Commun. Math. Phys. 170, 669 – 698 (1995)

Communications in
Mathematical
Physics

Two Dimensional Lattice Gauge Theory Based on a Quantum Group

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Received: 26 May 1994

Abstract: In this article we analyse a two dimensional lattice gauge theory based on a quantum group. The algebra generated by gauge fields is the lattice algebra introduced recently by A.Yu. Alekseev, H. Grosse and V. Schomerus in [1]. We define and study Wilson loops. This theory is quasi-topological as in the classical case, which allows us to compute the correlation functions of this theory on an arbitrary surface.

1. Introduction

Quantum groups appeared in the mid-eighties as hidden algebraic structures generalizing the notion of group symmetries in integrable systems [11]. There are now different definitions of quantum groups which include the local point of view (deformation of the Lie algebra) as well as the global point of view (deformation of the algebra of continuous functions on a Lie group).

The latter provides examples of quantum geometry, and the ordinary tools of differential geometry on Lie groups can be successfully defined and used to study, for example, harmonic analysis on quantum groups [19]. This success has encouraged people to apply these tools to build examples of quantum geometry where the notion of group symmetry is essential: quantum vector spaces, quantum homogeneous spaces, quantum principal fiber bundles [5]. It is then tempting to hope that quantum groups can be used in a much broader area than just integrable models, and could give, as an example, a Yang Mills type theory associated to a quantum group, leading hopefully to new Physics. There has been quite a lot of work dealing with q-deformed Yang Mills theory with a base space being a classical space or a quantum space. These works only deal with the study of what could be called classical configurations of the gauge fields, but do not study the path integral on the space of connections. The work of [5] although perfectly coherent for

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classical configurations requires a finer analysis when it studies path integrals, because the gauge fields, in their work, are living in the deformation of the envelopping algebra and it is not at all obvious to define the path integral on the space of connections taking values not in the Lie algebra but in the envelopping algebra and to show that when the parameter q goes to one, one recovers ordinary quantum Yang Mills.

Another way to cope with the problem of summing over all the configurations is to use a lattice regularization of the theory [7]. This is what has been first described by D.V. Boulatov [4]. He studied there the q-deformation of Wilson lattice formulation of gauge theories in dimension two and three and computed the lattice partition function of these models when q is a root of unity and when the Yang Mills coupling constant goes to 0. He argues that the partition function of this q-deformed lattice model is the Turaev–Viro invariant in dimension 3 and the partition function of $(G/G)_k$ models in dimension 2. This theory is unfortunately not consistent with the gauge invariance and moreover we have shown (unpublished) that it is equivalent, in the two dimensional case, to the undeformed Yang–Mills theory.

In their recent work [1], A. Alekseev & al. have studied a combinatorial quantization of the hamiltonian Chern–Simons theory. They were led to define an algebra of gauge fields on which the gauge quantum group acts. This algebra is an exchange algebra which appears to be a generalization of the discretization of the current algebra found by [2, 3]. The commutation relations between different link variables are the quantization of those found by V.V. Fock & A.A. Rosly [9].

In our present work we will define and study a quantum deformation of a lattice gauge theory on a triangulated surface. The gauge symmetry is described by a deformation of the algebra of continuous functions on a Lie group. The gauge fields living on the links generate an algebra on which coacts the gauge symmetry. Invariance under the gauge symmetry implies that this algebra is the algebra of gauge fields of A. Alekseev, H. Grosse and V. Schomerus. Our presentation of the gauge symmetry is dual to these authors and do not involve any involution. Using this point of view computations are greatly simplified.

Wilson loops in our formalism are obtained using the notion of quantum trace and have the usual properties: invariance under departure point, gauge invariance. We will use them as usual to define the Boltzmann weights of the theory. These Boltzmann weights satisfy the familiar convolution property of two-dimensional Yang Mills theory which implies that the theory is quasitopological, i.e. physical quantities depend only on the topology and the area of the surface. We are able to compute correlation functions in this model; they finally appear to be the q-analog of the correlation functions of ordinary two dimensional Yang Mills theory.

The present work is divided in four sections: in the first section of this work we recall standard properties of two dimensional Yang Mills theory. In the second section we derive and study the algebra of gauge fields using gauge covariance. The third section is devoted to the definition of Wilson loops and the study of their commutation properties. In the fourth section we define and study the analog of standard tools of quantum lattice gauge field theory, i.e. Boltzmann weights, Yang–Mills measure. We compute the correlation functions, i.e. partition functions on Riemann surfaces with punctures.

2. Algebraic Properties of Ordinary Lattice Gauge Theory

In this part we recall algebraic properties of ordinary lattice gauge theories associated to a compact group G.

Let R be a D-dimensional lattice, $R = \mathbf{Z}^D$ and denote by V the set of vertices of R and L the set of oriented links, i.e. the set of couple of nearest neighbour points.

A lattice gauge theory is defined by assigning to each link $l = (i,j)$ an element U_l of G satisfying

$$U_{(i,j)}U_{(j,i)} = 1 \,. \tag{1}$$

These link variables interact in a gauge invariant way. The group of gauge transformations is the group of maps g from V to G, which acts on the set of link variables by:

$$U^g_{(i,j)} = g_i U_{(i,j)} g_j^{-1} \,. \tag{2}$$

A particular set of gauge invariant functions are the Wilson loops.

If $C = (i_1, \dots, i_p, i_1)$ is a loop in R, with (i_j, i_{j+1}) element of L, we will define $U_C = \prod_{j=1}^p U_{(i_j, i_j+1)}$.

Let ϕ be any central function on G, (i.e. $\phi(xy) = \phi(yx), \forall x, y \in G$), then $\phi(U_C)$ is a gauge invariant function, which moreover does not depend on the departure point of C.

Let α be a representation of G, then the character χ_α is a central map.

When $G = SU(n), SO(n)$ it is the custom to associate to each plaquette, i.e. to each elementary loop $P = (i,j,k,l,i)$ a Boltzmann weight

$$w_\beta(U_P) = \exp\left(\frac{\beta}{n} Re(\chi_f(U_P) - n)\right) \,, \tag{3}$$

where f is the fundamental representation of G and n the dimension of f.

Let $d\mu$ be the normalized Haar measure of G; this measure is right and left invariant.

We can define a gauge invariant measure on the set of configurations by:

$$dv(U_{l\in L}) = \prod_P w_\beta(U_P) \prod_{l\in L} d\mu(U_l) \,, \tag{4}$$

where P exhausts the set of all plaquettes, l the set of links, and U_l satisfy relation (1).

One is then interested in the evaluation of mean values such as:

$\langle \psi((U_l)_{l\in I}) dv \rangle$, where I is a finite set of links and ψ a function of the variables $(U_l)_{l\in I}$.

It is also possible to define another Boltzmann weight [14] which includes all the equivalency classes of representations of G and is equivalent to the weight (3) when β goes to infinity or in the continuum limit. This weight is defined by:

$$w_{a,\beta}(U_P) = \sum_{\alpha\in Irr(G)} d_\alpha \chi_\alpha(U_P) e^{-\frac{aC_\alpha}{\beta n}} \,, \tag{5}$$

where we have used the notation $Irr(G)$ for the set of equivalency classes of irreducible representations of G, d_α for the dimension of α, and C_α for the value of the Casimir element in the representation α.

When β goes to infinity the two weights approach the δ function located at the unit element.

In two dimensions it is well known that Yang Mills theory is invariant under area preserving diffeomorphisms, and when the coupling constant of the theory goes to zero then this symmetry is enlarged to the group of all diffeomorphisms of the surface, i.e. Yang Mills theory in two dimensions is a topological field theory [17, 18].

The Boltzmann weight (5) is perfectly suited to describe this invariance nature of the theory because it satisfies an exact block spin transformation, as shown by Migdal [14] i.e.:

$$\int w_{a,\beta}(xy)w_{a',\beta}(y^{-1}z)d\mu(y) = w_{a+a',\beta}(xz)\,. \tag{6}$$

Following the general philosophy of quantum geometry we will now redefine algebraic properties of lattice gauge theory using only the algebra of functions on the set of link variables and on the gauge group.

Let us define for $z \in V$, the group $G_z = G \times \{z\}$ and $\hat{G} = \prod_{z\in V} G_z$.

$\hat{G}$ is the gauge group. Let us now define for $l \in L$, the set $G_l = G \times \{l\}$ and $\mathscr{L} = \prod_{l\in L} G_l$, $\mathscr{L}$ is the set of link variables. The gauge group $\hat{G}$ acts on the set $\mathscr{L}$ by gauge transformations:

$$\begin{aligned} \omega: \quad & \mathscr{L}\times\hat{G} \longrightarrow \mathscr{L} \\ & (U_{(xy)})_{(xy)\in L} \times (g_z)_{z\in V} \mapsto (g_x U_{(xy)} g_y^{-1})_{(xy)\in L}\,. \end{aligned} \tag{7}$$

The general principle of arrows reversing implies that if we define the algebras $\hat{\Gamma} = \mathscr{F}(\hat{G}) = \bigotimes_{z\in V} \mathscr{F}(G_z)$ and $\Lambda = \mathscr{F}(\mathscr{L})$, we have a coaction Ω of the Hopf algebra $\hat{\Gamma}$ on Λ:

$$\Omega : \Lambda \to \Lambda \otimes \hat{\Gamma}\,. \tag{8}$$

It satisfies the usual axiom of right coaction:

$$(\Omega \otimes id)\Omega = (id \otimes \Delta)\Omega\,. \tag{9}$$

By construction it is a morphism of algebras:

$$\Omega(AB) = \Omega(A)\Omega(B) \quad \forall A,B \in \Lambda\,. \tag{10}$$

The algebra $\mathscr{F}(G_{(xy)})$ has a Hilbert basis which consists in the matrix elements $\alpha(U_{(xy)})^j_i$, where α is in $Irr(G)$, and the coaction Ω on them can be written:

$$\Omega(\alpha(U_{(xy)})^i_j) = \alpha((g_x U_{(xy)} g_y^{-1}))^i_j = \sum_{p,q} \alpha(U_{(xy)})^p_q \alpha(g_x)^i_p \alpha(g_y^{-1})^q_j\,. \tag{11}$$

We will systematically use this point of view when dealing with the q-deformed version.

It is trivial to show that an element ϕ of Λ is a gauge invariant function if and only if:

$$\Omega(\phi) = \phi \otimes 1\,. \tag{12}$$

In the next section we study a q-analog of lattice gauge theory. The main principle which will guide us throughout this section is gauge covariance: in technical terms, we define a non-commutative deformation of the algebra $\Lambda = \mathscr{F}(\mathscr{L})$ such that a Hopf algebra $\hat{\Gamma}$ ("the quantum gauge group") coacts on it.

3. Gauge Symmetry and Algebra of Gauge Fields

Let $\mathscr{G}$ be a simple Lie algebra over $\mathbf{C}$ and let q be a complex number non-root of unity, then the Hopf algebra $A = U_q\mathscr{G}$ is quasitriangular. Let $Irr(A)$ be the set of all equivalency classes of finite dimensional irreducible representations. In each of these classes α we will pick out a particular representative α. Let V_α be the vector space on which the representation α acts.

We will denote by $\bar{\alpha}$ (resp. $\tilde{\alpha}$) the right (resp. left) contragredient representation associated to α acting on $V_\alpha^\star$ and defined by: $\bar{\alpha} = {}^t\alpha \circ S$ (resp. $\tilde{\alpha} = {}^t\alpha \circ S^{-1}$). We will also denote by 0 the representation of dimension 1 related to the counit ε.

Due to the quasitriangularity of A there exists an invertible element $R \in A \otimes A$ (the universal R-matrix) such that:

$$\Delta'(a) = R\Delta(a)R^{-1} \ \forall a \in A\,, \tag{13}$$

$$(\Delta \otimes id)(R) = R_{13}R_{23}\,, \tag{14}$$

$$(id \otimes \Delta)(R) = R_{13}R_{12}\,. \tag{15}$$

Let us write $R = \sum_i a_i \otimes b_i$, then the element $u = \sum_i S(b_i)a_i$ is invertible and satisfies [8]:

$$S^2(x) = uxu^{-1}\,, \tag{16}$$

$$uS(u) \text{ is a central element}\,,$$

$$\sum_i b_i u a_i = \sum_i S(b_i)uS(a_i) = 1\,, \tag{17}$$

$$\Delta(u) = (u \otimes u)(R_{21}R_{12})^{-1} = (R_{21}R_{12})^{-1}(u \otimes u)\,. \tag{18}$$

Moreover $U_q\mathscr{G}$ is a ribbon Hopf algebra [16], which means that an invertible element v exists such that:

$$v \text{ is a central element}\,,$$

$$v^2 = uS(u),\ \varepsilon(v) = 1,\ S(v) = v\,, \tag{19}$$

$$\Delta(uv^{-1}) = uv^{-1} \otimes uv^{-1}\,. \tag{20}$$

Let us define the group-like element $\mu = uv^{-1}$ and $\overset{\alpha}{\mu} = \alpha(\mu) \in \mathrm{End}(V_\alpha)$.

The q-dimension of α is defined by $[d_\alpha] = \mathrm{tr}_{V_\alpha}(\overset{\alpha}{\mu}) = \mathrm{tr}_{V_\alpha}(\overset{\alpha}{\mu}{}^{-1})$. We also denote by v_α the complex number $\alpha(v)$, and $\overset{\alpha}{u} = \alpha(u)$.

We define $\overset{\alpha\beta}{R} = (\alpha \otimes \beta)(R) \in \mathrm{End}(V_\alpha \otimes V_\beta)$. Let $(\overset{\alpha}{e}_i)_{i=1\cdots\dim V_\alpha}$ be a particular basis of V_α, and $(\overset{\alpha}{e}{}^i)_{i=1\cdots\dim V_\alpha}$ its dual basis. We will define the linear forms $\overset{\alpha}{g}{}_i^j = \langle \overset{\alpha}{e}{}^j | \alpha(\cdot) | \overset{\alpha}{e}_i \rangle$.

The existence of R implies that they satisfy the exchange relations:

$$\sum_{i,k} \overset{\alpha\beta}{R}{}^{pq}_{ik} \overset{\alpha}{g}{}^{i}_{j} \overset{\beta}{g}{}^{k}_{l} = \sum_{m,n} \overset{\beta}{g}{}^{q}_{n} \overset{\alpha}{g}{}^{p}_{m} \overset{\alpha\beta}{R}{}^{mn}_{jl} \tag{21}$$

which can be simply written:

$$\overset{\alpha\beta}{R}_{12} \overset{\alpha}{g}_1 \overset{\beta}{g}_2 = \overset{\beta}{g}_2 \overset{\alpha}{g}_1 \overset{\alpha\beta}{R}_{12}\,, \tag{22}$$

using the convenient notation: $\overset{\alpha}{g} = \sum_{i,j} \overset{\alpha}{e}_i \otimes \overset{\alpha}{e}{}^j \otimes \overset{\alpha}{g}{}^i_j$. This relation is also equivalent to:

$$\overset{\alpha\beta}{R}{}^{-1}_{21} \overset{\alpha}{g}_1 \overset{\beta}{g}_2 = \overset{\beta}{g}_2 \overset{\alpha}{g}_1 \overset{\alpha\beta}{R}{}^{-1}_{21}\,. \tag{23}$$

Let Γ be the restricted dual of $U_q(\mathcal{G})$; it is by construction a Hopf algebra, generated as a vector space by the elements $\overset{\alpha}{g}{}^i_j$.

