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On Continuity of Bowen–Ruelle–Sinai Measures in Families of One Dimensional Maps

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Abstract: Let us consider a family of maps $Q_a(x) = ax(1-x)$ from the unit interval $[0,1]$ to itself, where $a \in [0,4]$ is the parameter. We show that, for any $\beta < 2$, there exists a subset $E \ni 4$ in $[0,4]$ with the properties

(1) $\mathrm{Leb}([4-\varepsilon, 4] - E) < \varepsilon^\beta$ for sufficiently small $\varepsilon > 0$,
(2) Q_a admits an absolutely continuous BRS measure μ_a when $a \in E$, and
(3) μ_a converges to the measure μ_4 as a tends to 4 on the set E.

Also we give some generalization of this results.

1. Introduction

We consider (real) one dimensional dynamical systems, that is, iterations of smooth maps f from a closed interval (or a circle) to itself. The orbit of a point x is a sequence of points

$$x, f(x), f^2(x), f^3(x), \dots .$$

In describing the distribution of the orbit, we use a sequence of probability measures

$$\mu_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}, \quad n = 1, 2, \dots ,$$

and, if this sequence converges to a probability measure μ as $n \to \infty$, we call μ the asymptotic distribution of the orbit of x. Here the convergence is that in the sense of weak topology, that is,

$$\int \varphi d\mu_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) \to \int \varphi d\mu \quad \text{as } n \to \infty$$

for every continuous function φ on the interval. So the statistical properties of the orbit are given by the asymptotic distribution μ, if it exists.

We call a probability measure μ on the interval the Bowen–Ruelle–Sinai measure for f if the asymptotic distribution of the orbit exists and equals μ for almost every point in the interval with respect to the Lebesgue measure.

The problem we shall consider in this article is how the BRS measure depends on the parameter in families of maps. To fix our idea, let us consider the quadratic family $Q_a(x) = ax(1-x) : [0,1] \to [0,1]$, where $a \in [0,4]$. There are two typical classes of dynamics in this family. The first one is the so-called hyperbolic systems, that is, the class of Q_a which has a hyperbolic attracting periodic orbit. In this case, the attracting periodic orbit is unique and the invariant probability measure on it is the BRS measure for Q_a. The second is the class of maps which admits an absolutely continuous invariant probability measure (acim). In this case the acim is unique and it is the BRS measure for Q_a[1]. For example, Q_4 admits an absolutely continuous BRS measure $\mu_4 = 1/(\pi\sqrt{x(1-x)})dx$. The set of parameters corresponding to hyperbolic systems are open and hence has positive Lebesgue measure. On the other hand, Jakobson's theorem [3] tells that the set of parameters corresponding to the systems with acim has positive Lebesgue measure. It remains unknown whether the union of these two subsets of parameters has full Lebesgue measure in the parameter space $[0,4]$ or not.

Let us denote, by μ_a, the BRS measure for Q_a if it exists. If Q_b is hyperbolic for some parameter b, the nearby systems in the family are also hyperbolic and the BRS measure μ_a depends on a continuously in a neighborhood of b, because the attracting periodic orbit for Q_b survives under small perturbations. However, when the case Q_b admits an acim, the situation is quite different. Let us consider the case $b = 4$ where $Q_b = Q_4$ admits an acim μ_4. Though some numerical experiments seem to show that the distributions of the orbits for Q_a converge to μ_4 as $a \to 4$, the dependence of the BRS measure μ_a on the parameter a is quite irregular. For example we have

Theorem 1.1. *There exists a subset $F \subset [0,4]$ of parameters with the properties:*

(1) $\mathrm{Leb}(F \cap [4-\varepsilon, 4]) > c\varepsilon^2$ *for some constant c,*

(2) *Q_a is hyperbolic when $a \in F$, and*

(3) *the BRS measure μ_a for Q_a converges to the Dirac measure at the point* 0 *when a approaches to* 4 *on the set F.*

The proof is simple. Let a_n be a (unique) parameter with the kneading data

$$Q_{a_n}(0) > 0, \ Q_{a_n}^j(0) < 0 \quad \text{for } j = 2,3,\dots,n-1, \quad \text{and} \quad Q_{a_n}^n(0) = 0\,.$$

Then we can see that $4 - a_n \sim 4^{-n}$ as n goes to infinity. Let F_n be the interval containing a_n on which the attracting orbit for Q_{a_n} survives. Easy calculations show that $|F_n| \sim 4^{-2n}$. So if we put $F = \cup F_n$, F satisfies the conditions in the theorem.

