Scaling Limit for a Mechanical System of Interacting Particles

Kôhei Uchiyama

Department of Applied Physics, Tokyo Institute of Technology, Meguro-ku, Tokyo 152, Japan

Abstract: A system of a large number of classical particles moving on a one-dimensional segment with virtually reflecting boundaries is studied. The particles interact with one another through repulsive pair-potential forces and are subjected to resistance proportional to their velocities. Because of the latter it is only the number of particles that is conserved under the evolution of the system. It is proved that in the hydrodynamic limit of diffusion type scaling the normalized counting measure of particle locations converges and its limiting density is governed by a non-linear diffusion equation which in typical cases is of porous media equation type.

0. Introduction

This paper concerns the problem of a hydrodynamic limit for a system of a large number of particles that move on a one-dimensional segment according to a classical equation of motion. Particles interact by exerting upon each other repulsive potential forces given by a common pair-potential function that is unbounded at zero and may have an infinite range of influence. Also at both ends of the segment there act potential forces that repel particles approaching one of the endpoints and keep them in between. In addition the medium exerts on each particle a damping force, "resistance," whose magnitude is proportional to the velocity of the particle, so that nothing except the number of particles is conserved under the evolution of the system. We take the hydrodynamic scaling limit for the normalized counting measure of particle locations in the system and prove that its limiting density is governed by a non-linear diffusion equation that in typical cases is of porous media equation type.

An approach that has been recently developed for studying the hydrodynamic limits of stochastic systems is applied to the present model. Although our system is deterministic, the application goes well owing to the dissipative effect of resistance and the fact that our system is essentially of gradient type. The idea for the general

framework of the proof comes from Guo–Papanicolaou–Varadhan [GPV] and those
for technical devices from Varadhan [V], both of them dealing with stochastic sys-
tems. [V] as well as [GPV] employs a method that is based on an entropy bound
extensively, a method not available for deterministic models. However, on the one
hand, once it is established that the local states of the system converges in space-
time average to an equilibrium state, the entropy bound can be dispensed with for
the rest, especially for strong convergence of local density fields (the two-blocks
estimate), if the system is of gradient type and the spin values are non-negative, as
noted by Suzuki-Uchiyama [SU] and Uchiyama [U]. On the other hand, the equilib-
rium states assume a rather simple structure because of the existence of resistance:
in fact, every translation invariant equilibrium state for the infinite system to the
present model is degenerate at a steady state that is made up of phases such that
particles stand still with equal successive spacings. This result is established by
Lang [L] under a certain moment condition on the spacing. That condition however
is difficult to check in our case, and we shall show it is dispensable.

We shall verify the local equilibrium (in the sense mentioned above) by
computing the limit, under the hydrodynamic time scaling, of the time deriva-
tive of a certain functional of phases of the system. The rest of the proof is
similar to that of [SU] and [U] except for some technical details: at several
points nontrivial modifications must be done, necessitated for dealing with the un-
bounded potential function having a long tail and, in addition, with the boundary
term.

A stochastic version of the present system has been investigated by Olla-
Varadhan [OV] in which independent white noises are added as random forces
acting on particles. Their pair potential is smooth and has compact support. Some
of our key estimates (especially the two-blocks estimate) technically may be viewed
as an extension of theirs to unbounded potentials.

For systems of Newtonian particles (without damping term) the hydrodynamic
limit is obtained in [BDS] in the case of hard rods (moving on the line) and
discussed in [S] in the case when only nearest neighbour particles (moving on a
circle) interact by a hard core potential of finite range.

Mürmann [M] also investigates the hydrodynamic limit of a model closely related
to the present one: it deals with a nearest neighbour interaction model of determinis-
tic dynamics in which the velocity, instead of the acceleration, of a particle is given
as the gradient of a potential, with the potential function similar to ours. For such
a model there are two Lyapunov functionals, i.e., the kinetic and the potential ener-
gies taken separately, which are the main tools for the study of the hydrodynamic
limit in [M], whereas for ours there is only one Lyapunov functional, the total en-
ergy. An infinite particle system of similar nearest-neighbour interaction is studied in
[F, PS].

1. The Model and Main Result

Consider a system of N Newtonian particles of unit mass moving on the one di-
mensional open segment $(0, N)$ such that each particle is subjected to the resistance
equal in value to its momentum, interacts with the other particles through a pair
potential and is repelled with potential forces exerted by the two "walls" at 0 and

N. The equation of motion for the system is written as follows:

$$\frac{d}{dt}q_i(t) = p_i(t)\,, \tag{1.1a}$$

$$\frac{d}{dt}p_i(t) = -p_i(t) - \sum_{j \neq i} U'(q_i(t) - q_j(t))$$

$$-W'_-(q_i(t)) - W'_+(q_i(t) - N)\,, \tag{1.1b}$$

where $q_i(t)$ and $p_i(t)$ denote, respectively, the location and the momentum for the ith particle at time $t \geq 0$ and $U', W'_\pm$ the derivatives of functions $U, W_\pm$. The pair potential function $U(x)$, defined for $x \in \mathbf{R} - \{0\}$, is supposed to be continuously differentiable and satisfy

$$U(x) = U(-x);\quad U(x) \geq 0;\quad U(0+) = \infty;\quad U(\infty) = 0\,; \tag{1.2}$$

$U(x)$ is strictly convex on $0 < x < R$,

$$\text{where}\quad R := \sup\{x > 0 \mid U(x) > 0\} \leq \infty\,; \tag{1.3}$$

$$\int_1^\infty U(x)dx < \infty\,; \tag{1.4}$$

$$\liminf_{\delta \downarrow 0} \frac{1}{\psi(\delta)} \inf_{0 < x < \delta} \psi(x) > 0\,, \tag{1.5}$$

where

$$\psi(x) := -xU'(x),\quad x \neq 0\,.$$

From (1.2) and (1.3) it follows that $\psi(x) \geq 0$, i.e., the pair interaction is repulsive. The assumptions (1.2) through (1.4) are fairly natural for our main result to hold ((1.3) is needed for showing ergodicity of the equilibrium states: see Theorem 3 below), while the assumption (1.5) may be technical although hardly restrictive; it implies in particular that $\psi(x)$ either diverges to infinity or is bounded off both zero and infinity as $x \to 0$.

For the wall potentials $W(x) = W_-(x)$ or $W_+(-x), x > 0$, we suppose

$$W(0+) = \infty\,, \tag{1.6a}$$

$$W'(x) \leq 0,\qquad \lim_{x \to \infty} xW'(x) = 0\,. \tag{1.6b}$$

The initial configuration $q_i = q_i(0), i = 1, 2, \ldots, N$, is always supposed to be such that $q_i \neq q_j$ if $i \neq j$ and $0 < q_i < N$ for every i. The assumption $U(0+) = \infty$ together with (1.6a) guarantees that this condition for the initial configuration is preserved for all later times. The conditions (1.2) through (1.6) are assumed throughout the paper except in Theorem 4 of Sect. 4.

We are interested in a macroscopic picture for the particle configuration $(q_i(t))_{i=1}^N$, which is to be viewed under diffusion scaling. Let us introduce the macroscopic position variables

$$x_i(t) = \frac{1}{N}q_i(N^2t) \tag{1.7}$$

and the normalized counting measure

$$\alpha_t^N(d\theta) = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)}(d\theta), \quad \theta \in (0,1) \ :$$

for an open subset $A \subset (0,1)$ it takes the value $\alpha_t^N(A) = N^{-1}\#\{i|x_i(t) \in A\}$. Our main theorem that follows asserts that the limiting density for α_t^N as $N \to \infty$ is characterized by the following evolution equation:

$$\frac{\partial}{\partial t}\rho(\theta,t) = \frac{\partial^2}{\partial\theta^2}P(\rho(\theta,t)), \quad 0 < \theta < 1, t > 0, \tag{1.8a}$$

$$\frac{\partial}{\partial\theta}\rho(0,t) = \frac{\partial}{\partial\theta}\rho(1,t) = 0, \quad t > 0, \tag{1.8b}$$

where P is a function of one variable $u \geqq 0$ defined by

$$P(u) := -\sum_{k=1}^{\infty}kU'(k/u), \quad u > 0$$

and $P(0) = 0$. The infinite sum here converges due to the assumption (1.4), which is equivalent to $\int_1^{\infty}\psi(x)dx < \infty$. The function P may be naturally interpreted as pressure (see Remark 1 in Sect. 6); clearly P is continuous and non-decreasing.

To state the theorem we need the following bound for the total energy of the initial phase $(p_i, q_i)_{i=1}^{N}$: as $N \to \infty$,

$$\frac{1}{2}\sum_{i=1}^{N}p_i^2 + \frac{1}{2}\sum_{i,j(\neq)}U(q_i - q_j) + \sum_{i=1}^{N}[W_-(q_i) + W_+(q_i - N)] = o(N^3), \tag{1.9}$$

where the second sum extends to all ordered pairs (i,j) with $i \neq j, 1 \leqq i,j \leqq N$. This condition, allowing the initial energy per particle to become large up to $o(N^2)$, may appear very weak, but actually it turns out to be the right one for the present issue (cf. Remark 2 in Sect. 6). We will regard α_t^N as an element of the space of all probability measures on the closed interval $[0,1]$, which space we endow with the topology of weak convergence: according to it a sequence $(\alpha^N)_N$ in this space converges to an element α if and only if $\alpha^N(J) \to \alpha(J)$ as $N \to \infty$ for every continuous function J on $[0,1]$, where $\alpha(J)$ stands for the integral of J by α.

Theorem 1. *Suppose the sequence of initial phases* $(p_i, q_i)_{i=1}^{N} = (p_i(0), q_i(0))_{i=1}^{N}$, *$N = 1,2,\ldots$, satisfies (1.9). If α_0^N converges to a probability measure α_0 on $[0,1]$, then α_t^N converges uniformly for $t < T$ for each $T < \infty$, the limit measure is absolutely continuous relative to the Lebesgue measure $d\theta$, and the density, $\rho(\theta,t)$ say, is a unique weak solution of the non-linear boundary value problem (1.8) with the initial condition*

$$\rho(\theta,t)d\theta \text{ converges to } \alpha_0 \text{ as } t \downarrow 0. \tag{1.10}$$

The behaviour of $P(\rho)$ near $\rho = 0$ is determined by that of $U(x)$ at $x = \infty$. In particular if $U'(x) \sim Cx^{-m}$ $(m > 2)$ as $x \to \infty$, then $P(\rho) \sim [\sum_{k \geqq 1}k^{1-m}]C\rho^m$ as $\rho \downarrow 0$, so that Eq. (1.8) assumes virtually the same form as the equation of porous medium flow; according to that theory (cf. [A], e.g.) and in view of Theorem 1, the

macroscopic density for the system (1.1) therefore has a finite speed of propagation. By the way $P(\rho)$ behaves roughly like $\rho \int_{1/\rho}^{\infty} \psi(x)dx$ as $\rho \to \infty$.

