oplus\mathbb{C}$, (group of unitaries $G=SU(2)\times U(1)$) represented on *two* generations of leptons, $N=2$,

$$\mathscr{H}_L=\mathbb{C}^2\otimes\mathbb{C}^2,\qquad \mathscr{H}_R=\mathbb{C}\otimes\mathbb{C}^2\,.$$

With respect to the suggestive basis

$$\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L,\quad \begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}_L,\qquad e_R,\quad \mu_R$$

of $\mathscr{H}_L\oplus\mathscr{H}_R$, the representation has the following matrix form,

$$\rho(a,b)=\begin{pmatrix}a\otimes 1_N & 0\\ 0 & \bar{b}1_N\end{pmatrix},\quad a\in\mathbb{H},\ b\in\mathbb{C}\,.$$

The internal Dirac operator is

$$\mathscr{D}=\begin{pmatrix}0 & \mathscr{M}\\ \mathscr{M}^* & 0\end{pmatrix}=\begin{pmatrix}0 & \begin{pmatrix}0\\ M_e\end{pmatrix}\\ \begin{pmatrix}0 & M_e\end{pmatrix} & 0\end{pmatrix},$$

with

$$M_e:=\begin{pmatrix}m_e & 0\\ 0 & m_\mu\end{pmatrix},\quad m_e<m_\mu\,.$$

The most general positive 6×6 matrix z, that commutes with $\rho(\mathscr{A})$ and with $\mathscr{D}$ is

$$z=\begin{pmatrix}1_2\otimes\begin{pmatrix}y_1 & 0\\ 0 & y_2\end{pmatrix} & 0\\ 0 & \begin{pmatrix}y_1 & 0\\ 0 & y_2\end{pmatrix}\end{pmatrix}$$

with positive numbers y_1 and y_2. Consequently the coupling constants g_2 of $SU(2)$ and g_1 of $U(1)$ are related,

$$\cot^2\theta_w = \left(\frac{g_2}{g_1}\right)^2 = 2\ , \qquad \sin^2\theta_w = \frac{1}{3}\ .$$

Details are given in the appendix. In this model, Φ of Eq. (7) takes the form

$$\Phi = i\begin{pmatrix} 0 & \left[\begin{pmatrix} \varphi_1 & -\bar\varphi_2 \\ \varphi_2 & \bar\varphi_1 \end{pmatrix} \otimes 1_N\right]\mathcal{M} \\ \dots^* & 0 \end{pmatrix} \tag{13}$$

and is parameterized by two functions $\varphi_1, \varphi_2 : M \to \mathbb{C}$. Under gauge transformations, these transform as an $SU(2)$ doublet

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\ .$$

In terms of these parameters, the Higgs potential reads

$$\begin{aligned} V(\varphi) &= K(1-|\varphi|^2)^2\ , \\ K &:= \frac{3}{2}\left(y_1 m_e^4 + y_2 m_\mu^4\right) - \frac{3}{2}\frac{L^2}{y_1+y_2}\ , \\ L &:= y_1 m_e^2 + y_2 m_\mu^2\ . \end{aligned}$$

Note that the scalar fields φ_1 and φ_2 are not properly normalized, they are dimensionless. To get their normalization straight, we compute the factor in front of the kinetic term $\mathrm{tr}(\mathrm{d}\Phi^* * \mathrm{d}\Phi z)$ in the Klein–Gordon action $\mathrm{tr}(D\Phi^* * D\Phi z)$ as a function of the variable φ. By inserting Eq. (13) we obtain:

$$\mathrm{tr}(\mathrm{d}\Phi^* * \mathrm{d}\Phi z) = *2L|\partial\varphi|^2\ .$$

Likewise, we need the normalization of the gauge bosons and as shown in the appendix, we end up with the following mass relations:

$$\begin{aligned} m_W^2 &= \frac{L}{y_1+y_2} = \frac{y_1 m_e^2 + y_2 m_\mu^2}{y_1+y_2}\ , \\ m_H^2 &= \frac{2K}{L} = 3m_\mu^2\frac{y_2}{y_1}\frac{(1-m_e^2/m_\mu^2)^2}{(y_2/y_1 + m_e^2/m_\mu^2)(y_2/y_1+1)}\ . \end{aligned}$$