The action of the coproduct on these elements is:

$$\Delta(\overset{\alpha}{g}{}^i_j) = \sum_k \langle \overset{\alpha}{e}{}^i | \alpha(\cdot) | \overset{\alpha}{e}_k \rangle \langle \overset{\alpha}{e}{}^k | \alpha(\cdot) | \overset{\alpha}{e}_j \rangle = \sum_k \overset{\alpha}{g}{}^i_k \otimes \overset{\alpha}{g}{}^k_j\,. \tag{24}$$

V_α can be endowed with a structure of right comodule over Γ:

$$\Delta_\alpha(\overset{\alpha}{e}_i) = \sum_j \overset{\alpha}{e}_j \otimes \overset{\alpha}{g}{}^j_i\,. \tag{25}$$

We also have

$$\Delta_{\bar{\alpha}}(\overset{\alpha}{e}{}^i) = \sum_j \overset{\alpha}{e}{}^j \otimes S(\overset{\alpha}{g}{}^i_j) = \sum_j \overset{\alpha}{e}{}^j \otimes \overset{\bar{\alpha}}{g}{}^j_i\,. \tag{26}$$

Let α, β be two fixed elements of $Irr(A)$, because q is not a root of unity; finite dimensional representations are completely reducible:

$$\alpha \otimes \beta = \bigoplus_{\gamma \in Irr(A)} N^{\gamma}_{\alpha\beta}\gamma\,. \tag{27}$$

Let us define, for each γ, $(\psi^{\alpha,\beta}_{\gamma,m})_{m=1\cdots N^{\gamma}_{\alpha\beta}}$ a basis of $\mathrm{Hom}_A(V_\alpha \otimes V_\beta, V_\gamma)$ and $(\phi^{\gamma,m}_{\alpha,\beta})_{m=1\cdots N^{\gamma}_{\alpha\beta}}$ a basis of $\mathrm{Hom}_A(V_\gamma, V_\alpha \otimes V_\beta)$:

$$V_\alpha \otimes V_\beta \overset{\psi^{\alpha,\beta}_{\gamma,m}}{\rightarrow} V_\gamma \overset{\phi^{\gamma,m'}_{\alpha,\beta}}{\rightarrow} V_\alpha \otimes V_\beta\,. \tag{28}$$

We can always choose them such that they own the properties:

$$\sum_{m,\gamma} \phi^{\gamma,m}_{\alpha,\beta} \psi^{\alpha,\beta}_{\gamma,m} = id_{V_\alpha \otimes V_\beta} , \tag{29}$$

$$\psi^{\alpha,\beta}_{\gamma,m} \phi^{\gamma',m'}_{\alpha,\beta} = \delta^m_{m'} \delta^{\gamma}_{\gamma'} id_{V_\gamma} . \tag{30}$$

It is easy to check that we can always choose:

$$\phi^{\gamma,m}_{\beta,\alpha} = \lambda_{\alpha\beta\gamma} P_{12} \overset{\alpha\beta}{R}{}^{-1}_{21} \phi^{\gamma,m}_{\alpha,\beta} , \tag{31}$$

$$\psi^{\beta,\alpha}_{\gamma,m} = \lambda^{-1}_{\alpha\beta\gamma} \psi^{\alpha,\beta}_{\gamma,m} \overset{\alpha\beta}{R}_{21} P_{12} , \tag{32}$$

where $\lambda_{\alpha\beta\gamma} = (v_\alpha v_\beta v_\gamma^{-1})^{1/2}$.

We can define the Clebsch–Gordan coefficients by

$$\psi^{\alpha,\beta}_{\gamma,m}(\overset{\alpha}{e}_a \otimes \overset{\beta}{e}_b) = \sum_c \overset{\gamma}{e}_c \begin{pmatrix} c & \alpha & \beta \\ \gamma & a & b \end{pmatrix}^{\psi}_m , \tag{33}$$

$$\phi^{\gamma,m}_{\alpha,\beta}(\overset{\gamma}{e}_c) = \sum_{a,b} \overset{\alpha}{e}_a \otimes \overset{\beta}{e}_b \begin{pmatrix} a & b & \gamma \\ \alpha & \beta & c \end{pmatrix}^{\phi}_m . \tag{34}$$

We then obtain the relation:

$$\overset{\alpha}{g}{}^i_j \overset{\beta}{g}{}^k_l = \sum_{\gamma,m,p,q} \begin{pmatrix} i & k & \gamma \\ \alpha & \beta & p \end{pmatrix}^{\phi}_m \overset{\gamma}{g}{}^p_q \begin{pmatrix} q & \alpha & \beta \\ \gamma & j & l \end{pmatrix}^{\psi}_m . \tag{35}$$

This relation can also be written:

$$\overset{\alpha}{g}_1 \overset{\beta}{g}_2 = \sum_{\gamma,m} \phi^{\gamma,m}_{\alpha,\beta} \overset{\gamma}{g} \psi^{\alpha,\beta}_{\gamma,m} . \tag{36}$$

Let us now specialize $\mathcal{G}$ to be a Lie algebra of type A_n, B_n, C_n, D_n and f to be a fundamental representation of $U_q(\mathcal{G})$. In this case the restricted dual is the Hopf algebra generated by $\overset{f}{g}$ satisfying the relation:

$$\overset{ff}{R}_{12} \overset{f}{g}_1 \overset{f}{g}_2 = \overset{f}{g}_2 \overset{f}{g}_1 \overset{ff}{R}_{12} \text{ and } \begin{cases} \det_q(\overset{f}{g}) = 1 & \text{if } \mathcal{G} = A_n, \\ \overset{f}{g} C^t \overset{f}{g} C^{-1} = C^t \overset{f}{g} C^{-1} \overset{f}{g} = I & \text{if } \mathcal{G} = B_n, C_n, D_n , \end{cases} \tag{37}$$

where $\det_q$ is the quantum determinant and C is the quantum quadratic form defined, for example, in [15].

The antipode is defined to be the antimorphism of the algebra satisfying:

$$\sum_j \overset{f}{g}{}^i_j S(\overset{f}{g}{}^j_k) = \sum_j S(\overset{f}{g}{}^i_j) \overset{f}{g}{}^j_k = \delta^i_k . \tag{38}$$

An explicit description of S is obtained using the quantum analog of Cramers formulas [15]. The matrix elements $\overset{\alpha}{g}{}^j_i$ are expressed as polynomials in the matrix elements $\overset{f}{g}{}^j_i$; these polynomials can be computed using the Clebsch–Gordan coefficients.

Because of the relation (24) they satisfy:

$$\sum_j \overset{\alpha}{g}{}^i_j S(\overset{\alpha}{g}{}^j_k) = \sum_j S(\overset{\alpha}{g}{}^i_j)\overset{\alpha}{g}{}^j_k = \delta^i_k \,. \tag{39}$$

We will now define a non-commutative analog of the gauge symmetry Hopf algebra $\hat{\Gamma}$ and a non-commutative analog of the gauge field algebra Λ.

Let Σ be a compact connected triangulated oriented Riemann surface with boundary $\partial\Sigma$ and let $(F_i)_{i=1,\dots,n_F}$ be the oriented faces of Σ. Let us denote by L the set of edges counted with the orientation induced by the orientation of the corresponding faces. We have $L = L^i \cup L^b$, where L^i, L^b are respectively the set of interior edges and boundary edges ($\partial\Sigma = L^b$).

Finally let us also define V to be the set of points (vertices) of this triangulation, $V = V^i \cup V^b$, where V^i, V^b are respectively the set of interior vertices and boundary vertices.

If l is an oriented link it will be convenient to write $l = (x, y)$, where x is the departure point of l and y the end point of l. We will write $x = d(l)$ and $y = e(l)$.

It is important to notice that the definition of a triangulation imposes $x \neq y$. This property implies that for any link l incident to a vertex z it is possible to determine unambiguously if z is the departure or end point of l. We will moreover assume that a link l is completely characterized by its departure and end points.

If l is an interior edge then we define $\bar{l}$ to be the corresponding edge with opposite orientation.

Definition 1 (Gauge symmetry algebra). *Let us define for $z \in V$, the Hopf algebra $\Gamma_z = \Gamma \times \{z\}$ and $\hat{\Gamma} = \otimes_{z\in V}\Gamma_z$. This Hopf algebra will be called the gauge symmetry algebra.*

In order to define the non-commutative algebra of gauge fields we will have to endow the triangulation with an additional structure, an order on the set of links incident to each vertex, the *cilium* order. This can be achieved using the formalism of ciliated fat graph described by V.V. Fock and A.A. Rosly [9].

Definition 2 (Ciliation). *A ciliation of the triangulation is an assignment of a cilium c_z to each vertex z which consists in a non-zero tangent vector at z. The orientation of the Riemann surface defines a canonical cyclic order of the links admitting z as departure or end point. Let l_1, l_2 be links incident to a common vertex z, the partial cilium order $<$ is defined by: $l_1 < l_2$ if $l_1 \neq l_2, \bar{l}_2$ and the unoriented vectors c_z, l_1, l_2 appear in the cyclic order defined by the orientation of the surface.*

We will assume in the rest of this work that the triangulated Riemann surface Σ is endowed with a ciliation.

Definition 3 (Gauge fields algebra). *We shall now define the algebra of gauge fields Λ to be the algebra generated by the formal variables $\overset{\alpha}{U}(l)^i_j$ with $l \in L$, $\alpha \in Irr(A)$, $i, j = 1 \cdots dimV_\alpha$ and satisfying the following relations:*

Commutation rules.

$$\overset{\alpha}{U}(x,y)_1\,\overset{\beta}{U}(z,y)_2\overset{\alpha\beta}{R}_{12} = \overset{\beta}{U}(z,y)_2\,\overset{\alpha}{U}(x,y)_1\,, \tag{40}$$

$$\overset{\alpha}{U}(x,y)_1\,\overset{\alpha\beta}{R}{}^{-1}_{12}\,\overset{\beta}{U}(y,z)_2 = \overset{\beta}{U}(y,z)_2\,\overset{\alpha}{U}(x,y)_1\,, \tag{41}$$

$$\overset{\alpha\beta}{R}_{12}\,\overset{\alpha}{U}(y,x)_1\,\overset{\beta}{U}(y,z)_2 = \overset{\beta}{U}(y,z)_2\,\overset{\alpha}{U}(y,x)_1\,, \tag{42}$$

$$\forall (y,x),(y,z)\in L\ \ x\neq z\ \ \textit{and}\ \ (x,y)<(y,z)\,,$$

$$\overset{\alpha\beta}{R}_{12}\,\overset{\alpha}{U}(x,y)_1\,\overset{\beta}{U}(x,y)_2\,\overset{\alpha\beta}{R}{}^{-1}_{21} = \overset{\beta}{U}(x,y)_2\,\overset{\alpha}{U}(x,y)_1 \tag{43}$$

$$\forall (x,y)\in L\,,$$

$$\overset{\alpha}{U}(x,y)\,\overset{\alpha}{U}(y,x) = 1\,, \tag{44}$$

$$\forall (x,y)\in L^i\,,$$

$$\overset{\alpha}{U}(x,y)_1\,\overset{\beta}{U}(z,t)_2 = \overset{\beta}{U}(z,t)_2\,\overset{\alpha}{U}(x,y)_1$$

$$\forall x,y,z,t\ \ \textit{pairwise distinct in}\ \ V\,. \tag{45}$$

Decomposition rules.

$$\overset{\alpha}{U}(x,y)_1\,\overset{\beta}{U}(x,y)_2 = \sum_{\gamma,m}\phi^{\gamma,m}_{\alpha,\beta}\,\overset{\gamma}{U}(x,y)\psi^{\beta,\alpha}_{\gamma,m}P_{12}\,, \tag{46}$$

$$\overset{0}{U}(x,y) = 1,\ \forall (x,y)\in L\,, \tag{47}$$

where we have systematically used the notation:

$$\overset{\alpha}{U}(x,y) = \sum_{i,j}\overset{\alpha}{e}_i\otimes\overset{\alpha}{e}{}^j\otimes\overset{\alpha}{U}(x,y)^i_j\,. \tag{48}$$

This algebra was recently introduced by A.Yu. Alekseev, H. Grosse and V. Schomerus in [1].

The defining relations of the gauge fields algebra are obtained by demanding a property of covariance under the coaction of the gauge algebra $\hat{\Gamma}$ defined as follows:

Definition 4 (Gauge covariance). *Λ can be endowed with a right comodule structure $\Omega : \Lambda \to \Lambda\otimes\Gamma$ such that*

1. *Ω is a morphism of algebras.*
2. *The action of Ω on the elements $\overset{\alpha}{U}(x,y)^i_j$ is defined by the formula*:

$$\Omega(\overset{\alpha}{U}(x,y)) = (\Delta_\alpha\otimes\Delta_{\bar{\alpha}}\otimes id)(\overset{\alpha}{U}(x,y))\,. \tag{49}$$

This expression can be expressed in terms of components as:

$$\Omega(\overset{\alpha}{U}(x,y)^i_j) = \sum_{p,q}\overset{\alpha}{U}(x,y)^p_q\otimes(\overset{\alpha}{g}{}^x)^i_pS((\overset{\alpha}{g}{}^y)^q_j)\,, \tag{50}$$

where $(\overset{\alpha}{g}{}^{x})^{i}_{p}$ *is the image of the element* $\overset{\alpha}{g}{}^{i}_{p}$ *by the canonical injections* $\Gamma \hookrightarrow \Gamma_x \hookrightarrow \hat{\Gamma}$.

Proof. It is straightforward to show that the relations of definition of Λ are compatible with the definition of Ω. We shall verify in detail that relations (40, 46) are indeed covariant under the coaction of $\hat{\Gamma}$. Covariance of other relations can be checked using the same scheme.

