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On the Limiting Solution of the Bartnik-McKinnon Family

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Abstract: We analyze the limiting solution of the Bartnik-McKinnon family and show that its exterior is an extremal Reissner-Nordstrøm black hole and not a new type of non-abelian black hole as claimed in a recent article by Smoller and Wasserman.

The purpose of this short communication is to correct some erroneous statements made in a recent article by J.A. Smoller and A.G. Wasserman [1]. This article concerns the limiting behaviour of an infinite discrete family of regular, static, spherically symmetric solutions of the Einstein-Yang-Mills equations (gauge group *SU(2)*), whose first few members were discovered by Bartnik and McKinnon [2]. A general existence proof for this family was given by Smoller and Wasserman [3] and by the present authors together with P. Forgács [4].

In their article [1] the authors claim that a suitable subsequence of the infinite family converges to some limiting solution for all values of the radial coordinate $r \neq 1$. The part of this limit defined for $r > 1$ is interpreted as a new type of black hole solution with event horizon at $r = 1$. According to their claim the function $W(r)$ parametrizing the Yang-Mills potential is non-trivial, i.e., $W \not\equiv 0$ and tends to +1 or -1 for $r \longrightarrow \infty$. In contrast we claim that the limiting solution for $r > 1$ is given by the extremal Reissner-Nordstrøm (RN) solution with $W \equiv 0$. This can be easily derived from the results of our article [4] and is also strongly supported by numerical calculations. Subsequently we shall give a proof of this claim using the results of [4].

First we recall some definitions and results of [4]. The variables T, A, μ, w, and λ used in [1,3] correspond to the quantities $(AN)^{-1}$, μ, $2m$, W, and $2b$ in [4] and in this article. We parametrize the line element in the form

$$ds^2 = A^2(r)\mu(r)dt^2 - \frac{dr^2}{\mu(r)} - r^2 d\Omega^2 , \tag{1}$$

and use the 'Abelian gauge'

$$W^a_\mu T_a dx^\mu = W(r)(T_1 d\theta + T_2 \sin\theta d\varphi) + T_3 \cos\theta d\varphi \ , \tag{2}$$

for the static, spherically symmetric $SU(2)$ Yang-Mills field.

The field equations for A, μ, and W (see, e.g., Eqs. (6) in [4]) are singular at $r = 0$ and $r = \infty$ as well as for $\mu(r) = 0$. In order to desingularize them when $\mu \to 0$ we introduce $N = \sqrt{\mu}$, $U = NW'$, a new independent variable τ (with $\dot{} = d/d\tau$), and $\kappa = (\ln rAN)\dot{}$ as additional dependent variable. The field equations are then equivalent to the autonomous first order system

$$\dot{r} = rN \ , \tag{3a}$$

$$\dot{W} = rU \ , \tag{3b}$$

$$\dot{U} = \frac{W(W^2-1)}{r} - (\kappa - N)U \ , \tag{3c}$$

$$\dot{N} = (\kappa - N)N - 2U^2 \ , \tag{3d}$$

$$\dot{\kappa} = 1 + 2U^2 - \kappa^2 \ , \tag{3e}$$

$$(AN)\dot{} = (\kappa - N)AN \ , \tag{3f}$$

subject to the constraint

$$2\kappa N = 1 + N^2 + 2U^2 - (W^2-1)^2/r^2 \ . \tag{4}$$

If the initial data satisfy this constraint then it remains true for all τ.

There exists a one-parameter family of local solutions with regular origin where $W(r) = 1 - br^2 + O(r^4)$, $\mu(r) = 1 + O(r^2)$ such that $W(r)$ and $\mu(r)$ are analytic in r and b. If we adjust τ such that $\tau = \ln r + O(r^2)$ we obtain a one-parameter family of local solutions of the system (3) which satisfy the constraint (4) and are analytic in τ and b.

Similarly there exists a two-parameter family of local black hole solutions with $W(r) = W_h + O(r - r_h)$, $\mu(r) = O(r - r_h)$ such that $W(r)$ and $\mu(r)$ are analytic in r, r_h, and W_h. If we adjust τ such that $\tau = 0$ at the horizon we obtain a two-parameter family of solutions of (3, 4) analytic in τ, r_h, and W_h except for a simple pole in $\kappa(\tau)$ at the horizon.

