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The Algebra of Non-Local Charges in Non-Linear Sigma Models

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Abstract: We obtain the exact classical algebra obeyed by the conserved non-local charges in bosonic non-linear sigma models. Part of the computation is specialized for a symmetry group $O(N)$. As it turns out the algebra corresponds to a cubic deformation of the Kac–Moody algebra. We generalize the results for the presence of a Wess-Zumino term. The algebra is very similar to the previous one, now containing a calculable correction of order one unit lower. The relation with Yangians and the role of the results in the context of Lie-Poisson algebras are also discussed.

1. Introduction

In general, quantum field theoretic models where non-perturbative computations are known, contain an infinite number of conservation laws [1, 2]. In fact, the solvability of several exact S-matrices in two-dimensional models can be traced back to the Yang–Baxter relations [3, 4], which in turn are a direct consequence of the conservation of higher powers of the momentum. Alternatively, there is an infinite number of non-local conservation laws in most of these models as well [2, 5]. Both sets of conserved quantities can be related to the existence of a Lax pair in the theory: demanding compatibility of the Lax pair, one arrives at conserved charges as functions of the so-called spectral parameter implying, after Taylor expansion, an infinite number of conservation laws.

Another set of models containing an infinite number of conserved quantities are the two-dimensional conformally invariant theories [6, 7]. The Virasoro generators are a generalization of the energy-momentum-conserved charges. Defining

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a realization of the symmetry in terms of the null vectors implies a number of differential equations to be obeyed by the correlation functions, which can be integrated. In other words, a further knowledge of the underlying algebra obeyed by the conserved quantities, namely the Virasoro algebra, together with the differential representation of the conserved charges, permitted one to go one step further, i.e. to the complete computation of the correlators.

Our aim here is to obtain the algebra of conserved quantities for integrable theories. The algebra of local conservation laws is Abelian and therefore too simple. Massive perturbations of the conformal generators are also a possibility, since they also form a non-commuting algebra, and it would be worthwhile to understand the algebra, as well as the role played by the conservation laws surviving the mass perturbation [8]. For free fermions ($k = 1$ WZW models) the results conform to our expectation [9].

Non-local conserved charges, on the other hand, are very powerful objects. The first non-trivial one alone fixes almost completely the on-shell dynamics [5, 10].

Infinite algebras connected with non-trivial conserved quantities could thus be the key ingredient for the complete solvability of integrable models, and for the knowledge of their correlation functions. It is thus no wonder that the problem evaded solution in spite of several attempts. Indeed, it has been claimed long ago [11] that non-local charges might build up a Kac–Moody algebra, but the appearance of cubic terms found by several authors showed that the algebraic problem was much more involved [12–14]. For non-linear sigma models with a simple gauge group, the quantum non-local charges present no anomaly [15], and the monodromy matrix can be computed. Therefore the non-local charge algebra should be manageable; however, as it turns out, the complete algebra was not known, and there were hints that a possible break of the Jacobi identity might occur [12].

We show that there is a natural recombination of the standard non-local charges, whose algebra has an approachable structure, being composed of a linear part of the Kac–Moody form, and a calculable cubic term. Later we add a Wess–Zumino (WZ) term to the action, and show that both linear and cubic pieces of the algebra acquire a further contribution.

In order to find these results we adopt the following strategy. We explicitly compute the first few conserved charges generated by the procedure of Brézin et al. [16]: the Dirac brackets of those charges are rather obscure, as we verify (there are also examples in the literature [12–14]). Therefore we subsequently define an *improved* set of charges in order to simplify the algebra. By inspection, we propose an Ansatz for the general algebra of the improved charges. At this point we could argue, based on the Jacobi identity proved in the subsequent section, that once we have verified the algebra up to some order, there must be a set of charges whose algebra agrees with the Ansatz.

In order to verify the Jacobi identity, we introduce a set of (non-conserved) charges whose algebra is isomorphic to the Ansatz. In that case it is useful to start from the analysis of a kind of *chain algebra*, in the sense that we commute elements defined by chains of local currents tied by a non-local function in space. In terms of these objects we define a linear algebra, albeit with a much larger set of terms. Finally, by a sort of trace projection, we recover the original algebra in terms of the *saturated* charges, proving the Jacobi identity in an indirect way.