Verification of the covariance of relation (40):

$$
\begin{aligned}
\Omega(\overset{\alpha}{U}(x,y)_1 \overset{\beta}{U}(z,y)_2 \overset{\alpha\beta}{R}_{12}) &= \Omega(\overset{\alpha}{U}(x,y)_1)\Omega(\overset{\beta}{U}(z,y)_2)\overset{\alpha\beta}{R}_{12} \\
&= \overset{\alpha}{g}{}^{x}_{1}\overset{\alpha}{U}(x,y)_1 S(\overset{\alpha}{g}{}^{y})_1 \overset{\beta}{g}{}^{z}_{2} \overset{\beta}{U}(z,y)_2 S(\overset{\beta}{g}{}^{y})_2 \overset{\alpha\beta}{R}_{12} \\
&= \overset{\alpha}{g}{}^{x}_{1}\overset{\beta}{g}{}^{z}_{2}\overset{\alpha}{U}(x,y)_1 \overset{\beta}{U}(z,y)_2 S(\overset{\alpha}{g}{}^{y})_1 S(\overset{\beta}{g}{}^{y})_2 \overset{\alpha\beta}{R}_{12} \\
&= \overset{\alpha}{g}{}^{x}_{1}\overset{\beta}{g}{}^{z}_{2}\overset{\alpha}{U}(x,y)_1 \overset{\beta}{U}(z,y)_2 \overset{\alpha\beta}{R}_{12} S(\overset{\beta}{g}{}^{y})_2 S(\overset{\alpha}{g}{}^{y})_1 \\
&= \overset{\alpha}{g}{}^{x}_{1}\overset{\beta}{g}{}^{z}_{2}\overset{\beta}{U}(z,y)_2 \overset{\alpha}{U}(x,y)_1 S(\overset{\beta}{g}{}^{y})_2 S(\overset{\alpha}{g}{}^{y})_1 \\
&= \overset{\beta}{g}{}^{z}_{2}\overset{\beta}{U}(z,y)_2 S(\overset{\beta}{g}{}^{y})_2 \overset{\alpha}{g}{}^{x}_{1}\overset{\alpha}{U}(x,y)_1 S(\overset{\alpha}{g}{}^{y})_1 \\
&= \Omega(\overset{\beta}{U}(z,y)_2 \overset{\alpha}{U}(x,y)_1)\,.
\end{aligned}
$$

Verification of covariance of relation (46):

$$
\begin{aligned}
\Omega(\overset{\alpha}{U}(x,y)_1 \overset{\beta}{U}(x,y)_2) &= \Omega(\overset{\alpha}{U}(x,y)_1)\Omega(\overset{\beta}{U}(x,y)_2) \\
&= \overset{\alpha}{g}{}^{x}_{1}\overset{\alpha}{U}(x,y)_1 S(\overset{\alpha}{g}{}^{y})_1 \overset{\beta}{g}{}^{x}_{2}\overset{\beta}{U}(x,y)_2 S(\overset{\beta}{g}{}^{y})_2 \\
&= \overset{\alpha}{g}{}^{x}_{1}\overset{\beta}{g}{}^{x}_{2}\overset{\alpha}{U}(x,y)_1 \overset{\beta}{U}(x,y)_2 S(\overset{\alpha}{g}{}^{y})_1 S(\overset{\beta}{g}{}^{y})_2 \\
&= \overset{\alpha}{g}{}^{x}_{1}\overset{\beta}{g}{}^{x}_{2}\overset{\alpha}{U}(x,y)_1 \overset{\beta}{U}(x,y)_2 \overset{\alpha\beta}{R}_{21}{}^{-1} S(\overset{\alpha}{g}{}^{y}_{1}\overset{\beta}{g}{}^{y}_{2})\overset{\alpha\beta}{R}_{21}
\end{aligned}
$$

$$
= \sum_{\substack{\gamma,\gamma',\gamma'' \\ m,m',m''}} \phi^{\gamma,m}_{\alpha,\beta}\overset{\gamma}{g}\psi^{\alpha,\beta}_{\gamma,m}\phi^{\gamma',m'}_{\alpha,\beta}\overset{\gamma}{U}(x,y)\psi^{\beta,\alpha}_{\gamma',m'}P_{12}\overset{\alpha\beta}{R}_{21}{}^{-1}\phi^{\gamma'',m''}_{\alpha,\beta}S(\overset{\gamma}{g})\psi^{\alpha,\beta}_{\gamma'',m''}\overset{\alpha\beta}{R}_{21}
$$

$$
= \sum_{\substack{\gamma,\gamma',\gamma'' \\ m,m',m''}} \phi^{\gamma,m}_{\alpha,\beta}\overset{\gamma}{g}\psi^{\alpha,\beta}_{\gamma,m}\phi^{\gamma',m'}_{\alpha,\beta}\overset{\gamma}{U}(x,y)\psi^{\alpha,\beta}_{\gamma',m'}\lambda^{-1}_{\alpha\beta\gamma'}\phi^{\gamma'',m''}_{\alpha,\beta}S(\overset{\gamma}{g})\psi^{\alpha,\beta}_{\gamma'',m''}\overset{\alpha\beta}{R}_{21}
$$

$$
= \sum_{m,\gamma}\phi^{\gamma,m}_{\alpha,\beta}\Omega(\overset{\gamma}{U}(x,y))\psi^{\alpha,\beta}_{\gamma,m}\overset{\alpha\beta}{R}_{21}\lambda^{-1}_{\alpha\beta\gamma}
$$

$$
= \Omega\left(\sum_{m,\gamma}\phi^{\gamma,m}_{\alpha,\beta}\overset{\gamma}{U}(x,y)\psi^{\beta,\alpha}_{\gamma,m}P_{12}\right)\,. \qquad \square
$$

At this point we have many comments to make.

Remark 1. We just have seen that the braiding relation (22) is compatible with the exchange relation (40). But relation (22) can also be written (23), this would as well imply an exchange relation of the type (40) with R_{12} exchanged with R_{21}^{-1}. The aim of the ciliation is precisely to solve this ambiguity. Indeed if $i = (x, y)$ and $j = (z, y)$ with $i < j$, then the relation

$$\overset{\alpha}{U}(i)_1 \overset{\beta}{U}(j)_2 \overset{\alpha\beta}{R}_{12} = \overset{\beta}{U}(j)_2 \overset{\alpha}{U}(i)_1 \tag{51}$$

implies $\overset{\alpha}{U}(i)_1 \overset{\beta}{U}(j)_2 \overset{\alpha\beta}{R}{}_{21}^{-1} = \overset{\beta}{U}(j)_2 \overset{\alpha}{U}(i)_1$ for $i > j$. This can be easily checked by applying the permutation operator.

Remark 2. The commutation relations given in the definition of Λ are not minimal. The exchange relations of the type (40, 41, 42) are related to every other using relation (44) when links are interior links. If (x, y) is an interior link then we also have from relation (43):

$$\overset{\alpha}{U}(x, y)_1 \overset{\alpha\beta}{R}_{21} \overset{\beta}{U}(y,x)_2 = \overset{\beta}{U}(y,x)_2 \overset{\alpha\beta}{R}_{12} \overset{\alpha}{U}(x, y)_1 \,, \tag{52}$$

$$\overset{\alpha}{U}(x, y)_1 \overset{\alpha\beta}{R}{}_{12}^{-1} \overset{\beta}{U}(y,x)_2 = \overset{\beta}{U}(y,x)_2 \overset{\alpha\beta}{R}{}_{21}^{-1} \overset{\alpha}{U}(x, y)_1 \,. \tag{53}$$

It is also easy to see that (46) implies relations (43). Moreover we have a relation between $\overset{\alpha}{U}(y,x)$ and $\overset{\bar{\alpha}}{U}(x, y)$ which can be shown to hold using (46, 44):

Proposition 1. *If* (x, y) *is an interior link, we have*

$$\overset{\alpha}{U}(y,x) = \overset{\alpha}{\mu}{}^{-1t} \overset{\bar{\alpha}}{U}(x, y) \,. \tag{54}$$

(*This is the relation* (4.8) *of* [1].)

Proof. We first have to describe in detail the map $\psi_0^{\bar{\alpha},\alpha}$ and $\phi^0_{\bar{\alpha},\alpha}$, where as usual 0 denote the trivial representation of dimension 1. $\psi_0^{\bar{\alpha},\alpha}$ is the usual canonical map:

$$\begin{aligned} \psi_0^{\bar{\alpha},\alpha} : \quad & V_{\bar{\alpha}} \otimes V_\alpha \to \mathbf{C} \\ & \xi \otimes x \mapsto \langle \xi, x \rangle \,. \end{aligned} \tag{55}$$

$\phi^0_{\bar{\alpha},\alpha}$ is the map:

$$\begin{aligned} \phi^0_{\bar{\alpha},\alpha} : \quad & \mathbf{C} \to V_{\bar{\alpha}} \otimes V_\alpha \\ & \lambda \mapsto \lambda \frac{1}{\mathrm{tr}(u_\alpha^{-1})} \sum_i \overset{\alpha}{e}{}^i \otimes u_\alpha^{-1} \overset{\alpha}{e}_i \,. \end{aligned} \tag{56}$$

This is an easy consequence of the identity $S(x) = uS^{-1}(x)u^{-1}$. The normalization of $\phi^0_{\bar{\alpha},\alpha}$ is chosen such that $\psi_0^{\bar{\alpha},\alpha} \phi^0_{\bar{\alpha},\alpha} = id_{\mathbf{C}}$.

Let $(x, y) \in L$, the decomposition rule (46) implies that

$$\psi_0^{\bar{\alpha},\alpha} \overset{\bar{\alpha}}{U}(x, y)_1 \overset{\alpha}{U}(x, y)_2 = \psi_0^{\alpha,\bar{\alpha}} P_{12} = \psi_0^{\bar{\alpha},\alpha} \overset{\bar{\alpha},\alpha}{R}_{21} v_\alpha^{-1} \,, \tag{57}$$

in terms of components this can be written:

$$\sum_i \overset{\bar{\alpha}}{U}(x,y)^i_j \overset{\alpha}{U}(x,y)^i_l = \overset{\alpha}{\mu}{}^j_l \,. \tag{58}$$

This relation can also be written

$$\overset{\alpha}{U}(y,x)^l_i = \sum_j \overset{\bar{\alpha}}{U}(x,y)^i_j (\overset{\alpha}{\mu}{}^{-1})^l_j \,, \tag{59}$$

which is exactly relation (54). $\square$

Remark 3. The associativity "constraints" coming from exchange relations between elements of the algebra are automatically satisfied from quasitriangularity of $U_q(\mathcal{G})$.

Remark 4. In the classical case the variables U_{ij} are elements of the group G, in this case the gauge fields variables are of the same type as elements of the gauge group. In the q-deformed case this does not hold true as can be seen from the commutation relations in Λ. Let $X \subset L$ and Λ_X be the subalgebra of Λ generated by $\overset{\alpha}{U}(l)$ with $l \in X$ and $\alpha \in Irr(A)$. The q-deformed lattice theory is a nonultralocal theory in the sense that the algebras Λ_X and Λ_Y are pointwise commuting if X and Y have no link incident to the same vertex.

Let $M = \{l_1, \cdots, l_p\}$ be a subset of links such that M contains the boundary links and each of the interior links in one and only one orientation.

It is trivial to show using the commutation relations in the algebra Λ and the relation (54) that $\{\prod_{j=1}^p \overset{\alpha_j}{U}(l_j)^{m_j}_{n_j}\}, \alpha_j \in Irr(A), m_j, n_j = 1 \cdots d_j$ is a generating family of the vector space Λ.

We will assume in the rest of the paper that this family is a basis of Λ. This assumption is quite natural and perhaps it can be proved using a representation of the algebra Λ or using techniques like the Diamond Lemma. What can easily be shown is that this property is independent of the choice of M. We will use the assumption of independence of this family of vectors in order to build an analogue of the Haar measure in Sect. (5).

4. Wilson Loops

In this section we define elements of the algebra Λ, called "*Wilson loops*" attached to loops on the lattice and owning the following properties: gauge invariance and invariance under departure point (we will call this last property *cyclicity*).

A loop C of length $k = l(C)$ is said simple if $C = (x_1, \cdots x_k, x_{k+1} = x_1)$ with $x_1, \cdots, x_k$ pairwise distincts. From now we allow us to identify x_{n+k} and x_n for all n.

Let us define the sign $\varepsilon(x_i, C)$ to be 1 (resp. -1) if $(x_{i-1}, x_i) < (x_i, x_{i+1})$ (resp. $(x_{i-1}, x_i) > (x_i, x_{i+1})$) and denote $N(C)$ the cardinal of the set $\{i \in \{1 \ldots k\}/(x_{i-1}, x_i) < (x_i, x_{i+1})\}$.

If C is a contractile loop, $N(C)$ is simply the number of cilia located at the vertices of $\{x_1, \ldots, x_k\}$ and directed inside the domain enclosed by the loop C.

We have the relation: $2N(C) = l(C) + \sum_{x \in C} \varepsilon(x, C)$.

Let C_1 and C_2 be two simple loops of length greater than three with common edges such that C_1 and C_2 have opposite orientation on these edges. We will denote

$C_1 \# C_2$ to be the loop obtained by gluing C_1 and C_2 along their common part and removing these common edges.

Definition 5. *Let $\mathscr{C}_k$ be the set of loops of length $k(k \geqq 3)$, i.e. the set of $(k+1)-$uplets $C = (x_1, \cdots, x_k, x_{k+1} = x_1)$, where $(x_i, x_{i+1}) \in L$. The set of loops is defined by $\mathscr{C} = \bigcup_{k \geqq 2} \mathscr{C}_k$. We also define $\mathscr{C}^s$ to be the set of simple loops.*

Let us define for arbitrary adjacent links $(x,y),(y,z)$ the matrix:

$$\mathscr{R}^{\alpha\beta}(x,y,z) = \begin{cases} \overset{\alpha\beta}{R}{}_{12}^{-1} & if\,(x,y) < (y,z) \\ \overset{\alpha\beta}{R}_{21} & if\,(x,y) > (y,z)\,. \end{cases} \tag{60}$$

We define the Wilson loop in the representation α attached to $C \in \mathscr{C}^s$ by:

$$W^\alpha(C) = \omega_\alpha(C) v_\alpha \mathrm{tr}_{1\ldots k}\left(\overset{\alpha}{\mu}{}^{\otimes k}\sigma^{(k)}\left(\prod_{j=1}^{k-1} \overset{\alpha}{U}(x_j,x_{j+1})_j \mathscr{R}^{\alpha\alpha}_{jj+1}(x_j,x_{j+1},x_{j+2})\right)\overset{\alpha}{U}(x_k,x_1)_k\right), \tag{61}$$

where the expression $\mathrm{tr}_{1\ldots k}$ *means the trace over the space $V_\alpha^{\otimes k}, \sigma^{(k)}$ is the permutation operator $\sigma^{(k)} = P_{kk-1}\cdots P_{21}$, and $\omega_\alpha(C)$ is a non-zero complex number depending only on the distribution of cilia along the loop.*

Proposition 2. *Wilson loops $W^\alpha(x_1,\ldots,x_k)$ are gauge invariant and cyclic invariant, i.e. $W^\alpha(x_1,x_2,\ldots,x_k) = W^\alpha(x_2,\ldots,x_k,x_1)$. We have the following important relation:*

$$W^\alpha(C) = \rho_\alpha(x_1,C)\omega_\alpha(C)\mathrm{tr}_{V_\alpha}(\overset{\alpha}{\mu}\overset{\alpha}{U}(x_1,x_2)\cdots\overset{\alpha}{U}(x_k,x_1))\,, \tag{62}$$

where $\rho_\alpha(x_1,C) = v_\alpha^{1+\sum_{x \neq x_1}\varepsilon(x_1,C)}$.