Both types of initial data satisfy $\kappa \geq 1$ and this relation remains true for all τ due to the form of Eq. (3e).

In the following we exclude the case $W \equiv 0$ and can therefore assume $(W, U) \neq (0, 0)$ for all (finite) τ. Integrating Eqs. (3) with regular initial data $r(\bar{\tau}) > 0$, $N(\bar{\tau}) > 0$, $\kappa(\bar{\tau}) \geq 1$ satisfying the constraint (4) we obtain solutions analytic for all $\tau > \bar{\tau}$ as long as $N > -\infty$. There are three possible cases:

i) $N(\tau)$ has a zero at some $\tau = \tau_0$, the generic case. Then

$$(W^2(\tau_0) - 1)^2 = \left(1 + 2U^2(\tau_0)\right) r^2(\tau_0) \ , \tag{5}$$

and r has a maximum at $\tau = \tau_0$. For $\tau > \tau_0$ we find that $N < 0$ and r, W, U, κ, rN, and rAN remain analytic at least as long as $r \geq 0$.

ii) $N(\tau) > 0$ for all τ and $r(\tau)$ tends to infinity for $\tau \to \infty$. These are the asymptotically flat solutions with $(W, U, N, \kappa) \to (\pm 1, 0, 1, 1)$.

iii) $N(\tau) > 0$ for all τ and $r(\tau)$ remains bounded. This is a new type of 'oscillating' solution with $(r, W, U, N, \kappa, A) \to (1, 0, 0, 0, 1, \infty)$ for $\tau \to \infty$ first discussed in detail in [4].

Analyzing the solutions with regular origin and their dependence on b we have shown in [4]:

1. For each positive integer n there exists a globally regular and asymptotically flat solution with n zeros of W for at least one value $b = b_n$ and there is at most a finite number of such values b_n.
2. There exists an oscillating solution for at least one value $b = b_\infty$ and there is at most a finite number of such values b_∞.
3. The values b_n have at least one accumulation point for $n \to \infty$ and each such accumulation point is one of the values b_∞.

Completely analogous results hold for black hole solutions with fixed $r_h < 1$ and their dependence on W_h.

Let us analyze the oscillating solutions in some detail. Near the singular point $(r, W, U, N, \kappa) = (1, 0, 0, 0, 1)$ we introduce the parametrization (with $\overline{W} = \frac{W}{r}$ and $\overline{\kappa} = \kappa - 1$)

$$\overline{W}(\tau) = C_1 e^{-\frac{1}{2}\tau} \sin(\frac{\sqrt{3}}{2}\tau + \theta) , \tag{6a}$$

$$U(\tau) = C_1 e^{-\frac{1}{2}\tau} \sin(\frac{\sqrt{3}}{2}\tau + \frac{2\pi}{3} + \theta) , \tag{6b}$$

$$N(\tau) = C_2 e^{\tau} + \frac{2}{7}(\overline{W}^2 - U\overline{W} + 2U^2) , \tag{6c}$$

$$\overline{\kappa}(\tau) = C_4 e^{-2\tau} + \overline{W}^2 + 2U\overline{W} + 2U^2 , \tag{6d}$$

as in [4] and compute r from the constraint (4)

$$r^{-2} = \rho + \sqrt{\rho^2 - \overline{W}^4} , \quad \text{where} \quad \rho = \frac{1}{2}(1 - N)^2 + \overline{W}^2 + U^2 - \overline{\kappa} N . \tag{7}$$

The functions θ, C_1, C_2, and C_4 satisfy differential equations,

$$\dot{\theta} = f_0 , \tag{8a}$$

$$(C_1^2 e^{-\tau})^{\cdot} = C_1^2 e^{-\tau}(-1 + f_1) , \tag{8b}$$

$$(C_2 e^{\tau})^{\cdot} = C_2 e^{\tau} + f_2 , \tag{8c}$$

$$(C_4 e^{-2\tau})^{\cdot} = -2C_4 e^{-2\tau} + f_4 , \tag{8d}$$

with 'non-linear' terms f_i that can be expressed as homogeneous polynomials in $C_1^2 e^{-\tau}$, $C_2 e^{\tau}$, and $C_4 e^{-2\tau}$ of degree one for f_0 and f_1 and of degree two for f_2 and f_4 with (r, θ)-dependent coefficients that are bounded as long as r is bounded.