Remark 1.2. Similarly, we can show that there exists a subset F' of parameters satisfying the conditions (1) and (3) in the above theorem, with F replaced by F', and

(2′) Q_a admits absolutely continuous BRS measure μ_a when $a \in F'$.

In fact, the system Q_a with $a \in F_n$ is once renormalizable and thus there is an interval $F_n'' \supset F_n$ which consists of parameters such that Q_a admits the same type of renormalization. Let $F_n' \subset F_n''$ be the set of parameters $a \in F_n''$ such that Q_a admits an acim. Then applying Jakobson's theorem to the renormalized family, we can see that $\mathrm{Leb}(F_n')/\mathrm{Leb}(F_n'')$ is bounded away from 0 uniformly for n. $F'' = \cup F_n''$ is required.

Remark 1.3. See the work [2] of Hofbauer and Keller for more interesting results on the singular phenomena in families of unimodal maps. For example, they showed that there exists uncountable parameters for which Q_a has no BRS measure.

Now our result is

Theorem 1.4. *For any given $\beta < 2$, there exists a subset $E \ni 4$ of parameter space $[0, 4]$ with the properties*:

(1) $\mathrm{Leb}([4 - \varepsilon, 4] - E) < \varepsilon^\beta$ *for sufficiently small* $\varepsilon > 0$,
(2) Q_a *admits an absolutely continuous BRS measure* μ_a *when* $a \in E$, *and*
(3) μ_a *converges to the measure* μ_4 *as* a *tends to* 4 *on the set* E.

This result means that, though the BRS measure does not depend on the parameter continuously at $a = 4$, the set which gives the discontinuity such as F in Theorem 1.1 is relatively small. Actually this theorem follows from a more general result which we will state in the next section. We will see that similar results hold for many families and at many parameters.

The author learned from Hans Thunberg that the BRS measures for hyperbolic systems can behave regularly at most points in the parameter set constructed in the proof of Jakobson's theorem [6]. Also the author learned from him that M. Rychlik has already obtained a result similar to Theorem 1.4 [5, Proposition 1]. A result in [5] gives the continuity in $L^p (p \geqq 1)$ topology on the density functions of BRS measure, which is much finer than weak topology, and Theorem 1.4 follows from Rychlik's result except for the claim (1).

2. Main Result

Let f_0 be a C^2 map from $M = [0, 1]$ (or $M = S^1 := \mathbb{R}/\mathbb{Z}$) to itself and let $C(f_0)$ be the set of its critical points. We assume that f_0 satisfies the following four conditions:

(ND) All critical points of f_0 are non-degenerate.

(CE) There exist $\kappa_0 > 0$ and $r_0 > 0$ such that, for any critical point c and $n \geqq 0$.

(i) $|df_0^n(f_0(c))| > \exp(r_0 n - \kappa_0)$,

(ii) $|df_0^n(f_0(z))| > \exp(r_0 n - \kappa_0)$ for any $z \in f_0^{-n}(c)$.

(Hyp) All periodic points of f_0 are hyperbolic repelling.

(W) $\liminf_{n\to\infty} \frac{1}{n} \log |df_0(f_0^n(a))| = 0$ for any critical value a.

Let $F : M \times [0, 1] \to M$ be a one-parameter family of class C^2 with $F(x, 0) = f_0(x)$. We denote $f_t(\ \cdot\ ) = F(\ \cdot\ , t)$ for parameters $t \in [0, 1]$. Assume that the family F satisfies the condition

(NV) $\sum_{j=0}^{\infty} \dfrac{\frac{\partial}{\partial t} F(f_0^j(c), 0)}{df_0^j(f_0(c))} \neq 0$ for any critical point $c \in C(f_0)$.

The following is the main result in this paper.

Main Theorem. *For any given $\beta < 2$, there exists a subset $E \ni 0$ of parameter space with the properties*:

(1) $\mathrm{Leb}([0,\varepsilon] - E) < \varepsilon^{\beta}$ *for small* ε.