Proof. Let us show first the cyclicity property. From the definition of $\mathscr{R}$, the commutation rules (41) can be written:

$$\overset{\alpha}{U}(x_{j-1},x_j)_1\mathscr{R}^{\alpha\alpha}_{12}(x_{j-1},x_j,x_{j+1})\overset{\alpha}{U}(x_j,x_{j+1})_2 = \overset{\alpha}{U}(x_j,x_{j+1})_2\overset{\alpha}{U}(x_{j-1},x_j)_1\,. \tag{63}$$

We first have:

$$\begin{aligned}
&\prod_{j=1}^{k-1}(\overset{\alpha}{U}(x_j,x_{j+1})_j\mathscr{R}^{\alpha\alpha}_{jj+1}(x_j,x_{j+1},x_{j+2}))\overset{\alpha}{U}(x_k,x_1)_k \\
&= \overset{\alpha}{U}(x_1,x_2)_1\mathscr{R}^{\alpha\alpha}_{12}(x_1,x_2,x_3)\overset{\alpha}{U}(x_2,x_3)_2\cdots\mathscr{R}^{\alpha\alpha}_{k-1k}(x_{k-1},x_k,x_{k+1})\overset{\alpha}{U}(x_k,x_1)_k \\
&= \overset{\alpha}{U}(x_2,x_3)_2\overset{\alpha}{U}(x_1,x_2)_1\mathscr{R}^{\alpha\alpha}_{23}(x_2,x_3,x_4)\cdots\mathscr{R}^{\alpha\alpha}_{k-1k}(x_{k-1},x_k,x_{k+1})\overset{\alpha}{U}(x_k,x_1)_k \\
&= \overset{\alpha}{U}(x_2,x_3)_2\mathscr{R}^{\alpha\alpha}_{23}(x_2,x_3,x_4)\cdots\mathscr{R}^{\alpha\alpha}_{k-1k}(x_{k-1},x_k,x_{k+1})\overset{\alpha}{U}(x_1,x_2)_1\overset{\alpha}{U}(x_k,x_1)_k \\
&= \overset{\alpha}{U}(x_2,x_3)_2\cdots\mathscr{R}^{\alpha\alpha}_{k-1k}(x_{k-1},x_k,x_1)\overset{\alpha}{U}(x_k,x_1)_k\mathscr{R}^{\alpha\alpha}_{k1}(x_k,x_1,x_2)\overset{\alpha}{U}(x_1,x_2)_1 \\
&= \prod_{j=2}^{k}(\overset{\alpha}{U}(x_j,x_{j+1})_j\mathscr{R}^{\alpha\alpha}_{jj+1}(x_j,x_{j+1},x_{j+2}))\overset{\alpha}{U}(x_1,x_2)_1\,.
\end{aligned}$$

Using the identity

$$P_{kk-1}\cdots P_{32}P_{21} = P_{1k}P_{kk-1}\cdots P_{32}\,, \tag{64}$$

we finally obtain:

$$W^{\alpha}(x_1,x_2,\ldots,x_k) = W^{\alpha}(x_2,\ldots,x_k,x_1)\,, \tag{65}$$

which ends the proof of cyclicity.

Using the following formula:

$$\mathrm{tr}_{1\cdots k}(P_{kk-1}\cdots P_{32}P_{21}A_1^{(1)}\cdots A_k^{(k)}) = \mathrm{tr}(A^{(1)}\cdots A^{(k)})\,, \tag{66}$$

where $A^{(1)},\ldots,A^{(k)}$ are elements of $End(V_\alpha)$ and the usual properties on u recalled in (17, 19), we can easily write $W^{\alpha}(x_1,x_2,\ldots,x_k)$ in the more convenient form:

$$W^{\alpha}(C) = v_\alpha^{1+\sum_{x\neq x_1}\varepsilon(x,C)}\omega_\alpha(C)\mathrm{tr}_{V_\alpha}(\overset{\alpha}{\mu}\,\overset{\alpha}{U}(x_1,x_2)\cdots\overset{\alpha}{U}(x_k,x_1))\,. \tag{67}$$

Gauge invariance is more explicit in this form. Indeed we have:

$$\begin{aligned}
&\Omega(v_\alpha^{-1-\sum_{x\neq x_1}\varepsilon(x,C)}\omega_\alpha(C)^{-1}W^{\alpha}(x_1,x_2,\ldots,x_k)) \qquad (68)\\
&= \Omega\left(\overset{\alpha}{u}{}^{i_{k+1}}_{i_1}\prod_{j=1}^{k}\overset{\alpha}{U}(x_j,x_{j+1})^{i_j}_{i_{j+1}}\right)\\
&\text{(we have used an implicit sum over repeated indices)}\\
&= \overset{\alpha}{\mu}{}^{i_{k+1}}_{i_1}\prod_{j=1}^{k}(\overset{\alpha}{g}{}^{x_j})^{i_j}_{p_j}\,\overset{\alpha}{U}(x_j,x_{j+1})^{p_j}_{q_{j+1}}S(\overset{\alpha}{g}{}^{x_{j+1}})^{q_{j+1}}_{i_{j+1}}\\
&= (\overset{\alpha}{g}{}^{x_1})^{i_1}_{p_1}S(\overset{\alpha}{g}{}^{x_1})^{q_{k+1}}_{i_{k+1}}\,\overset{\alpha}{\mu}{}^{i_{k+1}}_{i_1}\prod_{j=2}^{k}S(\overset{\alpha}{g}{}^{x_j})^{q_j}_{i_j}(\overset{\alpha}{g}{}^{x_j})^{i_j}_{p_j}\prod_{j=1}^{k}\overset{\alpha}{U}(x_j,x_{j+1})^{p_j}_{q_{j+1}}\\
&= (\overset{\alpha}{g}{}^{x_1})^{i_1}_{p_1}\,\overset{\alpha}{\mu}{}^{q_{k+1}}_{i_{k+1}}S^{-1}(\overset{\alpha}{g}{}^{x_1})^{i_{k+1}}_{i_1}\prod_{j=2}^{k}\delta^{q_j}_{p_j}\prod_{j=1}^{k}\overset{\alpha}{U}(x_j,x_{j+1})^{p_j}_{q_{j+1}}\\
&= \overset{\alpha}{\mu}{}^{q_{k+1}}_{i_{k+1}}\delta^{i_{k+1}}_{p_1}\prod_{j=2}^{k}\delta^{q_j}_{p_j}\prod_{j=1}^{k}\overset{\alpha}{U}(x_j,x_{j+1})^{p_j}_{q_{j+1}}\\
&= v_\alpha^{-1-\sum_{x\neq x_1}\varepsilon(x,C)}\omega_\alpha(C)^{-1}W^{\alpha}(x_1,x_2\cdots,x_k)\otimes 1\,,
\end{aligned}$$

which shows that $W^{\alpha}(x_1,x_2\cdots,x_k)$ is a gauge invariant element □

Remark. We should make here a comment on the notion of trace and q-trace following the remark of Drinfeld in [8]. Let $\overset{\alpha}{g}$ be the matrix of elements of Γ. We can define $\mathrm{tr}(\overset{\alpha}{g}) = \sum_i \overset{\alpha}{g}{}^i_i$ (the ordinary trace) and $\mathrm{tr}_q(\overset{\alpha}{g}) = \mathrm{tr}(\overset{\alpha}{\mu}\,\overset{\alpha}{g})$ (the q-trace). It is a well known result that the trace is cyclic but not Ad-invariant and on the contrary the q-trace is Ad-invariant but not cyclic. We need both of these properties to build a consistent q-Yang Mills type theory. One of the great benefits of the algebra Λ is that the q-trace defined on it is both cyclic and Ad-invariant.

A *path* P of length k is a $(k+1)-$ uplet $P=(x_1,\dots,x_{k+1})$, where $(x_i,x_{i+1})\in L$. In the proof of the next proposition we will have to extend the definition of Wilson loop to a path P by:

$$M^\alpha(P)=\prod_{j=1}^{k}\overset{\alpha}{U}(x_j,x_{j+1})\,. \tag{69}$$

When P is a simple closed loop we of course have $W^\alpha(P)=\mathrm{tr}_{V^\alpha}(\overset{\alpha}{\mu}M^\alpha(P))$.

We will now study the commutation relations of the Wilson loops.

Let us consider two loops C and C'. Let $C\cap C'$ be the set of common vertices of C and C'. $C\cap C'$ is naturally a disconnected union of paths $P^{(1)},\dots,P^{(k)}$ (resp. $P'^{(1)},\dots,P'^{(k)}$) with the orientation induced by C (resp. C'). For each of these paths we define $L^{(1)},\dots,L^{(k)}$ (resp. $L'^{(1)},\dots,L'^{(k)}$) to be the paths obtained from $P^{(1)},\dots,P^{(k)}$ (resp. $P'^{(1)},\dots,P'^{(k)}$) by adding the neighbour vertices of the latter in C (resp. C'). Let us now consider a connected component $P^{(i)}$ of $C\cap C'$ and its related paths $L^{(i)}$ and $L'^{(i)}$. $L^{(i)}$ (resp. $L'^{(i)}$) is a set of links $(l_0^{(i)},l_1^{(i)},\dots,l_{n_i}^{(i)},l_{n_i+1}^{(i)})$ (resp. $(l'^{(i)}_0,l_1^{(i)},\dots,l_{n_i}^{(i)},l'^{(i)}_{n_i+1})$. $L^{(i)}$ is said to be a crossing zone if one of the following condition is fulfilled:

1. $n_i\neq 0, l_0^{(i)}<l'^{(i)}_0<l_1^{(i)}$ and $l_{n_i}^{(i)}<l_{n_i+1}^{(i)}<l'^{(i)}_{n_i+1}$ or cyclic.perm.
2. $n_i\neq 0, l'^{(i)}_0<l_0^{(i)}<l_1^{(i)}$ and $l_{n_i}^{(i)}<l'^{(i)}_{n_i+1}<l_{n_i+1}^{(i)}$ or cyclic.perm.
3. $n_i=0, l'^{(i)}_0<l_0^{(i)}<l'^{(i)}_1<l_1^{(i)}$ or cyclic.perm.
4. $n_i=0, l'^{(i)}_1<l_0^{(i)}<l'^{(i)}_0<l_1^{(i)}$ or cyclic.perm.

We shall declare that two simple loops C and C' do not cross if and only if they have no crossing zone.

We have the following important proposition:

Proposition 3. *If C and C' are non crossing simple loops then the corrresponding Wilson loop $W^\alpha(C)$ and $W^\beta(C')$ are commuting elements.*

Proof. Let us denote by $\mathcal{T}_{xyz}$ and $\mathcal{T}'_{xyz}$ the elements:

$$\mathcal{T}^{\alpha\beta}_{xyz}=\begin{cases}(\overset{\alpha\beta}{R}_{12}\overset{\alpha\beta}{R}_{21})^{-1} & \text{if } (x,y)<(y,z)\\ 1\otimes 1 & \text{if } (x,y)>(y,z)\end{cases} \tag{70}$$

and

$$\mathcal{T}'^{\alpha\beta}_{xyz}=\begin{cases}\overset{\alpha\beta}{R}_{12}\overset{\alpha\beta}{R}_{21} & \text{if} (x,y)>(y,z)\\ 1\otimes 1 & \text{if } (x,y)<(y,z)\end{cases} \tag{71}$$

It is easy to show the following relations using the basic commutation rules in Λ. Those are obtained by a tedious enumeration of all possible configurations of links and ciliations:

$$\overset{\alpha}{U}(w,y)_1\overset{\beta}{U}(x,y)_2(\mathcal{T}^{\alpha\beta}_{xyz})_{12}\overset{\beta}{U}(y,z)_2=\overset{\beta}{U}(x,y)_2\overset{\beta}{U}(y,z)_2\overset{\alpha}{U}(w,y)_1\,,$$

$$\overset{\alpha}{U}(y,w)_1\overset{\beta}{U}(x,y)_2\overset{\beta}{U}(y,z)_2=\overset{\beta}{U}(x,y)_2(\mathcal{T}^{\alpha\beta}_{xyz})_{12}\overset{\beta}{U}(y,z)_2\overset{\alpha}{U}(y,w)_1\,,$$

if $(y,z)<(x,y)<(w,y)$ or cyclic perm.,

$$\overset{\alpha}{U}(w,y)_1\,\overset{\beta}{U}(x,y)_2(\mathscr{T}'^{\alpha\beta}_{xyz})_{12}\,\overset{\beta}{U}(y,z)_2 = \overset{\beta}{U}(x,y)_2\,\overset{\beta}{U}(y,z)_2\,\overset{\alpha}{U}(w,y)_1\,,$$

$$\overset{\alpha}{U}(y,w)_1\,\overset{\beta}{U}(x,y)_2\,\overset{\beta}{U}(y,z)_2 = \overset{\beta}{U}(x,y)_2(\mathscr{T}'^{\alpha\beta}_{xyz})_{12}\,\overset{\beta}{U}(y,z)_2\,\overset{\alpha}{U}(y,w)_1\,.$$

if $(x,y) < (y,z) < (w,y)$ or cyclic perm.,

$$\overset{\alpha}{U}(z,y)_1\,\overset{\beta}{U}(x,y)_2(\mathscr{T}^{\alpha\beta}_{xyz})_{12}\,\overset{\beta}{U}(y,z)_2\,\overset{\beta}{U}(z,w)_2$$
$$= \overset{\beta}{U}(x,y)_2\,\overset{\beta}{U}(y,z)_2(\mathscr{T}^{\alpha\beta}_{yzw})_{12}\,\overset{\beta}{U}(z,w)_2\,\overset{\alpha}{U}(z,y)_1\,,$$

$$\overset{\alpha}{U}(z,y)_1\,\overset{\beta}{U}(x,y)_2(\mathscr{T}'^{\alpha\beta}_{xyz})_{12}\,\overset{\beta}{U}(y,z)_2\,\overset{\beta}{U}(z,w)_2$$
$$= \overset{\beta}{U}(x,y)_2\,\overset{\beta}{U}(y,z)_2(\mathscr{T}'^{\alpha\beta}_{yzw})_{12}\,\overset{\beta}{U}(z,w)_2\,\overset{\alpha}{U}(z,y)_1\,,$$

$$\overset{\alpha}{U}(y,z)_1\,\overset{\beta}{U}(x,y)_2\,\overset{\beta}{U}(y,z)_2(\mathscr{T}^{\alpha\beta}_{yzw})_{12}\,\overset{\beta}{U}(z,w)_2$$
$$= \overset{\beta}{U}(x,y)_2(\mathscr{T}^{\alpha\beta}_{xyz})_{12}\,\overset{\beta}{U}(y,z)_2\,\overset{\beta}{U}(z,w)_2\,\overset{\alpha}{U}(y,z)_1\,,$$

$$\overset{\alpha}{U}(y,z)_1\,\overset{\beta}{U}(x,y)_2\,\overset{\beta}{U}(y,z)_2(\mathscr{T}'^{\alpha\beta}_{yzw})_{12}\,\overset{\beta}{U}(z,w)_2$$
$$= \overset{\beta}{U}(x,y)_2(\mathscr{T}'^{\alpha\beta}_{xyz})_{12}\,\overset{\beta}{U}(y,z)_2\,\overset{\beta}{U}(z,w)_2\,\overset{\alpha}{U}(y,z)_1\,.$$

If C and C' have no common vertices then $W^{\alpha}(C)$ and $W^{\beta}(C')$ are trivially commuting.

Let $C \cap C'$ be the set of common vertices of C and C'. $C \cap C'$ is naturally a disconnected union of paths $P_1,\ldots,P_k$ (resp. $P'_1,\ldots,P'_k$) with the orientation induced by C (resp. C'). For each of these paths we define $L_1,\ldots,L_k$ (resp. $L'_1,\ldots,L'_k$) the paths defined as before.