A heuristic argument for the derivation of (1.8) may proceed as follows. A direct computation together with some formal manipulation will deduce from (1.1)

$$\alpha_t^N(J) - \alpha_0^N(J) = \int_0^t \frac{1}{2N} \sum_{i=1}^N \sum_{j \ne i} \psi(N(x_i(s) - x_j(s)))J''(x_i(s))ds + o(1), \quad (1.11)$$

where J is a smooth test function that is constant near each of the boundary points 0 and 1 (because of this boundary condition the wall-potential terms disappear). For the identification of the inner sum on the right side we turn back to the microscopic variables $q_i(N^2 s) = Nx_i(s)$. Equilibrium states for the infinite system corresponding to (1.1) are steady states in which each particle stands still in complete balance of forces exerted on it by the other particles; and any steady state must be a configuration of equal spacing, of which the common span is necessarily equal to the reciprocal of its density. By the damping effect due to the resistance working in (1.1) velocities would get very small after such long time as of order N^2, so that the equilibrium state must be locally built up in the limit under our scaling. If there exists a limiting density $\rho(\theta, s)$, this would allow one to replace $\sum_{j \ne i} \psi(N(x_i(s) - x_j(s)))$ by $2\sum_{k \ge 1} \psi(k/\rho(x_i(s), s))$, which, by passing to the limit in (1.11) and noticing $P(\rho) = \rho \sum_{k \ge 1} \psi(k/\rho)$, immediately yields the weak form of (1.8).

For this story to be justified, the following theorem (an extension of corresponding results in [OV, V and U] essentially to an unbounded potential) is fundamental at least in the present approach.

Theorem 2. *Under the same assumption as in Theorem 1 there exists a continuous even function* $\omega(x)$, $x \in \mathbf{R} - \{0\}$ *such that* $\omega \ge 0$; $\omega(x)$ *approaches, as* $x \to 0$, *a positive number or infinity according to whether* ψ *is bounded or unbounded (near zero); and*

$$\int_0^T \sum_{i=1}^N \sum_{j \ne i} \sum_{k \ne i} \psi(N(x_i(t) - x_j(t)))\omega(N(x_i(t) - x_k(t)))dt < CN,$$

where the constant C *depends on* T *but not on* N.

Theorem 2 will show that (1.11) is valid and α_t^N, $N = 1, 2, \ldots$, regarded as continuous measure-valued functions of $t \le T$, constitute a relatively compact set; moreover any limiting measure is absolutely continuous relative to the Lebesgue measure (see Corollary 1 of Sect. 3). Another, more essential application of Theorem 2 will be made, via Corollary 2 of Sect. 3, for deriving strong convergence of α_t^N (the two-blocks estimate) in Sect. 5.

What we need as the local equilibrium statement as mentioned in the second half of the heuristic argument given above is formulated in the next theorem. Let $\mu^N(d\mathbf{q})$ be the probability measure on the space of configurations $\mathbf{q} = (q_i)_{i=1}^N$ on $\mathbf{R}$ defined by

$$\mu^N(H) = \frac{1}{T} \int_0^T ds \int_0^1 H(N\mathbf{x}^\theta(s))d\theta, \quad (1.12)$$

where $\mathbf{x}^\theta$ stands for the configuration $\mathbf{x} = (x_i)_{i=1}^N$ viewed from $\theta : \mathbf{x}^\theta = (x_i - \theta)_{i=1}^N$ and $H = H(\mathbf{q})$ ranges over bounded, continuous local functions of locally finite configurations on $\mathbf{R}$.

Theorem 3. *Suppose that the same assumption as in Theorem* 1 *is fulfilled. Then any limit point μ of μ^N as $N \to \infty$ must be of the form*

$$\mu = \mu^o + \int_0^R \Phi_\rho \lambda(d\rho), \qquad \Phi_\rho(d\mathbf{q}) := \rho \int_0^{1/\rho} \delta_{\mathbf{e}(1/\rho,\theta)}(d\mathbf{q})d\theta .$$

Here $\mu^o = 0$ if $R = \infty$ and μ^o is a measure relative to which no particle is located within a range of potentials of the other particles a.e. if $R < \infty$; λ is a measure on $[0,R)$ such that $\int \rho\lambda(d\rho) \leqq 1$; $\mathbf{e}(r,\theta), r > 0$, stands for $(\ldots, -r + \theta, \theta, r + \theta, 2r + \theta, \ldots)$, the configuration of particles on $\mathbf{R}$ with equal successive spacings of common span r and having a particle at θ; and Φ_0 is understood to concentrate on the empty configuration.

The proof of Theorem 3 is based on a strengthening of a result by Lang [L].

Theorems 2 and 3 are proved in Sect. 3 and 4, respectively. The proof of Theorem 1 is given in Sect. 5. In Sect. 6 we present three remarks: the first one explains why $P(\rho)$ is pressure; the other two answer the illuminative question of what happens when the integrability condition (1.4) or the initial energy bound (1.9) is violated.

2. A Bound of the Kinetic Energy

We isolate this short section in which we note that for the system (1.1) the kinetic energy per particle approaches zero in the time average as $N \to \infty$ on account of assumption (1.9), the result being basic in all the later sections. Put $v_i(t) = p_i(N^2 t)$. Then the equation of motion (1.1) becomes

$$\frac{d}{dt}x_i(t) = Nv_i(t), \tag{2.1a}$$

$$\frac{d}{dt}v_i(t) = -N^2 v_i(t) - N^2 \sum_{j \neq i} U'(N(x_i(t) - x_j(t)))$$
$$-N^2 W'_-(Nx_i(t)) - N^2 W'_+(N(x_i(t) - 1))). \tag{2.1b}$$

For the total energy

$$\mathscr{E}^N(t) := \frac{1}{2}\sum_{i=1}^N v_i^2(t) + \frac{1}{2}\sum_{i,j(\neq)} U(N(x_i(t) - x_j(t)))$$
$$+ \sum_{i=1}^N [W_-(Nx_i(t)) + W_+(N(x_i(t) - 1)))],$$

we calculate its derivative to obtain

$$\frac{d}{dt}\mathscr{E}^N(t) = -N^2 \sum_{i=1}^N v_i^2(t) \leqq 0;$$

in particular $\mathscr{E}^N(t) \leqq \mathscr{E}^N(0)$ and

$$\int_0^T \frac{1}{N}\sum_{i=1}^N v_i^2(t)dt \leqq \frac{1}{N^3}\mathscr{E}^N(0), \tag{2.2}$$

of which the right side goes to zero as $N \to \infty$ under (1.9).

3. Estimation of $\sum \psi(N(x_i - x_j))$

This section is divided into four parts. The first three parts are devoted to the proof of Theorem 2, while in the last one we state three corollaries of Theorem 2 and its proof. The idea of the proof is essentially the same as that for the corresponding one in [V]. Our potential function, however, is not bounded, nor necessarily of compact support as in [V], and this necessitates a significant modification of the proof.

3.1. Let $g = g(x), x \neq 0$, be an even, continuous function such that g is non-increasing on $(0, \infty)$,

$$g(x) = \frac{a}{\sqrt{|x|}}, \qquad 0 < |x| < \frac{1}{2}$$

$$= 0, \qquad |x| > 1 ,$$

where a is a positive constant (the precise form of g as given above is unimportant) and

$$\int_{-\infty}^{\infty} g(x)dx = 1 .$$

Let $h(x), x \in \mathbf{R}$ be a smooth non-negative even function such that $\int h(x)dx = 1$ and $h(x) = 0$ for $|x| > 1$. We write $g_N(x) = Ng(Nx)$ and define a function $G_{(N)}$ on $\mathbf{R}$ by

$$G_{(N)}(x) = \int_{-\infty}^{x} dy \int_{-\infty}^{y} (g_N(u) - h(u))du$$

so that

$$G''_{(N)}(x) = g_N(x) - h(x) .$$

Both $|G_{(N)}|$ and $|G'_{(N)}|$ are bounded by 1, and vanish for $|x| > 1$. Put

$$B_N(x) := NW'_-(Nx) + NW'_+(N(x-1)), \quad |x| < 1 .$$

Then for $G = G_{(N)}$, Eq. (2.1) yields

$$\frac{d}{dt} \sum_{i,j(\neq)} G'(x_i - x_j)v_i = N \sum_{i,j(\neq)} G''(x_i - x_j)v_i(v_i - v_j) - N^2 \sum_{i,j(\neq)} G'(x_i - x_j)v_i$$

$$- N\sum_i \sum_{k \neq i} G'(x_i - x_k)\left[\sum_{j \neq i} NU'(N(x_i - x_j)) \right]$$

$$- N\sum_i \sum_{j \neq i} G'(x_i - x_j)B_N(x_i) .$$

Here (and below) the time variable is suppressed from functions $x_i(t), v_i(t)$, etc. for the sake of brevity. In the third term on the right side above we may replace $G'(x_i - x_k)$ by $\frac{1}{2}[G'(x_i - x_k) - G'(x_j - x_k)]$ if $j \neq k$, since U' is odd. Noticing $G'_{(N)}$ is also odd, we see that the first term on the right side equals $\frac{1}{2}N \sum (v_i - v_j)^2 G''(x_i - x_j)$ and the second $-\frac{1}{2}N(d/dt)\sum G(x_i - x_j)$. Integrating both sides of the resulting

equality on $[0, T]$ and then dividing them by N^3, we obtain after rearrangement of terms

$$\frac{1}{2N^2} \int_0^T \sum_{i,j,k(\pm)} [-NU'(N(x_i - x_j))][G'(x_i - x_k) - G'(x_j - x_k)]dt$$

$$+ \frac{1}{N^2} \int_0^T \sum_{i,j(\pm)} [-NU'(N(x_i - x_j))]G'(x_i - x_j)dt$$

$$= \frac{1}{N^3} \sum_{i,j(\pm)} v_i G'(x_i - x_j)\bigg|_{t=0}^T + \frac{1}{2N^2} \sum_{i,j(\pm)} G(x_i - x_j)\bigg|_{t=0}^T$$

$$- \frac{1}{2N^2} \int_0^T \sum_{i,j(\pm)} (v_i - v_j)^2 G''(x_i - x_j)dt$$

$$+ \frac{1}{N^2} \int_0^T \sum_{i,j(\pm)} B_N(x_i)G'(x_i - x_j)dt ,\qquad (3.1)$$

where the triple sum on the left side extends over all ordered triplets with three components different from one another. In view of (2.2) and the inequality

$$\frac{1}{N^2} \sup_t \sum |v_i(t)| \leqq \sup_t \left[N^{-3} \sum v_i^2(t) \right]^{1/2} \leqq [N^{-3} \mathscr{E}^N(0)]^{1/2} ,\qquad (3.2)$$

the sum of the first two terms on the right side of (3.1) may be estimated from above by $2\sqrt{\mathscr{E}^N(0)/N^3} + \|G\|_\infty$. Since $-G''_{(N)}(x) \leqq h(x)$, the third term is bounded above by $2\|h\|_\infty \mathscr{E}^N(0)/N^3$. For estimating the boundary term we make use of the relation (5.3) given in Sect. 5 (clearly its verification is made independently of any result obtained in this section). Taking $J(x) = (x - \frac{1}{2})^2$ in (5.3) and applying (2.2) and the assumption (1.6b) we see

$$\int_0^T \sum_{i=1}^N |B_N(x_i)|dt \leqq N + C_1 \frac{\mathscr{E}^N(0)}{N^2} + 2\int_0^T \sum_{i,j(\pm)} \psi(N(x_i - x_j))dt .\qquad (3.3)$$