(2) *Each map f_t with $t \in E$ admits a finite number of ergodic acim's. For Lebesgue almost every point on M, the asymptotic distribution of the orbit exists and coincides with one of the ergodic acim's.*

(3) *If $t(p) \in E$, $p = 1, 2, \ldots$, approaches to 0 and if μ_p, an acim for $f_{t(p)}$, converges to a measure μ as $p \to \infty$, then μ is an acim for f_0.*

Since Q_4 has a unique acim μ_4, Theorem 1.4 follows from the main theorem. Moreover, for a subset of $a \in [0,4]$ with positive Lebesgue measure, the families $f_t := Q_{t-a}$ and $f_t := Q_{a-t}$ satisfy the assumptions of the Main Theorem and a result similar to Theorem 1.4 holds for such families. (See [7, Theorem 1].)

The support of each ergodic acim for f_0, which is a union of finite number of closed intervals [4], contains at least one critical point, because if otherwise, the return map on a component of the support would be monotone and have a nonrepelling periodic point. So, if f_0 in the Main Theorem has only one critical point, f_0 admits a unique acim and, hence, a result similar to Theorem 1.4 holds for the family in the Main Theorem.

If each critical point of f_0 is not eventually periodic in the Main Theorem, it is not difficult to see that the support of each ergodic acim for f_0 has an absorbing neighborhood. So, under this additional assumption, we can obtain a result similar to Theorem 1.4 by restricting ourselves to one of the absorbing neighborhoods.

In [7], we have proved that there exists a subset of parameters satisfying the conditions (1) and (2) in the Main Theorem. Here we shall prove that the additional condition (8) holds for the parameter set constructed in [7]. We shall summarize some results in [7] which we will use in this article. But before mentioning it, let me explain the idea of the proof. The idea of the proof is similar to that in [8].

Let μ_p be the sequence of measures in (3) of the Main Theorem. Then each of them satisfies the entropy formula

$$h_{\mu_p}(f_{t(p)}) = \int \log |df_{t(p)}(x)| d\mu_p \,.$$

See [4]. Also we can prove easily that

$$\limsup_{p \to \infty} h_{\mu_p}(f_{t(p)}) \leqq h_{\mu}(f_0)\,. \tag{2.1}$$

A proof of (2.1) will be given in the appendix at the end of this paper.

Suppose that we have proved

$$\liminf_{p \to \infty} \int \log |df_{t(p)}(x)| d\mu_p \geqq \int \log |df_0(x)| d\mu \,. \tag{2.2}$$

It follows

$$h_{\mu}(f_0) \geqq \int \log |df_0(x)| d\mu \,.$$

Since the inequality in the converse direction (Ruelle inequality) holds for any invariant probability measures, we obtain the entropy formula for μ,

$$h_{\mu}(f_0) = \int \log |df_0(x)| d\mu \,.$$

Since the (upper) Lyapunov exponent for f_0 is positive at all points but the preimages of critical points [7, Corollary 4.4], this implies that μ is an acim [4, Theorem 3].

Therefore it is sufficient to prove (2.2) in order to prove the claim (3) of the Main Theorem. Equation (2.2) is equivalent to the condition

$$\lim_{\delta\to 0} \liminf_{p\to\infty} \int_{C(f_{t(p)},\delta)} \log |df_{t(p)}(x)| d\mu_p = 0 , \tag{2.3}$$

where $C(f, \delta)$ is the open δ-neighborhood of the set $C(f)$ of critical points of f. This is what we shall prove in the next section.

In the next section, we need the following fact which follows from the argument in [7] immediately. (See Remark 2.1 below.)

Fact. *For any given $\beta < 2$, there exists a subset E of parameters satisfying* (1) *and* (2) *in the Main Theorem. In addition, there exist $\eta > 0$ and $\delta_0 > 0$ such that the following hold for f_t with $t \in E$:*

(A) *If $x \in C(f_t, \delta_0)$, there exists a positive integer $q = q(x,t) < -\eta^{-1} \log |df_t(x)|$ such that*

$$\log |df_t^q(x)| \geqq -2\eta \log |df_t(x)| + \eta q + \kappa_0 + 1 ,$$

where κ_0 is that in the condition (*CE*) *on f_0. Also we have*

$$d(f_t^i(x), C(f_t)) > d(f_t^m(x), C(f_t)) \quad \textit{for } m < i \leqq m + q$$

and

$$\sum_{i=1}^{q(x,t)} \frac{|df_t^i(x)|}{|df_t(f_t^i(x))|} < \frac{1}{|df_t(x)|} . \tag{2.4}$$