It is sufficient to show that $M^{\alpha}(L_j)$ commute with $M^{\beta}(L'_j)$ to prove the commutation relation between Wilson loops attached to C and C' unless the marginal case where one of the P_j or P'_j is a closed curve (because in this case one of the L_j or L'_j is ill defined). This case is studied at the end of the proof.

Let us now consider a connected component of $C \cap C'$ and its related paths L and L'. Let us first suppose that L_j and L'_j have opposite orientation. We can write $L = (x_{n+1}, x_n, \ldots x_1, x_0)$ and the respective $L' = (x'_0, x_1, \ldots, x_n, x'_{n+1})$. The fact that C and C' do not cross implies that we fulfill one of the following conditions:

$$(x'_0,x_1) < (x_0,x_1) < (x_1,x_2) \text{ and } (x_{n-1},x_n) < (x_n,x_{n+1}) < (x_n,x'_{n+1})$$

or

$$(x_0,x_1) < (x'_0,x_1) < (x_1,x_2) \text{ and } (x_{n-1},x_n) < (x_n,x'_{n+1}) < (x_n,x_{n+1})$$

or cyclic perm.

Using the latter commutation properties and the latter remark, it is easy to show that $M^{\alpha}(L)$ commute with $M^{\beta}(L')$. This is more or less the standard "railway proof" of integrable models. In one of the latter situations we have for example:

$$M^{\alpha}(L)_1 M^{\beta}(L')_2$$
$$= \overset{\alpha}{U}(x_{n+1},x_n)_1 \cdots \overset{\alpha}{U}(x_1,x_0)_1\,\overset{\beta}{U}(x'_0,x_1)_2 \cdots \overset{\beta}{U}(x_n,x'_{n+1})_2$$

$$= \overset{\alpha}{U}(x_{n+1},x_n)_1 \cdots \overset{\alpha}{U}(x_2,x_1)_1 \overset{\beta}{U}(x_0',x_1)_2 (\mathscr{T}^{\alpha\beta}_{x_0'x_1x_2})_{12} \overset{\beta}{U}(x_1,x_2)_2$$

$$\cdots \overset{\beta}{U}(x_n,x_{n+1}')_2 \overset{\alpha}{U}(x_1,x_0)_1$$

$$= \overset{\alpha}{U}(x_{n+1},x_n)_1 \overset{\beta}{U}(x_0',x_1)_2 \cdots \overset{\beta}{U}(x_{n-1},x_n)_2 (\mathscr{T}^{\alpha\beta}_{x_{n-1}x_nx_{n+1}'})_{12} \overset{\alpha}{U}(x_n,x_{n+1}')_2$$

$$\times \overset{\alpha}{U}(x_n,x_{n-1})_1 \cdots \overset{\alpha}{U}(x_1,x_0)_1$$

$$= \overset{\beta}{U}(x_0',x_1)_2 \cdots \overset{\beta}{U}(x_n,x_{n+1}')_2 \overset{\alpha}{U}(x_{n+1},x_n)_1 \cdots \overset{\alpha}{U}(x_1,x_0)_1$$

$$= M^\beta(L')_2 M^\alpha(L)_1 \,.$$

If L and L' have the same orientation, we can easily modify the proof to the result:

$$M^\alpha(L)_1 M^\beta(L')_2$$

$$= \overset{\alpha}{U}(x_0,x_1)_1 \cdots \overset{\alpha}{U}(x_n,x_{n+1})_1 \overset{\beta}{U}(x_0',x_1)_2 \cdots \overset{\beta}{U}(x_n,x_{n+1}')_2$$

$$= \overset{\alpha}{U}(x_0,x_1)_1 \cdots \overset{\alpha}{U}(x_{n-1},x_n)_1 \overset{\beta}{U}(x_0',x_1)_2$$

$$\cdots \overset{\beta}{U}(x_{n-1},x_n)_2 (\mathscr{T}^{\alpha\beta}_{x_{n-1}x_nx_{n+1}'})_{12} \overset{\beta}{U}(x_n,x_{n+1}')_2 \overset{\alpha}{U}(x_n,x_{n+1})_1$$

$$= \overset{\alpha}{U}(x_{n+1},x_n)_1 \overset{\beta}{U}(x_0',x_1)_2 (\mathscr{T}^{\alpha\beta}_{x_0'x_1x_2})_{12} \overset{\beta}{U}(x_1,x_2)_2 \ldots \overset{\beta}{U}(x_{n-1},x_n)_2$$

$$\times \overset{\alpha}{U}(x_1,x_2)_1 \ldots \overset{\alpha}{U}(x_n,x_{n-1})_1$$

$$= \overset{\beta}{U}(x_0',x_1)_2 \ldots \overset{\beta}{U}(x_n,x_{n+1}')_2 \overset{\alpha}{U}(x_0,x_1)_1 \cdots \overset{\alpha}{U}(x_n,x_{n+1})_1$$

$$= M^\beta(L')_2 M^\alpha(L)_1 \,.$$

The latter proofs do not work if exceptionally $x_0 = x_n,\ x_{n+1} = x_1$. In this case the proof is a bit different:

$$M^\alpha(L)_1 M^\beta(L')_2$$

$$= \overset{\alpha}{U}(x_{n+1},x_n)_1 \cdots \overset{\alpha}{U}(x_1,x_0)_1 \overset{\beta}{U}(x_1,x_2)_2 \cdots \overset{\beta}{U}(x_n,x_1)_2$$

$$= \overset{\alpha}{U}(x_{n+1},x_n)_1 \cdots \overset{\alpha}{U}(x_2,x_1)_1 \mathscr{R}_{12}(x_0x_1x_2) \overset{\beta}{U}(x_1,x_2)_2 \cdots \overset{\beta}{U}(x_n,x_1)_2$$

$$\times \mathscr{R}_{12}(x_0x_1x_n)^{-1} \overset{\alpha}{U}(x_1,x_0)_1$$

$$= \overset{\alpha}{U}(x_{n+1},x_n)_1 \cdots \overset{\alpha}{U}(x_3,x_2)_1 \overset{\beta}{U}(x_1,x_2)_2 (\mathscr{T}^{\alpha\beta}_{x_1x_2x_3})_{12} \overset{\beta}{U}(x_2,x_3)_2$$

$$\cdots \overset{\beta}{U}(x_n,x_1)_2 \overset{\alpha}{U}(x_2,x_1)_1 \overset{\alpha}{U}(x_1,x_0)_1$$

$$= \overset{\alpha}{U}(x_{n+1},x_n)_1 \overset{\beta}{U}(x_1,x_2)_2 \cdots \overset{\beta}{U}(x_{n-1},x_n)_2 (\mathscr{T}^{\alpha\beta}_{x_{n-1}x_nx_1})_{12} \overset{\beta}{U}(x_n,x_1)_2$$

$$\times \overset{\alpha}{U}(x_n,x_{n-1})_1 \cdots \overset{\alpha}{U}(x_1,x_0)_1$$

$$= \overset{\beta}{U}(x_1,x_2)_2 \ldots \overset{\beta}{U}(x_n,x_1)_2 \overset{\alpha}{U}(x_{n+1},x_n)_1 \cdots \overset{\alpha}{U}(x_1,x_0)_1$$

$$= M^\beta(L')_2 M^\alpha(L)_1 \ .$$

In order to complete the proof of this theorem it remains to study the case where the two loops C and C' are equal up to orientation. Let $C = (x_1,\ldots,x_n,x_{n+1} = x_1)$. It is easy to show using an induction proof on $k \leqq n+1$ that we have:

$$\overset{\alpha\beta}{R}_{12} M^\alpha(x_2,\ldots,x_k)_1 M^\beta(x_2,\ldots,x_k)_2 = M^\beta(x_2,\ldots,x_k)_2 M^\alpha(x_2,\ldots,x_k)_1 \overset{\alpha\beta}{R}_{21} \ ,$$

$$\overset{\alpha\beta}{R}{}^{-1}_{21} M^\alpha(x_2,\ldots,x_k)_1 M^\beta(x_2,\ldots,x_k)_2 = M^\beta(x_2,\ldots,x_k)_2 M^\alpha(x_2,\ldots,x_k)_1 \overset{\alpha\beta}{R}{}^{-1}_{12} \ .$$

As a result we get:

$$\mathscr{R}^{\alpha\beta-1}(x_1,x_2,x_3)_{12} M^\alpha(x_1,\ldots,x_{n+1})_1 \mathscr{R}^{\alpha\beta}(x_n,x_1,x_2)_{12} M^\beta(x_1,\ldots,x_k)_2$$

$$= \mathscr{R}^{\alpha\beta-1}(x_1,x_2,x_3)_{12} M^\alpha(x_1,\ldots,x_n)_1 \overset{\beta}{U}(x_1,x_2)_2 \overset{\alpha}{U}(x_n,x_1)_1 M^\beta(x_2,\ldots,x_{n+1})_2$$

$$= \mathscr{R}^{\alpha\beta-1}(x_1,x_2,x_3)_{12} \overset{\alpha}{U}(x_1,x_2)_1 \overset{\beta}{U}(x_1,x_2)_2 \mathscr{R}^{\alpha\beta}(x_1,x_2,x_3)_{21}$$

$$\times M^\alpha(x_2,\ldots,x_{n+1})_1 M^\beta(x_2,\ldots,x_{n+1})_2$$

$$= \overset{\beta}{U}(x_1,x_2)_2 \overset{\alpha}{U}(x_1,x_2)_1 M^\alpha(x_2,\ldots,x_{n+1})_1 M^\beta(x_2,\ldots,x_{n+1})_2$$

$$= \overset{\beta}{U}(x_1,x_2)_2 \overset{\alpha}{U}(x_1,x_2)_1 \mathscr{R}^{\alpha\beta}(x_1,x_2,x_3)_{12} M^\beta(x_2,\ldots,x_{n+1})_2$$

$$\times M^\beta(x_2,\ldots,x_{n+1})_1 \mathscr{R}^{\alpha\beta-1}(x_1,x_2,x_3)_{21}$$

$$= M^\beta(x_1,\ldots,x_n)_2 \overset{\alpha}{U}(x_1,x_2)_1 \overset{\beta}{U}(x_n,x_1)_2 M^\alpha(x_2,\ldots,x_{n+1})_1 \mathscr{R}^{\alpha\beta-1}(x_1,x_2,x_3)_{21}$$

$$= M^\beta(x_1,\ldots,x_{n+1})_2 \mathscr{R}^{\alpha\beta}(x_n,x_1,x_2)_{12} M^\alpha(x_1,\ldots,x_k)_1 \mathscr{R}^{\alpha\beta-1}(x_1,x_2,x_3)_{21} \ .$$

Applying the permutation operator to this identity we obtain the equivalent equation:

$$\mathscr{R}^{\alpha\beta}(x_1,x_2,x_3)_{21} M^\alpha(x_1,\ldots,x_{n+1})_1 \mathscr{R}^{\alpha\beta}(x_n,x_1,x_2)_{12} M^\beta(x_1,\ldots,x_k)_2$$

$$= M^\beta(x_1,\ldots,x_{n+1})_2 \mathscr{R}^{\alpha\beta}(x_n,x_1,x_2)_{12} M^\alpha(x_1,\ldots,x_k)_1 \mathscr{R}^{\alpha\beta}(x_1,x_2,x_3)_{12} \ .$$

These two equivalent relations can be recast in the reflection equation:

$$\mathscr{R}^{\alpha\beta-1}_{12}(x_n,x_1,x_2) M^\alpha(C)_1 \mathscr{R}^{\alpha\beta}_{12}(x_n,x_1,x_2) M^\beta(C)_2$$

$$= M^\beta(C)_2 \mathscr{R}^{\alpha\beta}_{21}(x_n,x_1,x_2) M^\alpha(C)_1 \mathscr{R}^{\alpha\beta-1}_{12}(x_n,x_1,x_2) \ . \qquad (72)$$

Let us denote $\mathscr{R}^{-1}(x_n,x_1,x_2) = \sum_i a_i \otimes b_i$, from quasitriangularity properties we have:

$$\sum_{i,j} a_i a_j \otimes S^{-2}(b_j) S^{-1}(b_i) = 1 \otimes 1 \ . \qquad (73)$$

Therefore:

$$\begin{aligned}
&\mathrm{tr}_{V_\alpha}(\overset{\alpha}{\mu}M^\alpha(x_1,\ldots,x_{n+1}))\mathrm{tr}_{V_\beta}(\overset{\beta}{\mu}M^\beta(x_1,\ldots,x_{n+1}))\\
&\quad=\sum_{i,j}\mathrm{tr}_{12}(\overset{\alpha}{\mu}_1\overset{\beta}{\mu}_2M^\alpha(x_1,\ldots,x_{n+1})_1a_ia_j\otimes S^{-2}(b_j)S^{-1}(b_i)M^\beta(x_1,\ldots,x_{n+1})_2)\\
&\quad=\mathrm{tr}_{12}(\overset{\alpha}{\mu}_1\overset{\beta}{\mu}_2\mathscr{R}_{12}^{\alpha\beta-1}(x_n,x_1,x_2)M^\alpha(C)_1\mathscr{R}_{12}^{\alpha\beta}(x_n,x_1,x_2)M^\beta(C)_2)\\
&\quad=\mathrm{tr}_{12}(\overset{\alpha}{\mu}_1\overset{\beta}{\mu}_2M^\beta(C)_2\mathscr{R}_{21}^{\alpha\beta}(x_n,x_1,x_2)M^\alpha(C)_1\mathscr{R}_{12}^{\alpha\beta-1}(x_n,x_1,x_2))\\
&\quad=\mathrm{tr}_{V_\beta}(\overset{\beta}{\mu}M^\beta(x_1,\ldots,x_{n+1}))\mathrm{tr}_{V_\alpha}(\overset{\alpha}{\mu}M^\alpha(x_1,\ldots,x_{n+1}))\,.
\end{aligned}$$

When $C' = \bar{C}$ a similar proof applies as well.

This ends the proof of the theorem. □

All what we have previously done is a study of a q-analog of the space of configurations of lattice gauge theory and its gauge covariance. We have not included for the present time the quantum fluctuations of the gauge fields. This will be achieved in the next section.

5. Quantum q-Deformed Lattice Gauge Theory

In ordinary lattice gauge field theory, integration over the fields uses as a central tool the Haar measure. This has been already recalled in Sect. 1. We will now study a q-analog of this notion, i.e. we will study integrals over Λ invariant under the coaction of the gauge algebra $\hat{\Gamma}$.

Proposition 4 (Invariant measure). *There exists a unique linear form $h\colon \Lambda \to \Lambda_{\partial\Sigma}$ such that*:

1. *(invariance)* $(h\otimes id)\Omega(A) = h(A)\otimes 1\forall A\in\Lambda$,
2. *(factorisation)* $h(AB) = h(A)h(B)$

$$\forall A\in\Lambda_X, \forall B\in\Lambda_Y, \forall X, Y\subset L, (X\cup\bar{X})\cap(Y\cup\bar{Y}) = \emptyset\,,$$

3. $h_{|\partial\Sigma} = id_{\partial\Sigma}$.