This paper is divided as follows: in Sect. 2 we review the algebra obeyed by Noether local currents based on Refs. [17, 18]. In Sect. 3 we consider the canonical construction of higher non-local conservation laws. We introduce the improved charges and write the Ansatz for their complete algebra. We also define the algebra of saturated charges, which turns out to be isomorphic to the algebra of conserved charges. We derive the chain algebra structure, the corresponding Jacobi identity, and relate the results to the case of non-local charges. In Sect. 4 we review some results on Yangian algebras and verify that the first pair of charges provides a concrete realization of that structure. In Sect. 5 we introduce the WZ interaction to derive the corresponding algebra. We leave Sect. 6 for conclusions.

2. Current Algebra of Non-Linear Sigma Models

The current algebra of classical non-linear sigma models on arbitrary Riemannian manifolds (M) is known [17]. Indeed, consider a non-linear sigma model on M, with metric $g_{ij}(\varphi)$, and the maps $\varphi^i(x)$ from two-dimensional Minkowski space Σ to M. The sigma model action is given by

$$S = \frac{2}{2\lambda^2} \int_{\Sigma} d^2 x \eta^{\mu\nu} g_{ij}(\varphi) \partial_\mu \varphi^i \partial_\nu \varphi^j \, . \tag{1}$$

The phase space consists of pairs $(\varphi^i(x), \pi_i(x))$, where π is a section of the pull-back $\varphi^*(T^*M)$ of the cotangent bundle of M to the Minkowski space via φ, and the canonical equal-time Poisson brackets read

$$\begin{aligned} \{\varphi^i(x), \varphi^j(y)\} &= \{\pi_i(x), \pi_j(y)\} = 0 \, , \\ \{\varphi^i(x), \pi_j(y)\} &= \delta^i_j \delta(x-y) \, . \end{aligned} \tag{2}$$

From the action (1) we find the canonically conjugated momenta, given by the expression

$$\pi_i = \frac{1}{\lambda^2} g_{ij} \dot{\varphi}^j \, . \tag{3}$$

We suppose that there is a connected Lie group G acting on M by isometries, such that a generator of the Lie algebra $\mathbf{g}$ of G is represented by a fundamental vector field

$$X_M(m) = \frac{d}{dt} e^{tX} \cdot m|_{t=0} \tag{4}$$

on M; the Noether current may be defined as

$$(j_\mu, X) = -\left(\frac{1}{\lambda^2} g_{ij}(\varphi) \partial_\mu \varphi^i X^j_M(\varphi) \right) . \tag{5}$$

We define also the symmetric scalar field j as

$$(j, X \otimes Y) = \frac{1}{\lambda^2} g_{ij}(\varphi) X^i_M Y^j_M \, . \tag{6}$$

In terms of basis t^a of $\mathbf{g}$, such that $[t^a, t^b] = f^{abc} t^c$, we have

$$\begin{aligned} j_\mu &= j^a_\mu t^a \\ j &= j^{ab} t^a \otimes t^b \, , \end{aligned} \tag{7}$$

and we find the current algebra

$$\begin{aligned}
\{j_0^a(x), j_0^b(y)\} &= -f^{abc} j_0^c(x)\delta(x-y)\,,\\
\{j_0^a(x), j_1^b(y)\} &= -f^{abc} j_1^c(x)\delta(x-y) + j^{ab}(y)\delta'(x-y)\,,\\
\{j_1^a(x), j_1^b(y)\} &= 0\,,\\
\{j_0^a(x), j^{bc}(y)\} &= -(f^{abd} j^{cd}(x) + f^{acd} j^{bd}(x))\delta(x-y)\,,\\
\{j_1^a(x), j^{bc}(y)\} &= 0\,,\\
\{j^{ab}(x), j^{cd}(y)\} &= 0\,.
\end{aligned} \tag{8}$$