(B) *If $0 < \delta \leqq \delta_0$, $f_t^i(x) \notin C(f_t, \delta)$ for $i = 0, 1, 2, \ldots, n-1$ and $f_t^n(x) \in C(f_t, \delta)$, then*

$$\log |df_t^n(x)| \geqq \eta n - \kappa_0 - 1 .$$

Remark 2.1. Let me explain briefly how we can get the Fact from [7]. Note that E and η are not those in [7]. We put $E = \bigcup_{L > L_0} Z_L$, where Z_L is the set of parameters defined in [7] and let L_0 be sufficiently large. Then, from Proposition 6.1 and 6.2 in [7], the properties (1) and (2) in the Main Theorem hold for E (under an appropriate choice of constants). The first, second and third claim of (A) for small η follow from claims (c), (d) and (e) of [7, Proposition 7.2] and their proof respectively. Actually, the proof of [7, Proposition 7.2] is given only for x which belongs to the orbits of critical points but, clearly, it holds for every point sufficiently close to the critical points. The property (B) in the case $\delta = \delta_0$ follows from [7, Lemma 5.1 (2)] if δ_0 and η are sufficiently small. But once we have the property (B) for $\delta = \delta_0$, one can combine it with (A) and get (B) in the case $0 < \delta \leqq \delta_0$.

In the proof of the Main Theorem, we shall assume $M = S^1$. This does not violate the generality because, in the case $M = [0, 1]$, we can extend the maps f_t as maps from a circle to itself in an appropriate way. (See [7, Sect. 2].) By changing the parameter if necessary, we assume, as in [7], that f_t with $t \in [0, 1]$ has a constant number of non-degenerate critical points. Let κ_1 be a large constant such that

$$|df_t(x)| < \kappa_1 \quad \text{and} \quad \kappa_1^{-1} < |df_t(x)|/d(x, C(f_t)) < \kappa_1 \quad \text{for } x \in M \text{ and } t \in [0, 1] ,$$

where $d(x, C(f_t))$ denotes the distance from a point x to the set $C(f_t)$ of critical points.

3. Proof of the Main Theorem

Let η, δ_0 and E be those in the Fact in the previous section. For $t \in E$, $\delta > 0$, $h > 0$ and a positive integer n, we define

$$B_n(h, \delta, t) = \left\{ x \in M \;\middle|\; \sum_{i=0}^{n-1} \log |df(f_t^i(x))| \cdot \chi_{C(f_t,\delta)}(f_t^i(x)) < -hn \right\},$$

where $\chi_{C(f_t,\delta)}$ is the indicator function of the set $C(f_t, \delta)$. Note that we have $B_n(h, \delta, t) \subset B_n(h, \delta', t)$ if $\delta < \delta' < \kappa_1^{-1}$. We shall prove

Proposition 3.1. *For any given $h_0 > 0$, there exist positive numbers $\delta < \kappa_1^{-1}$, ε and M such that*

$$\mathrm{Leb}(B_n(h, \delta, t)) < M \exp(-\varepsilon h n)$$

for any $h \geqq h_0$, $n \geqq 1$ and $t \in E$.

We indicate first how (2.3) follows from Proposition 3.1. Suppose that t belongs to E and that ν is an ergodic acim for f_t. Let X be the set of points for which the distribution of the orbit exists and equals ν. Then X has positive Lebesgue measure. Let λ_0 be a measure defined by $\lambda_0(Y) = (\mathrm{Leb}(X))^{-1}\mathrm{Leb}(Y \cap X)$ for any measurable set Y, and put $\lambda_n = (1/n)\sum_{i=0}^{n-1} f^i(\lambda_0)$. For any given $h_0 > 0$, we can choose ε, δ and M as in Proposition 3.1 and obtain

$$\int \sum_{i=0}^{n-1} \log |df_t(f_t^i(x))| \cdot \chi_{C(f_t,\delta)}(f_t^i(x))\, d\lambda_0(x) = \int_{B_n(h_0,\delta,t)^c} + \int_{B_n(h_0,\delta,t)}$$

$$\geqq -h_0 n - (\mathrm{Leb}(X))^{-1} \int_{h_0 n}^{\infty} M \exp(-\varepsilon s)\, ds\,.$$

