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Convex Delay Endomorphisms

A. Rovella[1], **F. Vilamajó**[2]

[1] Centro de Matemática, Universidad de la República, Ed. Acevedo 1139, and IMERL-Facultad de Ingeniería, Universidad de la República CC 30, Montevideo 11200, Uruguay.
e-mail: leva@cmat.edu.uy

[2] Departament de Matemática Aplicada 2, Escola Tecnica Superior d'Enginyers Industrials, Colom 11, 08222 Terrassa, Barcelona, Espanya. e-mail: vilamajor@ma2.upc.es.

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Abstract: In this paper delay equations $x_{n+k} = f(x_n,\dots,x_{n+k-1})$ are considered, where the function f is supposed to be convex, having a unique point of maximum. It is proved that if there are no stationary solutions then all solutions must diverge. Considering the one parameter family $f_\mu = \mu + f$ and associating to it a family of two dimensional maps F_μ it is shown that the set of points having bounded orbit under F_μ is homeomorphic to the product of a Cantor set and a circle, and is hyperbolic and stable.

1. Introduction

Any delay equation of order k:

$$x_{n+k} = f(x_n,\dots,x_{n+k-1}) \tag{1}$$

can be associated with a transformation of R^k given by

$$F(x_1,\dots,x_k) = (x_2,\dots,x_k, f(x_1,\dots x_k))\,. \tag{2}$$

Any orbit of the map F is in one to one correspondence with a solution of the delay equation (1). Here we will deal with delay equations where the function f is *convex*, in the sense that f is a C^2 function such that the quadratic form associated with the second derivative is definite at every point. In this case Eq. (1) is called a convex delay equation and the map F defined in (2) is called a convex delay endomorphism. In the rest of this work, we will take this quadratic form negatively definite, so that f could have at most one critical point that should be a maximum. A stationary solution of the delay equation (1) is a constant solution $x_n = x$ for every n; the existence of such an x is equivalent to have a solution of the equation $f(x,\dots,x) = x$. Moreover, the fixed points of F are the points $(x,\dots,x)$, where x is a solution of $f(x,\dots,x) = x$. So when f is convex the delay equation associated would have at most two stationary solutions, or, which is the same, the endomorphism F would have at most two fixed points. We will prove the following result:

Theorem 1.1. *Let f be convex and suppose that F has no fixed points. Then the ω limit set under F of any point in R^k is empty.*

In terms of delay equations this says that if f is convex and there are no stationary solutions, then all the solutions must diverge.

Consider a convex first order equation given by $f: R \to R$, and suppose that f is not only convex but there is a negative constant such that f'' is less than this constant. If we push up the graph of f vertically, we will obtain a one parameter family $f_\mu = \mu + f$; for this one dimensional map it is easy to see that for every large parameter the function f_μ will have two fixed repelling points and that the set of preimages of any one of these points accumulates in a Cantor hyperbolic set which is the complement in the line of the basin of attraction of ∞ (or, what is the same, the set of points with an empty ω limit set). Under some new conditions on the function f that will be defined in Sect. 3, this result remains true for second order equations; these are open conditions, define a set $\mathscr{U}$, and imply that F is convex.

Theorem 1.2. *There exists an open set $\mathscr{U}$ in $C^2(R^2)$ such that for any $f \in \mathscr{U}$ the family of endomorphisms $F_\mu(x, y) = (y, \mu + f(x, y))$ has the following properties, for every μ sufficiently large:*

a) *F_μ has two fixed saddle points.*

b) *The closure of the stable manifold of one of these points is diffeomorphic to the product of a Cantor set K with a circle S^1.*

c) *The basin of ∞ is the complementary set in R^2 of the closure of the stable manifold.*

As a corollary of the proof of this theorem a description of the dynamics of F_μ restricted to the closure of the stable manifold ($= K \times S^1$) can be obtained. Each circle of $K \times S^1$ is mapped into an unclosed curve contained in another circle, so this defines a one dimensional map on K, that becomes equivalent to a shift:

Theorem 1.3. *Let W^s_μ be the stable manifold of one of the fixed points of F_μ, and $\overline{W^s_\mu}$ its closure. Consider the set: $\Lambda = \bigcap_{n \geqq 0} F^n_\mu(\overline{W^s_\mu})$. Then Λ is compact, F_μ-invariant, hyperbolic and coincides with the closure of the periodic points of F_μ. Two different cases can occur: either Λ is a horseshoe and F_μ/Λ is a homeomorphism, or it is contained in the unstable manifold of each one of the fixed points, which in this case are equal.*

The second alternative of the last theorem it is not generic: the usual case is the first. Now the dynamics of the maps F_μ are completely described for every large parameter value.

The results of the first theorem were shown to hold for a particular family of quadratic delay endomorphisms in Whitley [W], where the dynamics of the family for large parameter values is also studied; however, their example does not satisfy the hypothesis of our Theorems 1.2 and 1.3.

A very interesting reference on the subject of delay equations is the book of P. Montel [Mon], where the theory of delay maps is treated from a general viewpoint.

2. Absence of Fixed Points

As was explained in the introduction the hypothesis of Theorem one is equivalent to the non-existence of solutions of the equation $f(x,\dots,x) = x$ or, which is the same, the graph of f does not intersect the diagonal of R^{k+1}. Let $f''(x)$ be the Hessian matrix of f at the point x. By hypothesis, f is convex, which means that if Q_x is the quadratic form associated with $f''(x)$, then $Q_x(v) = vf''(x)v^t < 0$ for each nonzero vector v.