Consequently

$$m_e < m_W < m_\mu\ , \qquad m_H < \sqrt{3}(m_\mu - m_e)\ .$$

We obtain a model with less constrained weak angle by slightly modifying this example. Let us represent the electro-weak algebra on one generation of leptons and one generation of (uncoloured) quarks,

$$\mathcal{H}_L = \mathbb{C}^2 \oplus \mathbb{C}^2, \qquad \mathcal{H}_R = (\mathbb{C}\oplus\mathbb{C})\oplus\mathbb{C}$$

with suggestive basis

$$\begin{pmatrix} u \\ d \end{pmatrix}_L , \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L , \quad \begin{matrix} u_R, \\ d_R, \end{matrix} \quad e_R ,$$

and representation

$$\rho(a,b) := \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & \bar{b} \end{pmatrix} , \quad (a,b) \in \mathbb{H} \oplus \mathbb{C}, \; B := \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix} .$$

We choose the internal Dirac operator:

$$\mathcal{D} := \begin{pmatrix} 0 & 0 & \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} & 0 \\ 0 & 0 & 0 & \begin{pmatrix} 0 \\ m_e \end{pmatrix} \\ \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} & 0 & 0 & 0 \\ 0 & \begin{pmatrix} 0 & m_e \end{pmatrix} & 0 & 0 \end{pmatrix} .$$

All indicated fermion masses are supposed positive and different. Now, the most general scalar product on the differential algebra $\Omega_{\mathcal{D}}(\mathbb{H} \oplus \mathbb{C})$ is defined with the 7×7 matrix

$$z = \begin{pmatrix} x 1_2 & 0 & 0 & 0 \\ 0 & y 1_2 & 0 & 0 \\ 0 & 0 & x 1_2 & 0 \\ 0 & 0 & 0 & y \end{pmatrix}$$

with positive numbers x and y. In this example we get:

$$\sin^2\theta_w = \frac{x+y}{5x+3y} ,$$

implying

$$\frac{1}{5} < \sin^2\theta_w < \frac{1}{3} .$$

This z is in the image of the center of $\mathbb{H} \oplus \mathbb{C}$ under ρ if and only if $x = y$ and we have $\sin^2\theta_w = 0.4$.

The drawback of these two examples – electrically charged neutrinos and up- and down-quark with opposite charges – is corrected by adding strong interactions. As strong interactions are vector-like, this addition is immediate except for the fact that the representation of the left-handed quarks, $(3, 2, \frac{1}{3})$ in Eq. (1), is a tensor product. However, this is a tensor product of two representations of *two unrelated* algebras ($M_3(\mathbb{C})$ and $\mathbb{H}$) and as such, it can be included in the CL scheme by generalizing the representations to bimodules [1,22]. A bimodule is a pair of algebras, each represented on a common Hilbert space, such that the two representations commute. The constraints indicated in Sect. 4 remain otherwise unaffected and for the standard model, they can be stated as follows. The scalar representation is one weak isospin doublet, implying a mass ratio for the W and Z bosons given by the ρ factor

$$\rho := \frac{m_W^2}{m_Z^2 \cos^2\theta_w} = 1 .$$

With the general scalar product (6), the other constraints read [23],

$$m_t > \sqrt{3} m_W > \sqrt{3} m_e ,$$

$$m_H = \sqrt{3\frac{(m_t/m_W)^4 + 2(m_t/m_W)^2 - 1}{(m_t/m_W)^2 + 3}}\, m_W ,$$

$$\sin^2\theta_w < \frac{2}{3}\left(1 + \frac{1}{g}\left(\frac{g_2}{g_3}\right)^2\right)^{-1} .$$

For the more restricted scalar product coming from the center, the constraints are tighter:

$$m_t = 2\, m_W , \qquad m_H = 3.14\, m_W , \qquad \sin^2\theta_w < \frac{8}{15} = 0.533 .$$

It is a pleasure to acknowledge the help and advice of Robert Coquereaux, Vaughan Jones, Daniel Kastler, John Madore and Stanislaw Woronowicz.