Thus from condition (1.9) or, what is the right one here, $\mathscr{E}^N(0) = O(N^3)$ it follows that

$$\frac{1}{2N^2} \int_0^T \sum_{i,j,k(\pm)} [-NU'(N(x_i - x_j))][G'(x_i - x_k) - G'(x_j - x_k)]dt$$

$$+ \frac{1}{N^2} \int_0^T \sum_{i,j(\pm)} [-NU'(N(x_i - x_j))]G'(x_i - x_j)dt$$

$$\leqq C_2 + \frac{2}{N} \int_0^T \sum_{i,j(\pm)} \psi(N(x_i - x_j))dt .\qquad (3.4)$$

The remainder of the proof of Theorem 2 concerns inequalities that are valid for an arbitrary configuration $(x_1, x_2, \ldots, x_N)$ and have nothing to do with the dynamics (1.1).

3.2. Let us estimate the left side of (3.4) from below. For simplifying the arguments that follow we assume $R > 1$. To begin with we notice the following two trivial inequalities:

$$\frac{G'(x_i - x_k) - G'(x_j - x_k)}{x_i - x_j} = \frac{1}{x_i - x_j} \int_{x_j}^{x_i} [g_N - h](y - x_k) dy$$

$$\geq \frac{1}{x_i - x_j} \int_{x_j}^{x_i} g_N(y - x_k) dy - \|h\|_\infty \, ; \qquad (3.5a)$$

$$\frac{G'(x_i - x_j)}{x_i - x_j} \geq g_N(x_i - x_j) - \|h\|_\infty \, . \qquad (3.5b)$$

It is here in the proof of the next lemma that we need assumption (1.5). We put

$$\psi^*(x) = \psi(x)\chi\{|x| \leq 1/4\} \, ,$$

where $\chi\{|x| \leq a\} = 1$ or 0 according as $|x| \leq a$ or $|x| > a$.

Lemma 1. *There exists an even function $\omega(x)$, $x \in \mathbf{R} - \{0\}$ such that $\omega \geq 0$; $\omega(x)$ is positive and continuous for $0 < |x| < 1/4$; $\omega(x)$ converges to a positive number or diverges to infinity as $x \to 0$ according as ψ^* is bounded or unbounded; and*

$$\sum_{i,j,k(\neq)} \psi(N(x_i - x_j)) \frac{1}{x_i - x_j} \int_{x_j}^{x_i} g_N(y - x_k) dy \geq N \sum_{i,j,k(\neq)} \psi^*(N(x_i - x_j)) \, \omega(N(x_i - x_k)) \, .$$

Proof. Since $(y - x)^{-1} \int_x^y g_N(u - z) du \geq \min\{g_N(x - z), g_N(y - z)\}$, the inequality required in the lemma is implied by

$$\max_{\pi \in \mathfrak{S}(3)} \left[\psi(q_{\pi(i)} - q_{\pi(j)}) \min\{g(q_{\pi(i)} - q_{\pi(k)}), g(q_{\pi(j)} - q_{\pi(k)})\} \right]$$

$$\geq 6 \max_{\pi \in \mathfrak{S}(3)} \left[\psi^*(q_{\pi(i)} - q_{\pi(j)}) \omega(q_{\pi(i)} - q_{\pi(k)}) \right] \, , \qquad (3.6)$$

where $\mathfrak{S}(3)$ is the permutation group of three letters i, j, k and q_i, q_j and q_k are arbitrary three real numbers different from one another. If we define $\tilde{\omega}(x), 0 < x \leq 1/2$, by

$$\tilde{\omega}(x) = \inf_{0 < u < x} \tilde{g}(u), \qquad \tilde{g}(x) = \psi(x) \inf_{u : x \leq u \leq 1/2} \frac{g(u)}{\psi(u)}$$

(recall we have supposed $R > 1$), then $\tilde{\omega}$ is non-increasing; $\tilde{\omega}(x)$ approaches infinity or a positive number as $x \downarrow 0$ according as ψ^* is unbounded or bounded by virtue of assumption (1.5); moreover if $0 < s \leq t \leq 1/4$,

$$\psi(s)g(s + t) \geq \frac{1}{\sqrt{2}} \max\{\psi(s + t)\tilde{\omega}(s), \psi(t)\tilde{\omega}(s), \psi(s)\tilde{\omega}(t)\} \, . \qquad (3.7)$$

Finally define $\omega(x) = (6\sqrt{2})^{-1}\tilde{\omega}(|x|)\chi\{|x| < 1/4\}$. A little reflection then shows that (3.6) follows from (3.7), finishing the proof of Lemma 1.

Substituting the inequality (3.5) and that given in Lemma 1 into (3.4) we conclude:

$$\frac{1}{N} \int_0^T \sum_{i,j(\neq)} \left[\psi^*(N(x_i - x_j)) \sum_{k \neq i} \omega(N(x_i - x_k)) \right] dt$$

$$\leq C + (4 + \|h\|_\infty) \frac{1}{N} \int_0^T \sum_{i,j(\neq)} \psi(N(x_i - x_j)) dt . \qquad (3.8)$$

3.3. We wish to dominate the integral on the right side of (3.8) by a small fraction of that on the left side. We continue to assume $R > 1$ and choose the function ω in Lemma 1 so that $\omega \leq g$ as in the proof of it.

Lemma 2. *There are constants C_1 and C_2 that depend only on ψ such that*

$$\sum_{i,j(\neq)} \psi(N(x_i - x_j)) \leq C_1 N + C_2 \sum_{i,j(\neq)} \psi^*(N(x_i - x_j)) .$$

Proof. For $\ell > 0$ put

$$n_\ell(\theta) = n_{N,\ell}(\theta; \mathbf{x}) := \#\left\{ j \,\Big|\, |x_j - \theta| \leq \frac{\ell}{2N} \right\} .$$

Then, by noticing that $\sup_{a \leq x \leq b} \psi(x) \leq (b/a)\psi(a)$ if $0 < a < b$,

$$\psi_i^{**} := \sum_{j \neq i} \psi(N(x_i - x_j)) \chi\{N|x_i - x_j| > 1/4\}$$

$$\leq 2 \sum_{k \neq 0} \psi\left(\frac{k}{4}\right) n_{1/4}\left(x_i + \frac{k}{4N} + \frac{k/|k|}{8N}\right) . \qquad (3.9)$$

But for every integer k,

$$\sum_i n_{1/4}\left(x_i + \frac{k}{4N} + \frac{1}{8N}\right)$$

$$\leq \sum_m n_{1/4}\left(\frac{m}{4N}\right) \left[n_{1/4}\left(\frac{m}{4N} + \frac{k}{4N}\right) + n_{1/4}\left(\frac{m}{4N} + \frac{k-1}{4N}\right) \right]$$

$$\leq 2 \sum_m n_{1/4}^2\left(\frac{m}{4N}\right) . \qquad (3.10)$$

Thus

$$\sum_i \psi_i^{**} \leq 4 \left[\sum_{k \neq 0} \psi\left(\frac{k}{4}\right) \right] \sum_m n_{1/4}^2\left(\frac{m}{4N}\right) . \qquad (3.11)$$

On the other hand, by putting

$$\psi_i^* := \sum_{j \neq i} \psi(N(x_i - x_j)) \chi\{N|x_i - x_j| \leq 1/4\} ,$$

$$\sum_i \psi_i^* \geq \left(\inf_{|u| \leq 1/4} \psi(u) \right) \sum_i (n_{1/2}(x_i) - 1)$$

$$\geq \left(\inf_{|u| \leq 1/4} \psi(u) \right) \left[\sum_m n_{1/4}^2\left(\frac{m}{4N}\right) - N \right] . \qquad (3.12)$$

We can therefore find two constants C_1 and C_2 for the inequality of the lemma to hold.

Lemma 3. *For every $\delta > 0$ there can be found a constant M so that*

$$\sum_{i,j(\neq)} \psi(N(x_i - x_j)) \leqq NM + \delta \sum_{i,j(\neq)} \left[\psi^*(N(x_i - x_j)) \sum_{k \neq i} \omega(N(x_i - x_k)) \right].$$

Proof. Put

$$\omega_i = \sum_{j \neq i} \omega(N(x_i - x_j)),$$

$$\gamma(u) = \inf\left\{ \sum_k \omega(s_k) \;\middle|\; (s_k)_k \text{ is a finite sequence} \right.$$

$$\left. \text{of real numbers such that } \sum_k \psi^*(s_k) \geqq u \right\}.$$

Then $\gamma(u) \uparrow \infty$ as $u \uparrow \infty$, and, since $\gamma(\psi_i^*) \leqq \omega_i$,

$$\psi_i^* \chi\{\psi_i^* \geqq u\} \leqq \frac{1}{\gamma(u)} \psi_i^* \omega_i ; \tag{3.13}$$

hence $\sum \psi_i^* \leqq Nu + \sum \psi_i^* \chi\{\psi_i^* \geqq u\} \leqq Nu + (1/\gamma(u))\sum \psi_i^* \omega_i$. Substituting the last inequality into Lemma 2 and choosing u so large that $C_2/\gamma(u) < \delta$, we obtain the estimate claimed in the lemma.