Proof of Theorem 1.1. As the graph of f does not intersect the diagonal of R^{k+1}, there is a positive number α and a unique point $x_0 \in R^n$ such that the graph of $f + \alpha$ intersects the diagonal of R^{k+1} at $(x_0,\dots,x_0)$. Without loss of generality it can be assumed that $x_0 = 0$; then, using Taylor's expansion around 0, we obtain:

$$f(x) = -\alpha + v \cdot x + xHx + Rx\,, \tag{3}$$

where $v = f'(0)$, $H = f''(0)$ and $R : R^k \to R$ is a C^2 function such that $\lim_{x\to 0} R(x)/|x|^2 = 0$. Denoting $v = (v_1,\dots,v_k)$ observe that the vector $(v_1,\dots,v_k,-1)$ is orthogonal to the tangent space of the graph of f at 0, which by assumption contains the diagonal of R^{k+1}, so that $\sum_{i=1}^k v_i = 1$. Now define the following Lyapunov function:

$$L(x_1,\dots,x_k) = v_1x_1 + (v_1 + v_2)x_2 + \cdots + (v_1 + \cdots + v_{k-1})x_{k-1} + x_k\,. \tag{4}$$

As it is well known, to prove the theorem it is sufficient to show that for every $x \in R^2$, $L(F(x)) - L(x) < 0$. Then, using (3), (4) and that $\sum v_i = 1$, we obtain:

$$\begin{aligned} L(F(x)) - L(x) &= v_1x_2 + (v_1 + v_2)x_3 + \cdots + (v_1 + \cdots + v_{k-1})x_k + f(x) - L(x) \\ &= -\alpha + xHx + R(x)\,. \end{aligned} \tag{5}$$

Now define the function $\varphi : R^k \to R$ by $\varphi(x) = xHx + R(x)$ and observe that $\varphi(0) = 0$, $\varphi'(0) = 0$ and $\varphi''(x) = f''(x)$. So φ'' is negative definite from which it follows that $\varphi(x) < 0$ for every $x \in R^k$, x not zero. This implies that $L(F(x)) - L(x) \leqq -\alpha < 0$ in (5), and the theorem is proved.

3. Dynamics for Large Parameter Values

We will begin by describing the C^2-open set $\mathcal{U}$ for which the theorems are valid.

Let

$$B = -\sup\{\partial_{22}f(x,y) : (x,y) \in R^2\}\,,$$

$$A = -\inf\{\partial_{11}f(x,y) : (x,y) \in R^2\}\,,$$

$$A' = -\sup\{\partial_{11}f(x,y) : (x,y) \in R^2\}\,.$$

Definition 3.1. *Let $\mathcal{U}$ be the set of C^2 functions $f : R^2 \to R$ such that the following conditions hold*:

$$(P1) \quad B \geqq KA\,;$$

where K is a positive number to be defined later

$$(P2) \quad -\partial_{11}f(x,y) \geqq |\partial_{12}f(x,y)| \ \forall (x,y) \in R^2 \, ,$$

$$(P3) \quad A' > 0 \, .$$

Remarks.

1. (P1) and (P2) together imply that f is convex. Using also (P3) it follows that $\lim_{|(x,y)|\to\infty} f(x,y) = -\infty$.

2. It is clear that this set $\mathscr{U}$ is open in the C^2 topology.

3. Theorems 1.2 and 1.3 are not valid in general if $B < A$: take for example $f(x,y) = -Ax^2 - By^2$ with $A > B$, calculate the eigenvalues of the fixed points of F_μ, and observe that they are not saddles.

Now define the one parameter family to be considered: take $f \in \mathscr{U}$, and define: $f_\mu(x,y) = \mu + f(x,y)$ and $F_\mu : R^2 \to R^2$ by $F_\mu(x,y) = (y, f_\mu(x,y))$.

Now let's introduce some elementary curves that will play an important role. The critical curves of f_μ are:

$$l_1 = \{(x,y) : \partial_1 f_\mu(x,y) = 0\} \, ,$$

$$l_2 = \{(x,y) : \partial_2 f_\mu(x,y) = 0\} \, .$$

These curves are in fact independent of μ; l_1 is the graph of a function of y, so that $l_1 = \{(\tilde{x}(y), y) : y \in R\}$, with

$$\tilde{x}'(y) = -\frac{\partial_{12}f(\tilde{x}(y), y)}{\partial_{11}f(\tilde{x}(y), y)} \, .$$

l_2 is the graph of a function of x, so that $l_2 = \{(x, \tilde{y}(x)) : x \in R\}$, with

$$\tilde{y}'(x) = -\frac{\partial_{12}f(x, \tilde{y}(x))}{\partial_{22}f(x, \tilde{y}(x))} \, .$$

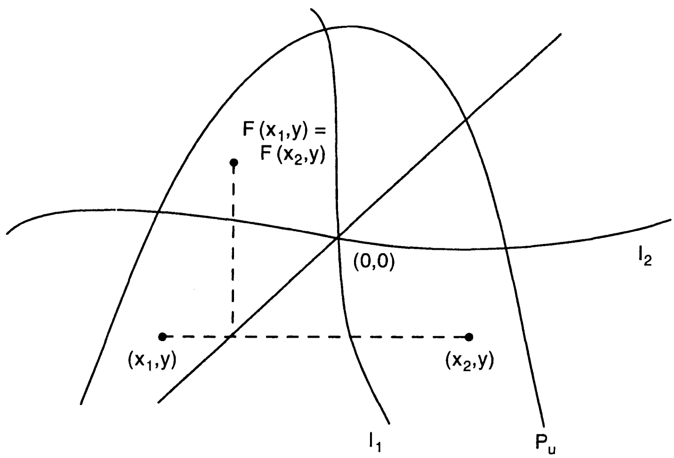
By properties (P1) and (P2) we have that:

$$|\tilde{x}'(y)| < 1/K \ \forall y \quad \text{and} \quad |\tilde{y}'(x)| < 1/K^2 \ \forall x \, .$$

So $K > 1$ implies that l_1 and l_2 have one and only one point of intersection that will be supposed to be $(0,0)$ by making a translation. From this it follows that f_μ takes its maximum at $(0,0)$.

Also observe that l_1 is the set of critical points of F_μ. The image P_μ of l_1 under F_μ is the graph of a function $\tilde{z}_\mu(x) = f_\mu(\tilde{x}(x), x)$, that has negative second derivative, as is easy to check using (P1) and (P2). So the complementary set of P_μ contains two connected components, one of which, $\tilde{P}_\mu$, is convex; actually, $F_\mu(R^2) = P_\mu \cup \tilde{P}_\mu$. Any point outside $P_\mu \cup \tilde{P}_\mu$ has no preimages under F_μ; a point in P_μ has only one preimage lying on l_1; and points in $\tilde{P}_\mu$ have two preimages, having the same second coordinate and located one at each side of l_1.

Denote by $\xi_\alpha(\mu)$ the α-level curve of f_μ, that is, $\xi_\alpha(\mu) = \{(x, y) : f_\mu(x, y) = \alpha\}$.



Lemma 3.1. *For every μ sufficiently large a function s of μ is defined such that*:

a) $(s(\mu), s(\mu))$ *is a fixed saddle point of* f_μ,
b) $s(\mu) \to -\infty$ *as* $\mu \to +\infty$,
$s'(\mu) \to 0$ *as* $\mu \to +\infty$,
c) *A local stable manifold of* $(s(\mu), s(\mu))$ *is transversal to* $\xi(\mu)$, *the family of level curves of* f_μ.

