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Period Doubling for $C^{2+\epsilon}$ Mappings

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Abstract: The rigorous treatment of period-doubling cascades, developed by Lanford for analytic families, is extended to cover $C^{2+\alpha}$ families for any $\alpha > 0$. This requires spectral analysis of the linearised doubling operator on spaces of differentiable mappings, and a version of stable manifold theory which takes account of the non-differentiability of the doubling operator on these spaces.

1. Introduction

The theory of cascades of period-doubling bifurcations for one-parameter families of one-dimensional mappings has been developed by Feigenbaum [1, 2] and rigorously justified by Lanford [4] for analytic families. The purpose of the present paper is to extend this theory to cover $C^{2+\alpha}$ families for any $\alpha > 0$.

The Feigenbaum–Lanford theory is based on the study of a non-linear "doubling operator" T on a suitable space of analytic functions (see Sect. 2 for more details). The main ingredients of the theory are the existence of a fixed point g of T, analysis of the spectrum of the Fréchet derivative of T at g, and the use of stable manifold theory to deduce information about the behaviour of T near g. We study the behaviour of T on a larger function space consisting of C^γ functions, where $\gamma = 2 + \alpha$. The two main problems are to extend the spectral theory of the derivative $DT(g)$ to this larger space, and to extend the stable manifold theory.

The analysis of the spectrum of $DT(g)$ on the larger space, given the analytic theory, can be reduced to the determination of its essential spectrum (i.e. the spectrum modulo compact operators). In some ways this is the most natural approach. However we have chosen a different method, involving the introduction of localised norms, which, although not any simpler, is somewhat more elementary in that it avoids the theory of compact perturbations, and may be more suitable for extending the results to other renormalisation problems. The details are given in Sects. 3 and 4.

The study of the nonlinear map T and the existence of the stable manifold are complicated by the fact that T is not differentiable on the larger space – even the derivative $DT(g)$ at the fixed point is not a Fréchet derivative (although it is a Gateau derivative). The resulting technical problems are dealt with in Sects. 5 and 6.

It seems reasonable to expect that the $C^{2+\alpha}$ theory could be developed directly, rather than via the analytic theory as in the present work. Indeed, recent developments due to Sullivan (of which an account can be found in [5]; see also [3]) may well achieve this. However, the methods developed here are fairly general and it is hoped that they may be applicable to other renormalisation problems for which a direct approach might not be available.

While this work was being written up, the author became aware work on the same problem by Lanford, who also considered period n-tupling. The author is grateful to Prof. Lanford for providing details of his work and for valuable discussions.

2. Analytic Theory

Lanford [4] established the following result: consider the Banach space A_0 of functions of the form

$$f(x) = x^2 \sum (\alpha_n + \beta_n x)(x^2 - 1)^n, \quad \sum |\alpha_n| + |\beta_n| < \infty ,$$

and let A_1 be the set of functions of the form $1 + f$, $f \in A_0$. Then there is an even function $g \in A_1$ such that g is monotone on $[0, 1]$, $g''(0) < 0$ and

$$g(x) = \lambda^{-1} g(g(\lambda x)) , \tag{1}$$

where λ is a constant ($\lambda = -0.3995...$). The operator T given by $Tf(x) = f(1)^{-1} f(f(f(1)x))$ is defined and infinitely differentiable (in the Fréchet sense) in a neighbourhood U of g in A_1, and T maps U into A_1. By (1), $g(1) = \lambda$ and g is a fixed point of T.

If $f \in U$ then the Fréchet derivative $DT(f)$ is a bounded linear operator on A_0, given by

$$\begin{aligned} DT(f)\phi(x) &= f(1)^{-1}\{\phi(f(f(1)x)) + f'(f(f(1)x))\phi(f(1)x)\} \\ &\quad + \phi(1)\{-f(1)^{-2} f(f(f(1)x)) + x f(1)^{-1} f'(f(f(1)x)) f'(f(1)x)\} . \end{aligned}$$

The Fréchet derivative at the fixed point $DT(g)$ has a simple eigenvalue $\delta = 4.669\ldots$, with corresponding eigenvector h, and the rest of its spectrum is contained in the unit disc $\{\lambda\colon |\lambda| < 1\}$. The spectral projection P corresponding to the eigenvalue λ can be written $P\phi = \sigma(\phi)h$, where σ is a bounded linear functional on A_0.

From stable manifold theory there is a local one-dimensional unstable manifold at g for T, which can be parametrised by a real-analytic mapping $t \to h_t \in A$, defined for $t \in \mathbf{R}, |t|$ small, such that $h_0 = g$ and $Th_t = h_{\delta t}$ for $|t|$ small. The family h_t undergoes a cascade of period-doubling bifurcations. We can suppose the parametrisation is chosen so that the first period-doubling occurs at $t = 1$; then the n^{th} bifurcation occurs at $t = \delta^{-n}$.