In order to give explicit examples although without loss of generality, we specialize to the $O(N)$ case, with Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu \varphi_i \partial^\mu \varphi_i, \quad \sum_{i=1}^{N} \varphi_i^2 = 1\,, \tag{9}$$

and Hamiltonian density

$$\mathscr{H} = \frac{1}{2}(\pi_i^2 + \varphi_i'^2)\,, \tag{10}$$

where $\pi_i = \dot\varphi_i$. We have to impose the constraints

$$\varphi_i^2 - 1 = 0 \quad \text{and} \quad \varphi_i \pi_i = 0\,. \tag{11}$$

Dirac brackets can be easily calculated and read

$$\begin{aligned}
\{\varphi_i(x), \varphi_j(y)\} &= 0\,,\\
\{\varphi_i(x), \pi_j(y)\} &= (\delta_{ij} - \varphi_i\varphi_j)(x)\delta(x-y)\,,\\
\{\pi_i(x), \pi_j(y)\} &= -(\varphi_i\pi_j - \varphi_j\pi_i)(x)\delta(x-y)\,.
\end{aligned} \tag{12}$$

In terms of phase space variables the conserved current components may be written as

$$(j_0)_{ij} = \varphi_i\pi_j - \varphi_j\pi_i\,, \tag{13a}$$

$$(j_1)_{ij} = \varphi_i\varphi_j' - \varphi_j\varphi_i'\,. \tag{13b}$$

Notice that j_μ is an antisymmetric matrix-valued field. On the other hand the intertwiner field given in (7) is symmetric,

$$(j)_{ij} = \varphi_i\varphi_j\,. \tag{13c}$$

We observe that the Hamiltonian (10) can be written in the Sugawara form,

$$\mathscr{H} = -\frac{1}{4}\operatorname{tr}(j_0^2 + j_1^2)\,. \tag{14}$$

It is convenient to present the current algebra in terms of matrix components, which follows from the elementary brackets (12):

$$\begin{aligned}
\{(j_0)_{ij}(x), (j_0)_{kl}(y)\} &= (\delta\circ j_0)_{ij,kl}(x)\delta(x-y)\,,\\
\{(j_1)_{ij}(x), (j_0)_{kl}(y)\} &= (\delta\circ j_1)_{ij,kl}(x)\delta(x-y) + (\delta\circ j)_{ij,kl}(x)\delta'(x-y)\,,\\
\{(j_1)_{ij}(x), (j_1)_{kl}(y)\} &= 0\,,
\end{aligned}$$

$$\{(j)_{ij}(x), (j)_{kl}(y)\} = 0\,,$$
$$\{(j)_{ij}(x), (j_1)_{kl}(y)\} = 0\,,$$
$$\{(j)_{ij}(x), (j_0)_{kl}(y)\} = -(\delta \star j)_{ij,kl}(x)\delta(x-y)\,, \tag{15}$$

where

$$(\delta \circ A)_{ij,kl} \equiv \delta_{ik}A_{jl} - \delta_{il}A_{jk} + \delta_{jl}A_{ik} - \delta_{jk}A_{il}\,, \tag{16}$$

$$(\delta \star A)_{ij,kl} \equiv \delta_{ik}A_{jl} - \delta_{il}A_{jk} - \delta_{jl}A_{ik} - \delta_{jk}A_{il}\,. \tag{17}$$

Further useful properties of the product defined in (16) are listed in Appendix A. The algebra of components (8) can be easily re-derived from (15) using the property (A.8).

3. Improved Non-local Charges and Their Algebra

Non-local charges may be generated by a very simple algorithm [16], starting out of a current j_μ obeying

$$\partial^\mu j_\mu = 0\,,$$
$$\partial_\mu j_\nu - \partial_\nu j_\mu + 2(j_\mu, j_\nu] = 0\,. \tag{18}$$

Given a conserved current $J_\mu^{(n)}$, one defines the associated non-local potential $\chi^{(n)}$ through the equation

$$J_\mu^{(n)} = \varepsilon_{\mu\nu}\partial^\nu \chi^{(n)}\,, \tag{19}$$

and build the $(n+1)^{\text{th}}$ order non-local current

$$J_\mu^{(n+1)} \equiv D_\mu \chi^{(n)} = \partial_\mu \chi^{(n)} + 2[j_\mu, \chi^{(n)}]\,. \tag{20}$$