Proof. As was explained before, the fixed points of F_μ are the points (x,x) for which $f_\mu(x,x) = x$. Let $g(x) = f(x,x)$. Using (P1), (P2) and (P3) it is easy to see that g has negative second derivative bounded below from zero which implies that the graph of g intersects any line $y = x - \mu$ for μ large enough. As g has its maximum at zero, one of these points will have negative coordinates; let's denote this point by $(s(\mu), s(\mu))$. It is clear that $s(\mu) \to -\infty$ as $\mu \to +\infty$ and that $s'(\mu) = (1 - g'(s_\mu))^{-1}$, which implies part b. Let's prove that (s_μ, s_μ) is a saddle point. The eigenvalues are given by

$$\lambda_\pm = 1/2(E \pm \sqrt{E^2 + 4D}) ,$$

where $E = E_\mu = \partial_2 f(s_\mu, s_\mu)$ and $D = D_\mu = \partial_1 f(s_\mu, s_\mu)$.

Now observe that:

$$D_\mu = \int_{s_\mu}^{0} - \partial_{12} f(x,x) - \partial_{11} f(x,x) dx$$

$$= \int_{s_\mu}^{0} - \partial_{11} f(x,x) \left(1 + \frac{\partial_{12} f(x,x)}{\partial_{11} f(x,x)}\right) dx \leqq A(1 + K^{-1})(-s_\mu) ,$$

where (P2) was used. Similarly, using (P1) and (P2) we obtain that

$$E_\mu = \int_{s_\mu}^{0} - \partial_{22} f(x,x) \left(1 + \frac{\partial_{12} f(x,x)}{\partial_{22} f(x,x)}\right) dx \geqq B(1 - 1/K^2)(-s_\mu) .$$

Therefore $E_\mu/D_\mu > 1$ which implies that $\lambda_- \in (-1,0)$. In addition it follows from the facts above that $\lambda_+ \to +\infty$ when $\mu \to +\infty$. This proves part a) of the lemma. To prove part c) it is enough to observe that an eigenvector associated to λ_- is $(1,\lambda_-)$, while a tangent vector to $\xi_{s(\mu)}(\mu)$ at $(s(\mu),s(\mu))$ is $(1,-D/E)$, and it is easy to check that $\lambda_- > -D/E$.

The proof of Theorems 2 and 3 is based on the study of the behavior of the stable manifold of $S_\mu = (s_\mu, s_\mu)$ (that is defined locally as for a diffeomorphism and then by taking preimages). Denote by W^s_μ the stable manifold of S_μ. We will prove that W^s_μ has infinitely many connected components, each one diffeomorphic to a circle. We begin with the following simple fact:

Remark. Let γ be a C^1 1–1 curve such that it intersects P_μ transversally at two points. Then $F_\mu^{-1}(\gamma)$ is a C^1 Jordan curve. The proof of this fact is easy using that any point in P_μ has a double preimage. The transversality is used to obtain that $F_\mu^{-1}(\gamma)$ is C^1 at the points of intersection with l_1.

This is the procedure that makes W^s_μ contain a closed curve: it is enough to prove that the local stable manifold of S_μ intersects P_μ in a pair of points to imply that W^s_μ contains a C^1 simple closed curve. It will be shown that this curve has, in fact, four points of intersection with P_μ; taking the preimage under F_μ of this curve we will obtain another closed simple curve, which will also intersect P_μ at four points. Automatically, the following preimages under F_μ give a sequence of closed curves each one having four points of intersection with P_μ. To prove these facts we will first show that W^s_μ is transversal to $\xi(\mu)$ before its intersection with l_1 or l_2; this, as we will see, implies that these intersections actually occur. And secondly, a technique will be developed permitting us to study the set W^s_μ as it was a level curve of f_μ.

As f is convex, every level curve $\xi_\alpha(\mu)$ is a Jordan C^2 curve that encloses a convex region. In general, if ξ is a Jordan curve then $i(\xi)$ will denote the bounded component and $e(\xi)$ the unbounded component of $R^2\backslash\xi$. As the maximum of each f_μ is taken at $(0,0)$ we have that $\xi_\alpha(\mu) = \phi$ for $\alpha > \mu + f(0,0)$, and that $(0,0) \in i(\xi_\alpha(\mu))$ for $\alpha < \mu + f(0,0)$; in this case, $\xi_\alpha(\mu)$ intersects both l_1 and l_2, the intersections with l_1 correspond to the horizontal tangents of $\xi_\alpha(\mu)$ and those with l_2 to the vertical tangents of $\xi_\alpha(\mu)$. For any fixed μ, the level curves $\xi_\alpha(\mu)$ form a foliation of $R^2\backslash(0,0)$, that we have denoted by $\xi(\mu)$. Let γ be any C^1 curve that is transversal to the family $\xi(\mu)$; then we will say that γ is *entering* $\xi(\mu)$ at t if $(f\circ\gamma)'(t) > 0$ and that is *leaving* $\xi(\mu)$ at t if $(f\circ\gamma)(t) < 0$.

Let's denote by Q_1 the connected component of $R^2\backslash l_1 \cup l_2$ which contains S_μ. Let $\alpha = \alpha_\mu$ be a curve parametrizing the connected component of $W^s_\mu \cap Q_1$ which contains the point S_μ, and with the following properties, where we take μ large and drop the subindex:

- $\alpha(0) = S_\mu$.
- $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ with $\alpha_1(t) > 0$ *for* t *small.*

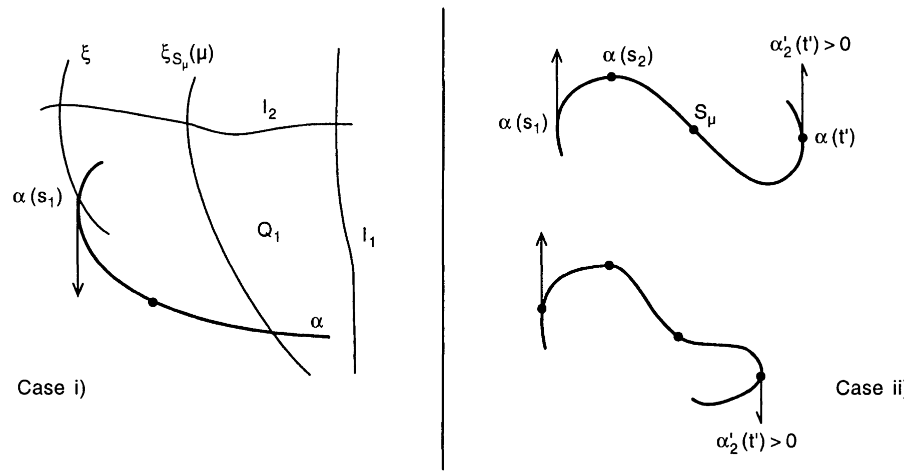
It follows from Lemma 3.1 that α is entering $\xi(\mu)$ at $t = 0$.