Lemma 4. *Let ψ_i^{**} be as in (3.9). Then there are constants C_1 and C_2 such that*

$$\sum_i [\psi_i^{**}]^2 \leqq C_1 N + C_2 \sum_{i,j(\neq)} \left[\psi^*(N(x_i - x_j)) \sum_{k \neq i} \omega(N(x_i - x_k)) \right].$$

Proof. Denote by S^N the sum on the right side. As in the proof of Lemma 2 we see

$$S^N \geqq \left(\inf_{|u| \leqq 1/8} \psi(u)\omega(u) \right) \sum_i (n_{1/4}(x_i) - 1)^2$$

$$\geqq \left(\inf_{|u| \leqq 1/8} \psi(u)\omega(u) \right) \left[\sum_m n_{1/8}^3 \left(\frac{m}{8N} \right) - 6 \sum_m n_{1/4}^2 \left(\frac{m}{4N} \right) \right].$$

The second term within the square bracket on the last line being dominated by a constant multiple of $S^N + N$ according to (3.12) and Lemma 3, this inequality yields

$$\sum_m n_{1/4}^3 \left(\frac{m}{4N} \right) \leqq C_1' N + C_2' S^N . \tag{3.14}$$

On the other hand applying the Schwarz inequality to the right side of (3.9) and then making the same argument as in (3.10) we see

$$\sum_i [\psi_i^{**}]^2 \leqq 4 \sum_i \left[\sum_{k \neq 0} \psi\left(\frac{k}{4} \right) \right] \left[\sum_{k \neq 0} \psi\left(\frac{k}{4} \right) n_{1/4}^2 \left(x_i + \frac{k}{4N} + \frac{k/|k|}{8N} \right) \right]$$

$$\leqq 4 \cdot 4 \left[\sum_{k \neq 0} \psi\left(\frac{k}{4} \right) \right]^2 \sum_m n_{1/4}^3 \left(\frac{m}{4N} \right) ,$$

of which, owing to (3.14), the last member is dominated by $C_1 N + C_2 S^N$ as required.

The proof of Theorem 2 is now completed in a few lines. By applying the Schwarz inequality and Lemma 4 in turn we see

$$\sum_{i,j(\neq)} \left[\psi(N(x_i - x_j)) \chi\{N|x_i - x_j| > 1/4\} \sum_{k \neq i} \omega(N(x_i - x_k)) \right]$$

$$\leq C_3 + C_4 \sum_{i,j(\neq)} \left[\psi^*(N(x_i - x_j)) \sum_{k \neq i} \omega(N(x_i - x_k)) \right] .$$

The asserted estimate is obtained by applying this one after substituting that of Lemma 3 into (3.8).

3.4. We state three corollaries of Theorem 2 for convenience of later citations.

Corollary 1. *For each $T > 0$,*

$$\lim_{\delta \downarrow 0} \sup_{\substack{0 \leq s < t \leq T \\ |t-s| < \delta}} \sup_N \int_s^t \frac{1}{N} \sum_{i,j(\neq)} \psi(N(x_i(u) - x_j(u))) du = 0 .$$

Proof. Immediate from Lemma 3 and Theorem 2.

Corollary 2. *The functions* $\Psi^N(t,i) := \sum_{j \neq i} \psi(N(x_i(t) - x_j(t))), N = 1, 2, \ldots,$ *are uniformly integrable relative to the respective product measures $N^{-1} \sum_{j=1}^N \delta_j(\cdot) \times dt \in \mathfrak{M}(\{1, 2, \ldots, N\} \times [0, T])$.*

Proof. In the decomposition $\Psi^N(\cdot, i) = \psi_i^* + \psi_i^{**}$ (where ψ_i^{**} is defined in the proof of Lemma 2), the uniform integrability of the first term, ψ_i^*, follows from (3.13) and Theorem 2. As for the second term Lemma 4 and Theorem 2 together verify that its second moment is bounded; it is in particular uniformly integrable.

By making the same argument as leading to (3.11) we deduce from Theorem 2, Lemma 3 and (3.11) the following

Corollary 3. *Let φ be a continuous, integrable positive function on $\mathbf{R}$ and suppose $\varphi(y)/\varphi(x)$ is bounded on $\{|x| < |y|\}$. Then there exists a constant C such that for all $N = 1, 2, \ldots$ and all $M \geq 0$,*

$$\int_0^T \sum_{i,j(\neq)} \varphi(N(x_i(t) - x_j(t))) \chi\{N|x_i(t) - x_j(t)| \geq M\} dt \leq CN \sum_{|k| \geq M} \varphi(k) .$$

4. Local Equilibrium

Recall μ^N denotes the probability measure on the set of N-particle configurations determined by (1.12). Since the expectation of the density of configurations under μ^N is (uniformly) bounded, the family $(\mu^N)_{N \geq 1}$ is relatively compact in the space of probability measures on the set of locally finite configurations on $\mathbf{R}$ under the usual topology determined by the convergence of the integral of bounded, continuous local functions. In this section we prove Theorem 3, namely that any limit point of μ^N, which is necessarily translation invariant, concentrates on configurations of equal spacing. We note that if μ is a limit point of μ^N, then $\mu\{\mathbf{q} \mid q_i \neq q_j \text{ if } i \neq j\} = 1$, as is verified by Theorem 2.

Lemma 5. *For every limit point μ of μ^N,*

$$\mu\left\{\mathbf{q} = (q_i) \,\middle|\, \sum_{j \neq i} U'(q_i - q_j) = 0 \text{ for all } i\right\} = 1 . \tag{4.1}$$

Proof. In a formal level the result of Lemma 5 is inferred from an identity (set $g \equiv 1$ below) obtained by calculating the right side of

$$N^{-3}\sum v_i \sum_{j \neq i} U'(x_i - x_j)\bigg|_{t=0}^{T} = N^{-3}\int_0^T \frac{d}{dt}\left[\sum v_i \sum_{j \neq i} U'(x_i - x_j)\right] dt$$

according to (2.1) and then passing to the limit as $N \to \infty$. In the identity however there arise singular terms that should be avoided for the latter procedure of taking the limit. In this proof we truncate them by means of two functions g and φ whose introduction we start with.

For each $K > 3$, let $g(u) = g(u, K)$ be a smooth function on $\mathbf{R}$ such that

$$g(u) = 0 \quad \text{if } |u| > K; \quad = 1 \text{ if } |u| \leq K - 2; \quad \text{and} \quad |g|, |g'| \leq 1 .$$

Let φ be a smooth function on $\mathbf{R} - \{0\}$ that satisfies the following conditions:

$$|U'(x)| \leq \varphi(x) \quad \text{and} \quad -x\varphi'(x) > 0 \text{ for } x \neq 0; \quad \int_{|x|>1} \varphi(x)dx < \infty ,$$

$$|\varphi'(x)| \leq \varphi(x) \text{ for } |x| > 1; \quad \text{and} \quad |\varphi'(y)| \leq 2|\varphi'(x)| \text{ if } |y| > |x| > 0 .$$

Such a function φ may be constructed by modifying U as follows. The problem is how to define it for large x. Clearly we can suppose $R = \infty$. Put $f(x) = -U'(x-1)/U(x-1)$, $x \geq 2$. If $\sup f \leq 1$, we can take $\varphi(x) = \text{const } U(x-1)$ for $x \geq 2$, since $U(x-1) \geq -U'(x)$, $x \geq 2$. If $\sup f > 1$, putting $x_o = \inf\{x \geq 2 \mid f(x) > 1\}$, we define $\varphi(x) = U(x-1)$ for $2 \leq x \leq x_o$ and

$$\varphi(x) = U(x_o - 1)\exp\{-(x - x_o)/2\} \quad \text{for } x_o < x \leq x_1 ,$$

where x_1 is the least value of $x \geq x_o$ at which $U(x_o - 1)\exp\{-(x - x_o)\}$ coincides with $U(x-1)$. Necessarily $f(x_1) \leq 1/2$ if $x_1 < \infty$. Now starting from x_1 instead of 2 we continue the same procedure as above to define φ on $[2, K]$ for every K and hence on $[2, \infty)$. The function φ thus defined fulfills all the requirements as is easily checked.

Let $\tilde{U}'_{(\varepsilon)}(x)$ be a smooth approximation of $U'(x)$ such that $|\tilde{U}'_{(\varepsilon)}(x)| \leq 2\varphi(x)$, $|\tilde{U}''_{(\varepsilon)}(x)| \leq C_\varepsilon|\varphi'(x)|$ and $\tilde{U}'_{(\varepsilon)}(x) \to U'(x)$ ($\varepsilon \downarrow 0$) for each $x \neq 0$. Now, putting

$$\varphi_i(t) = \sum_{j \neq i} \varphi(N(x_i(t) - x_j(t))),$$

$$F_i(t) = -\sum_{j \neq i} U'(N(x_i(t) - x_j(t))),$$

$$\tilde{F}_{i,\varepsilon}(t) = -\sum_{j \neq i} \tilde{U}'_{(\varepsilon)}(N(x_i(t) - x_j(t))) ,$$

we see

$$\frac{1}{N^3}\frac{d}{dt}\left[\sum_{i=1}^{N}v_i(t)\tilde{F}_{i,\varepsilon}(t)g(\varphi_i(t))\right] = -\frac{1}{N}\sum_i v_i\tilde{F}_i g(\varphi_i) + \frac{1}{N}\sum_i F_i\tilde{F}_i g(\varphi_i)$$

$$-\frac{1}{N^2}\sum_i B_N(x_i)\tilde{F}_i g(\varphi_i) - \frac{1}{N}\sum_{i,j(+)} v_i(v_i - v_j)\tilde{U}''(N(x_i - x_j))g(\varphi_i)$$

$$+\frac{1}{N}\sum_i v_i\tilde{F}_i g'(\varphi_i)\sum_{j+i}\varphi'(N(x_i - x_j))(v_i - v_j)\,.$$

Here the argument t and the parameter ε are suppressed on the right side. We integrate both sides of this identity with respect to $t \in [0, T]$ and take the limit as $N \to \infty$. We are to show that the contribution of every term except the second one on the right side vanishes in the limit to have

$$\lim_{N \to \infty}\int_0^T \frac{1}{N}\sum_i F_i(t)\tilde{F}_{i,\varepsilon}(t)g(\varphi_i(t); K)dt = 0 \tag{4.2}$$

for every $K > 0$ (with ε fixed). By noticing that $|\tilde{F}_i g(\varphi_i)| \leq 2K$ and by making use of (2.2), (3.2), (3.3), Lemma 3 and Theorem 2, the left side as well as the first and third terms on the right side are readily disposed of to this end. For dealing with the rest, i.e., the last two terms on the right side we choose, for each K, $\delta = \delta(K) > 0$ so that $\varphi(2\delta) > K$, which is possible since $\varphi(x) \to \infty$ as $x \to 0$. Then

$$g(\varphi_i) = g'(\varphi_i) = 0 \quad \text{if } \min\{x_i - x_{i-1}, x_{i+1} - x_i\} \leq \frac{2\delta}{N}\,, \tag{4.3}$$

where x_i are numbered in the increasing order: $x_1 < x_2 < \cdots < x_N$ and $x_0 = -\infty$ and $x_{N+1} = \infty$. It would be clear from the properties imposed on φ that our task for the proof of (4.2) is reduced to showing

$$\int_0^T \sum_i v_i g(\varphi_i)\sum_{j+i}\tilde{U}''(N(x_i - x_j))v_j = o(N)\,, \tag{4.4a}$$

$$\int_0^T \sum_i v_i\tilde{F}_i g'(\varphi_i)\sum_{j+i}\varphi'(N(x_i - x_j))v_j = o(N)\,. \tag{4.4b}$$