It can be evaluated on any element using the formula:

$$h(\overset{\alpha}{U}(x,y)^i_j) = \delta_{\alpha,0}\,, \tag{74}$$

where 0 *denotes the trivial representation of dimension* 1, *i.e.* 0 *is the counit, and* (x,y) *is an interior edge.*

It can be recursively computed on any element of Λ *using the formula*

$$h(A\overset{\alpha}{U}(x,y)^i_jB) = \delta_{\alpha,0}h(AB) \tag{75}$$

with (x,y) *an interior edge and* $A,B\in\Lambda_X,(x,y)\notin\Lambda_X,(y,x)\notin\Lambda_X$.

Proof. From the assumed invariance of h we get that

$$h(\overset{\alpha}{U}(x,y)^l_j)1\otimes 1 = (\overset{\alpha}{g}{}^x)^l_m h(\overset{\alpha}{U}(x,y)^m_n)S((\overset{\alpha}{g}{}^y)^n_j)\,. \tag{76}$$

Using the independence of the vectors $(\overset{\alpha}{g}{}^x)^l_m$ we get that $h(\overset{\alpha}{U}(x,y)^l_j) = 0$, except in the case $\alpha = 0$. Let $M = L^b \cup \{l_1,\dots,l_p\}$ be a subset of links such that M contains the boundary links and each of the interior links in one and only one orientation. From the assumption on the basis of Λ we can write any element A of Λ as a unique linear combination:

$$A = \sum_{\alpha_1,\dots,\alpha_p} U_{L^b}^{\alpha_1,\dots,\alpha_p}\operatorname{tr}\left(a_{\alpha_1,\dots,\alpha_p}\prod_{j=1}^{p}\overset{\alpha_j}{U}(l_j)_j\right), \tag{77}$$

where $a_{\alpha_1,\dots,\alpha_p} \in End(\otimes_{j=1}^p V_{\alpha_j})$ and $U_{L^b}^{\alpha_1,\dots,\alpha_p} \in \Lambda_{\partial\Sigma}$.

From $h_{|\partial\Sigma} = id_{\partial\Sigma}$ and the assumed factorisation property we get that $h(A) = a_{0,\dots,0}U_{L^b}^{0,\dots,0}$. This shows uniqueness of h.

It is straightforward to show that h defined by the last formula is invariant under the coaction of $\hat{\Gamma}$.

The factorisation property is proved using the commutation relations in Λ and the relation $\overset{0\alpha}{R} = \overset{\alpha 0}{R} = id_{V_\alpha}$. Relation (75) is proved similarly. □

It will be convenient in the rest of this article to use the notation $\int dh$ instead of h.

Proposition 5. *Let* (x,y) *be an interior link, we have the important formula*:

$$h(\overset{\alpha}{U}(y,x)^l_j\overset{\alpha}{U}(x,y)^k_l) = \frac{1}{[d_\alpha]}(\overset{\alpha}{\mu}{}^{-1})^k_j\delta^i_l\,, \tag{78}$$

which can also be written:

$$h(\overset{\alpha}{U}(y,x)_1\overset{\alpha}{\mu}_2\overset{\alpha}{U}(x,y)_2) = \frac{1}{[d_\alpha]}P_{12}\,. \tag{79}$$

Proof. Let (x,y) be an interior edge. From the decomposition rule we get:

$$h(\overset{\bar\alpha}{U}(x,y)_1\overset{\alpha}{U}(x,y)_2) = \phi^0_{\bar\alpha,\alpha}\psi_0^{\alpha,\bar\alpha}P_{12} = \psi_0^{\bar\alpha,\alpha}\overset{\bar\alpha\alpha}{R}_{21}v_\alpha^{-1}\,. \tag{80}$$

Using the known expressions of the right-hand side (see Proposition (1)) we obtain:

$$h(\overset{\bar\alpha}{U}(x,y)^l_j\overset{\alpha}{U}(x,y)^k_l) = \frac{1}{[d_\alpha]}(\overset{\alpha}{u}{}^{-1})^k_i\overset{\alpha}{u}{}^j_l\,. \tag{81}$$

Using relation (54) we finally prove the relation (79). □

We will now define an analog of the Yang–Mills action which is, as usual, built with elementary Wilson loops attached to faces of the triangulation.

Definition 6 (Boltzmann weights). *Let F be a face of the triangulation, we define a Boltzmann weight element of Λ:*

$$W_F = \sum_{\alpha \in Irr(A)} [d_\alpha] W^\alpha(\partial F) e^{-a_F C_\alpha} , \tag{82}$$

where a_F is the area of the face F and C_α are fixed non-negative numbers.

This element is the non-commutative analog of the Boltzmann weight (5). If F and F' are two faces, Proposition (3) implies that W_F and $W_{F'}$ commute. We can then define unambiguously the gauge invariant element: $\prod_{i=1}^{n_F} W_{F_i}$ which does not depend on the labelling of faces.

Proposition 6 (Yang Mills measure). *The Yang Mills measure h_{YM} on Λ is defined as follows:*

$$h_{YM}(A) = h\left(A \prod_{i=1}^{n_F} W_{F_i}\right) \forall A \in \Lambda . \tag{83}$$

It is an invariant measure in the sense that:

$$(h_{YM} \otimes id)\Omega(A) = h_{YM}(A) \otimes 1 \quad \forall A \in \Lambda . \tag{84}$$

It will be convenient to use the notation:

$$h_{YM}(a) = \int_\Sigma a \; dh_{YM} . \tag{85}$$

Proof. Let A be an element of Λ. We have:

$$\begin{aligned}(h_{YM} \otimes id)\Omega(A) &= (h \otimes id)\left(\Omega(A)\left(\prod_{i=1}^{n_F} W_{F_i} \otimes 1\right)\right) \\ &= (h \otimes id)\left(\Omega(A)\Omega\left(\prod_{i=1}^{n_F} W_{F_i}\right)\right) \\ &= (h \otimes id)\left(\Omega\left(A\prod_{i=1}^{n_F} W_{F_i}\right)\right) = h_{YM}(A) \otimes 1 . \quad \square\end{aligned}$$

Definition 7 (Correlation functions). *Let Σ be a triangulated Riemann surface with boundary $\partial\Sigma = \bigcup_{i=1}^n C_i$, where C_i are nonintersecting simple loops. We define the partition function $Z(\Sigma; C_1, \ldots, C_n)$ to be the element of $\bigotimes_{i=1}^n \Lambda_{C_i}$ defined by:*

$$Z(\Sigma; C_1, \ldots, C_n) = \int_\Sigma 1 \; dh_{YM} . \tag{86}$$

Proposition 7 (Locality). *Let Σ be the same surface as before, and let us consider C a simple loop (consisting in links belonging to the triangulation) which divides Σ in two pieces Σ_1 and Σ_2.*

We have the usual Markov (or locality) property:

$$Z(\Sigma; C_1, \ldots, C_n) = \int Z(\Sigma_1; C_1, \ldots, C_i, C) Z(\Sigma_2; C_{i+1}, \ldots, C_n, \bar{C}) \prod_{l \in C} dh(l) . \tag{87}$$

Proof. Commutation of Boltzmann weights implies obviously the Markov property. $\square$

We will later on use this locality property to compute any correlation functions using only two and three point correlation functions.

In the rest of this work we will choose a precise expression (see Proposition 8) for the functions $\omega_\alpha(C)$. This implies that the theory is quasitopological. In order to simplify expressions in the proofs we will moreover assume that W_F is independent of the area of the face, i.e. we are choosing $C_\alpha = 0$. We could as well compute the correlation functions of the continuum limit of the theory in the case where $C_\alpha \neq 0$. The relationships between correlation functions of the theory where $C_\alpha \neq 0$ and the topological one is the same as those described by [17, 18] in the undeformed case.

Recall the theorem of Alexander [10] in the two dimensional case: Let K, L be compact simplicial complexes of dimension two and Σ_K, (resp. Σ_L) the topological surfaces defined by K (resp. L), Σ_K is homeomorphic to Σ_L if and only if K and L admit a common subdivision obtained by a finite number of stellar moves.

The stellar move can be conveniently replaced by an equivalent set of two moves called Matveev moves M_1, M_2 which consist respectively in:

1. replacing any triplet of triangular faces described by their boundary links: $(x,a,y), (\bar{y},b,\bar{z}), (z,c,\bar{x})$ by the triangular face (a,b,c).
2. replacing a couple of neighbour triangular faces, described by their boundary links, $(a,b,x), (\bar{x},c,d)$ by the couple $(a,y,d), (\bar{y},b,c)$.

It is important to remark that each move applied to the triangulation leaves the set of boundary links fixed.

In order to simplify the notations inside the following proofs, we have made a convenient abuse of notation and have used the same letter l to denote the link l and the gauge variable U_l attached to it. We also have used the convention of labelling the loops by their links and not by their vertices.

Proposition 8 (Topological invariance). *If ω_α satisfies the following property*:

$$\omega_\alpha(C_1 \# C_2) = \omega_\alpha(C_1)\omega_\alpha(C_2) \tag{88}$$

for any simple loops C_1, C_2 of length greater than three with common edges, then the correlation functions do not depend on the triangulation of the surface with fixed boundaries.

A particular set of ω_α is defined as follows: *let t_α be any complex numbers,*

$$\omega_\alpha(C) = v_\alpha^{t_\alpha(1+\frac{1}{2}\sum_{x\in C}\varepsilon(x,C))} \, . \tag{89}$$

Proof. Let us first prove invariance under the first Matveev move:

Let us glue three Boltzmann weights associated to triangular faces labelled by their boundary $C_1 = (x,a,y), C_2 = (\bar{y},b,\bar{z}), C_3(z,c,\bar{x})$ around the same vertex m. The result after integration on the common links is simply the Boltzmann weight of the triangular plaquette $C = (a,b,c)$. Indeed we have:

$$\int dh(x)dh(y)dh(z)W(x,a,y)W(y^{-1},b,z^{-1})W(z,c,x^{-1})$$

$$= \int dh(x)dh(y)dh(z)\sum_{\alpha,\beta,\gamma}[d_\alpha][d_\beta][d_\gamma]\rho_\alpha(m,C_1)\rho_\beta(m,C_2)\rho_\gamma(m,C_3)$$

$$\times\, \omega_\alpha(C_1)\omega_\beta(C_2)\omega_\gamma(C_3)\mathrm{tr}_1(\overset{\alpha}{\mu}_1\overset{\alpha}{x}_1\overset{\alpha}{a}_1\overset{\alpha}{y}_1)\mathrm{tr}_2(\overset{\beta}{\mu}_2\overset{\beta}{y}{}_2^{-1}\overset{\beta}{b}_2\overset{\beta}{z}{}_2^{-1})\mathrm{tr}_3(\overset{\gamma}{\mu}_3\overset{\gamma}{z}_3\overset{\gamma}{c}_3\overset{\gamma}{x}{}_3^{-1})\,,$$

but we have, using the integration formula (79), that:

$$
\begin{aligned}
&\int dh(x)dh(y)dh(z)\mathrm{tr}_{123}(\overset{\alpha}{\mu}_1\overset{\alpha}{x}_1\overset{\alpha}{a}_1\overset{\alpha}{y}_1\overset{\beta}{\mu}_2\overset{\beta}{y}{}_2^{-1}\overset{\beta}{b}_2\overset{\beta}{z}{}_2^{-1}\overset{\gamma}{\mu}_3\overset{\gamma}{z}_3\overset{\gamma}{c}_3\overset{\gamma}{x}{}_3^{-1})\\
&\quad = \int dh(x)\mathrm{tr}_{123}(\overset{\alpha}{\mu}_1\overset{\alpha}{x}_1\overset{\alpha}{a}_1\delta_{\alpha,\beta}[d_\alpha]^{-1}P_{12}\overset{\beta}{b}_2\delta_{\beta,\gamma}[d_\beta]^{-1}P_{23}\overset{\gamma}{c}_3\overset{\gamma}{x}{}_3^{-1})\\
&\quad = \delta_{\alpha,\beta,\gamma}[d_\alpha]^{-2}\int dh(x)\mathrm{tr}_{123}(P_{12}P_{23}\overset{\alpha}{\mu}_3\overset{\alpha}{x}_3\overset{\alpha}{a}_3\overset{\alpha}{b}_3\overset{\alpha}{c}_3\overset{\alpha}{x}{}_3^{-1})\\
&\quad = \delta_{\alpha,\beta,\gamma}[d_\alpha]^{-2}\int dh(x)\mathrm{tr}(\overset{\alpha}{\mu}\overset{\alpha}{x}\overset{\alpha}{a}\overset{\alpha}{b}\overset{\alpha}{c}\overset{\alpha}{x}{}^{-1})\,.
\end{aligned}
$$

In order to integrate over the link x, we can use Proposition (2) and the relation

$$
\prod_{i=1}^{3}\rho_\alpha(m,C_i) = \rho_\alpha(m,(x,a,b,c,x^{-1}))\,, \tag{90}
$$

so that we can write:

$$
\begin{aligned}
&\int dh(x)dh(y)dh(z)W(x,a,y)W(y^{-1},b,z^{-1})W(z,c,x^{-1})\\
&\quad = \int dh(x)\sum_\alpha[d_\alpha]\omega_\alpha(C)\mathrm{tr}_{1\cdots5}(\overset{\alpha}{\mu}{}^{\otimes5}P_{54}\cdots P_{21}\overset{\alpha}{x}_1\mathscr{R}_{12}\overset{\alpha}{a}_2\mathscr{R}_{23}\overset{\alpha}{b}_3\mathscr{R}_{34}\overset{\alpha}{c}_4\mathscr{R}_{45}\overset{\alpha}{x}{}_5^{-1})\\
&\quad = \int dh(x)\sum_\alpha[d_\alpha]\omega_\alpha(C)\mathrm{tr}_{1\cdots5}(\overset{\alpha}{\mu}_1\overset{\alpha}{\mu}_2\overset{\alpha}{\mu}_3\overset{\alpha}{\mu}_5P_{54}\cdots P_{21}\overset{\alpha}{a}_2\mathscr{R}_{23}\overset{\alpha}{b}_3\mathscr{R}_{34}\overset{\alpha}{x}_1\overset{\alpha}{\mu}_5\overset{\alpha}{x}{}_5^{-1}\overset{\alpha}{c}_4)\\
&\quad = \sum_\alpha\omega_\alpha(C)\mathrm{tr}_{12345}(\overset{\alpha}{\mu}_1\overset{\alpha}{\mu}_2\overset{\alpha}{\mu}_3\overset{\alpha}{\mu}_5P_{15}P_{54}\cdots P_{21}\overset{\alpha}{a}_2\mathscr{R}_{23}\overset{\alpha}{b}_3\mathscr{R}_{34}\overset{\alpha}{c}_4)\\
&\quad = \sum_\alpha[d_\alpha]\omega_\alpha(C)\mathrm{tr}_{234}(\overset{\alpha}{\mu}_2\overset{\alpha}{\mu}_3\overset{\alpha}{\mu}_4P_{43}P_{32}\overset{\alpha}{a}_2\mathscr{R}_{23}\overset{\alpha}{b}_3\mathscr{R}_{34}\overset{\alpha}{c}_4)\\
&\quad = W(a,b,c)\,,
\end{aligned}
$$

which shows that M_1 is satisfied.