Lemma 3.2. α_μ *is transversal to* $\xi(\mu)$.

Proof. Observe first that if at a point t, α is tangent to ξ, then $f\circ\gamma$ has a critical point at t, so that $F\circ\gamma$ has horizontal tangent at t, and this implies that $F^2\circ\gamma$ has vertical tangent at t. Reasoning by contradiction, suppose that at a point $s < 0$,

α is tangent to some curve of ξ; let $s_0 = \max\{s < 0 : \alpha \textit{ is tangent to } \xi \textit{ at } s\}$. Then, at s_0, $F \circ \alpha$ has horizontal tangent and $F^2 \circ \gamma$ has vertical tangent. Now, as α is part of W^s, which is invariant, it follows that there exists $s_1 \in (s_0, 0)$, such that α has a vertical tangent at s_1 (that is, $\alpha_1'(s_1) = 0$). Redefine, if necessary s_1 as maximum with this property. Obviously $s_0 < s_1 < 0$, and we have to distinguish between two cases:

i) $\alpha_2'(s_1) < 0$ and ii) $\alpha_2'(s_1) > 0$.



In case i), observe that α is leaving ξ at s_1, because α is contained in Q_1; as it was entering ξ at zero there must occur a tangency between α and ξ in the interval $(s_1, 0)$, which is a contradiction with the definition of s_0.

In case ii), there must exist a point $s_2, s_1 < s_2 < 0$, such that $\alpha_2'(s_2) = 0$. Take s_2 maximum with this property. If $\alpha_1'(s_2) < 0$, we conclude that α is leaving ξ at s_2, so as in case 1 a contradiction appears. If $\alpha_1'(s_2) > 0$, define $t' > 0$ such that $F(\alpha(s_2)) = \alpha(t')$ (so $\alpha_1'(t') = 0$). Now $\alpha_2'(t') > 0$ implies that there exists $t'' \in (0, t')$, such that $\alpha_2'(t'') = 0$; thus, taking the image of $\alpha(t')$ we find a point of vertical tangency between α and ξ which corresponds to an $s \in (s_1, 0)$, in contradiction with the definition of s_1. Therefore $\alpha_2'(t') < 0$, so there exists $t''' \in (0, t')$ such that ξ and α are tangent at t'''; it follows that α has horizontal tangent at a point in $(s_2, 0)$, which contradicts the definition of s_2.

The following two lemmas, that will be used often later, imply that the level curve of f_μ passing through the fixed point S_μ must intersect the set P_μ; this, together with the previous result will imply that W_μ^s also intersects P_μ; then, using the remark above Lemma 1 forces W_μ^s to contain a C^1 Jordan curve.

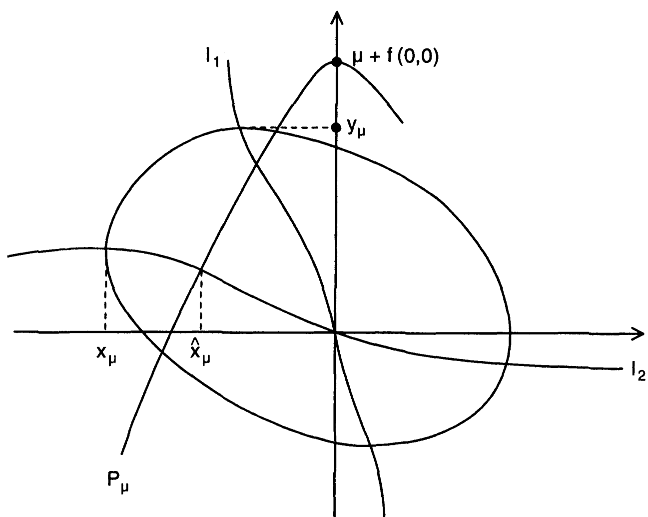
Lemma 3.3. *Let τ be a C^1 function of μ such that $\tau'(\mu) \to 0$ as $\mu \to \infty$. Then for all μ sufficiently large $\xi_{\tau(\mu)}(\mu)$ has four points of intersection with P_μ.*

Proof. Let's first calculate $y_\mu = \max\{y : (x, y) \in \xi_{\tau(\mu)}(\mu)\}$. As it is easy to see, this maximum must be taken at point of intersection of $\xi_{\tau(\mu)}(\mu)$ with l_1 so that y_μ satisfies: $f_\mu(\tilde{x}(y_\mu), y_\mu) = \tau(\mu)$. This implies that $y_\mu \to \infty$ as $\mu \to \infty$ because

$f(\tilde{x}(y_\mu), y_\mu) = \tau(\mu) - \mu$ which tends to $-\infty$ as $\mu \to \infty$ by hypothesis. Therefore, as $\partial_1 f_\mu(\tilde{x}(y_\mu), y_\mu) = 0$, it follows that:

$$y'_\mu = \frac{\tau'(\mu) - 1}{\partial_2 f(\tilde{x}(y_\mu), y_\mu)} \,.$$

From this we obtain that $y'_\mu \to 0$ as $\mu \to \infty$ because $\partial_2 f(\tilde{x}(y_\mu), y_\mu) \to +\infty$. In addition, the maximum second coordinate of points in P_μ is $\mu + f(0,0)$, which results in greater than y_μ for every μ large, because $y'_\mu \to 0$. This shows that P_μ crosses $\xi_{\tau(\mu)}(\mu)$ vertically.



Now let x_μ be the first coordinate of the left point of intersection of l_2 with $\xi_{\tau(\mu)}(\mu)$ and $\hat{x}_\mu$ the first coordinate of the left point of intersection of l_2 with P_μ. We claim that $|x_\mu| > |\hat{x}_\mu|$. Observe that x_μ satisfies the equation:

$$f_\mu(x_\mu, \tilde{y}(x_\mu)) = \tau_\mu \,,$$

so that $x_\mu \to -\infty$ as $\mu \to +\infty$, which can be proved as above.