3. Hölder Spaces

Let $I = [a, b]$ be a closed interval in $\mathbf{R}$. We denote by $C^0(I)$ the Banach space of real-valued continuous functions on I with the supremum norm $\|f\|_0 = \sup_I |f(x)|$. If $0 < \alpha < 1$, then we denote by $C^\alpha(I)$ the space of real-valued continuous functions on I satisfying a Hölder condition with exponent α; $C^\alpha(I)$ is a Banach space with norm

$$\|f\|_\alpha = \max\left\{\|f\|_0,\ \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}\right\}$$

for $\alpha > 0$, where $\|f\|_0$ denotes the supremum norm. We also define, for $\delta > 0$, an associated semi-norm

$$\|f\|_{\alpha,\delta} = \sup_{0 < |x-y| < \delta} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Now suppose $0 < \alpha < \beta < 1$ and let $\phi_i \in C^\beta(I)$ and $\psi_i \in C^1(I)$ for $i = 1, \ldots, n$ and suppose $\psi_i(I) \subseteq I$. Define a bounded linear operator S on $C^\alpha(I)$ by

$$Sf(x) = \sum_{i=1}^{n} \phi_i(x) f(\psi_i(x)). \tag{2}$$

Let

$$M = \sup_{x \in I} \sum_{i=1}^{n} |\phi_i(x)||\psi_i'(x)|^\alpha.$$

Then we have:

Lemma 1. (i) *Let $\varepsilon > 0$. Then we can find $\delta > 0$ such that*

$$\|Sf\|_{\alpha,\delta} \leqq (M + \varepsilon)\|f\|_\alpha$$

for all $f \in C^\alpha(I)$.

(ii) *Let $\varepsilon > 0$. Then we can find $\eta > 0$ such that if $\tilde{\phi}_i \in C^\beta(I)$ and $\tilde{\psi}_i \in C^1(I)$ with $\|\psi_i - \tilde{\psi}_i\|_1 < \eta$, $\|\phi_i - \tilde{\phi}_i\|_\beta < \eta$ and $\tilde{\psi}_i(I) \subseteq I$ for $i = 1, \ldots, n$, then the corresponding operator $\tilde{S}$ on $C^\alpha(I)$ satisfies $\|\tilde{S} - S\| < 2(M + \varepsilon)$.*

Proof. If $x, y \in I$ with $|x - y| < \delta$, then

$$Sf(x) - Sf(y) = \sum_{i=1}^{n} \phi_i(x)[f(\psi_i(x)) - f(\psi_i(y))] + \sum_{i=1}^{n} [\phi_i(x) - \phi_i(y)] f(\psi_i(y)).$$

The second term is bounded by

$$\|f\|_0 |x - y|^\beta \sum \|\phi_i\|_\beta \leqq \|f\|_0 \delta^{\beta-\alpha} |x - y|^\alpha \sum \|\phi_i\|_\beta$$

and, using the mean-value theorem, the first term is bounded by

$$\sum |\phi_i(x)||\psi_i'(x_i)|\|f\|_\alpha |x - y|^\alpha,$$

where $x \leqq x_i \leqq y$ for $i = 1, \ldots, n$. Hence it suffices to choose δ so that $\delta^{\beta-\alpha} \sum \|\phi_i\|_\beta < \varepsilon/2$ and so that $|x_i - x| < \delta$ implies $\sum |\phi_i(x)||\psi_i'(x_i)| < M + \varepsilon/2$.

(ii) From the last sentence of the proof of (i) it follows that if η is small enough then the conclusion (i) applies to $\tilde{S}$ with the same choice of δ. Hence

$$\|Sf - \tilde{S}f\|_{\alpha,\delta} < 2(M + \varepsilon)\|f\|_\alpha.$$

Moreover, if η is small enough we have $\|Sf - \tilde{S}f\|_0 \leqq \delta\varepsilon\|f\|_\alpha$ and applying the inequality

$$\frac{|g(x) - g(y)|}{|x - y|^\alpha} \geqq 2\delta^{-1}\|g\|_0, \quad |x - y| \geqq \delta$$

with $g = Sf - \tilde{S}f$, the desired result follows. □

If k is a positive integer and $0 < \alpha < 1$, we let $C^{k+\alpha}$ denote the Banach space of real functions on I having k^{th} derivative in C^α, with norm

$$\|f\|_{k+\alpha} = \max\{\|f\|_0, \|f^{(k)}\|_\alpha\}.$$

Lemma 2. *Let $\gamma = k + \alpha$, where k and α are as above. Suppose $\phi_i \in C^{\gamma_1}(I)$ and $\psi_i \in C^{\gamma_2}(I)$ for $i = 1, \ldots, n$, where $\gamma_1, \gamma_2 > \gamma$ and $\gamma_2 \geqq 1$. Let*

$$M = \sup_{x \in I} \sum_{i=1}^{n} |\phi_i(x)||\psi_i'(x)|^\gamma.$$

Then for any $\varepsilon > 0$ one can find $\delta > 0$ such that

$$\|(Sf)^{(k)}\|_{\alpha,\delta} \leqq (M + \varepsilon)\|f\|_\gamma.$$

Proof. We can write

$$(Sf)^{(k)} = \sum \phi_i(x)\psi_i'(x)^k f^{(k)}(\psi_i(x)) + Rf,$$

where Rf involves derivatives of f of order less than k. The result then follows readily from Lemma 1(i). □

Lemma 3. *Let S be as in Lemma 2. Define an operator S_γ on $C(I)$ by*

$$S_\gamma f(x) = \sum_{i=1}^{n} |\phi_i(x)||\psi_i'(x)|^\gamma f(x).$$

Suppose $\rho > 0$ and that the spectral radius of S_γ is less than ρ. Then for any $\varepsilon > 0$ we can find a positive integer m and $\delta > 0$ so that

$$\|S^m f\|_{\gamma,\delta} \leqq \varepsilon\rho^m\|f\|_\gamma, \quad f \in C^\gamma(I).$$

Proof. Choose m so that the norm of S_γ^m as an operator on $C(I)$ (with the supremum norm) is less than $\varepsilon\rho^m/3$. We can write S^m in the form $S^m f(x) = \sum_{i=1}^{r} \theta_i(x) f(\sigma_i(x))$, where $r = n^m$, $\theta_i, \sigma_i \in C^\beta(I)$ for some $\beta > \gamma$ and $\sigma_i(I) \subseteq I$. Applying Lemma 2 to S^m and noting that

$$S_\gamma^m f(x) = \sum_{i=1}^{r} |\theta_i(x)||\sigma_i'(x)|^\gamma$$

gives the desired result. □

The following lemma relates the C^γ norm to the associated semi-norm.