Let us now show the invariance under the second Matveev move. Consider two triangles $C_1=(a,b,x), C_2=(\bar{x},c,d)$ and denote by m the vertex common to a,x and d,

$$
\begin{aligned}
&\int dh(x)W(a,b,x)W(x^{-1},c,d)\\
&\quad = \int dh(x)\sum_{\alpha,\beta}[d_\alpha][d_\beta]\rho\alpha(m,C_1)\rho\beta(m,C_2)\omega_\alpha(C_1)\omega_\beta(C_2)\\
&\qquad \times\mathrm{tr}_{12}(\overset{\alpha}{\mu}_1\overset{\alpha}{a}_1\overset{\alpha}{b}_1\overset{\alpha}{x}_1\overset{\beta}{\mu}_2\overset{\beta}{x}{}_2^{-1}\overset{\beta}{c}_2\overset{\beta}{d}_2)\\
&\quad = \sum_{\alpha,\beta}[d_\alpha][d_\beta]\rho\alpha(m,C)\omega_\alpha(C)\mathrm{tr}_{12}(\overset{\alpha}{\mu}_1\overset{\alpha}{a}_1\overset{\alpha}{b}_1\delta_{\alpha,\beta}[d_\alpha]^{-1}P_{12}\overset{\beta}{c}_2\overset{\beta}{d}_2)\\
&\quad = \sum_\alpha[d_\alpha]\rho_\alpha(m,C)\omega_\alpha(C)\mathrm{tr}(\overset{\alpha}{\mu}\overset{\alpha}{a}\overset{\alpha}{b}\overset{\alpha}{c}\overset{\alpha}{d})\\
&\quad = W(a,b,c,d)\,.
\end{aligned}
$$

We just have to use cyclicity of $W(a,b,c,d)$ and to invert the previous computation to conclude the proof. □

Remark. Let C_1 and C_2 be simple loops of length greater than three with common edges, let $C_1 \cap C_2$ be the set of common edges, then it is easy to show the convolution property:

$$W(C_1 \# C_2) = \int W(C_1)W(C_2) \prod_{l \in C_1 \cap C_2} dh(l)\,. \tag{91}$$

In the rest of this work we will assume that

$$\omega_\alpha(C) = v_\alpha^{t_\alpha(1+\frac{1}{2}\sum_{x \in C}\varepsilon(x,C))}\,. \tag{92}$$

Proposition 9 (characters). *Let C be a simple closed curve on Σ. We will define a "character"*

$$\overset{\alpha}{\chi}(C) = v_\alpha^{-1-t_\alpha} W_\alpha(C)\,. \tag{93}$$

These elements satisfy the following usual orthonormality property:

$$\int \overset{\alpha}{\chi}(C)\overset{\beta}{\chi}(\bar{C}) \prod_{l \in C} dh(l) = \delta_{\alpha,\beta}\,. \tag{94}$$

Proof. The orthogonality relation is proved as follows:

$$\begin{aligned}
&\int \overset{\alpha}{\chi}(C)\overset{\beta}{\chi}(\bar{C}) \prod_{l \in C} dh(l) \\
&= \int v_\alpha^{-1-t_\alpha} v_\beta^{-1-t_\beta} \rho_\alpha(C)\rho_\beta(C^{-1}) \\
&\quad \mathrm{tr}_{12}\left(\overset{\alpha}{\mu}_1 \prod_{i=1}^{k} \overset{\alpha}{U}(l_i)_1 \omega_\alpha(C)\omega_\beta(C^{-1}) \overset{\beta}{\mu}_2 \prod_{i=k}^{1} \overset{\beta}{U}(\bar{l}_i)_2 \right) \prod_{l \in C} dh(l) \\
&= \int \mathrm{tr}_{12}\left(\overset{\alpha}{\mu}_1 \prod_{i=1}^{k-1} \overset{\alpha}{U}(l_i)_1 [d_\alpha]^{-1} \delta_{\alpha,\beta} P_{12} \prod_{i=k-1}^{1} \overset{\beta}{U}(\bar{l}_i)_2 \right) \prod_{i=1}^{k-1} dh(l_i) \\
&= \int \delta_{\alpha,\beta}[d_\alpha]^{-1} \mathrm{tr}_{12}\left(P_{12} \overset{\alpha}{\mu}_1 \prod_{i=1}^{k-1} \overset{\alpha}{U}(l_i) \right)_1 \prod_{i=k-1}^{1} \overset{\alpha}{U}(l_i)_1^{-1}) \prod_{i=1}^{k-1} dh(l_i) \\
&= \delta_{\alpha,\beta}[d_\alpha]^{-1} \mathrm{tr}_{12}(P_{12}\overset{\alpha}{\mu}_1) \\
&= \delta_{\alpha,\beta}\,.
\end{aligned}$$

Proposition 10 (Two and three points correlation functions). *Let $S_{2,2}$ be the two holed sphere with boundary loops C_1, C_2 and $S_{2,3}$ be the three holed sphere with boundary loops C_1, C_2, C_3. The expression of the two points and three points correlation functions is expressed in terms of characters as*:

$$Z(S_{2,2}; C_1, C_2) = \sum_{\alpha \in Irr(A)} \overset{\alpha}{\chi}_{S_{2,2},C_1} \overset{\alpha}{\chi}_{S_{2,2},C_2}\,, \tag{95}$$

$$Z(S_{2,3}; C_1, C_2, C_3) = \sum_{\alpha \in Irr(A)} v_\alpha^{-1-t_\alpha} [d_\alpha]^{-1} \overset{\alpha}{\chi}_{S_{2,3},C_1} \overset{\alpha}{\chi}_{S_{2,3},C_2} \overset{\alpha}{\chi}_{S_{2,3},C_3}\,. \tag{96}$$

Proof. Let us compute the two point correlation function with the loops C_1 and C_2 as boundaries.

To simplify the proof we consider the cylinder obtained by gluing two opposite edges of an hexagon $C = (a, b, c, d, e, \bar{c})$. Let us denote by m (resp. n) the departure point of b (resp. e). We have $C_1 = (a, b)$ and $C_2 = (d, e)$,

$$\int dh(c)W(a,b,c,d,e,c^{-1})$$

$$= \int dh(c)\sum_{\alpha}[d_\alpha]\rho_\alpha(m,C)\omega_\alpha(C)\mathrm{tr}(\overset{\alpha}{\mu}\overset{\alpha}{b}\overset{\alpha}{c}\overset{\alpha}{d}\overset{\alpha}{e}\overset{\alpha}{c}{}^{-1}\overset{\alpha}{a})$$

$$= \int dh(c)\sum_{\alpha}[d_\alpha]v_\alpha\omega_\alpha(C)\mathrm{tr}_{1\cdots6}(\overset{\alpha}{\mu}{}^{\otimes6}\sigma^{(6)}\overset{\alpha}{b}_1\mathcal{R}_{12}\overset{\alpha}{c}_2\mathcal{R}_{23}\overset{\alpha}{d}_3\mathcal{R}_{34}\overset{\alpha}{e}_4\mathcal{R}_{45}\overset{\alpha}{c}_5{}^{-1}\mathcal{R}_{56}\overset{\alpha}{a}_6)$$

$$= \int dh(c)\sum_{\alpha}[d_\alpha]v_\alpha\omega_\alpha(C)\mathrm{tr}_{1\cdots6}(\overset{\alpha}{\mu}_1\overset{\alpha}{\mu}_2\overset{\alpha}{\mu}_3\overset{\alpha}{\mu}_5\overset{\alpha}{\mu}_6\sigma^{(6)}\overset{\alpha}{b}_1\mathcal{R}_{12}\overset{\alpha}{d}_3\mathcal{R}_{34}\overset{\alpha}{c}_2\overset{\alpha}{\mu}_5\overset{\alpha}{c}_5{}^{-1}\overset{\alpha}{e}_4\mathcal{R}_{56}\overset{\alpha}{a}_6)$$

$$= \sum_{\alpha}v_\alpha\omega_\alpha(C)\mathrm{tr}_{1\cdots6}(\overset{\alpha}{\mu}_1\overset{\alpha}{\mu}_2\overset{\alpha}{\mu}_3\overset{\alpha}{\mu}_5\overset{\alpha}{\mu}_6\sigma^{(6)}P_{25}\overset{\alpha}{b}_1\mathcal{R}_{15}\mathcal{R}_{56}\overset{\alpha}{a}_6\overset{\alpha}{d}_3\mathcal{R}_{34}\overset{\alpha}{e}_4)\,.$$

Using the relation $\sigma^{(6)}P_{25} = P_{43}P_{32}P_{65}P_{51}$, the previous equation can also be written:

$$= \sum_{\alpha}v_\alpha\omega_\alpha(C)\mathrm{tr}_{156}(\overset{\alpha}{\mu}_1\overset{\alpha}{\mu}_5\overset{\alpha}{\mu}_6P_{65}P_{51}\overset{\alpha}{b}_1\mathcal{R}_{15}\mathcal{R}_{56}\overset{\alpha}{a}_6)\mathrm{tr}_{34}(\overset{\alpha}{\mu}_3\overset{\alpha}{\mu}_4P_{34}\overset{\alpha}{d}_3\mathcal{R}_{34}\overset{\alpha}{e}_4)$$

$$= \sum_{\alpha}v_\alpha^{-2-2t_\alpha}\omega_\alpha(C_1)\omega_\alpha(C_2)\rho_\alpha(m,C_1)\rho_\alpha(n,C_2)\mathrm{tr}(\overset{\alpha}{\mu}\overset{\alpha}{b}\overset{\alpha}{a})\mathrm{tr}(\overset{\alpha}{\mu}\overset{\alpha}{d}\overset{\alpha}{e})\,.$$

Therefore we have proven (95) when C_1, C_2 have length two. The structure of the proof is the same when C_1, C_2 have different lengths.

Let us now compute the three point correlation function with the loops C_1, C_2, C_3 as boundaries. To simplify the proof we consider the three holed sphere obtained from a decagon $C = (b^{(2)}, a, c^{(1)}, e, d^{(1)}, d^{(2)}, \bar{e}, c^{(2)}, \bar{a}, b^{(1)})$. We will denote by x, y, z respectively the departure points of $b^{(1)}, a^{(1)}, c^{(1)}$,

$$\int dh(a)dh(e)W(b^{(2)},a,c^{(1)},e,d^{(1)},d^{(2)},e^{-1},c^{(2)},a^{-1},b^{(1)})$$

$$= \int dh(a)dh(e)\sum_{\alpha}[d_\alpha]v_\alpha\omega_\alpha(C)\mathrm{tr}_{0\cdots9}(\overset{\alpha}{\mu}{}^{\otimes10}P_{09}\cdots P_{21}\overset{\alpha}{b}{}^{(2)}_1\mathcal{R}_{12}\overset{\alpha}{a}_2\mathcal{R}_{23}\overset{\alpha}{c}{}^{(1)}_3$$

$$\times\mathcal{R}_{34}\overset{\alpha}{e}_4\mathcal{R}_{45}\overset{\alpha}{d}{}^{(1)}_5\mathcal{R}_{56}\overset{\alpha}{d}{}^{(2)}_6\mathcal{R}_{67}\overset{\alpha}{e}_7{}^{-1}\mathcal{R}_{78}\overset{\alpha}{c}{}^{(2)}_8\mathcal{R}_{89}\overset{\alpha}{a}_9{}^{-1}\mathcal{R}_{90}\overset{\alpha}{b}{}^{(1)}_0)$$

$$= \int dh(a)dh(e)\sum_{\alpha}[d_\alpha]v_\alpha\omega_\alpha(C)\mathrm{tr}_{0\cdots9}(\overset{\alpha}{\mu}_0\overset{\alpha}{\mu}_9\overset{\alpha}{\mu}_7\overset{\alpha}{\mu}_5\cdots\overset{\alpha}{\mu}_1P_{09}\cdots P_{21}\overset{\alpha}{b}{}^{(2)}_1\mathcal{R}_{12}\overset{\alpha}{c}{}^{(1)}_3$$

$$\times\mathcal{R}_{34}\overset{\alpha}{d}{}^{(1)}_5\mathcal{R}_{56}\overset{\alpha}{e}_4\overset{\alpha}{\mu}_7\overset{\alpha}{e}_7{}^{-1}\overset{\alpha}{d}{}^{(2)}_6\mathcal{R}_{78}\overset{\alpha}{a}_2\overset{\alpha}{\mu}_9\overset{\alpha}{a}_9{}^{-1}\overset{\alpha}{c}{}^{(2)}_8\mathcal{R}_{90}\overset{\alpha}{b}{}^{(1)}_0)$$

$$= \sum_{\alpha}[d_\alpha]^{-1}\omega_\alpha(C)v_\alpha\mathrm{tr}_{0\cdots9}(\overset{\alpha}{\mu}_0\overset{\alpha}{\mu}_9\overset{\alpha}{\mu}_7\overset{\alpha}{\mu}_5\cdots\overset{\alpha}{\mu}_1P_{09}\cdots P_{21}P_{47}P_{29}\overset{\alpha}{b}{}^{(2)}_1\mathcal{R}_{19}\overset{\alpha}{c}{}^{(1)}_3$$

$$\times\mathcal{R}_{37}\overset{\alpha}{d}{}^{(1)}_5\mathcal{R}_{56}\overset{\alpha}{d}{}^{(2)}_6\mathcal{R}_{78}\overset{\alpha}{c}{}^{(2)}_8\mathcal{R}_{90}\overset{\alpha}{b}{}^{(1)}_0)$$

$$= \sum_\alpha [d_\alpha]^{-1} v_\alpha \omega_\alpha(C) \mathrm{tr}_{019}(\overset{\alpha}{\mu}_0 \overset{\alpha}{\mu}_1 \overset{\alpha}{\mu}_9 P_{09} P_{91} \overset{\alpha}{b}{}_1^{(2)} \mathcal{R}_{19} \mathcal{R}_{90} \overset{\alpha}{b}{}_0^{(1)})$$

$$\times \mathrm{tr}_{2378}(\overset{\alpha}{\mu}_2 \overset{\alpha}{\mu}_3 \overset{\alpha}{\mu}_7 P_{87} P_{73} P_{32} \overset{\alpha}{c}{}_3^{(1)} \mathcal{R}_{37} \mathcal{R}_{78} \overset{\alpha}{c}{}_8^{(2)})$$

$$\times \mathrm{tr}_{456}(\overset{\alpha}{\mu}_4 \overset{\alpha}{\mu}_5 P_{65} P_{54} \overset{\alpha}{d}{}_5^{(1)} \mathcal{R}_{56} \overset{\alpha}{d}{}_6^{(2)})$$

$$= \sum_\alpha [d_\alpha]^{-1} v_\alpha^{-4-4t_\alpha} \prod_{i=1}^{3} \omega_\alpha(C_i) \rho_\alpha(C_i) \mathrm{tr}(\overset{\alpha}{\mu} \overset{\alpha}{b}{}^{(1)} \overset{\alpha}{b}{}^{(2)}) \mathrm{tr}(\overset{\alpha}{\mu} \overset{\alpha}{c}{}^{(1)} \overset{\alpha}{c}{}^{(2)}) \mathrm{tr}(\overset{\alpha}{\mu} \overset{\alpha}{d}{}^{(1)} \overset{\alpha}{d}{}^{(2)}) .$$

Therefore we have proven (96) when C_1, C_2, C_3 have length two. The structure of the proof is the same when C_1, C_2, C_3 have different lengths. □

Proposition 11 (Partition function of a general Riemann surface). *The partition function Z_g of a compact connected orientable Riemann surface of genus g without boundary is given by the formula:*

$$Z_g = \sum_\alpha (v_\alpha^{1+t_\alpha}[d_\alpha])^{2-2g} . \tag{97}$$

Proof. This result is easily obtained as in the undeformed case by using the expression of the two and three point correlation functions and the orthogonality of the characters [17, 18]. □

Remark 1. When $C_\alpha \neq 0$ the last formula is modified as follows:

$$Z_g = \sum_\alpha (v_\alpha^{1+t_\alpha}[d_\alpha])^{2-2g} e^{-A_\Sigma C_\alpha} , \tag{98}$$

where A_Σ is the area of the Riemann surface Σ.