We can apply a general result for perturbed linear systems (see, e.g., [5] p.330) stating the existence of a stable manifold. The system (8) has one unstable mode, $C_2 e^{\tau}$, and hence there exists a three-dimensional stable manifold of initial data, i.e., quadruples $Y = (\overline{W}, U, N, \overline{\kappa})$ such that $Y \to 0$ for $\tau \to \infty$. Eliminating the freedom to add a constant to τ we are left with a two-parameter family of oscillating solutions. In [4] we have derived the stronger result that θ and C_1 have a limit for $\tau \to \infty$ (with $C_1(\infty) \neq 0$) whereas $C_2 e^{2\tau} \to 0$ and $C_4 e^{-\tau} \to 0$ for each member of this two-parameter family. Consequently these oscillating solutions have infinitely many zeros of W and inifinitely many minima of N as $r \to 1$.

Conversely there exists a one-dimensional 'unstable manifold' (i.e., stable manifold for decreasing τ) of initial data such that $Y \to 0$ for $\tau \to -\infty$. These initial data $Y = (0, 0, N, 0)$ describe the extremal RN black hole with $r = (1 - N)^{-1}$.

In the following we analyze the behaviour of solutions for b near (one of the values) b_∞ and in particular the behaviour of globally regular solutions with n zeros of W in the limit $b_n \longrightarrow b_\infty$ for $n \longrightarrow \infty$. In view of the analytic dependence of the solutions on b and τ the trajectories reach any given neighbourhood of the singular point $Y = 0$ for b sufficiently close to b_∞. Trajectories missing the singular point cannot stay near it, they must start to 'run away.' They will, however, remain close to the unstable manifold. In the limit $b_n \longrightarrow b_\infty$ they converge to the unstable manifold, i.e., extremal RN solution.

We can decompose Y into its parts parallel and perpendicular to the unstable manifold and measure the distance from the singular point $Y = 0$ by

$$|Y| = \max(|Y_{\|}|, |Y_{\perp}|) , \quad \text{with} \quad |Y_{\|}| = |N| , \quad |Y_{\perp}| = \max(C_1^2 e^{-\tau}, |\overline{\kappa}|) . \tag{9}$$

Using the distance function $|\cdot|$ we get from the smooth dependence of the solutions on b and τ that all solutions with $b \approx b_\infty$ must come close to the singular point $Y = 0$ for some $\tau = \tau_0$.

Lemma 1. *Given b_∞ and any $\epsilon > 0$ there exist some $\delta > 0$ and τ_0 such that all solutions with $|b - b_\infty| < \delta$ satisfy $|Y|(\tau_0) < \epsilon$ and $0 < 1 - r(\tau_0) < \epsilon$.*

Let us analyze the behaviour of these trajectories in the neighbourhood of $Y = 0$. The general result [5] also states the existence of some $\eta > 0$ such that trajectories missing the singular point cannot stay within $|Y| < \eta$ for all τ. Due to the structure of Eqs. (3), resp. (8) this runaway is caused by the growth of N. The trajectories can therefore be characterized by three possibilities: They either run into the singular point $Y = 0$ or miss it on one or the other side; in the latter case either N stays positive and r grows beyond $r = 1$ or N has a zero while $r < 1$ and r runs back to $r = 0$. This is expressed by

Lemma 2. *There exists some $\eta > 0$ such that for any solution of Eqs. $(3a - e, 4)$ with $|Y| < \epsilon \ll \eta$ and $0 < 1 - r < \epsilon$ at some $\tau = \tau_0$ there are three possible cases:*
a) $r < 1$, $N > 0$ for all $\tau > \tau_0$ and $Y \longrightarrow 0$ for $\tau \longrightarrow \infty$,
b) $r = 1$ for some $\bar{\tau} > \tau_0$, $N = \eta$ for some $\tau_1 > \bar{\tau}$, $\dot{N}(\tau_1) > 0$, and $0 < N < \eta$, $|Y_\perp| < \epsilon$ for $\tau_0 < \tau < \tau_1$,
c) $N = 0$ for some $\bar{\tau} > \tau_0$, $N = -\eta$ for some $\tau_1 > \bar{\tau}$, $\dot{N}(\tau_1) < 0$, and $r < 1$, $|N| < \eta$, $|Y_\perp| < \epsilon$ for $\tau_0 < \tau < \tau_1$.