Lemma 4. *Let $\gamma > 0$. Then for all $f \in C^\gamma$ we have*

$$\|f\|_\gamma \leqq 4\max\left\{\|f\|_{\gamma,\eta}, \frac{2k!}{\eta^\gamma}\|f\|_0\right\}.$$

Proof. Suppose the R.H.S. $\leqq 4$. Then we claim that $\|f^{(k)}\|_0 \leqq 2\eta^\alpha$. To prove this, suppose the contrary. Let J be any subinterval of length η. Then since $\|f\|_{\gamma,\eta} \leqq 1$, we see that $|f^{(k)}| > \eta^\alpha$ on J. Then we have $\sup_J |f| > \eta^\gamma/(2k!)$, which contradicts the assumed bound on $\|f\|_0$. It then follows that $\|f\|_\gamma \leqq 4$ as required. □

We conclude this section with an elementary interpolation result which will be needed later.

Lemma 5. *Let* $I = [-1, 1]$ *and let* $f \in C^{1+\alpha}(I)$ *where* $0 < \alpha < 1$. *Then*

$$\|f'\|_0 \leqq 4\|f\|_{1+\alpha}^{1/(1+\alpha)}\|f\|_0^{\alpha/(1+\alpha)}.$$

Proof. Let $M = \|f\|_{1+\alpha}$ and define $\eta > 0$ by $\eta^{1+\alpha} = 2\|f\|_0/M$. Then $\eta > 2$. Suppose there is $x \in I$ such that $|f'(x)| > 2M\eta^\alpha$. Then we can find an interval $J \subseteq I$ containing x and with length η. By the definition of M we must then have $|f'(y)| > M\eta^\alpha$ for $y \in J$ and so $|\int_J f'| > M\eta^{1+\alpha}$. This implies that $\|f\|_0 > M\eta^{1+\alpha}/2$, contradicting the definition of η. Hence $\|f'\|_0 \leqq 2M\eta^\alpha$, and the result follows using the definitions of M and η. □

4. Spectral Estimates for the Linearised Doubling Operator

We now extend to $C^{2+\alpha}$, where $0 < \alpha < 1$, the Feigenbaum–Lanford results on the spectrum of the derivative at the fixed point g of the doubling operator T. This is done by using Lemma 3 to reduce the problem to the analytic case considered in Sect. 1.

Let $I = [-1, 1]$. For $\gamma \geqq 2$, let C_0^γ be the set of functions f in $C^\gamma(I)$ satisfying $f(0) = f'(0) = 0$, and let C_1^γ be the set of functions of the form $1 + f$, where $f \in C_0^\gamma$. Then if f is close to g in C_1^γ, it follows that f maps I into itself and we can define $Tf \in C_1^\gamma$ by $Tf(x) = f(1)^{-1}f(f(f(1)x))$. Let S be the formal derivative of T at g:

$$S\phi(x) = \lambda^{-1}[\phi(g(\lambda x)) + g'(g(\lambda x))\phi(\lambda x) + \phi(1)\{-g(x) + xg'(x)\}].$$

Then S is a bounded linear operator on C_0^γ. We also define, for $\gamma > 0$, a bounded linear operator S_γ on $C(I)$ by

$$S_\gamma\phi(x) = |\lambda|^{\gamma-1}[|g'(\lambda x)|^\lambda \phi(g(\lambda x)) + |g'(g(\lambda x))|\phi(\lambda x)].$$

To apply Lemma 3 we require an estimate for the spectral radius of S_γ, which is given by the following lemma.

Lemma 6. *If* $\gamma > 1$ *then* $\rho(S_\gamma) \leqq |\lambda|^{\gamma-2}$.

Proof. We use the following facts about g which follow, for example, from Lanford's polynomial approximation: $g''(x) < 0$ for $x \in I$ and $g'(g(\lambda)) < -1$. Then differentiating the functional equation (1) and letting $x \to 0$ gives $g'(1) = \lambda^{-1}$. Since g' is monotone we deduce that $1 \leqq |g'(g(\lambda x))| \leqq |\lambda|^{-1}$.

Now choose $\eta > 0$ so that $2 + \eta < |\lambda|^{-1}$ and let $M(x) = \eta + |g'(x)|^\gamma$. We shall show that

$$|\phi(x)| \leqq M(x), \quad x \in I \Rightarrow |S_\gamma\phi(x)| \leqq |\lambda|^{\gamma-2}M(x), \ x \in I. \qquad (3)$$

Assuming (3) we have, for any $\phi \in C(I)$, $|\phi(x)| \leqq \eta^{-1}\|\phi\|_0 M(x)$ which by (3) implies

$$\|S_\gamma^m \phi\|_0 \leqq \eta^{-1}\lambda^{m(\gamma-2)}\|\phi\|_0 \sup M(x)\,,$$

so $\|S_\gamma^m\|_0 \leqq \mathrm{const}\,\lambda^{m(\gamma-2)}$ for any positive integer m, whence $\rho(S_\gamma) \leqq \lambda^{\gamma-2}$.