7. Appendix

This appendix collects our normalization conventions of a YMH model in a spacetime of signature $+---$. Let φ, ψ, and W be complex fields of spin 0, 1/2, and 1. The kinetic terms determine the normalization of the fields in the Lagrangian and the masses and coupling constants are defined with respect to this normalization. With $\hbar = c = 1$, the Lagrangian is

$$\begin{aligned}\mathscr{L} = & \frac{1}{2}\partial_\mu\varphi^*\partial^\mu\varphi - \frac{1}{2}m_\varphi^2\varphi^*\varphi + \bar\psi i\,\partial\!\!\!/\psi - m_\psi\bar\psi\psi \\ & - \frac{1}{2}\partial_\mu W_\nu^*\partial^\mu W^\nu + \frac{1}{2}\partial_\mu W^{*\mu}\partial_\nu W^\nu + \frac{1}{2}m_W^2 W_\mu^* W^\mu .\end{aligned}$$

Note the one half in front of the scalar Lagrangian, i.e. we decompose the complex scalar into real fields as $\varphi = \mathrm{Re}\,\varphi + i\mathrm{Im}\,\varphi$. We use the following definitions:

$$\psi = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix} , \qquad \partial\!\!\!/\psi := \gamma^\mu\partial_\mu\psi, \qquad \bar\psi := (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)\gamma^0 .$$

Our gamma matrices are,

$$\gamma^0 := \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} \qquad \gamma^1 := \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}$$

$$\gamma^2 := \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix} \qquad \gamma^3 := \begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix} .$$

They satisfy the anticommutation relation $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}1_4$ with the flat Minkowski metric

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

We take

$$\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

such that $\gamma_5^2 = 1_4$. γ_5 anticommutes with all other gamma matrices, $\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0$. With the definitions

$$\psi_L := \frac{1_4 - \gamma_5}{2}\psi, \qquad \psi_R := \frac{1_4 + \gamma_5}{2}\psi ,$$

the free Dirac Lagrangian reads

$$\mathscr{L}_\psi = \overline{\psi_L} i \,\partial\!\!\!/ \psi_L + \overline{\psi_R} i \,\partial\!\!\!/ \psi_R - m_\psi \overline{\psi_L}\psi_R - m_\psi \overline{\psi_R}\psi_L .$$

In Euclidean spacetime, the Dirac Lagrangian written in this chiral form vanishes identically and the fermions have to be doubled. With

$$W_{\mu\nu} := \partial_\mu W_\nu - \partial_\nu W_\mu ,$$

the free part of the Yang–Mills Lagrangian becomes

$$\mathscr{L}_W = -\frac{1}{4} W^*_{\mu\nu} W^{\mu\nu} + \frac{1}{2} m_W^2 W^*_\mu W^\mu .$$

The couplings of the gauge bosons to scalars and fermions in their respective representations are introduced through the covariant derivatives, while the self couplings of the gauge bosons come from the field strength. All their coupling constants derive from the choice of one invariant scalar product on the Lie algebra. Amazingly enough, the parametrization of this scalar product seems uniform in the literature, at least for the classical groups,

$$(b, b') := \frac{1}{g_1^2}\,\bar{b}b', \quad b, b' \in u(1) ,$$

$$(a, a') := \frac{2}{g_n^2}\,\mathrm{tr}(a^* a'), \quad a, a' \in su(n) .$$

The gauge bosons sit in a 1-form $A = A_\mu \,\mathrm{d}x^\mu$ with values in the Lie algebra and the Yang–Mills Lagrangian reads

$$\mathscr{L}_{YM} = -\frac{1}{4}(F_{\mu\nu}, F^{\mu\nu})$$

with the field strength $F = 1/2 F_{\mu\nu}\,\mathrm{d}x^\mu \,\mathrm{d}x^\nu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

As an illustration, let us consider the standard model of electro-weak interactions with $G = SU(2) \times U(1)$, one doublet of scalars φ and Higgs potential

$$V(\varphi) = \lambda(\varphi^*\varphi)^2 - \frac{\mu^2}{2}(\varphi^*\varphi) . \tag{14}$$

First we choose the electric charge generator Q:

$$iQ := i\left(g_2 \sin\theta_w \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, g_1 \cos\theta_w \right),$$

a normalized vector in the Cartan subalgebra of $\mathfrak{g} = su(2) \oplus u(1)$ spanned by the weak isospin and hypercharge,

$$I_3 := i\left(g_2 \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, 0 \right), \quad Y := i(0, g_1).$$

We complete iQ to an orthonormal bias of $\mathfrak{g}^{\mathbb{C}}$ of eigenvectors of $[Q, \cdot\,]$,

$$\tilde{Z} := i\left(g_2 \cos\theta_w \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, -g_1 \sin\theta_w \right),$$

$$I^+ := i\left(\frac{g_2}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0 \right), \quad I^- := i\left(\frac{g_2}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0 \right).$$

The eigenvalues are 0 and $\pm g_2 \sin\theta_w =: \pm e$. The multiplet of gauge bosons is now written as

$$A_\mu(x) := \gamma_\mu(x) iQ + Z_\mu(x)\tilde{Z} + \frac{1}{\sqrt{2}}(W_\mu(x) I^+ + W_\mu^*(x) I^-),$$

where the photon $\gamma_\mu(x)$ and the $Z_\mu(x)$ are real fields while the W is complex.