Remark 2. The previous proposition clearly shows that the partition function is independent of the choice of the ciliation we choose to define the algebra of gauge fields.

Proposition 12 (Correlation functions). *The correlation functions $Z_{g,n}(C_1, \cdots, C_n)$ of a Riemann surface of genus g with boundary $\partial\Sigma = \cup_i C_i$ is given by the formula:*

$$Z_g = \sum_\alpha (v_\alpha^{1+t_\alpha}[d_\alpha])^{2-2g-n} \prod_{i=1}^{n} \overset{\alpha}{\chi}_{\Sigma,C_i} . \tag{99}$$

Proof. The proof is the same as in the classical case. It suffices to cut the surface in three holed spheres glued along their boundaries and to use Proposition 10. □

Up to this point the choices of the t_α were completely arbitrary. From Propositions (11–12) we clearly see that the choice $t_\alpha = -1$ is particularly important. In that case final formulas exhibit a complete symmetry in the exchange of q and q^{-1}. If we take $\mathcal{G} = sl_2$ and formally set q being a root of unity and truncate the spectrum according to the value of q the partition function Z_g is equal to the Turaev–Viro invariant of $\Sigma \times [0, 1]$. This tends to support the connection between our theory at $t_\alpha = -1$ with hamiltonian Chern–Simons theory.

In the rest of this work we will set $t_\alpha = -1$ and prove the fusion identities for characters. But it is nice to introduce a new notation:

Definition 8. *From now the holonomy $\mathscr{H}$ associated to a simple closed curve $C = (x_1, x_2, \dots, x_{n+1} = x_1)$ will be a renormalised version of the previous definition*:

$$\mathscr{H}^{\alpha}(x_1, x_2, \dots, x_{n+1} = x_1) = v_{\alpha}^{\frac{1}{2}((\sum_{x \neq x_1} \varepsilon(x,C)) - \varepsilon(x_1,C))} \prod_{i=1}^{n} \overset{\alpha}{U}(x_i, x_{i+1}) . \tag{100}$$

Then we have:

$$\chi_{\alpha}(C) = \mathrm{tr}(\overset{\alpha}{\mu}\mathscr{H}^{\alpha}(C)) \tag{101}$$

and for each plaquettes F:

$$W_F = \sum_{\alpha} [d_{\alpha}] \chi_{\alpha}(\partial F) . \tag{102}$$

Proposition 13 (Fusion of characters). *Let C be a simple loop, we have the following fusion identity on characters*:

$$\chi_{\alpha}(C)\chi_{\beta}(C) = \sum_{\gamma} N_{\alpha\beta}^{\gamma} \chi_{\gamma}(C) . \tag{103}$$

Proof. We will prove it when C is of length three. The generalization to arbitrary length is straightforward. Let $C = (x, y, z)$, where x, y, z are the links of C, where $x = (m, n), y = (n, p), z = (p, m)$; using the trick of the last part of the proof of Proposition (3) we can write:

$$\begin{aligned}
\overset{\alpha}{x}_1\overset{\alpha}{y}_1\overset{\alpha}{z}_1\overset{\beta}{x}_2\overset{\beta}{y}_2\overset{\beta}{z}_2 &= \sum_{i,j} \overset{\alpha}{x}_1\overset{\alpha}{y}_1\overset{\alpha}{z}_1 a_1^{(i)} a_1^{(j)} S^{-2}(b_2^{(j)}) S^{-1}(b_2^{(i)}) \overset{\beta}{x}_2\overset{\beta}{y}_2\overset{\beta}{z}_2 \\
&= \sum_{i,j} S^{-2}(b_2^{(j)}) \overset{\alpha}{x}_1\overset{\alpha}{y}_1\overset{\alpha}{z}_1 \mathscr{R}_{12}(p,m,n) \overset{\beta}{x}_2\overset{\beta}{y}_2\overset{\beta}{z}_2 a_1^{(j)} \\
&= \sum_{i,j} S^{-2}(b_2^{(j)}) \overset{\alpha}{x}_1\overset{\alpha}{y}_1\overset{\beta}{x}_2\overset{\alpha}{z}_1\overset{\beta}{y}_2\overset{\beta}{z}_2 a_1^{(j)} \\
&= \sum_{i,j} S^{-2}(b_2^{(j)}) \overset{\alpha}{x}_1\overset{\beta}{x}_2 \mathscr{R}_{21}(m,n,p) \overset{\alpha}{y}_1\overset{\beta}{y}_2 \mathscr{R}_{21}(n,p,m) \overset{\alpha}{z}_1\overset{\beta}{z}_2 a_1^{(j)} \\
&= \sum_{\substack{i,j \\ \gamma, m}} S^{-2}(b_2^{(j)}) \phi_{\alpha\beta}^{\gamma,m} \overset{\gamma}{x}\overset{\gamma}{y}\overset{\gamma}{z} \psi_{\gamma,m}^{\beta,\alpha} P_{12} a_1^{(j)} \lambda_{\alpha\beta\gamma}^{-\varepsilon(n)-\varepsilon(p)} .
\end{aligned}$$

As a result we get:

$$\begin{aligned}
&\mathrm{tr}_{12}(\overset{\alpha}{\mu}_1\overset{\beta}{\mu}_2\overset{\alpha}{x}_1\overset{\alpha}{y}_1\overset{\alpha}{z}_1\overset{\beta}{x}_2\overset{\beta}{y}_2\overset{\beta}{z}_2) \\
&= \sum_{\substack{i,j \\ \gamma, m}} \mathrm{tr}_{12}(\overset{\alpha}{\mu}_1\overset{\beta}{\mu}_2 S^{-2}(b_2^{(j)}) \phi_{\alpha\beta}^{\gamma,m} \overset{\gamma}{x}\overset{\gamma}{y}\overset{\gamma}{z} \psi_{\gamma,m}^{\beta,\alpha} P_{12} a_1^{(j)}) \lambda_{\alpha\beta\gamma}^{-\varepsilon(n)-\varepsilon(p)} \\
&\times \sum_{\gamma, m} \mathrm{tr}_{12}(\overset{\alpha}{\mu}_1\overset{\beta}{\mu}_2 \phi_{\alpha\beta}^{\gamma,m} \overset{\gamma}{x}\overset{\gamma}{y}\overset{\gamma}{z} \psi_{\gamma,m}^{\beta,\alpha} P_{12} \mathscr{R}_{12}^{-1}(p,m,n)) \lambda_{\alpha\beta\gamma}^{-\varepsilon(n)-\varepsilon(p)} \\
&\times \sum_{\gamma} N_{\alpha\beta}^{\gamma} \mathrm{tr}(\overset{\gamma}{\mu}\overset{\gamma}{x}\overset{\gamma}{y}\overset{\gamma}{z}) \lambda_{\alpha\beta\gamma}^{-\varepsilon(n)-\varepsilon(p)+\varepsilon(m)} ,
\end{aligned}$$

which implies straightforwardly the fusion relation:

$$\chi_\alpha(C)\chi_\beta(C) = \sum_\gamma N^\gamma_{\alpha\beta}\chi_\gamma(C)\,. \quad \square \tag{104}$$

Finally it is possible to show that W_F is in a certain sense a delta function which constrains the holonomy of the plaquettes to be equal to one.

Proposition 14 (Delta function and flat connections). *The Boltzmann weight is a delta function located at the unit element for the holonomy around the plaquette C:*

$$W_C \times (\mathcal{H}^\beta(C) - 1) = 0\,. \tag{105}$$

Proof. In fact we will show the latter property in a slightly different form:

$$W_C(\mathrm{tr}(\overset{\beta}{\mu}\mathcal{H}^\beta(C)V) - \mathrm{tr}(\overset{\beta}{\mu}V)) = 0 \tag{106}$$

for any V in $End(V_\beta)$.

With the same method as before we have:

$$\begin{aligned} W_F\mathrm{tr}(\overset{\beta}{\mu}\mathcal{H}^\beta(C)V^\beta) &= (\sum_\alpha [d_\alpha]\chi_\alpha(C))\mathrm{tr}(\overset{\beta}{\mu}\mathcal{H}^\beta(C)V) \\ &= \sum_\alpha [d_\alpha]\mathrm{tr}_{12}(\overset{\alpha}{\mu}_1\mathcal{H}^\alpha(C)_1\overset{\beta}{\mu}_2\mathcal{H}^\beta(C)_2V_2) \\ &= \sum_{\gamma,m}\mathrm{tr}_{V_\gamma}\left(\overset{\gamma}{\mu}\mathcal{H}^\gamma(C)\left(\sum_\alpha [d_\alpha]\psi^{\beta\alpha}_\gamma V\phi^\gamma_{\beta\alpha}\right)\right)\,. \end{aligned}$$

In order to simplify the last part of the latter equation, we must establish some relations between Clebsch–Gordan coefficients. There exist complex numbers $A^{\beta\alpha}_\gamma$ and $B^\gamma_{\beta\alpha}$ such that:

$$\begin{aligned} \psi^{\beta,\alpha}_{\gamma,m} &= \sum_{m'} M^{m'}_m A^{\beta\alpha}_\gamma(\psi^{\beta,\bar\beta}_0 \otimes id_{V_\gamma})(id_{V_\beta}\otimes\phi^{\bar\beta,\alpha}_{\gamma,m'})\,, \\ \phi^{\gamma,m}_{\beta,\alpha} &= \sum_{m'}(M^{-1})^m_{m'}B^\gamma_{\beta\alpha}(id_{V_\beta}\otimes\psi^{\gamma,m'}_{\bar\beta,\alpha})(\phi^0_{\beta,\bar\beta}\otimes id_{V_\gamma}) \end{aligned}$$

with:

$$A^{\beta\alpha}_\gamma B^\alpha_{\bar\beta,\gamma} = [d_\beta]\,. \tag{107}$$

Using the scalar product it is also easy to obtain the relations:

$$B^\alpha_{\beta,\gamma} = [d_\alpha]B^{\bar\gamma}_{\bar\alpha,\beta}\,, \tag{108}$$

$$A^\gamma_{\beta,\alpha} = [d_\gamma]B^{\bar\gamma}_{\bar\alpha,\bar\beta}\,. \tag{109}$$

The latter formulas imply the identity:

$$\begin{aligned} \psi^{\beta,\alpha}_\gamma V^\beta V'^\alpha\phi^\gamma_{\beta,\alpha} = {}& [d_\alpha]^{-1}[d_\beta][d_\gamma](\psi^{\beta,\bar\beta}_0\otimes id_{V_\gamma})(id_{V_\beta}\otimes\phi^{\bar\beta,\alpha}_\gamma) \\ &\times V^\beta V'^\alpha(id_{V_\beta}\otimes\psi^\gamma_{\bar\beta,\alpha})(\phi^0_{\beta,\bar\beta}\otimes id_{V_\gamma})\,. \end{aligned}$$

with $V' \in End(V_\alpha)$

We are now able to conclude the proof:

$$\sum_{\alpha}[d_\alpha]\psi_{\gamma}^{\beta,\alpha}V^{\beta}\phi_{\beta,\alpha}^{\gamma}$$

$$= \sum_{\alpha,\gamma,m}[d_\beta][d_\gamma](\psi_0^{\beta,\bar\beta}\otimes id_{V_\gamma})(id_{V_\beta}\otimes\phi_{\gamma}^{\bar\beta,\alpha})V^{\beta}(id_{V_\beta}\otimes\psi_{\bar\beta,\alpha}^{\gamma})(\phi_{\beta,\bar\beta}^{0}\otimes id_{V_\gamma})$$

$$= [d_\gamma]id_{V_\gamma}([d_\beta]\psi_0^{\beta,\bar\beta}V^{\beta}\phi_{\beta,\bar\beta}^{0})$$

$$= [d_\gamma]id_{V_\gamma}\mathrm{tr}_{V_\beta}(\overset{\beta}{\mu}V^{\beta}). \quad \square$$

This property is essential in the context of quantum gauge theory as described before [14], but it also allows us to understand why this theory is related to Chern–Simons theory.

In our next work [6] we will study more deeply the algebra of gauge invariant elements and develop the properties of $P = \prod_i W_{F_i}$. We will define gauge invariant and cyclic invariant objects associated to intersecting curves and will show that the expectation values of these Wilson loops are in fact 3-dimensional knot invariants related to Reshetikhin–Turaev ribbon invariants in the three manifold $\Sigma \times [0,1]$.

6. Conclusion

In this work we have defined and studied a two dimensional quantum gauge field theory where the gauge symmetry is described by a quantum group. This model is a quasitopological field theory with an infinite number of fields. We still have to understand the situation where q is a root of unity (this was not allowed in our setting). In order to solve this problem one should certainly use the formalism of weak quasi Hopf algebras [13] to ensure truncation of the spectrum. This study has already been started in [1]. Our formalism does not use any involution but it is highly desirable to introduce one to ensure reality properties of the theory. It seems that when q is real there is no problem of this type and the corresponding theory is a deformation of two dimensional Yang Mills theory associated to compact classical Lie groups.

This gauge theory appears to describe a three dimensional topological theory, i.e. that all correlation functions of Chern–Simons theory can be obtained by a finite dimensional path integral formalism in a two dimensional space time [6]. One of the remaining challenging problems is to extend the present formalism to higher dimensional space time where the corresponding theories should be far more interesting. In two dimensions the definition of Λ uses as a central tool a cilium order at each vertex. This is easy to define using the natural cyclic order on a two dimensional oriented surface. In higher dimension it seems that there is no problem of definition of Λ using as well a cilium at each vertex, but one has to find clever permutation invariant objects in order to compensate non-commutation of Boltzmann weights.

Another important problem which still remains open is to find representations of the gauge field algebra Λ. Representations of lattice Kac Moody algebras have already been found in [3, 2]. Constructions inspired by these works should lead to representations of Λ. Representations of this algebra is an important step to understand the precise continuum limit of those theories and to compare them with Hamiltonian Chern–Simons theory.

Acknowledgements. We would like to thank M. Petropoulos for interesting discussions at the beginning of this work as well as A.Yu. Alekseev, B. Enriquez, L. Freidel, J.M. Maillet, V. Pasquier, N. Reshetikhin and V. Schomerus for discussions and encouragements.

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Communicated by R.H. Dijkgraaf