It remains to prove (3). Suppose $|\phi(x)| \leqq M(x)$, $x \in I$. Then

$$\begin{aligned}|S_\gamma\phi(x)| \leqq S_\gamma M(x) &= |\lambda|^{\gamma-1}[|g'(x)|^\gamma + |g'(g(\lambda x))|^{1-\gamma}|g'(x)|^\gamma \\ &\quad + \eta[|g'(\lambda x)|^\gamma + |g'(g(\lambda x))|)] \\ &\leqq |\lambda|^{\gamma-1}[(2+\eta)|g'(x)|^\gamma + \eta|\lambda|^{-1}] \leqq |\lambda|^{\gamma-2}M(x)\end{aligned}$$

as required; the second last inequality used the estimate $1 \leqq |g'(g(\lambda x))| \leqq |\lambda|^{-1}$. □

An alternative proof of Lemma 6, which applies more generally to n-tupling operators, has been found by O. Lanford (personal communication).

The next lemma gives the spectral estimates which we require. The proof is based on Lemmas 3 and 6.

Lemma 7. *Suppose $\gamma > 0$ and $\rho(S_\gamma) < \rho$, where $1 \leqq \rho < \delta$. Then the linear functional σ on A_0 extends continuously to C_0^γ. Moreover, given $\varepsilon > 0$, one can find a positive integer m such that, for all $f \in C_0^\gamma$,*

$$\|S^m f - \delta^m \sigma(f)h\|_\gamma \leqq \varepsilon\rho^m\|f\|_\gamma\,.$$

Proof. By Lemma 3 we can find a positive integer r and $\eta > 0$ so that $\|S^r f\|_{\gamma,\eta} < \rho^r\|f\|_\gamma/16$ for all $f \in C_0^\gamma$. We may also suppose that Q^r has spectral radius $\leqq 1/2$, regarded as an operator on A_0. Now, using the compactness of the unit ball of C_0^γ in $C(I)$ and the density of A_0 in C_0^γ, we can find a finite set $f_1,\ldots,f_N$ of functions in A_0, with $\|f_i\|_\gamma < 1$, such that, for any $f \in C_0^\gamma$ with $\|f\|_\gamma \leqq 1$, there is an i such that $\|f - f_i\|_0 < \eta^\gamma/(16Mk!)$, where M is the norm of S^r as an operator on $C(I)$. It then follows, using Lemma 4, that $\|S^r(f - f_i)\|_\gamma \leqq \rho^r/2$. In other words, given $f \in C_0^\gamma$ with $\|f\|_\gamma \leqq 1$ we can write $S^r f = S^r f_i + \rho^r\psi/2$, where $\psi \in C_0^\gamma$ with $\|\psi\|_\gamma \leqq 1$. We can then apply S^r again and treat $S^r\psi$ in the same way. After finitely many steps we obtain

$$S^{kr}f = S^{kr}f_{i_1} + (\rho^r/2)S^{(k-1)r}f_{i_2} + \cdots + (\rho^r/2)^{k-1}S^r f_{i_k} + (\rho^r/2)^k\psi_k\,,$$

where $\|\psi_k\|_\gamma \leqq 1$. We can rewrite this as

$$\begin{aligned}S^{kr}f =& \delta^{kr}\left\{\sigma(f_{i_1} + (\rho^r/2\delta^r)\sigma(f_{i_2}) + \cdots + (\rho^r/2\delta^r)^{k-1}\sigma(f_{i_k})\right\}h \\ &+ Q^{kr}f_{i_1} + (\rho^r/2)Q^{(k-1)r}f_{i_2} + \cdots + (\rho^r/2)^{k-1}Q^r f_{i_k} + (\rho^r/2)^k\psi_k\,.\end{aligned}$$

Now $\|Q^{kr}f_i\|_\gamma \leqq C2^{-k}$, $i = 1,\ldots,N$, since $f_i \in A_0$. Then we see that, as $k \to \infty$, $\delta^{-kr}S^{kr}f$ tends to a limit which we can write as $\sigma(f)h$, thereby defining σ as a bounded linear functional on C_0^γ. The required estimate then follows by taking $m = kr$ for k large enough. □

The conclusion of Lemma 7 implies that, apart from the simple eigenvalue δ, the spectrum of S as an operator on C_0^γ lies strictly inside the disc with centre 0 and radius ρ. Lemma 6 shows that, if $\gamma > 2$, we can take $\rho = 1$.

5. Behaviour of T Near g

We now apply the preceding linearised theory to study the non-linear mapping T on C_1^γ, where $\gamma = 2 + \alpha$, $0 < \alpha < 1$.

Lemma 8. *We can find a positive integer m and $\eta > 0$ so that if $t \in \mathbf{R}$ and $\phi \in C_0^\gamma$ with $|t| < \eta$ and $\|\phi\|_\gamma < \eta$, then*

$$\|T^m(h_t + \phi) - h_s\|_\gamma < \frac{1}{2}\|\phi\|_\gamma ,$$

where $s = \delta^m(t + \sigma(\phi))$.