For sufficiently large n, we have

$$\int \sum_{i=0}^{n-1} \log |d_t(f_t^i(x))| \cdot \chi_{C(f_t,\delta)}(f_t^i(x))\, d\lambda_0(x) > -2h_0 n\,.$$

Since λ_n converges to ν as n goes to infinity, we have

$$\lim_{n\to\infty} \int (d/n) \sum_{i=0}^{n-1} \log |df(f_t^i(x))| \cdot \chi_{C(f_t,\delta)}(f_t^i(x))\, d\lambda_0(x)$$

$$= \lim_{n\to\infty} \int \log |df_t(x)| \cdot \chi_{C(f_t,\delta)}(x)\, d\lambda_n(x)$$

$$\leqq \int \log |df_t(x)| \cdot \chi_{C(f_t,\delta)}(x)\, d\nu(x)\,.$$

Therefore

$$0 \geqq \int \log |df_t(x)| \cdot \chi_{C(f_t,\delta)}(x)\, d\nu(x) \geqq -2h_0\,.$$

Since every acim is a convex combination of ergodic acim's and since we can take h_0 arbitrarily small, this implies (2.3).

Let us begin the proof of Proposition 3.1. For a C^1 map φ from an interval $[a,b]$ to $\mathbb{R}$, put

$$\mathrm{Dist}(\varphi,[a,b]) = \sup_{x,y\in[a,b]} \log \frac{|d\varphi(x)|}{|d\varphi(y)|} .$$

Then we have

Lemma 3.2 (*cf. Lemma* 3.1 *in* [7]). *There exists* $\kappa > 0$ *such that if we put*

$$a = a(x,n,t) = \left[\kappa \sum_{i=0}^{n-1} \frac{|df_t^i(x)|}{|df_t(f_t^i(x))|}\right]^{-1}$$

for $n \geqq 1, t \in [0,1]$ *and* $x \in M - \bigcup_{i=0}^{n-1} f_t^{-i}(C(f_t))$, *it holds*

$$\mathrm{Dist}(f_t^n,[x-a,x+a]) \leqq \sum_{i=0}^{n-1} \mathrm{Dist}(f_t, f_t^i([x-a,x+a])) < 1 .$$

If $x \in \bigcup_{i=0}^{n-1} f_t^{-i}(C(f_t))$, we regard $a(x,n,t) = 0$. Put $a(x,0,t) = \kappa_1^{-1}$ for $x \in M$ and $t \in [0,1]$.

Remark 3.3. From this lemma and the definition of $a(\,\cdot\,)$, we have $a(y,n,t) > e^{-1}\, a(x,n,t)$ for $y \in [x - a(x,n,t), x + a(x,n,t)]$.

Remark 3.4. The length of the interval $f_t^n([x-a(x,n,t),x+a(x,n,t)])$ is smaller than $2e|df_t^n(x)|a(x,n,t) \leqq 2e\kappa_1^2\kappa^{-1}$ from the lemma above. So, taking κ large, we assume that the interval $f_t^n([x-a(x,n,t),x+a(x,n,t)])$ is shorter than half of the smallest distance between the critical points of f_t for any $x \in [0,1], n \geqq 0$ and $t \in [0,1]$. (We will use this in the proof of Lemma 3.6.) Also we assume $a(x,n,t) \leqq 1$ by taking κ larger if necessary.

From now on, we consider the maps f_t with $t \in E$ and choose δ and other constants uniformly for $t \in E$. For simplicity of notation, we will write $f, q(x), a(x,n)$, $B_n(h,\delta)$ instead of $f_t, q(x,t), a(x,n,t), B_n(h,\delta,t)$.

The following is the main step of the proof of Proposition 3.1.