(These estimates are ready to verify if we assume $\mathscr{E}^N(0) = o(N^2)$; see also (6.1).)
   We prove (4.4b) first. Put

$$\chi_{i,m} = \chi\{(m - 1)\delta \leq Nx_i < m\delta\}\,.$$

Then

$$\left|\sum_i v_i\tilde{F}_i g'(\varphi_i)\sum_{j+i}\varphi'(N(x_i - x_j))v_j\right| \leq 2K\sum_i\sum_m \chi_{i,m}|v_i g'(\varphi_i)|\sum_{j+i}|\varphi'(N(x_i - x_j))v_j|\,. \tag{4.5}$$

If $\chi_{i,m} = 1$, i.e., $(m - 1)\delta \leq Nx_i < m\delta$, then by the last condition imposed on φ,

$$\sum_{j+i}|\varphi'(N(x_i - x_j))v_j| \leq 2\sum_{j>i}|\varphi'(m\delta - Nx_j)v_j| + 2\sum_{j<i}|\varphi'((m - 1)\delta - Nx_j))v_j|\,.$$

Thus the contribution of the sum over $j > i$ to the right side of (4.5) is at most

$$4K\sum_i\sum_m\chi_{i,m}|v_ig'(\varphi_i)|\sum_{j>i}|\varphi'(m\delta - Nx_j)v_j|$$

$$\leqq 8K\sum_{k\geqq1}|\varphi'(k\delta)|\sum_i\sum_m\chi_{i,m}|v_ig'(\varphi_i)|\sum_j\chi_{j,m+k+1}|v_j| \ . \tag{4.6}$$

Here we have applied (4.3) again. Observe that

$$\chi_{i,m}g'(\varphi_i) = 0 \text{ if } \sum_j\chi_{j,m+k+1} > \frac{K}{\varphi((k+2)\delta)} \text{ for some } k \geqq 1 \ .$$

Then by applying the Schwarz inequality both to the double sum over i,m and to the sum over j we can further dominate the sum on the right side of (4.6) by

$$\sum_{k\geqq1}\frac{|\varphi'(k\delta)|\sqrt{K}}{\sqrt{\varphi((k+2)\delta)}}\sqrt{\sum_i\sum_m\chi_{i,m}v_i^2}\sqrt{\sum_m\sum_i\chi_{i,m}|g'(\varphi_i)|^2\sum_j\chi_{j,m+k+1}v_j^2}$$

$$\leqq \sqrt{K}\sum_{k\geqq1}\frac{-\varphi'(k\delta)}{\sqrt{\varphi((k+2)\delta)}}\sum_j|v_j|^2 \ .$$

We can suppose δ is chosen small enough that $\varphi(y + 2\delta) > \frac{1}{2}\varphi(y - \delta)$ for $y > 2$, which implies $-\varphi'(k\delta)/\sqrt{\varphi((k+2)\delta)} < -4\varphi'(x\delta)/\sqrt{\varphi(x\delta)}$ for $x \in [k-1,k]$, $k > 2/\delta$, and hence that the infinite sum over k above converges. We have the same bound for the contribution of the sum over $j < i$ in (4.5). In view of (2.2) this proves (4.4b). Since $\tilde{U}''(x) < C_\varepsilon|\varphi'(x)|$, the same proof works for (4.4a). Thus the proof of (4.2) is finished.

From (4.2) it clearly follows that

$$\lim_{N\to\infty}\int_0^T\frac{1}{N}\sum_iF_i(t)\tilde{F}_{i,\varepsilon}(t)g(\varphi_i(t);K)\int_0^1h_N(x_i(t) - \theta)d\theta dt = 0 \ ,$$

where $h_N(x) = Nh(Nx)$ as before. The integral under the limit may be written

$$T\int\sum_i\sum_{j\neq i}U'(q_i - q_j)\sum_{j\neq i}\tilde{U}'_{(\varepsilon)}(q_i - q_j)g\left(\sum_{j\neq i}\varphi(q_i - q_j);K\right)h(q_i)\mu^N(d\mathbf{q}) \ .$$

Hence an application of Corollary 3 shows that for each limit point μ,

$$\int\sum_i\sum_{j\neq i}U'(q_i - q_j)\sum_{j\neq i}\tilde{U}'_{(\varepsilon)}(q_i - q_j)g\left(\sum_{j\neq i}\varphi(q_i - q_j);K\right)h(q_i)\mu(d\mathbf{q}) = 0 \ .$$

Finally, noticing $\int\sum_ih(q_i)\mu(d\mathbf{q}) < \infty$, we let $\varepsilon \downarrow 0$ to conclude

$$\int\sum_i\left[\sum_{j\neq i}U'(q_i - q_j)\right]^2g\left(\sum_{j\neq i}\varphi(q_i - q_j);K\right)h(q_i)\mu(d\mathbf{q}) = 0.$$

Since $\sum_{j\neq i}\varphi(q_i - q_j) < \infty$ for all i a.s.(μ) (this simply follows from the fact that μ is a translation invariant probability on the space of locally finite configurations), letting $K \to \infty$ in this relation proves $\sum_{j\neq i}U'(q_i - q_j) = 0$ for all i a.s.(μ). The proof of Lemma 5 is complete.

Theorem 3 immediately follows from Lemma 5 if we prove the next theorem, where neither (1.4) nor (1.5) is needed.

Theorem 4. *Let $((q_i)_{i\in\mathbf{Z}}); \mu)$ be a translation invariant point process on* $\mathbf{R}$. *If* (4.1) *holds with U satisfying* (1.2) *and* (1.3), *then with μ-probability one either (q_i) has equal successive spacings of a common span less than R or all the spans between nearest-neighbour particles are greater or equal to R.*

For the proof of Theorem 4 we suppose $(q_i)_{i\in\mathbf{Z}}$ is numbered in an increasing order, i.e., $q_i < q_{i+1}$ for all $i \in \mathbf{Z}$, and put $z_i = q_i - q_{i-1}$. Let $\eta(z), z > 0$, be a non-increasing function such that $0 < \eta(x) < R$ and

$$F(\eta(z)) \geqq 3F(z), \quad \text{where} \quad F(x) := |U'(x)|. \tag{4.7}$$

Clearly such a function η exists. We prepare a simple lemma.

Lemma 6. *Suppose U satisfies* (1.2) *and* (1.3) *and let η and z_i be as above. If $\sum_{j\neq i} U'(q_i - q_j) = 0$ for all i, then $\min\{z_{i+1}, z_{i-1}\} \geqq \eta(z_i)$ for all i.*

Proof. By applying $\sum_{j\neq i} U'(q_i - q_j) = 0$ first for $i = 0$ and then for $i = 1$ together with the monotonicity of $F(x) = |U'(x)|, x > 0$, we see

$$\sum_{j=-2}^{-\infty} F(q_0 - q_j) + F(z_0) = F(z_1) + \sum_{j=2}^{\infty} F(q_j - q_0)$$

$$\leqq F(z_1) + \sum_{j=2}^{\infty} F(q_j - q_1) = 2F(z_1) + \sum_{j=-1}^{-\infty} F(q_1 - q_j)$$

$$\leqq 3F(z_1) + \sum_{j=-2}^{-\infty} F(q_0 - q_j);$$

hence $F(z_0) \leqq 3F(z_1)$, showing $z_0 \geqq \eta(z_1)$ as desired.

Proof of Theorem 4. Let δ be an arbitrary positive number. Let $g(x), x \geqq 0$ be a bounded continuous function such that $g(x) = 0$ for $x \leqq \delta$ and strictly increasing for $x > \delta$. Then the condition (4.1) implies, in view of Lemma 6, that with μ-probability one

$$\min\{z_0, z_1, z_2\} \geqq \eta(\delta) \quad \text{if} \quad g(z_1) > 0 \tag{4.8}$$

and, by symmetry of $F = |U'|$, that with μ-probability one

$$0 = g(z_1)\left[\sum_{n\neq 1} U'(q_1 - q_n) - \sum_{n\neq 0} U'(q_0 - q_n)\right]$$

$$= g(z_1)\sum_{n=2}^{\infty} [F(q_n - q_1) - F(q_{n-1} - q_0)]$$

$$-g(z_1)\sum_{n=2}^{\infty} [F(q_1 - q_{-n+2}) - F(q_0 - q_{-n+1})]. \tag{4.9}$$

From (4.8) it follows that

$$g(z_1)\sum_{n\neq 1} \left(|U'(q_1 - q_n)| + |U'(q_0 - q_{n-1})|\right) \leqq \sum_{n\neq 1} [\varphi(q_1 - q_n) + \varphi(q_0 - q_{n-1})]$$

with φ a bounded even function having a monotone integrable tail. Since the density $\rho := \lim_{K+L\to\infty, K,L\geq 0} (K+L)^{-1}\#\{i \mid -L \leq q_i \leq K\}$ exists a.s.(μ), we may suppose $\int \rho d\mu < \infty$ (otherwise one has only to restrict μ on the set $\{\rho < M\}$ and afterwards let $M \to \infty$). This makes the right side of the inequality above (and hence the left side) integrable relative to μ, so that we can change the order of the summation in (4.9) even under the integration by μ, which we now carry out. The translation invariance of μ then shows that

$$
\begin{aligned}
0 &= \sum_{n=2}^{\infty} \int [g(z_1)(F(q_n - q_1) - F(q_{n-1} - q_0)) \\
&\quad -g(z_n)(F(q_n - q_1) - F(q_{n-1} - q_0))]\,\mu(d\mathbf{q}) \\
&= \sum_{n=2}^{\infty} \int [g(z_n) - g(z_1)][-F(q_n - q_1) + F(q_{n-1} - q_0)]\,\mu(d\mathbf{q}) .
\end{aligned}
$$

Since $(q_n - q_1) - (q_{n-1} - q_0) = z_n - z_1$, and since both g and F are monotone, all the integrands have the same sign and hence must vanish a.s.(μ); in particular $[g(z_2) - g(z_1)][F(z_2) - F(z_1)] = 0$ a.s. (μ). Finally letting the constant δ, on which the support of g depends, tend to zero, we arrive at the conclusion of Theorem 4.