Proof. Using Lemmas 6 and 7 we can choose m so that Q^m has norm $< 1/8$ as an operator on C_0^γ and S_γ^m has norm $< 1/16$ as an operator on C^0. Then for f near g in C_1^γ we can define a bounded linear operator $A(f)$ on C^α by

$$A(f)\psi(x) = f^r(0)\sum_{k=1}^{r} \left\{\prod_{s=k}^{r-1} f'(f^s(f^r(0)x))\prod_{s=0}^{k-2} f'(f^s(f^r(0)x))^2\right\}\psi(f^{k-1}(f^r(0)x)) ,$$

where $r = 2^m$. If also $f \in C_1^{1+\gamma}$ and $\phi \in C_0^\gamma$ we have

$$(DT^m(f)\phi)'' = A(f)\phi'' + B(f)\phi' + D(f)\phi ,$$

where $B(f)$ and $D(f)$ are linear operators of the form (2) (with coefficients depending nonlinearly on f and its first 3 derivatives). We then find, taking $f = h_t$, that

$$(T^m(h_t + \phi) - T^m h_t)'' = A(h_t + \phi)\phi'' + B(h_t)\phi' + D(h_t)\phi + E(h_t, \phi) ,$$

where the last term $E(h_t, \phi)$ is of second order in ϕ, and for t and $\|\phi\|_\gamma$ small has C^α norm bounded by $C\|\phi\|_\gamma^2$. Moreover, provided $|t|$ is small enough, $\|(B(h_t) - B(g))\phi'\|_\alpha$ and $\|(D(h_t) - D(g))\phi\|_\alpha$ will both be less than $\frac{1}{8}\|\phi\|_\gamma$. Next, we apply Lemma 1(ii) to $A(g)$; the corresponding M is $\|S_\gamma^m\| < 1/16$. We conclude that if $|t|$ and $\|\phi\|_\gamma$ are small enough then $A(h_t + \phi) - A(g)$ has norm less than 1/8 as an operator on C^α.

Putting everything together, and noting that

$$(S^m\phi)'' = A(g)\phi'' + B(g)\phi' + D(g)\phi ,$$

we see that

$$\|T^m(h_t + \phi) - T^m h_t - S^m\phi\|_\gamma < \frac{1}{4}\|\phi\|_\gamma ,$$

provided $|t|$ and $\|\phi\|_\gamma$ are small enough. Since also

$$\|S^m\phi - \delta^m\sigma(\phi)h\|_\gamma < \|Q^m\phi\|_\gamma < \frac{1}{8}\|\phi\|_\gamma ,$$

we deduce that

$$\|T^m(h_t + \phi) - T^m h_t - \delta^m\sigma(\phi)h\|_\gamma < \frac{3}{8}\|\phi\|_\gamma .$$

Now let $\tau = \delta^m t$ and with $s = \delta^m(t + \sigma(\phi))$ as in the statement of the theorem we see that, for $|t|$ and $\|\phi\|_\gamma$ small enough,

$$\|h_s - h_\tau - \delta^m \sigma(\phi) h\|_\gamma < C\sigma(\phi)^2 < \frac{1}{8}\|\phi\|_\gamma ,$$

and since $T^m h_t = h_\tau$, the desired inequality follows. □

Lemma 8 implies that the action of T^m moves points in C_1^γ closer to the unstable manifold, as long as they are close enough to the fixed point g.

We observe that the proof of Lemma 8, applied with $m = 1$, will show that there is a positive constant B such that whenever $|t| < \eta$ and $\|\phi\|_\gamma < \eta$ we have

$$\|T^k(h_t + \phi) - h_s\|_\gamma < B\|\phi\|_\gamma, \quad k = 0, 1, \ldots, m , \tag{4}$$

where $s = \delta^k(t + \sigma(\phi))$.

We also note that since $\sigma(h_t) = t + O(t^2)$ we can suppose η chosen so that

$$\|\sigma(h_t) - t\| < \|t\|/4 \quad \text{for } \|t\| < \eta . \tag{5}$$

6. Stable Manifold

We next apply Lemma 8 to define the local stable manifold for T in C_1^γ. Choose positive numbers ε and ε' so that $\varepsilon' \max(B, \delta^m + 1) < \eta$ and $\varepsilon \max(1, \|\sigma\|_\gamma) < \varepsilon'$. Suppose $\phi \in C_0^\gamma$ with $\|\phi\|_\gamma < \varepsilon$. We then define recursively sequences $\{t_k\}$ and $\{\phi_k\}$ for $k = 0, 1, 2, \ldots$ by the relations $T^{mk}(g + \phi) = h_{t_k} + \phi_k$ and $t_{k+1} = \delta^m(t_k + \sigma(\phi_k))$, starting with $t_0 = 0$ and $\phi_0 = \phi$. We continue as long as $|t_k| < \eta$. Then by Lemma 8 we have $\|\phi_k\|_\gamma < 2^{-k}\varepsilon$. Moreover, if $|t_k| \leqq \varepsilon'$ then $t_{k+1} < \eta$. Hence there are 3 mutually exclusive possibilities:

(A) for some k, $\varepsilon' < t_k < \eta$ and $|t_j| \leqq \varepsilon'$ for $j < k$,
(B) for some k, $\varepsilon' < -t_k < \eta$ and $|t_j| \leqq \varepsilon'$ for $j < k$,
(C) $|t_j| \leqq \varepsilon'$ for all $k = 0, 1, 2, \ldots$.

Let $W = \{g + \phi\colon\ \phi \in C_0^\gamma,\ \|\phi\|_\gamma < \varepsilon\}$, and let W_+, W_- and W_0 denote the subsets of W, where respectively (A), (B) and (C) hold. The map $\phi \to t_k$ is continuous on W, so W_+ and W_- are open, and hence W_0 is a relatively closed subset of W.

If $f = g + \phi \in W_0$, then $\|\phi_k\|_\gamma < 2^{-k}\varepsilon$ and so $|t_{k+1} - \delta^m t_k| < 2^{-k}\varepsilon$ for all k, which implies $|t_k| < 2^{-k}\varepsilon$ and so $\|T^{km} f - g\|_\gamma < C2^{-k}$ for all k, where C is a constant. This justifies the interpretation of W_0 as the local stable manifold for T; that it is indeed a C^1 manifold will follow from subsequent estimates.