Lemma 3.5. *For any given* $K > 1$, *we can take* $\delta > 0$ (*uniformly for* $t \in E$) *so small that, for every point* $x \in B_n(h,\delta)$ *with* $n \geqq 1$ *and* $h > 0$, *there exist a sequence of positive integers*

$$0 \leqq n_1 < n_2 < \cdots < n_d < n$$

and points $z_i \in M$ *with* $f^{n_i}(z_i) \in C(f)$ *satisfying*

$$k_i := \left[\log \frac{a(z_i,n_i)}{|x-z_i|}\right] > K+1 \quad \textit{for } 1 \leqq i \leqq d$$

and

$$\sum_{i=1}^{d} (k_i - 1) > \varepsilon' hn ,$$

where $\varepsilon' = \{4(1+\eta^{-1})(1+\eta^{-1}\log\kappa_1)\}^{-1}\eta$.

First we show that this lemma proves Proposition 3.1. Let us put

$$J_{m,k} = \bigcup_{z \in f^{-m}(C(f))} [z - \exp(-k)a(z,m), z + \exp(-k)a(z,m)] \quad \text{for } m,k \geqq 0\,.$$

Then, from Remark 3.4, the right side is a disjoint union.

Lemma 3.6. *If* $0 \leqq n_1 \leqq n_2 < \cdots < n_d$ *and* $k_i > 1$ *for* $i = 1, 2, \ldots, d$, *it holds*

$$\mathrm{Leb}\left(\bigcap_{i=1}^{d} J_{n_i,k_i}\right) < 3\exp\left(-\sum_{i=1}^{d}(k_i - 1)\right)\,.$$

Proof. Let $1 \leqq i \leqq d-1$ and consider two points $w_1 \in f^{-n_i}(C(f))$ and $w_2 \in f^{-n_{i+1}}(C(f))$. Put $a_1 = a(w_1, n_i)$ and $a_2 = a(w_2, n_{i+1})$. From Lemma 3.2, w_1 is not contained in $[w_2 - a_2, w_2 + a_2]$. So, if $[w_2 - e^{-k_2}a_2, w_2 + e^{-k_2}a_2]$ has non-empty intersection with $[w_1 - e^{-k_1}a_1, w_1 + e^{-k_1}a_1]$, we can see that $[w_2 - a_2, w_2 + a_2]$ is contained in $[w_1 - e^{-k_1+1}a_1, w_1 + e^{-k_1+1}a_1]$.

Let us denote, by $J^*_{n_i,k_i}$, the union of connected components of J_{n_i,k_i} that intersect $\bigcap_{i=1}^{d} J_{n_i,k_i}$. Also we denote, by $J^*_{n_i,k}$, the union of connected components of $J_{n_i,k}$ that intersect $J^*_{n_i,k_i}$. From the above argument, $J^*_{n_{i+1},0} \subset J^*_{n_i,k_i-1}$. Hence we have

$$\mathrm{Leb}(J^*_{n_{i+1},k_{i+1}}) = e^{-k_{i+1}}\mathrm{Leb}(J^*_{n_{i+1},0}) \leqq e^{-k_{i+1}}\mathrm{Leb}(J^*_{n_i,k_i-1}) = e^{-k_{i+1}+1}\mathrm{Leb}(J^*_{n_i,k_i})\,.$$

Applying this for $i = d-1, d-2, \ldots$ in turn, we obtain the claim. □

Now, let δ be that taken in Lemma 3.5 for some $K > 0$. Then, taking the combinations of n_i and k_i into account, we get, from Lemma 3.5,

$$\lambda(B_n(h,\delta)) < \sum_{l > \varepsilon' hn}\sum_{d < l/K} C(n,d)C(l+d-1, d-1)\exp(-l)\,.$$

We can get Proposition 3.1 by using the approximation $\log C(p,q) = q(1 + \log(p/q)) + \mathcal{O}(1)$, which follows from $\log n! = n\log n - n + \mathcal{O}(1)$, and by taking K sufficiently large.

Let us prove Lemma 3.5. Let x be a point in $B_n(h,\delta)$. From the definition of $B_n(h,\delta)$, the set $\mathcal{N}_0 := \{0 \leqq m < n; f^m(x) \in C(f,\delta)\}$ is not empty. The point in the proof is to choose the numbers n_i with the required properties from $\mathcal{N}_0$. We will do this in two steps. First let us define a sequence $0 \leqq n'_1 < n'_2 < \cdots < n'_{d'} < n$ by

$$n'_1 = \text{minimum element of } \mathcal{N}_0\,,$$

$$n'_{i+1} = \text{minimum element of } (\mathcal{N}_0 \cap \{m \in \mathbb{N}; n_i + q(f^{n_i}(x)) < m < n\})$$

and

$$\mathcal{N}_0 \cap \{m \in \mathbb{N}; n'_{d'} + q(f^{n'_{d'}}(x)) < m < n\} = \emptyset\,.$$

Let us put $\mathcal{N}_1 = \{n'_1, n'_2, \ldots, n'_{d'}\}$, $I_i = [n'_i + 1, n'_i + q(f^{n'_i}(x))] \cap [0, n-1] \cap \mathbb{N}$ and $I = \bigcup_i I_i$. Let us denote $Z(m) = -\log|df(f^m(x))|$.