Using (P3) it follows that:

$$f(x_\mu, \tilde{y}(x_\mu)) = \int_0^{x_\mu} \partial_1 f(t, \tilde{y}(t))dt + f(0,0) \,,$$

$$\partial_1 f(t, \tilde{y}(t)) = \int_0^t \partial_{11} f(s, \tilde{y}(s)) - \frac{(\partial_{12} f(s, \tilde{y}(s)))^2}{\partial_{22} f(s, \tilde{y}(s))} ds \leqq -A'(1 - 1/K^3)t$$

similarly, but now using (P2), it follows that:

$$\partial_1 f(t, \tilde{y}(t)) \geqq -A(1 + 1/K^3)t \,,$$

and this implies that:

$$\frac{A'}{2}(1-1/K^3)x_\mu^2 \leqq \mu - \tau(\mu) \leqq \frac{A}{2}(1+1/K^3)x_\mu^2 ,$$

and therefore

$$\liminf_{\mu\to\infty} \frac{|x_\mu|}{\sqrt{A_0^{-1}\mu}} \geqq 1 , \tag{6}$$

where $A_0 = \frac{A}{2}(1+1/K^3)$.

Now let's estimate the point $\hat{x}_\mu$. It is easy to see that $\tilde{z}_\mu(x) \leqq -B_0x^2+\mu$, where $B_0 = \frac{B}{2}(1-1/K^3)$, from which it follows that P_μ can be substituted by the parabola $y = -B_0x^2+\mu$.

This, together with the fact that l_2 is contained in the cone $|y| \leqq x/K^2$, imply that:

$$|\hat{x}_\mu| \leqq \frac{1/K^2 + \sqrt{1/K^2 + 4B_0\mu}}{2B_0} ,$$

from which it follows that

$$\limsup_{\mu\to+\infty} \frac{|\hat{x}_\mu|}{\sqrt{B_0^{-1}\mu}} \leqq 1 . \tag{7}$$

As $B_0 > A_0$, (6) and (7) imply the claim. Observe that this should be repeated for right intersections. So this shows that P_μ crosses $\xi_{\tau(\mu)}(\mu)$ also horizontally. This finishes the proof of the lemma.

Let τ be a C^1 function of μ such that $\tau'(\mu) \to 0$ as $\mu \to \infty$. Then the lemma just proved implies that for any point in $\tilde{P}\backslash i(\xi_{\tau(\mu)}(\mu))$ the partial derivative with respect to the second variable is not zero. We will need now to find a lower bound for this derivative and, more than this, we will show that a relation between the partial derivative with respect to the first and second variables exists. This will be used later to obtain stable foliations in $\tilde{P}_\mu\backslash i(\xi_{\tau(\mu)}(\mu))$.

Lemma 3.4. *There exists λ (for example, $\lambda = 10$) such that, if $(x,y) \in e(\xi_{\tau(\mu)}(\mu)) \cap \tilde{P}_\mu$ and μ is sufficiently large then*:

$$\left|\frac{\partial_2 f_\mu(x,y)}{\partial_1 f_\mu(x,y)}\right| \geqq \lambda .$$

Proof. Firstly observe that

$$|\partial_2 f(x,y)| = \left|\partial_2 f(x,\tilde{y}(x)) + \int_{\tilde{y}(x)}^{y} \partial_{22} f(x,s)ds\right| \geqq B|y-\tilde{y}(x)| .$$

And in the same manner:

$$|\partial_1 f(x,y)| \leqq A|\tilde{x}(y) - y| .$$

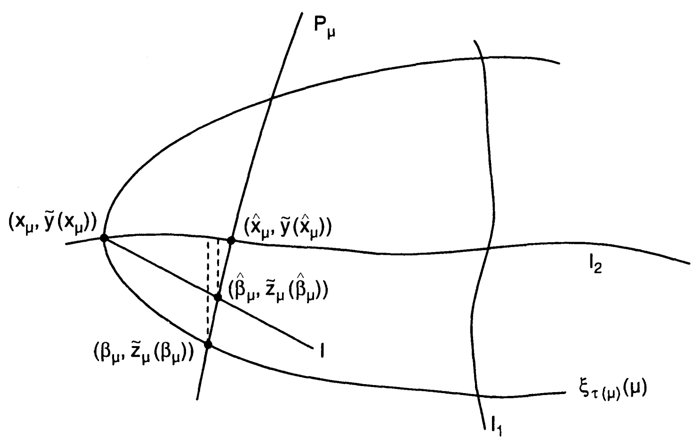
From this we obtain:

$$\left|\frac{\partial_2 f(x,y)}{\partial_1 f(x,y)}\right| \geqq \frac{B}{A}\frac{|y-\tilde{y}(x)|}{|\tilde{x}(y)-x|} . \tag{8}$$

Now suppose that a constant λ independent of μ was found such that:

$$\left|\frac{\tilde{y}(x)-y}{\tilde{x}(y)-x}\right| \geqq \frac{A\lambda}{B} \tag{9}$$

for any point (x, y) of intersection of P_μ with $\xi_{\tau(\mu)}(\mu)$. It follows that the same estimate is valid for any other point in $P_\mu \cap \xi_{\tau(\mu)}(\mu)$ (this can easily be seen using that the tangent vector to P_μ is almost vertical at points not approaching l_1, see the figure). In fact, what we will show is that (9) is valid for $(x, y) = (\beta_\mu, \tilde{z}_\mu(\beta_\mu))$, the point of intersection of P_μ with $\xi_{\tau(\mu)}(\mu)$ located at Q_1. For the other points in $P_\mu \cap \xi_{\tau(\mu)}(\mu)$ the reasoning is similar.