Proof. The general result [5] mentioned above states the existence of some $\eta > 0$ such that that either $Y \longrightarrow 0$ (case **a**) or $|Y| = \eta$ for some τ_1 (case **b** and **c**). Choosing η small enough, Eq. (8b) shows that the 'amplitude' $|C_1|e^{-\tau/2}$ decreases as long as $|Y| < \eta$. Moreover Eq. (3e) implies that $|\overline{\kappa}| < \epsilon$ remains true as long $U^2 < \epsilon$. Therefore $|Y_\perp| < \epsilon$ as long as $|N| < \eta$.

Next, if $|N| \gg |Y_\perp|$ then $\dot{N} \approx (1 - N)N$ due to Eq. (3d) and $r \approx (1 - N)^{-1}$ due to Eq. (7), i.e., $r > 1$ and $\dot{N} > 0$ for $N \gg \epsilon$, resp. $r < 1$ and $\dot{N} < 0$ for $N \ll -\epsilon$. Finally, Eq. (5) implies that N can vanish only when $r < 1$.

To conclude the argument we analyze what happens to the solutions in the limit $b \longrightarrow b_\infty$.

Proposition 3. *Given b_∞ and η as defined above there exists some $\delta > 0$ such that the solutions with regular origin and $|b - b_\infty| < \delta$ satisfy:*
*1. Case **a** of Lemma 2 holds if and only if $b = b_\infty$. There exist continuous functions $\bar{\tau}(b) < \tau_1(b)$ defined for $b \neq b_\infty$ such that the same case either **b** or **c** holds for all*

$b < b_\infty$ and for all $b > b_\infty$ (with $|b - b_\infty| < \delta$); case **b** *holds in particular for the globally regular solutions with n zeros of W as $b_n \to b_\infty$ for $n \to \infty$.*

2. *In the limit $b \to b_\infty$ both $\bar\tau$ and $\tau_1 - \bar\tau$ diverge. The part of the solution defined for $\tau < \bar\tau$ converges for any fixed τ or $r < 1$ to the oscillating solution. The part defined for $\tau > \bar\tau$ converges for any fixed $\tau - \tau_1$ or $r \neq 1$ to the exterior, resp. interior of the extremal RN solution with $W \equiv 0$ in case* **b**, *resp.* **c**.

Proof.

1. Since an oscillating solution exists only for finitely many values of b, we can choose $\delta > 0$ in Lemma 1 such that the interval $|b - b_\infty| < \delta$ contains only one of them, namely b_∞. The existence of $\bar\tau$ and τ_1 for $b \neq b_\infty$ was shown in Lemma 2. The rest follows from the continuity of the solutions in b.
2. The convergence of the solutions follows from the convergence of the initial data, i.e., quadruples Y at an arbitrary regular point. The initial data for any fixed τ converge to those of the oscillating solution. At the same time $\bar\tau$ (with $r(\bar\tau) = 1$, resp. $N(\bar\tau) = 0$) diverges. On the other hand $Y(\tau_1)$ converges to $(0, 0, \pm\eta, 0)$, i.e., to initial data for the exterior or interior of the extremal RN black hole and $\bar\tau - \tau_1 \to -\infty$. Convergence for fixed r requires in addition $N \neq 0$; given $r \neq 1$ this is satisfied for b sufficiently close to b_∞.

Using exactly the same arguments one obtains

Corollary. *Analogous results hold true for black hole solutions with any fixed $r_h < 1$ and W_h, W_{hn}, $W_{h\infty}$ replacing b, b_n, b_∞.*

Having shown the incorrectness of the statements made by Smoller and Wasserman in [1] about the limiting solution one may ask for the source of this error. Looking at their arguments one finds that they use Prop. 3.2 of their earlier work [3] in an essential way. This proposition is, however, wrong as it stands; its validity requires the further assumption of a uniformly bounded rotation number (as made for their Prop. 3.1). This additional assumption is not satisfied for the Bartnik-McKinnon family.

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