Claim 1. $\sum_{i=1}^{d'} Z(n'_i) > \dfrac{hn}{1 + \eta^{-1}\log\kappa_1}$.

Proof. Since $|df^q(f^{n_i'}(x))| > 1$ for $q = q(f^{n_i'}(x)) < -\eta^{-1}\log|df(f^{n_i'}(x))|$ from (A) of the Fact, it holds

$$0 < \log|df^q(f^{n_i'}(x))| = \sum_{j=n_i'}^{n_i'+q+1} \log|df(f^j(x))| < -\sum_{m\in\mathcal{N}_0\cap I_i} Z(m) + q\log\kappa_1 .$$

This implies

$$\sum_{m\in\mathcal{N}_0\cap I_i} Z(m) < (\eta^{-1}\log\kappa_1)Z(n_i') .$$

Since $\mathcal{N}_0 = \mathcal{N}_1\cup(I\cap\mathcal{N}_0)$, we obtain the claim from the definition of $B_n(h,\delta)$. □

For $0 \leqq m < n$, let us put

$$X(m) = -\log|df(f^m(x))| + \log|df^m(x)| - \eta m .$$

Let $\mathcal{N}_2 = \{0 \leqq n_1 < n_2 < \cdots < n_d\}$ be the set of $m\in\mathcal{N}_1$ satisfying

$$X(m) - \eta Z(m) > \max_{\substack{j\in\mathcal{N}_1\\ j<m}} X(j) .$$

Then we have

Claim 2. $\sum_{m\in\mathcal{N}_2} Z(m) > \dfrac{hn}{(1+\eta^{-1})(1+\eta^{-1}\log\kappa_1)}$.

Proof. Let $l_1 < l_2$ be two adjacent elements in $\mathcal{N}_2\cup\{n\}$ and let $l_1 < m_1 < \cdots < m_r < l_2$ be the elements of $\mathcal{N}_1$ between them. Then since $m_i \notin \mathcal{N}_2$, we have

$$X(m_r) \leqq \eta Z(m_r) + \max_{\substack{j\in\mathcal{N}_1\\ j<m_r}} X(j) \leqq \eta\sum_{i=1}^{r} Z(m_i) + X(l_1) .$$

On the other hand we have

$$\begin{aligned} X(m_r) - X(l_1) &= \log|df^{m_r-l_1}(f^{l_1}(x))| + Z(m_r) - Z(l_1) - \eta(m_r - l_1) \\ &\geqq 2\eta\sum_{i=1}^{r-1} Z(m_i) + Z(m_r) - Z(l_1) \end{aligned}$$

because

$$\log|df^{m_r-l_1}(f^{l_1}(x))| \geqq \sum_{i=1}^{r-1} 2\eta Z(m_i) + \eta(m_r - l_1)$$

from (A) and (B) in the Fact. Therefore we obtain

$$Z(l_1) \geqq \eta\sum_{i=1}^{r} Z(m_i) .$$

This and Claim 1 prove Claim 2. □

Claim 3. *There exists a constant C (which does not depend on the choice of δ) such that, for each $m\in\mathcal{N}_2$, it holds*

$$\frac{|df(f^m(x))|}{a(x,m)|df^m(x)|} < C\exp(-\eta Z(m)) .$$

Proof. The left side is written as

$$\sum_{i<m} \exp(X(i) - X(m) - \eta(m-i)) .$$

We estimate this sum. Consider the following cases for $0 \leqq i < m$:

(1) $i \in \mathcal{N}_1$
(2) $i \in I$,
(3) otherwise.