Let's begin estimating the numerator of (9): The level curve $\xi_{\tau(\mu)}(\mu)$ is given by the equation $f_\mu(x, y) = \tau(\mu)$ which defines a function $X(y)$ in a neighborhood of the point $(x_\mu, \tilde{y}(x_\mu))$ such that: $X(\tilde{y}(x_\mu)) = x_\mu$, $f_\mu(X(y), y) = \tau(\mu)$ and therefore:

$$X'(y) = -\frac{\partial_2 f(X(y), y)}{\partial_1 f(X(y), y)} \,. \tag{10}$$

Derivating once more we can easily obtain that $X''(y) < 0$; thus, we can assume that

$$\left|\frac{\partial_2 f(X(y), y)}{\partial_1 f(X(y), y)}\right| \leqq \lambda \,, \tag{11}$$

because the contrary assumption trivially implies the lemma. As $X''(y) > 0$, Eqs. (10) and (11) imply that $X'(y) \leqq \lambda$, for every $|y - \tilde{y}(x_\mu)| \leqq X^{-1}(\hat{x}_\mu)$, where for $X^{-1}(\hat{x}_\mu)$ we denote that preimage of $\hat{x}_\mu$ contained in Q_1. Now this implies that for $y \in (\tilde{y}(x_\mu), X^{-1}(x_\mu))$:

$$|X(y) - x_\mu| \leqq \lambda |y - \tilde{y}(x_\mu)| \,. \tag{12}$$

Let l be the line $x - x_\mu = -\lambda(y - \tilde{y}(x_\mu))$. It follows that the vertical distance from $(\hat{x}_\mu, \tilde{y}(\hat{x}_\mu))$ to l is

$$\tilde{y}(\hat{x}_\mu) - y = \frac{\hat{x}_\mu - x_\mu}{\lambda} \,. \tag{13}$$

Now, if $(\hat{\beta}_\mu, \tilde{z}_\mu(\hat{\beta}_\mu))$ is the point of intersection of P_μ with l, then it follows from (12) that

$$\tilde{y}(\beta_\mu) - \tilde{z}_\mu(\beta_\mu) \geqq \tilde{y}(\hat{\beta}_\mu) - \tilde{z}_\mu(\hat{\beta}_\mu) . \tag{14}$$

But $\hat{\beta}_\mu$ can be estimated easily, because P_μ can be substituted by the line $y - \tilde{y}(\hat{x}_\mu) = -2B_0\hat{x}_\mu(x - \hat{x}_\mu)$ (this follows from the fact that $|\tilde{z}'_\mu(x)| > -2B_0\hat{x}_\mu$ for $x < \hat{x}_\mu$), and this gives, just intersecting this line with l:

$$\hat{\beta}_\mu - \hat{x}_\mu \leqq \frac{y - \tilde{y}(\hat{x}_\mu)}{-2B_0\hat{x}_\mu} = \frac{\tilde{y}(x_\mu) - \tilde{y}(\hat{x}_\mu) - 1/\lambda(\hat{\beta}_\mu - x_\mu)}{-2B_0\hat{x}_\mu} ,$$

and following:

$$|\hat{\beta}_\mu - \hat{x}_\mu| = \left| \frac{\tilde{y}(x_\mu) - \tilde{y}(\hat{x}_\mu) + 1/\lambda(x_\mu - \hat{x}_\mu)}{2B_0\hat{x}_\mu(1 + 1/\lambda)} \right| \leqq \frac{(1/\lambda + 1/K^2)|x_\mu - \hat{x}_\mu|}{|2B_0\hat{x}_\mu(1 + 1/\lambda)|} . \tag{15}$$

Finally, using (13) and (15) we obtain:

$$\begin{aligned} \tilde{y}(\hat{\beta}_\mu) - \tilde{z}_\mu(\hat{\beta}_\mu) &\geqq 1/\lambda(\hat{x}_\mu - x_\mu) - (1/K^2 + 1/\lambda)(\hat{x}_\mu - \hat{\beta}_\mu) \\ &\geqq \left(1/\lambda - \frac{(1/K^2 + 1/\lambda)^2}{|2B_0\hat{x}_\mu(1 + 1/\lambda)|}\right)(\hat{x}_\mu - x_\mu) . \end{aligned}$$

Therefore we can take μ large in such a way that

$$\tilde{y}(\hat{\beta}_\mu) - \tilde{z}_\mu(\hat{\beta}_\mu) \geqq \frac{\hat{x}_\mu - x_\mu}{2\lambda} .$$

This provides, using also (14), an estimate for $\tilde{y}(\beta_\mu) - \tilde{z}(\beta_\mu)$.

Now join this with (8) and the fact that the horizontal distance from $(\beta_\mu, \tilde{z}_\mu(\beta_\mu))$ to l_1 is less than $|x_\mu|$ to obtain that:

$$\left| \frac{\partial_2 f(\beta_\mu, \tilde{z}_\mu(\beta_\mu))}{\partial_1 f(\beta_\mu, \tilde{z}_\mu(\beta_\mu))} \right| \geqq \frac{B}{2A\lambda} \frac{\hat{x}_\mu - x_\mu}{-x_\mu} = \frac{B}{2A\lambda}\left(1 - \frac{\hat{x}_\mu}{x_\mu}\right) .$$

Thus, using the estimate for x_μ and $\hat{x}_\mu$ obtained in the previous lemma it follows that, for μ sufficiently large,

$$\left| \frac{\partial_2 f(\beta_\mu, \tilde{z}_\mu(\beta_\mu))}{\partial_1 f(\beta_\mu, \tilde{z}_\mu(\beta_\mu))} \right| \geqq \frac{B}{2A\lambda}(1 - \sqrt{B_0/A_0}) \geqq \frac{B}{4A\lambda} > K/4\lambda > \sqrt{K}/4 .$$

For the last step to work, we make $\lambda < \sqrt{K}$, so for any λ satisfying this, the lemma is proved (recall (9)). In particular, we can take $\lambda = 10$ if K is large enough.

This provides the necessary techniques to obtain stable foliations.

Lemma 3.5. *Let τ be a C^1 function of μ such that $\tau'(\mu) \to 0$ at infinity. Let $R_\mu = \tilde{P}_\mu \cap e(\xi_{\tau(\mu)}(\mu))$ and define $G_\mu = \bigcap_{n \geqq 0} F_\mu^{-n}(R_\mu)$. Then, if μ is sufficiently large, there exists a C^1 stable foliation of G_μ invariant under F_μ.*