We note that if $\varepsilon' < t_k < \eta$ then, using (5), we have

$$\begin{aligned}\sigma(T^{mk}(g + \phi)) = \sigma(h_{t_k} + \phi_k) &> \sigma(h_{t_k}) - \|\sigma\|_\gamma \|\phi_k\|_\gamma \\ &> 3|t_k|/4 - \varepsilon'/2 \geqq \varepsilon'/4 .\end{aligned} \tag{6}$$

We wish now to study the behaviour under the action of T^r of elements of W close to W_0. As preparation for this, the next lemma describes the behaviour of the derivative of T^r at points of W_0. Note first that, by Lemmas 1(ii) and 6, we may suppose that m in Lemma 8 and ε are chosen so that the norm of $S^m - DT^m(f)$ as an operator on C^1 is less than $\delta^m/8$.

Lemma 9. *There is a continuous mapping $f \to \theta_f$ from W_0 to $(C_0^1)^*$, such that*

$$\|DT^r(f)\phi - \delta^r\theta_f(\phi)h\|_1 \leqq C\delta^r 2^{-r/m}\|\phi\|_1$$

for all $f \in W_0$ and $\phi \in C_0^1$, $r = 1, 2, \ldots$, where C is a constant.

Proof. Fix $f \in W_0$. Let $R_k = \delta^{-mk}DT^{mk}(f)$; R_k acts as a bounded linear operator on C_0^1. Let $\phi \in C_0^1$ with $\|\phi\|_1 \leqq 1$. Then $R_{k+1}\phi = \delta^{-m}DT^m(f_k)R_k\phi$, where $f_k = T^{mk}f$. We write $R_k\phi = \alpha_k h + \psi_k$, where $\alpha_k \in \mathbf{R}$ and $\psi_k \in C_0^1$ are defined recursively as follows: $\alpha_0 = 0$, $\psi_0 = \phi$ and if α_k and ψ_k have been defined then

$$\begin{aligned} R_{k+1}\phi &= \delta^{-m}DT^m(f_k)(\alpha_k h + \psi_k) \\ &= \alpha_k h + \alpha_k\delta^{-m}(DT^m(f_k) - S^m)h \\ &\quad + \delta^{-m}(DT^m(f_k) - S^m)\psi_k + \sigma(\psi_k)h + \delta^{-m}Q^m\psi_k \\ &= \alpha_{k+1}h + \psi_{k+1}\,, \end{aligned}$$

where $\alpha_{k+1} = \alpha_k + \sigma(\psi_k)$ and

$$\psi_{k+1} = \alpha_k\delta^{-m}(DT^m(f_k) - S^m)h + \delta^{-m}(DT^m(f_k) - S^m)\psi_k + \delta^{-m}Q^m\psi_k\,.$$

Since $\|f_k - g\|_\gamma = O(2^{-k})$ and the mapping from C_1^γ to C_1^1 given by $f \to DT^m(f)h$ is differentiable at g, we have $\|(DT^m(f_k) - S^m)h\|_1 = O(2^{-k})$ as $k \to \infty$. Since also the norms of the two operators $DT^m(f_k) - S^m$ and Q^m on C_0^1 are each less than $\delta^m/8$ we deduce that

$$\|\psi_{k+1}\|_1 \leqq \frac{1}{4}\|\psi_k\|_1 + c_1|\alpha_k|2^{-k}\,.$$

Hence $\|\psi_k\|_1 = O(2^{-k})$ and $|\alpha_k - \alpha_{k+1}| = O(2^{-k})$. Thus α_k converges to a limit $\theta_f(\phi)$ and

$$\|R_k\phi - \theta_f(\phi)h\|_1 \leqq c_2 2^{-k}\|\phi\|_1$$

for any $\phi \in C_0^1$, from which the desired estimate follows.

To show that θ is continuous, note first that, since σ is a bounded linear functional on C_0^γ for some $\gamma < 1$, the mapping $f \to V_{f,k} \in (C_0^1)^*$, where $V_{f,k}\phi = \sigma(R_k\phi)$ is continuous on W_0 for any k. Now for any $\phi \in C_0^1$ we have

$$\|\theta_f(\phi) - V_{f,k}\phi\|_1 = \|\sigma(\theta_f(\phi)h - R_k\phi)\|_1 \leqq c_3 2^{-k}\|\phi\|_1\,,$$

so that $V_{f,k} \to \theta_f$ uniformly on W_0, whence θ is continuous. $\square$

In applying Lemma 9 we meet the usual difficulty that $DT(f)$ is not a Fréchet derivative. The following lemma is an adequate substitute for our purposes.

Lemma 10. *There is a constant $c_0 > 0$ such that if $f, f_1 \in W$ then*

$$\|Tf_1 - Tf - DT(f)\phi\|_1 \leqq c_0\|\phi\|_1^\rho\,,$$

where $\phi = f_1 - f$ and $\rho = 1 + \alpha/(1+\alpha) > 1$.

Proof. If we write out $(T(f+\phi) - Tf - DT(f)\phi)'$ explicitly and use the fact that the hypothesis $f, f_1 \in W$ gives bounds for $\|f\|_\gamma$ and $\|\phi\|_\gamma$ we find that

$$\|Tf_1 - Tf - DT(f)\phi\|_1 \leqq \text{const}(\|\phi\|_0\|\phi\|_2 + \|\phi\|_1^2)\,.$$

Now Lemma 5 applied to ϕ' gives $\|\phi\|_2 \leqq 4\|\phi\|_\gamma^{1/(1+\alpha)}\|\phi\|_1^{\alpha/(1+\alpha)}$ and the desired result follows. □

The next lemma gives the main technical estimate of the paper; the proof is based on Lemmas 8,9 and 10.