Remark. The same result as Theorem 4 is obtained by Lang [L] under the additional assumption that $\int |U'(z_0)| d\mu < \infty$. The proof given above is essentially the same as his except for the use of Lemma 6 in place of the integrability of $U'(z_0)$.

5. Proof of Theorem 1

This section is divided into four subsections. In the last one we show that the weak convergence of α_t^N implies the strong convergence of its mollified local density $\alpha_t^N(h_{N/\ell}(\,\cdot - \theta))$ as $N \to \infty$ and $\ell \to \infty$ in this order. Taking this result for granted we deduce Theorem 1 from Theorems 2 and 3 in the first three subsections.

5.1. First we carry out some calculation to see that our system is essentially of gradient type. This part of the proof is very similar to the corresponding one in [OV]. Let J be a smooth function on the closed interval $[0,1]$. By (2.1) we obtain

$$
\frac{d}{dt}\alpha_t^N(J) = \sum_{i=1}^{N} J'(x_i(t))v_i(t) \tag{5.1}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\sum J'(x_i)v_i &= N\sum J''(x_i)v_i^2 - N^2\sum J'(x_i)v_i - N\sum J'(x_i)B_N(x_i) \\
&\quad -N^2\sum_i J'(x_i)\sum_{j\neq i} U'(N(x_i - x_j)) \tag{5.2}
\end{aligned}
$$

as well, where, as before, $B_N(x) = N[W'_-(Nx) + W'_+(N(x-1))]$ and the time parameter is suppressed. Since U' is odd, the last term in (5.2) may be rewritten as

$$
-\frac{1}{2}N^2 \sum_{i,j\,(\neq)} (J'(x_i) - J'(x_j))U'(N(x_i - x_j)).
$$

Notice that N^2 times the right side of (5.1) appears as the second term on the right side of (5.2), substitute its expression accordingly obtained into (5.1) and then integrate both sides of the resulting equality on a time interval $[0,t]$ and you see

$$\alpha_t^N(J) - \alpha_0^N(J) = \frac{1}{N} \int_0^t \sum J''(x_i) v_i^2 ds - \frac{1}{N^2} \sum J'(x_i) v_i \Big|_{s=0}^t$$

$$- \frac{1}{N} \int_0^t \sum J'(x_i) B_N(x_i) ds$$

$$+ \int_0^t \frac{1}{2N} \sum_{i,j(\neq)} \frac{J'(x_i) - J'(x_j)}{x_i - x_j} \psi(N(x_i - x_j)) ds. \quad (5.3)$$

By (2.2) and (3.2) the first and the second terms on the right side converge to zero as $N \to \infty$. Suppose that $J'(0) = J'(1) = 0$. Then the third term also converges to zero due to the assumption (1.6b) (see the estimate (3.3) that is an immediate consequence of (5.3)). If ψ decays for large x sufficiently fast, then the ratio $(J'(x_i) - J'(x_j))/(x_i - x_j)$ in the last term might be replaced by $J''(x_i)$. Corollary 3 in Sect. 3 ensures such replacement is valid under the assumption (1.4), so that

$$\alpha_t^N(J) - \alpha_0^N(J) = \int_0^t \frac{1}{2N} \sum_{i,j(\neq)} \psi(N(x_i(s) - x_j(s))) J''(x_i(s)) ds + o(1), \quad (5.4)$$

where $o(1)$ is locally uniform in t.

5.2. For each smooth J satisfying $J'(0) = J'(1) = 0$ the family of continuous functions $(\alpha_t^N(J), 0 \leq t \leq T), N = 1, 2, \ldots$, is equi-continuous according to Corollary 1 and (5.4). Clearly it is uniformly bounded. Hence it is relatively compact in the space of continuous functions under the topology of uniform convergence. This shows the family of continuous measure-valued functions $(\alpha_t^N, 0 \leq t \leq T), N = 1, 2, \ldots$, also is relatively compact; by Corollary 3 any limit point of it is absolutely continuous for each t. We can identify the limiting density, $\rho(\theta, t)$ say, and thereby complete the proof of Theorem 1, if we show that the integral on the right side of (5.4) converges to

$$\int_0^t \int_0^1 P(\rho(\theta, s)) J''(\theta) d\theta ds, \quad (5.5)$$

since $\rho(\theta, t)$ is then revealed to be a weak solution of the Cauchy problem (1.8) and (1.10) that is unique by virtue of the following theorem. (See also Lemma 8 below.)

Theorem 5. *Let α_0 be a finite Borel measure on $[0,1]$. Then for every $T > 0$ there is at most one non-negative weak solution $\rho = \rho(\theta, t)$ to the problem (1.8) on $(0,T]$ that satisfies the integrability condition $\int_0^T dt \int_0^1 P(\rho(\theta, t)) d\theta < \infty$ and the initial condition (1.10), as well.*

Proof. Theorem 6 in [U] asserts a similar uniqueness for the nonlinear diffusion equation (1.8a) with the periodic boundary condition. Its proof given there is available also for the case of the reflecting boundary condition (1.8b) without any alteration except that we here use the fundamental solution of the heat equation (that plays as a test function in the proof) with the reflecting boundary condition instead of the periodic boundary condition.

5.3. Let h be the same smooth even function as introduced in Sect. 3. Write $h_K(\theta) := Kh(K\theta)$ for $K > 0$ as before. Then for $\ell > 0$,

$$\int_0^t \frac{1}{2N} \sum_{i,j(\neq)} \psi(N(x_i - x_j))J''(x_i)ds$$

$$= \int_0^t \left[\int_{-\ell/N}^{1+\ell/N} \frac{1}{2N} \sum_{i,j(\neq)} h_{N/\ell}(x_i - \theta)\psi(N(x_i - x_j))J''(\theta)d\theta \right] ds + O\left(\frac{\ell}{N}\right) ,$$

where J'' is smoothly extended outside $[0,1]$. Introducing the function

$$H_\psi^\ell(\mathbf{q}) = \frac{1}{2} \sum_{i,j(\neq)} \psi(q_i - q_j)h_{1/\ell}(q_i), \quad \mathbf{q} = (q_i)_{i=1}^N ,$$

we can rewrite the inner integral on the right side above in a concise form:

$$\int_{-\ell/N}^{1+\ell/N} H_\psi^\ell(N\mathbf{x}^\theta(s))J''(\theta)d\theta .$$

The contribution from the boundary region $[-\ell/N, 0) \cup (1, 1 + \ell/N]$ to this integral is negligible since $\Psi^N(t,i) := \sum_{j \neq i} \psi(N(x_i(t) - x_j(t)))$ is uniformly integrable as asserted in Corollary 2 of Sect. 3. For the proof of Theorem 1 it therefore suffices to show

$$\lim_{\ell \to \infty} \limsup_{m \to \infty} \int_0^T ds \int_0^1 |H_\psi^\ell(N_m\mathbf{x}^\theta(s)) - P(\rho(\theta,s))|d\theta = 0 , \qquad (5.6)$$

where N_m is an arbitrary subsequence of $\{N\}$ along which α_t^N converges to $\rho(\theta,t)d\theta$. Again by virtue of the uniform integrability of $\Psi^N(t,i)$ we can replace $H_\psi^\ell(\mathbf{q})$ in (5.6) by

$$H_{\psi,M}^\ell(\mathbf{q}) = \frac{1}{2} \sum_i \left(\left[\sum_{j \neq i} \psi(q_i - q_j) \right] \wedge M \right) h_{1/\ell}(q_i) .$$

On the other hand, since $\rho(\theta,s)P(\rho(\theta,s))$ is integrable with respect to $d\theta ds$ on $[0,1] \times [0,T]$ as we will see in Lemma 7 at the end of this section, we can also replace $P(\rho)$ in (5.6) by

$$P_M(\rho) := \left(\left[\sum_{k=1}^\infty \psi(k/\rho) \right] \wedge M \right) \rho .$$

Put

$$\rho^\ell(\mathbf{q}) := \sum h_{1/\ell}(q_i) = \frac{1}{\ell} \sum h(q_i/\ell) .$$

Then $P_M(\rho(\theta,s))$, in turn, can be replaced by $P_M(\rho^{N\varepsilon}(N\mathbf{x}^\theta(s)))$ since $\rho^{N\varepsilon}(N\mathbf{x}^\theta(s)) = \alpha_t^N(h_{1/\varepsilon}(\cdot - \theta))/2\varepsilon$ well approximates the limiting density $\rho(\theta,s)$ as $N \to \infty$ along N_m and $\varepsilon \downarrow 0$ in this order. According to Theorem 6 below we can further replace $P_M(\rho^{N\varepsilon}(N\mathbf{x}^\theta(s)))$ by $P_M(\rho^\ell(N\mathbf{x}^\theta(s)))$. As a consequence of these replacements of functions the problem is reduced to showing that for all $M \geq 1$,

$$\lim_{\ell \to \infty} \limsup_{N \to \infty} \int_0^T ds \int_0^1 |H_{\psi,M}^\ell(N\mathbf{x}^\theta(s)) - P_M(\rho^\ell(N\mathbf{x}^\theta(s)))|d\theta = 0 ,$$

or by rewriting it by means of μ^N,

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} \int |H^\ell_{\psi M}(\mathbf{q}) - P_M(\rho^\ell(\mathbf{q}))|\, \mu^N(d\mathbf{q}) = 0 \,. \tag{5.7}$$

But this follows from the local equilibrium assertion Theorem 3. (Noticing that both $H^\ell_{\psi M}(\mathbf{q})$ and $P_M(\rho^\ell(\mathbf{q}))$ are the same order as $\rho^\ell(\mathbf{q})$ with M fixed, one should accordingly make a truncation argument that may be based on Corollary 3.)

Thus the proof of Theorem 1 is complete if we prove the following

Theorem 6.

$$\lim_{\ell\to\infty} \limsup_{\varepsilon\downarrow 0} \limsup_{N\to\infty} \int_0^T ds \int_0^1 |\rho^{N\varepsilon}(N\mathbf{x}^\theta(s)) - \rho^\ell(N\mathbf{x}^\theta(s))|\, d\theta = 0 \,.$$

5.4. The proof of Theorem 6 is done by means of the Young measure. Define a finite measure $\pi^{N,\ell}$ on $[0,T]\times[0,1]\times[0,\infty)$ by

$$\pi^{N,\ell}(dt\,d\theta\,du) = dt\,d\theta\,\delta_{\rho^\ell(N\mathbf{x}^\theta(t))}(du) \,.$$

Then the family $(\pi^{N,\ell}(dt\,d\theta\,du); N = 1,2,\ldots,\ell = 1,2,\ldots)$ constitutes a relatively compact set of measures. Our task was to identify every limit point of α_t^N. For any subsequence of N along which α_t^N converges, we can choose its subsequence N^* such that $\pi^{N,\ell}$ converges along N^* for all ℓ. Hence it would suffice to study limit points of $\pi^{N,\ell}$ along such a subsequence N^*. Recall that any limit point of $\alpha_t^N(d\theta)$ possesses a density.