Proof. Fix any μ large enough and drop the index μ. Observe first that $F(G) \subset G$. Define, for each $x \in G$ a cone $C_x = \{(u,v) : |v/u| < \varepsilon\}$ where ε is a positive number to be chosen. Now, for $(u, v) \in C_{F(x)}$ we have:

$$DF^{-1}_{F(x)}(u,v) = \frac{-1}{\partial_1 f}(u\partial_2 f - v, -u\partial_1 f) = (u_1, v_1)\,, \tag{16}$$

where the derivatives are calculated at $F(x)$. Furthermore

$$\left|\frac{v_1}{u_1}\right| = \left|\frac{u\partial_1 f}{u\partial_2 f - v}\right| = \left|\frac{\partial_1 f}{\partial_2 f - v/u}\right| \leqq \left|\frac{\partial_1 f}{\partial_2 f/2}\right|$$

if $\varepsilon < |\partial_2 f|/2$. But $F(x) \in G \subset e(\xi_{\tau(\mu)}(\mu))$ so that the previous lemma can be applied to obtain:

$$\left|\frac{v_1}{u_1}\right| \leqq 2/\lambda < \varepsilon\,,$$

if $\varepsilon = 3/\lambda$. This ε also satisfies $\varepsilon < |\partial_2 f|/2$ if μ is sufficiently large, because $\lambda (= 10)$ is independent of μ, while $|\partial_2 f| \to \infty$ for points in $e(\xi_{\tau(\mu)}(\mu))$. This proves that $(u_1, v_1) \in C_x$ if $(u, v) \in C_F(x)$. In addition, using (16):

$$\begin{aligned} |(u_1, v_1)| = |u_1| + |v_1| &= \frac{|u\partial_2 f - v| + |u\partial_1 f|}{|\partial_1 f|} \\ &\geqq \frac{|u|(|\partial_2 f| - |u/v| + |\partial_1 f|)}{|\partial_1 f|} \geqq \frac{|u|}{2}\left|\frac{\partial_2 f}{\partial_1 f}\right| \\ &\geqq \frac{\lambda}{2}|u| \geqq \frac{\lambda}{2}\frac{|u| + |v|}{1+\varepsilon} = \frac{\lambda}{2(1+\varepsilon)}|(u,v)| > 2|(u,v)|\,. \end{aligned}$$

This proves that DF^{-1} leaves the family of cones invariant and expands the length. As it is known this implies the existence of the foliation (see [HPS]), thus proving the lemma.

Proof of Theorem 1.2.

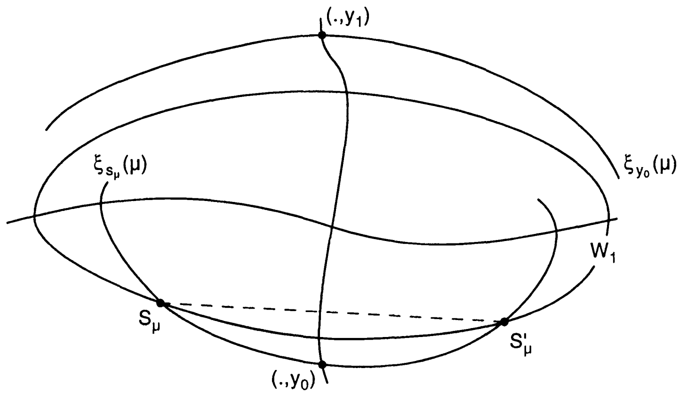
Step 1. W^s_μ has infinitely many connected components.

It is known, by Lemma 3.2, that the connected component of $W^s_\mu \cap Q_1$ containing S_μ (parametrized by the curve α), is transversal to the family of level curves ξ. This means that $\alpha(t) \in e(\xi_{s_\mu}(\mu))$ for $t < 0$, because $f(\alpha(0)) = s_\mu$. In addition, by Lemma 3.1, it follows that $\lim_{\mu\to\infty} s'_\mu = 0$, and thus Lemma 3.3 (with s_μ in place of τ), can be applied to obtain that $\xi_{s_\mu}(\mu)$ intersects P_μ in Q_1. Joining these facts it follows that α also intersects P_μ unless it doesn't reach l_2 or P_μ. But in this latter case we will find a contradiction: firstly, this implies that there is a two periodic orbit $\{p_1, p_2\}$ such that p_1 and p_2 are the extreme points of α. Now it follows that the direction given by the tangent to α at p_1, is non-contracting. On the other hand, observe that:

$$\left|\frac{\alpha'_2(t_1)}{\alpha'_1(t_1)}\right| < \left|\frac{\partial_1 f}{\partial_2 f}\right| < \lambda^{-1}\,,$$

where t_1 is such that $\alpha(t_1) = p_1$ and the last inequality follows from Lemma 3.4. Now the equation above implies that the tangent direction to α at p_1 is contained in the stable cones as defined in the previous lemma: so this direction is contracting, and we find a contradiction.

Until now we have thus proved that α (and so also W_μ^s) intersect P_μ at one point. Let's denote by α_1 the curve $F^{-1}(\alpha)\backslash\alpha$ and let's show that it also intersects P_μ: in fact, let S_μ' be the preimage of S_μ which is not S_μ. The image of that part of α_1 that lies between l_1 and S_μ', is located above S_μ, and this implies that α_1 is outside $\xi_{s_\mu}(\mu)$ between l_1 and S_μ'. At S_μ', α_1 intersects $\xi_{s_\mu}(\mu)$, and after this, α_1 is contained in $e(\xi_{s_\mu}(\mu))$, so that Lemmas 3.3 and 3.4 can be used as before to obtain that α_1 also intersects P_μ. Therefore, we have proved that W_μ^s contains a C^1 curve intersecting P_μ transversally at a pair of points, which implies that W_μ^s contains a closed simple C^1 curve that contains the point S_μ, and that will be denoted by W_1.



Let y_0 be the second coordinate of the intersection of $\xi_{s_\mu}(\mu)$ with l_1. It is clear by Lemma 3.2 that W_1 is contained in $\{(x,y) : y > y_0\}$. As the image of W_1 is contained in W_1, it follows that $W_1 \subset i(\xi_{y_0}(\mu))$. Now let's calculate the dependence of y_0 on μ : y_0 must satisfy the equation $f_\mu(\tilde{x}(y_0), y_0) = s_\mu$, hence it follows that:

$$y_0'(\mu) = -\frac{1}{\partial_2 f_\mu(\tilde{x}(y_0), y_0)} .$$

This implies, as in the proof of Lemma 4.2, that $y_0'(\mu) \to 0$ as $\mu \to \infty$. Therefore Lemma 4.2 can be applied to y_0 in place of τ to obtain that $\xi_{y_0}(\mu)$ intersects P_μ at four points and so W_1 also intersects P_μ in four points. This means that the preimage $F^{-1}(W_1)$ contains another closed simple C^1 curve that will be denoted by W_2. Now we will prove that W_2 also intersects P_μ at four points. To do this apply the same idea as before: first observe that $W_1 \subset \{(x,y) : y < y_1\}$, where y_1 is the maximum of the second coordinates of points in $\xi_{y_0}(\mu)$, then it follows that W_2 has to be contained in $e(\xi_{y_1}(\mu))$, so it suffices to show that $y_1' \to 0$ and use Lemma 3.3. In fact y_1 satisfies the equation $f_\mu(\tilde{x}(y_1), y_1) = y_0$ so that $1 + \partial_2 f(\tilde{x}(y_1), y_1) y_1' = y_0'$, which implies that $y_1'(\mu) \to 0$ as $\mu \to \infty$, thus Lemma 3.3 says that $\xi_{y_1}(\mu)$ (and so also W_2) intersects P_μ at four points. Thus the preimage of W_2 has also two simple closed C^1 curves as preimages, which, by simple inspection of the location of preimages must be both contained in $e(W_2)$ and $i(W_1)$. Furthermore each one of these new curves must intersect P_μ at four points, and so each one has a pair of curves as preimage, and so on. This implies that W_μ^s has infinitely many components, each one of which is a closed C^1 curve.