Note first that $\theta_g = \sigma$ and so by the continuity of θ we can find ε_0 with $0 < \varepsilon_0 < \varepsilon$ such that if $f \in W_0$ and $\|f - g\|_\gamma < \varepsilon_0$, then $|\theta_f(h) > 1/2$ and $\|\theta_f\|_1 \leqq 2\|\sigma\|_1$. Let V_0 be the set of $f \in W_0$ such that $\|f - g\|_\gamma < \varepsilon_0$.

Lemma 11. *We can find positive numbers β, τ and c such that if r is a sufficiently large positive integer and $f \in V_0$, $f_1 \in W$ with $\|f_1 - f\|_1 \leqq \beta\delta^{-r}$ then*

$$\|T^r f_1 - h_{\delta^r\theta_f(f_1-f)}\|_\gamma \leqq c\delta^{-\tau r}\,.$$

Proof. By Lemma 9 there is a constant c_1 such that if $f \in W_0$ then the norm of $DT^k(f)$ as an operator on C_0^1 is bounded by $c_1\delta^k$ for any positive integer k. Choose $v > 0$ so that $v^{\rho-1} = (1-\delta^{1-\rho})/(c_0(2c_1)^\rho)$ and let $c_2 = v(1-\delta^{1-\rho})$, where ρ and c_0 are as in Lemma 10. Let $\beta = \min(v, \eta)$.

Fix f and f_1 satisfying the hypotheses above and let $\psi = f_1 - f$. Let k_0 be the first positive integer value of k such that either $k > r$ or $\|T^k f_1 - g\| \geqq \eta$. We now define a sequence χ_k inductively by

$$T^k f_1 = T^k f + DT^k(f)\psi + \sum_{j=1}^{k} DT^{k-j}(f)\chi_j\,,$$

and by Lemma 10 we have

$$\|\chi_{k+1}\|_1 \leqq c_0\left\|DT^k(f)\psi + \sum_{j=1}^{k} DT^{k-j}(f)\chi_j\right\|_1^\rho$$

for $k < k_0$. We now show by induction that $\|\chi_j\|_1 \leqq c_2\delta^{\rho(j-r)}$ for $j \leqq k_0$. Supposing this holds for $j = 1, \ldots, k$ we deduce that

$$\begin{aligned}\|\chi_{k+1}\|_1 &\leqq c_0\left\{c_1 v\delta^{k-r} + c_1c_2\sum_{j=1}^{k}\delta^{k-j}\delta^{\rho(j-r)}\right\}^\rho \\ &\leqq c_0\left\{c_1v\delta^{k-r} + c_1c_2\delta^{\rho(k-r)}/(1-\delta^{1-\rho})\right\}^\rho = c_0(2c_1v)^\rho\delta^{\rho(k-r)} = c_2\delta^{\rho(k-r)}\end{aligned}$$

as required. It now follows that, if $k \leqq k_0$,

$$\begin{aligned}\|T^k f_1 - T^k f - DT^k(f)\psi\|_1 &\leqq \sum_{j=1}^{k} c_1\delta^{k-j}c_2\delta^{\rho(j-r)} = c_1c_2\sum_{j=1}^{k}\delta^{(\rho-1)j} \\ &\leqq \delta^{\rho(k-r)}/(1-\delta^{1-\rho}) = c_1v\delta^{\rho(k-r)}\,.\end{aligned}$$

We also have, by Lemma 9,

$$\|DT^k(f)\psi - \delta^k\theta_f(\psi)h\|_1 \leqq c_2 2^{-k/m}\delta^{k-r}\,,$$

and, since $f \in W_0$,

$$\|T^k f - g\| \leqq c_3 2^{-k/m} .$$

We deduce that, for $k \leqq k_0$, we have

$$\|T^k f_1 - g - \delta^k \theta_f(\psi)h\|_1 \leqq (c_2 + c_3)2^{-k/m} + c_1 v \delta^{\rho(k-r)} . \tag{7}$$

Equation (7) implies in particular that, for $k \leqq k_0$,

$$|\sigma(T^k f_1 - g)| \leqq c_4(\delta^{k-r} + 2^{-k/m}) . \tag{8}$$

Now let $\mu = \rho \log \delta/(m^{-1} \log 2 + \rho \log \delta)$ and let l be the nearest integer to $\mu r/m$, so that 2^{-l} is comparable to $\delta^{\rho(lm-r)}$. It follows from (8) that if r is large enough then $|\sigma(T^k f_1 - g)| < \varepsilon'/4$ for $k \leqq \min(ml, k_0)$. We now assert that $lm \leqq k_0$. To prove this, suppose on the contrary that $lm > k_0$ and let $j \leqq l$ be the smallest positive integer such that $jm > k_0$. Then by (6) we have $\|T^{im} f_1 - g\|_\gamma < \varepsilon'$ for $i < j$, and hence by (4), $\|T^k f_1 - g\|_\gamma < B\varepsilon'$ for $k \leqq mj$, contradicting the definition of k_0. So $lm \leqq k_0$ as asserted.