Theorem 7. *Let N^* be a subsequence of N along which $\pi^{N,\ell}$ converges for all $\ell = 1,2,\ldots$. Let $\rho(\theta,t)d\theta$ and $\pi(dt\,d\theta\,du)$ be limit points of $\alpha_t^N(d\theta)$ and $\pi^{N,\ell}(dt\,d\theta\,du)$, respectively, as first $N\to\infty$ along N^* and then $\ell\to\infty$. Then the Young measure $\pi(dt\,d\theta\,du)$ is degenerate at $\rho(\theta,t)$, namely $\pi(dt\,d\theta\,du) = dt\,d\theta\,\delta_{\rho(\theta,t)}(du)$.*

The deduction of Theorem 6 from Theorem 7 is straightforward, since the second moment $\int u^2\pi^{N,\ell}(dt\,d\theta\,du)$ is uniformly bounded according to Corollary 3. (Cf. Eq. (4.5) in [U].) The proof of Theorem 7 is similar to that given in [V] for its Theorem 7.6. The proof given there is modified according to [SU] and [U] so that we can dispense with the entropy technique as applied in [V]. In both [U] and [SU] the fundamental solution on the torus is used in place of a function $g_{N,\varepsilon,\lambda} = h_{N\varepsilon} - h_\lambda$ that is used in [V], for simplicity in [U] and since the model is multidimensional in [SU]. This time we use the fundamental solution for the heat equation on the unit interval with reflecting boundary condition not for simplicity but for necessity for dealing with the boundary term.

Proof of Theorem 7. Let $p_t(x,y), 0 \leqq x,y \leqq 1, t > 0$, be the fundamental solution for the heat equation $\partial_t u = (\partial^2/\partial x^2)u$ on $[0,1]$ with the reflecting boundary condition. We will apply the following representation of it:

$$p_t(x,y) = \sum_{n=-\infty}^{\infty} [g_t(-x+y+2n) + g_t(x+y+2n)] \,, \tag{5.8}$$

where $g_t(x) = (4\pi t)^{-1/2} \exp\left(-x^2/(4t)\right)$. We carry out the same calculation as that leading to (3.1). This time however we compute, instead of $(d/dt)\sum_{i,j(\neq)} G'(x_i - x_j)v_i$,

$$\frac{d}{dt}\sum_{i,j(\neq)} \partial_x p_\tau(x_i, x_j)v_i$$

with τ fixed. Here ∂_x denotes the partial differentiation with respect to the first argument of p_τ. Since p_τ is symmetric (with respect to x and y),

$$N \sum_{i,j(\neq)} \partial_x p_\tau(x_i, x_j)v_i = \frac{1}{2}\frac{d}{dt}\sum_{i,j(\neq)} p_\tau(x_i, x_j).$$

In place of (3.1) we accordingly obtain

$$\frac{1}{2N^2}\int_0^T \sum_{i,j,k(\neq)} [-NU'(N(x_i - x_j))][\partial_x p_\tau(x_i, x_k) - \partial_x p_\tau(x_j, x_k)]dt$$

$$+\frac{1}{N^2}\int_0^T \sum_{i,j(\neq)} [-NU'(N(x_i - x_j))]\partial_x p_\tau(x_i, x_j)dt$$

$$= \frac{1}{N^3}\sum_{i,j(\neq)} v_i \partial_x p_\tau(x_i, x_j)\bigg|_{t=0}^T + \frac{1}{2N^2}\sum_{i,j(\neq)} p_\tau(x_i, x_j)\bigg|_{t=0}^T$$

$$-\frac{1}{2N^2}\int_0^T \sum_{i,j(\neq)} [v_i^2 \partial_x^2 p_\tau(x_i, x_j) + v_i v_j \partial_x \partial_y p_\tau(x_i, x_j)]dt$$

$$+\frac{1}{N^2}\int_0^T \sum_{i,j(\neq)} B_N(x_i)\partial_x p_\tau(x_i, x_j)dt.$$

Integrate both sides of this identity with respect to τ on an interval $[K^2/N^2, b]$, where b and K are constants. Since

$$\int_a^b [\partial_x p_\tau(x_i, x_k) - \partial_x p_\tau(x_j, x_k)]d\tau = \int_{x_i}^{x_j} [p_b(u, x_k) - p_a(u, x_k)]du,$$

two times the first term on the left side then becomes

$$\frac{1}{N^2}\int_0^T \sum_{i,j,k(\neq)} \psi(N(x_i - x_j))\frac{1}{x_i - x_j}\int_{x_i}^{x_j} [p_b(u, x_k) - p_{K^2/N^2}(u, x_k)]du. \tag{5.9}$$

We are to see that this is essentially non-negative in the limit as first $N \to \infty$ and then $b \downarrow 0, K \to \infty$, by examining the other terms one by one in the following items 0) through iv).

0) Since $\partial_x p_\tau(x_i, x_j)$ has the same sign as $U'(N(x_i, -x_j))$, the second term on the left side is non-positive.

i) $|\int_a^b \partial_x p_\tau(x, y)d\tau|$ is bounded whenever $0 < a < b < 1$.

ii) $\sup_{x,y}\int_0^b p_\tau(x, y)d\tau \to 0$ as $b \downarrow 0$.

iii) Since $\partial_x \partial_y p_\tau(x, y) = (\partial/\partial\tau) p_\tau^o(x, y)$, where p_t^o is the fundamental solution of the heat equation with the absorbing boundary condition (as being clear from the Fourier expansion of $p_\tau(x, y)$), the contribution of the third term on the right side

is bounded below by

$$-\frac{\|p_b\|_\infty}{N} \int_0^T \sum_i v_i^2 dt + \frac{1}{N^2} \int_0^T \sum_{i,j(\neq)} [v_i^2 \, p_{K^2/N^2}(x_i, x_j) - v_i v_j \, p_{K^2/N^2}^o(x_i, x_j)] dt \,.$$

The first term vanishes as $N \to \infty$ and the second term is non-negative since $p_a \geq p_a^o$ and $p_a^o(x, y) = p_a^o(y, x)$.

iv) The boundary term vanishes in the limit as $N \to \infty$ and $b \downarrow 0$ in this order as is verified as follows. By symmetry we have only to look at the contribution of $x_i \leq 1/2$. For every $\varepsilon > 0$ this part is bounded by

$$\frac{1}{N^2} \int_0^T \sum_{i,j(\neq)} |B_N(x_i)| \chi\left(x_i < \frac{\varepsilon}{N}\right) \left| \int_{K^2/N^2}^b \partial_x p_\tau(x_i, x_j) d\tau \right| dt$$

$$+ \left[2 \sup_{x \geq \varepsilon} |W_-'(x)| \right] \frac{1}{N} \int_0^T \sum_{i,j(\neq)} \left| \int_{K^2/N^2}^b \partial_x p_\tau(x_i, x_j) d\tau \right| dt$$

at least for large N. For each ε the second term is dominated by a constant multiple of

$$\int_0^T \int_0^1 [\rho^1(N\mathbf{x}^\theta(t))]^2 d\theta dt \times \sup_x \sup_{a:0<a<b} \int_0^1 \left| \int_a^b \partial_x p_\tau(x, y) d\tau \right| dy$$

as is easily shown as in the proof of Lemma 2 in Sect. 3. The first factor is uniformly bounded and the second one converges to zero as $b \downarrow 0$. In view of (3.3) it therefore suffices for our purpose at present to show the following

Lemma 7.

$$\lim_{\varepsilon \to 0} \sup_{0<b<1} \sup_{K \geq 1} \sup_{N > K/\sqrt{b}} \sup_{0<y<1} \sup_{0<x<\varepsilon/N} \left| \int_{K^2/N^2}^b \partial_x p_\tau(x, y) d\tau \right| = 0 \,.$$

Proof. The contribution to $\partial_x p_\tau$ of the sum over $n \neq 0$ in the representation (5.8) causes no problem so that we have only to compute the one coming from $n = 0$, i.e.,

$$S_{b,K,N}(x, y) := \int_{K^2/N^2}^b [g_\tau'(x - y) + g_\tau'(x + y)] d\tau \,.$$

Observing

$$\int_a^b g_\tau'(z) d\tau = \frac{1}{\sqrt{4\pi\tau}} \int_a^b \frac{z}{2\tau} e^{-z^2/4\tau} d\tau = \int_{z/\sqrt{b}}^{z/\sqrt{a}} g_1(u) du \,,$$

we see that if $y > x$ and $x < \varepsilon/N$, then

$$S_{b,K,N}(x, y) \leq 2 \int_{Ny/K-\varepsilon/K}^{Ny/K+\varepsilon/K} g_1(u) du + \int_{y/\sqrt{b}-\varepsilon/\sqrt{bN}}^{y/\sqrt{b}+\varepsilon/\sqrt{bN}} g_1(u) du \,,$$

which with $N > K/\sqrt{b}$ clearly converges to zero as $\varepsilon \to 0$ uniformly in y as well as in the other variables. The case $y < x$ is easier, since then $|x \pm y|/\sqrt{a} \leq 2\varepsilon/K$.

We have observed that the limit supremum of the formula in (5.9) as $N \to \infty$ and then $b \downarrow 0$, $K \to \infty$ is non-negative, which implies

$$I_1 := \liminf_{K \to \infty} \liminf_{N \to \infty}{}^* \int_0^T \frac{1}{N^2} \sum_{i,j,k \, (\mp)} \psi(N(x_i - x_j)) \frac{1}{x_i - x_j} \int_{x_i}^{x_j} p_{K^2/N^2}(u, x_k) du \, dt$$

$$\leqq \limsup_{b \downarrow 0} \limsup_{N \to \infty}{}^* \int_0^T \frac{1}{N^2} \sum_{i,j,k \, (\mp)} \psi(N(x_i - x_j)) \frac{1}{x_i - x_j} \int_{x_i}^{x_j} p_b(u, x_k) du \, dt =: I_2 \,,$$

where $*$ indicates that the limit is taken along the subsequence N^* specified in Theorem 7. From now on we can proceed as in [U] or [SU]. Here is given an outline. The integrand on the left side is very singular, but we need only a lower bound for it and can apply Fatou's lemma to deduce from Theorem 3,

$$\int_0^T dt \int_0^1 d\theta \int_0^\infty u P(u) \pi_{t,\theta}(du) \leqq I_1 < \infty \,, \tag{5.10}$$

where we have applied also the fact that $\pi(dt d\theta du)$ can be written $dt d\theta \pi_{t,\theta}(du)$.