The scalar fields sit in a $SU(2)$ doublet with hypercharge $y_S = -1/2$:

$$\tilde{\rho}_S(a,b)\varphi = (a + y_S b 1_2)\varphi, \quad a \in su(2),\ b \in u(1).$$

$\tilde{\rho}_S$ denotes the Lie algebra representation. In order to keep the photon massless, we must choose g_1 such that one of the scalars has zero electric charge,

$$\frac{1}{i}\tilde{\rho}_S(iQ) = \begin{pmatrix} 0 & 0 \\ 0 & -e \end{pmatrix}.$$

This implies

$$\frac{g_1}{g_2} = \frac{\sin\theta_w}{\cos\theta_w}.$$

The gauge bosons masses come from the absolute value squared of the covariant derivative of the vacuum v. Since v satisfies $|v|^2 = \mu^2/(4\lambda)$ we choose

$$v = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{\mu^2}{\lambda}} \\ 0 \end{pmatrix}$$

and obtain

$$\frac{1}{2}|\tilde{\rho}_S(A_\mu)v|^2 = \frac{1}{2}m_Z^2 Z_\mu Z^\mu + \frac{1}{2}m_W^2 W_\mu^* W^\mu$$

with

$$m_W = g_2\frac{\mu}{4\sqrt{\lambda}} \quad \text{and} \quad m_W = \cos\theta_w m_Z.$$

To compute the mass of the physical, real Higgs scalar H, we change variables in the Higgs potential,

$$\varphi = v + \begin{pmatrix} H(x) + ih_Z(x) \\ h_W(x) \end{pmatrix} ,$$

and obtain

$$V(\varphi(x)) = V(v) + \frac{1}{2} m_H^2 H^2(x) + \text{terms of order 3 and 4 in } H(x), h_Z(x), h_W(x) ,$$

with

$$m_H = \sqrt{2}\mu .$$

We come back to the first CL example of Sect. 6. If we write $\omega = 1/2\omega_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \in \Omega^2 M$ then $\omega * \omega = 1/2\omega_{\mu\nu}\omega^{\mu\nu}\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$. Consider the Yang–Mills Lagrangian in Eq. (9) on Minkowski space,

$$-\mathrm{tr}[\rho(F) * \rho(F) z] = \frac{1}{2}\,\mathrm{tr}[\rho(F_{\mu\nu})\rho(F^{\mu\nu}) z]\,\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 .$$

This term is nothing else but $-1/4\ (F_{\mu\nu}, F^{\mu\nu})$. Hence

$$\frac{1}{2}\,\mathrm{tr}(\rho(a,b)^* \rho(a',b')z) = \frac{1}{2}\left(\mathrm{tr}(a^*a')(y_1 + y_2) + b\bar{b}'(y_1 + y_2)\right)$$

is by comparison equal to

$$\frac{1}{4}((a,b),(a',b')) = \frac{1}{4}\left(\frac{2}{g_2^2}\,\mathrm{tr}(a^*a') + \frac{1}{g_1^2}\,\bar{b}b'\right) .$$

Consequently

$$g_1^2 = \frac{1}{2}\frac{1}{(y_1 + y_2)}, \qquad g_2^2 = \frac{1}{(y_1 + y_2)} ,$$

and $\sin^2\theta_w = 1/3$. The remaining two pieces of the Euclidean Lagrangian (9) read in Minkowski space

$$2L|(\partial_\mu + \rho(A_\mu))\varphi|^2 - K(1 - |\varphi|^2)^2 ,$$

and after the proper rescaling of the scalar

$$\frac{1}{2}|(\partial_\mu + \rho(A_\mu))\varphi|^2 - K + \frac{1}{2}\frac{K}{L}|\varphi|^2 - \frac{K}{16L^2}|\varphi|^4 .$$

Comparing with Eq. (14) we have

$$\lambda = \frac{K}{16L^2}, \qquad \mu^2 = \frac{K}{L}$$

and

$$m_W^2 = g_2^2\frac{\mu^2}{16\lambda} = \frac{L}{y_1 + y_2}, \qquad m_H^2 = 2\mu^2 = 2\frac{K}{L} .$$

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Communicated by A. Connes