Let us denote, by $\Sigma_1, \Sigma_2, \Sigma_3$, the sum of $\exp(X(i) - X(m) - \eta(m-i))$ over $0 \leqq i < m$ satisfying (1), (2), (3) respectively.

In case (1), we have

$$X(i) - X(m) < -\eta Z(m) .$$

Hence

$$\Sigma_1 < C_1 \exp(-\eta Z(m))$$

for some constant C_1 which depends only on η. From (2.4), we can see that $\Sigma_2 < \Sigma_1$. Consider case (3). Note that $f^i(x) \notin C(f, \delta)$ and $|df(f^i(x))| > \kappa_1^{-2}|df(f^m(x))|$ in this case. If $f^i(x) \in C(f, \delta_0)$, we have, from (A) and (B) of the Fact,

$$\begin{aligned} X(i) - X(m) &< -\log|df(f^i(x))| + \log|df(f^m(x))| + 2\eta \log|df(f^i(x))| \\ &< 2\log\kappa_1 + 2\eta\log|df(f^m(x))| . \end{aligned}$$

If $f^i(x) \notin C(f, \delta_0)$, we have

$$X(i) - X(m) < -\log(\kappa_1^{-1}\delta_0) + \log|df(f^m(x))| - \kappa_0 - 1 < 2\eta\log|df(f^m(x))| ,$$

provided δ is much smaller than δ_0. So we have

$$\Sigma_3 < C_2|df(f^m(x))|^{2\eta} = C_2\exp(-2\eta Z(m))$$

for some constant C_2 which does not depend on the choice of δ, provided that δ is sufficiently small. From all these, we obtain Claim 4. □

Now we finish the proof of Lemma 3.5. From Claim 4, we have

$$\frac{d(f^{n_i}(x), C(f))}{a(x, n_i)|df^{n_i}(x)|} < C'\exp(-\eta Z(m)) \quad \text{for each } 1 \leqq i \leqq d ,$$

where C' is a constant which does not depend on the choice of δ. If the right side is smaller than e^{-1}, we can find a point $z_i \in f^{-n_i}(C(f))$ such that

$$|x - z_i| < eC'\exp(-\eta Z(n_i))a(x, n_i)$$

from Lemma 3.2.

We have $[\eta Z(n_i)/2] > K$ for given K if we take δ sufficiently small. From Claim 5 and the inequality above, we have

$$\sum_{i=1}^{d} [\eta Z(n_i)/2] > \frac{\eta h n}{4(1+\eta^{-1})(1+\eta^{-1}\log\kappa_1)}$$

and

$$|x - z_i| < \exp(-[\eta Z(n_i)/2])a(z_i, n_i) \quad (cf. \text{ Remark } 3.3)$$

when δ is sufficiently small. These show Lemma 3.5.

Appendix: Proof of the Inequality (2.1)

Let ζ_p be the partition of M by the critical points of $f_{t(p)}$. Then on each element of $\bigvee_{i=0}^{m-1} f_{t(p)}^{-i}\zeta_p$, $f_{t(p)}^m$ is monotone. Let ζ' be any partition of M into intervals. Then the partition $\bigvee_{i=0}^{m-1} f_{t(p)}^{-i}(\zeta_p \bigvee \zeta')$ divides each element of $\bigvee_{i=0}^{m-1} f_{t(p)}^{-i}\zeta_p$ into at most Nm parts, where N is the number of the elements of ζ'. So we obtain $h_{\mu_p}(f_{t(p)}, \zeta_p) \geqq h_{\mu_p}(f_{t(p)}, \zeta')$ and $h_{\mu_p}(f_{t(p)}) = h_{\mu_p}(f_{t(p)}, \zeta_p)$. Similarly we get $h_\mu(f_0) = h_\mu(f_0, \zeta_0)$, where ζ_0 is the partition of M by the critical points of f_0. Since μ_p converges to μ, we have

$$\lim_{p\to\infty} H_{\mu_p}\left(\bigvee_{i=0}^{m-1} f_{t(p)}^{-i}\zeta_p\right) = H_\mu\left(\bigvee_{i=0}^{m-1} f^{-i}\zeta_0\right)$$

for each m. Therefore we obtain (2.1) from the definition of the metric entropy: $h_\mu(f, \zeta) = \inf_m m^{-1} H_\mu(\bigvee_{i=0}^{m-1} f^{-i}\zeta)$.

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