We then deduce from (7) that

$$\|T^{ml} f_1 - g - \delta^{ml} \theta_f(\psi)h\|_1 \leqq c_5 \delta^{-\rho(r-ml)} , \tag{9}$$

and by Lemma 8 we have, for some t,

$$\|T^{lm} f_1 - h_t\|_\gamma \leqq \text{const}\, 2^{-l} \leqq c_6 \delta^{-\rho(r-ml)} . \tag{10}$$

Writing $s = \delta^{ml} \theta_f(\psi)$ we deduce from (9) and (10) that

$$\|h_t - g - sh\|_1 \leqq (c_5 + c_6)\delta^{-\rho(r-ml)} ,$$

and so

$$|t - s| \leqq c_7(\delta^{-\rho(r-ml)} + s^2) \leqq c_8 \delta^{-\rho(r-ml)} ,$$

where the last inequality uses $|s| \leqq \text{const}\, \delta^{lm-r}$. Using (10) we then conclude that

$$\|T^{ml} f_1 - h_s\|_\gamma \leqq c_9 \delta^{-\rho(r-lm)} . \tag{11}$$

Now let p be the integer part of $\frac{r}{m} - l$. Using Lemma 8 we can write, for $j = 0, 1, \ldots, p$, $T^{m(l+j)} f_1 = h_{s_j} + \phi_j$, where $s_0 = s = \delta^{ml} \theta_f(\psi)$, $s_{j+1} = \delta^m(s_j + \sigma(\phi_j))$ and

$$\|\phi_j\|_\gamma \leqq 2^{-j} \|\phi_0\| \leqq 2^{-j} c_9 \delta^{-\rho(r-ml)} .$$

We have then $s_p = \delta^{mp} s + \sum_{j=0}^{p-1} \delta^{m(p-j-1)} \sigma(\phi_j)$ from which we deduce that $|s_p - \delta^{m(l+p)} \theta_f(\psi)| < c_{10} \delta^{(1-\rho)(1-\mu)r}$. The desired result now follows from (4). □

We can now prove the main results.

Theorem 1. *V_0 is a C^1 submanifold of C_1^γ.*

Proof. We note first that if $f \in V_0$, then $\theta_f(h) > 0$, so that by Lemma 11 we see that for t small and positive $f + th \in W^+$, while for t small and negative $f + th \in W^-$.

Hence if we fix $f_0 \in V_0$, then in a small neighbourhood of f_0 there can be at most one member of V_0 on a line $\{f + th\colon t \in \mathbf{R}\}$. Let $E = \{\phi \in C_0^\gamma\colon \theta_{f_0}(\phi) = 0\}$. Then for $\phi \in E$ with $\|\phi\|_\gamma$ sufficiently small, we have that $f_0 + \phi + th$ is in $W_\pm$ for $t = \pm\|\sigma\|_\gamma$, by Lemma 11. Hence for each such ϕ there is a unique small real number $\chi(\phi)$ such that $f_0 + \phi + \chi(\phi)h \in V_0$.

Now let ϕ_1 and ϕ_2 belong to E with small norms, and let $f_i = f_0 + \phi_i + \chi(\phi_i)$ for $i = 1, 2$, so that $f_1, f_2 \in V_0$. Then by Lemma 11 we have $\theta_{f_1}(f_2 - f_1) = O(\|f_2 - f_1\|_\gamma^\tau$. So

$$|\theta_{f_1}(\phi_2 - \phi_1)| + \theta_{f_1}(h)\|\chi(\phi_2) - \chi(\phi_1)\|_\gamma = O\left(\|\phi_2 - \phi_1\|_\gamma + \|\chi(\phi_2) - \chi(\phi_1)\|_\gamma^{1+\tau}\right)$$

and on letting $\phi_2 \to \phi_1$ we conclude that χ is differentiable at ϕ_1 and that

$$D\chi(\phi_1) = -\theta_{f_1}(h)^{-1}\theta_{f_1}\ .$$

From the continuity of θ we then deduce the continuity of $D\chi$ on a neighbourhood of 0 in E, and hence that V_0 is a C^1 manifold in a neighbourhood of f. □

Now consider a one-parameter family $\{f_\mu\}$ such that $f_0 \in V_0$ and $F_\mu \in W$ for μ near 0. We suppose also that the mapping $\mu \to f_\mu$ is differentiable at 0 as a mapping into C^1. The latter statement would be true, for example, if the function $f(\mu, x) = f_\mu(x)$ was C^2 as a function of two real variables. Then we have the following:

Theorem 2. *Suppose that* $\theta_{f_0}(\phi) = a \neq 0$, *where* $\phi = [d/d\mu f_\mu]_{\mu=0}$. *Let* $\varepsilon > 0$. *Then for* n *sufficiently large,* f_μ *has an attracting orbit of period* 2^n *for*

$$\delta^{-n-1}(1+\varepsilon) < a\mu < \delta^n(1-\varepsilon)\ .$$

Proof. The hypotheses imply that $\theta_{f_0}(f_\mu - f_0) = a\mu + o(|\mu|)$. It then follows from Lemma 11 that there exists $k > 0$ such that $\|T^r f_{\delta^{-r}\mu} - h_{a\mu}\|_\gamma \to 0$ as $r \to \infty$, uniformly in $\mu \leqq k$. Now let l be a positive integer such that $\delta^{-l} < ak$ and suppose $\delta^{-l-1}(1+\varepsilon) \leqq a\mu \leqq \delta^{-l}(1-\varepsilon)$. Then $h_{a\mu}$ has a uniformly (w.r.t. μ) attracting orbit of period 2^l, and hence, if r is large enough, so also does $T^r f_{\mu\delta^{-r}}$ for each μ in this range. This implies that $f_{\delta^{-r}\mu}$ has an attracting orbit of period 2^{l+r}, which completes the proof. □

We remark that, since we do not require f_μ to be C^3, we cannot expect "clean" period-doubling bifurcations at single parameter values μ_n; rather, the transition from an attracting orbit of period 2^n to one of period 2^{n+1} can take place over a μ-interval whose length is $o(\delta^{-n})$ as $n \to \infty$.

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