As for the right side we first replace the integral under the double limit supremum by $\int dt \int H_\psi^1(N\mathbf{x}^0(t)) N^{-1} \sum p_b(\theta, x_k) d\theta$, which readily justified by Corollary 3 and the smoothness of p_b. Since $p_b(\theta, \theta')$ can be uniformly approximated by a finite sum of products $a_k(\theta) b_k(\theta')$ of smooth functions and by Theorem 3 together with the same truncation argument based on Corollary 2 as made just after (5.6) we can further replace the latter by

$$\int_0^T dt \int P_M(\rho^\ell(N\mathbf{x}^0(t))) \alpha_t^N(p_b(\theta, \cdot)) d\theta = \int_0^T \iint P_M(u) \alpha_t^N(p_b(\theta, \cdot)) \pi^{N,\ell}(dt d\theta du)$$

if we let $\ell \to \infty$ and $M \to \infty$ in this order after letting $N^* \to \infty$. Noticing that $\alpha_t^N(p_b(\theta, \cdot))$ approaches $\int p_b(\theta, \theta') \rho(\theta', t) d\theta'$ uniformly in (θ, t) (with b fixed) and that $\rho(\theta, t) = \int u \pi_{t,\theta}(du)$ we accordingly conclude

$$I_2 = \limsup_{b \downarrow 0} \int_0^T \iint dt d\theta \int p_b(\theta, \theta') \int u' \pi_{t,\theta'}(du') d\theta' \int P(u) \pi_{t,\theta}(du) \,.$$

We can take the limit within the outer double integral sign $\iint dt d\theta$. In fact by applying both the semi-group property and symmetry of p_τ with the help of Fubini's theorem we see that its integrand may be replaced with

$$F_b(t, \theta) := \int u' \pi_{t,\theta}^{(b)}(du') \int P(u) \pi_{t,\theta}^{(b)}(du) \,,$$

where $\pi_{t,\theta}^{(b)}(du) = \int_0^1 d\eta \, \pi_{t,\eta}(du) p_{b/2}(\theta, \eta)$, and that $F_b(t, \theta)$ is uniformly integrable by $dt d\theta$ owing to the integrability result in (5.10) (see [SU]: the argument leading to (7.8) of it). Therefore

$$I_2 = \int_0^T dt \int d\theta \int u' \pi_{t,\theta}(du') \int P(u) \pi_{t,\theta}(du) \,. \tag{5.11}$$

It would be standard to deduce $\pi_{t,\theta}(du) = \delta_{\rho(\theta,t)}(du)$ from (5.10), (5.11) and the monotonicity of P. The proof of Theorem 6 is finished.

By (5.10) and Theorem 6 we incidentally obtain

Lemma 8. *If $\rho(\theta,t)d\theta$ is a limit point of $\alpha_t^N(d\theta)$, then*

$$\int_0^T dt \int_0^1 \rho(\theta,t)P(\rho(\theta,t))d\theta < \infty.$$

6. Concluding Remarks

Remark 1. The function $P(\rho)$ is, at least formally, identified with the pressure in a standard manner based on the virial theorem: indeed $\frac{1}{2}\sum\psi(N(x_i - x_j))$ is nothing but the interior virial $-\sum q_i U'(q_i - q_j)$ that, since the kinetic energy is negligible, should be balanced with the boundary virial, which in turn must equal the pressure times the volume. This argument however is legitimate only in an equilibrium situation (or in the time average). The following reasoning, though the underlying idea is the same as above, applies to non-equilibrium states. Namely, under the assumption of Theorem 1.1, we deduce from (5.3) that there exist weak limits

$$p_-(t) = \lim_{N\to\infty} \sum_{i=1}^N - W'_-(Nx_i(t)) \quad \text{and} \quad p_+(t) = \lim_{N\to\infty} \sum_{i=1}^N W'_+(Nx_i(t) - N)$$

(the pressure at $\theta = 0$ and $\theta = 1$, respectively) and that for every smooth function J on $[0,1]$,

$$\int_0^1 \rho(t,\theta)J(\theta)d\theta - \alpha_o(J) = J'(0)\int_0^t p_-(s)\,ds - J'(1)\int_0^t p_+(s)ds$$

$$+\int_0^t \int_0^1 P(\rho(\theta,s))J''(\theta)d\theta ds.$$

This identity with J suitably taken and varied verifies that $p_-(t) = P(\rho(0,t))$ and $p_+(t) = P(\rho(1,t))$ a.s., where the version of the weak solution $\rho(\theta,t)$ should be properly chosen.

Remark 2. The initial energy bound (1.9), $\mathscr{E}^N(0) = o(N^3)$, is natural and indispensable for the conclusion of Theorem 1 to hold. If it is violated, then the so-called initial layer appears in general, i.e., the limiting density of α_t^N, even if it exists and satisfies the non-linear diffusion equation (1.8), can assume an initial trace $\lim_{t\downarrow 0} \rho(\theta,t)d\theta$ different from $\alpha_0 := \lim \alpha_0^N$. For example consider the initial phases $q_i(0) = i/2$; $p_i(0) = N/2$, $i = 1,\ldots,N$. Then within a macroscopically negligible time, e.g., $t = N$, each particle $q_i(t)$ travels a distance of about $N/2$, a macroscopically significant distance of $1/2$, which results in the initial discontinuity of the macroscopic mass distribution.

Even if (1.9) does not hold, α_t^N can still converge uniformly for $t \in [1/T, T]$ for each $T < \infty$ along suitable subsequences and any limit point may admit a density $\rho(\theta,t)$ that solves (1.8) (though it is then impossible to determine the initial trace from the given datum α_0 only), which is always the case if

$$U(x) \leq c\psi(x) \quad \text{for } |x| < 1 \quad \text{and} \quad \mathscr{E}^N(0) = O(N^\gamma)$$

with some constants c and $\gamma (\geq 3)$. To see this we observe that, in view of (2.2), (3.1), Lemma 3 and Theorem 2, the supplementary assumption on U made above

implies

$$\int_0^T \mathscr{E}^N(t)dt \leqq AN + B\frac{\mathscr{E}^N(0)}{N^2}$$

for some constants A and B. Since $\mathscr{E}^N(t)$ is decreasing in t, the substitution of $T = N^{-\lambda}\tau$ gives $\tau\mathscr{E}^N(N^{-\lambda}\tau) \leqq AN^{1+\lambda} + BN^{-2+\lambda}\mathscr{E}^N(0)$. Applying this inequality, repeatedly according to the magnitude of γ, we deduce from $\mathscr{E}^N(0) = O(N^\gamma)$ that

$$\mathscr{E}^N(N^{-\lambda}) = O(N^{1+\lambda}) \quad \text{for } 0 < \lambda < 2 . \tag{6.1}$$

Taking $\lambda = 1$, e.g., and applying Theorem 1.1 to the system (2.1) started at $t = N^{-1}$, we obtain the asserted convergence of α_t^N with a limiting density satisfying (1.8).

Remark 3. If the condition (1.4) fails to hold or equivalently $\int^\infty \psi(x)dx = \infty$, then the inner sum on the right side of (1.11) diverges to infinity in the hydrodynamic limit and we need to scale the time by a factor smaller than N^2 for obtaining a non-trivial limit. A heuristic argument suggests that instead of (1.7) we should define

$$x_i(t) = N^{-1}q_i((N^{-1}\log N)t) \quad \text{if } \psi(x) \sim x^{-1} \text{ as } x \to \infty,$$

$$x_i(t) = N^{-1}q_i(N^{-1+\gamma}t) \qquad \text{if } \psi(x) \sim x^{-1+\gamma} \text{ as } x \to \infty,$$

where $0 < \gamma < 1$: in the former case we would have Eq. (1.8) with $P(\rho) = \rho^2$ for the limiting density field, while in the latter case (1.8) must be replaced by

$$\frac{\partial}{\partial t}\rho(\theta, t) = \frac{\partial}{\partial\theta}\left[\rho(\theta, t)\int\frac{\rho(\theta+u, t) - \rho(\theta-u, t)}{u}|u|^{-\gamma}du\right] .$$

References

[A] Aronson, D.G.: The porous medium equation. In: Nonlinear Diffusion Problems. A. Fasano, M. Primicerio (eds), Vol. **1224**, Springer Lecture Notes in Math., Berlin, Heidelberg, New York: Springer, 1986

[BDS] Boldrighini, C., Dobrushin, R.N., Suhov, Yu.M.: One-dimensional hard rod caricature of hydrodynamics. J. Stat. Phys. **31**, 577–615 (1983)

[F] Fritz, J.: On the asymptotic behaviour of Spitzer's model for evolution of one-dimensional point systems. J. Stat. Phys. **38**, 615–645 (1985)

[GPV] Guo, M.Z., Papanicolaou, G.C., Varadhan, S.R.S.: Non-linear diffusion limit for a system of nearest neighbor interactions. Commun. Math. Phys. **118**, 31–59 (1988)

[L] Lang, R.: On the asymptotic behaviour of infinite gradient systems. Commun. Math. Phys. **65**, 129–149 (1979)

[M] Mürmann, M.G.: The hydrodynamic limit of a one-dimensional nearest neighbor gradient system. J. Stat. Phys. **48**, 769–788 (1987)

[OV] Olla, S., Varadhan, S.R.S.: Scaling limit for interacting Ornstein-Uhlenbeck processes. Commun. Math. Phys. **135**, 355–378 (1991)

[PS] Presutti, E., Scacciatelli, E.: The evolution of a one-dimensional point system: A note on Fritz's paper. J. Stat. Phys. **38**, 647–653 (1985)

[S] Sinai, Ya.G.: Dynamics of local equilibrium Gibbs distributions and Euler equations. The one-dimensional case. Selecta Math. Sov. **7**, 279–289 (1988)

[SU] Suzuki, Y., Uchiyama, K.: Hydrodynamic limit for a spin system on a multidimensional lattice. Probab. Theory Relat. Fields **95**, 47–74 (1993)

[U] Uchiyama, K.: Scaling limits of interacting diffusions with arbitrary initial distributions. Probab. Theory Relat. Fields **99**, 97–110 (1994)

[V] Varadhan, S.R.S.: Scaling limit for interacting diffusions. Commun. Math. Phys. **135**, 313–353 (1991)