Step 2. The complementary set of the closure of W_μ^s is the basin of ∞, that is, the set of points with empty ω limit set. If we prove that $e(W_1)$ is contained in the basin of ∞ then it will follow that $i(W_2) = F^{-1}(e(W_1))$ is also contained in the basin of ∞. Now the preimage of this open disc is an annulus whose boundary is the preimage of W_2. It follows that W_μ^s accumulates on the complementary set of the basin of ∞; as this is an open set, Step 2 is proved; so what we must show is that $e(W_1)$ is contained in the basin of ∞. Every point in $e(W_1)$ must also lie in $e(\xi_{y_0}(\mu))$ so that Lemma 3.5 can be applied to obtain a stable foliation each of whose leaves intersect P_μ. This induces a one dimensional map from P_μ into itself, that has a fixed point corresponding to S_μ, and either carries every point to ∞ or has another fixed point. But the latter case is impossible because it would imply the existence of another fixed point of F_μ with negative coordinates (recall Lemma 3.1).

To finish the proof of Theorem 1.2 it remains to show that the closure of W_μ^s is a Cantor set of closed curves. To do this we will need an unstable foliation defined outside the curve W_2.

Lemma 3.6. *Let μ be sufficiently large and define $H = \bigcap_{n\geqq 0} F_\mu^n(\tilde{P}_\mu)\setminus \bigcup_{n\geqq 0} F^n(i(W_2))$. Then there exists an unstable, almost vertical, C^1 foliation defined on H and invariant under F.*

Proof. First observe that if $x \in H$, then a preimage of x is contained in H. For each point in H define a cone $C = \{(u,v) : u/v < \varepsilon\}$, where ε is a small number to be defined. Take $(u,v) \in C$ and $x \in H$; then, calculating $DF_x(u,v) = (u_1, v_1)$, we obtain:

$$|u_1/v_1| = \left|\frac{v}{u\partial_1 f + v\partial_2 f}\right| \leqq \frac{1}{|\partial_2 f| - |\partial_1 f||u/v|} \leqq \frac{1}{|\partial_2 f| - \varepsilon\lambda^{-1}|\partial_2 f|}$$

$$\leqq \frac{1}{|\partial_2 f|/2} \leqq \varepsilon\,, \tag{17}$$

where Lemma 3.4, was used and $\varepsilon = 3/B$. This proves that $(u_1, v_1) \in C_{F(x)}$ for $(u,v) \in C_x$. Furthermore:

$$|(u_1, v_1)| = |u_1| + |v_1| = |v| + |u\partial_1 f + v\partial_2 f| \geqq |v|(1 + |\partial_2 f| - |\partial_1 f||u/v|)$$

$$\geqq \frac{|v||\partial_2 f|}{2} > \frac{|\partial_2 f|}{2(1+\varepsilon)}|(u,v)|\,. \tag{18}$$

It follows that DF expands the length of vectors in the cones and the lemma follows by the results of [HPS].

Define $I_1 = \overline{i(W_1)} \cap P_\mu$ and $I_2 = F(I_1) \cap \overline{i(W_1)}$, ($\bar{A}$ denotes the closure of A). I_1 is the union of two curves and I_2 is the union of at most four curves. What we must show is that $\overline{W_\mu^s} \cap I_1$ is a Cantor set.

Observe that the stable foliation obtained in Lemma 3.5 can be extended to $\tilde{P}_\mu \setminus \bigcup_{n\geqq 0} F_\mu^{-n}(i(W_2)) = \tilde{P}_\mu \cap \overline{W_\mu^s}$ because $i(W_2) \supset i(\xi_{y_1}(\mu))$ and $y_1'(\mu) \to 0$ as $\mu \to \infty$, which was shown in Step 1. This defines a map π which carries points in $\overline{W_\mu^s} \cap I_2$ to I_1 along the leaves of the stable foliation. Now the proof will be completed by observing the three following facts:

1. The map F restricted to $I_1 \cap F^{-1}(I_2)$ is an expansive map because I_1 and I_2 are almost vertical lines and Lemma 3.6 can be applied. This implies that this

restriction of F satisfies bounded distortion properties and so it preserves cross ratios of intervals (this is a well known fact).

2. The map π has been defined as induced by a stable foliation of a C^2 map, F_μ. This implies that π also has to satisfy bounded distortion properties (this is an observation of Newhouse that can be found in [PT]). Now, as above, the map π also preserves cross ratios.

3. Maps which preserve cross ratios of intervals define Cantor sets (this is a simple fact).

The proof of Theorem 1.2 is complete.

Proof of Theorem 1.3. Fix any large value of μ, suppose first that there exists some integer $n > 0$ such that F restricted to $F^n(R^2)$ is one to one. Then obviously F/Λ is a homeomorphism. (Recall that $\Lambda = \bigcap_{n \geqq 0} F^n(\overline{W_\mu^s})$.) To prove that F/Λ is a shift we proceed as for a horseshoe: first give an itinerary $j(x) \in 2^z$ to each x in Λ and then prove that j conjugates F/Λ with the shift. To obtain the hyperbolicity just use the foliations shown to exist in Lemmas 3.5 and 3.6. If there is no $n > 0$ such that $F/F^n(R^2)$ is one to one, then it follows that the unstable manifolds of the fixed points must coincide because there is a contraction in the horizontal direction. Now Λ is contained in the unstable manifold of S_μ (and of the other fixed point). Finally, the hyperbolicity follows from Lemma 3.6 and the fact that these unstable manifolds have to be contained in the unstable foliation.

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