Such a current is also conserved as a consequence of Eq. (18). Here we have to mention that for the first non-local current $J_\mu^{(1)}$ the coefficient in front of the cummutator in (20) must be taken as 1 instead of 2. We call the corresponding conserved integrals,

$$\hat{Q}^{(n)} = \int dx\, J_0^{(n)}(x)\,, \tag{21}$$

the *standard* charges, since they constitute the usual set found in the literature. We use the "hat" notation to distinguish these charges from a new set to be defined later on. It is worth writing the explicit formulae of the first pair,

$$\hat{Q}^{(0)} = \int dx j_0\,, \tag{22}$$

$$\hat{Q}^{(1)} = \int dx\,(j_1 + 2j_0\partial^{-1}j_0)\,, \tag{23}$$

where the operator ∂^{-1} is defined by Eq. (B.1) in Appendix B.

However, it turns out that the algebra satisfied by this standard set of charges is not transparent enough [12–14]. In the search for a more suitable basis of charges we find out an algebraic algorithm, where the charge $\hat{Q}^{(1)}$ plays a fundamental role, generating an *improved* set of conserved charges $\{Q^{(n)}\}$, $n = 0, 1, \ldots$. Indeed the first pair coincides with the standard one,

$$Q^{(0)} = \hat{Q}^{(0)}, \quad Q^{(1)} = \hat{Q}^{(1)}\,. \tag{24}$$

The remaining charges are defined iteratively by means of the Dirac brackets with $Q^{(1)}$: we verify that the bracket $\{Q^{(1)}, Q^{(n)}\}$ always produces a term of the form $(\delta \circ A)$ for some A, which we call the linear piece; and other essentially different terms as $(B \circ C)$, with B and C different from the identity matrix (δ), coming from surface contributions, which we refer to as the non-linear piece (n.l.p.). Therefore we can take A as a definition of the charge $Q^{(n+1)}$,

$$(\delta \circ Q^{(n+1)}) \equiv \{Q^{(1)}, Q^{(n)}\} - (\text{n.l.p.})\,. \tag{25}$$

While the standard charges are defined through an integro-differential algorithm the improved ones are generated by an algebraic procedure (where $Q^{(1)}$ plays the role of a "step" generator). The reader may find in Appendix B the expression of the first six improved charges. We remark though that the new set can actually be related to the standard one. Nevertheless the following examples

$$Q^{(2)} \equiv \frac{2}{3}\hat{Q}^{(2)} + \frac{4}{3}\hat{Q}^{(0)} - \frac{1}{3}(\hat{Q}^{(0)})^3\,,$$

$$Q^{(3)} \equiv \frac{1}{3}\hat{Q}^{(3)} + \frac{8}{3}\hat{Q}^{(1)} - \frac{2}{3}\hat{Q}^{(1)}(\hat{Q}^{(0)})^2 - \frac{4}{3}\hat{Q}^{(0)}\hat{Q}^{(1)}\hat{Q}^{(0)} - \frac{2}{3}(\hat{Q}^{(0)})^2\hat{Q}^{(1)} \tag{26}$$

show that we have a non-linear transformation of basis, with wide consequences for the algebraic structure. Indeed from the above definitions and the current algebra given in (15) we obtain, after a rather tedious calculation, the following Ansatz:

$$\{Q_{ij}^{(m)}, Q_{kl}^{(n)}\} = (\delta \circ Q^{(n+m)})_{ij,kl} - \sum_{p=0}^{m-1}\sum_{q=0}^{n-1}(Q^{(p)}Q^{(q)} \circ Q^{(m+n-p-q-2)})_{ij,kl}\,, \tag{27}$$

which is the first main result of this paper. It consists of a cubic deformation of a Kac-Moody algebra. Such simplicity opposes the standard algebra available in the literature. In fact, taking the results from [13] for instance, one learns that the non-linear part of the algebra of standard charges is not so simple as a cubic term: quadratic, as well as cubic and even higher powers of charges come out, so that in general one has a polynomial whose order increases with the order of the charges involved. One can easily verify this behavior by substituting the transformations (26) into the Ansatz (27).

After defining the algebra (27) we can make the definition (25) of the improved charges more precise,

$$(n-2)Q_{ij}^{(n+1)} \equiv \{Q_{ai}^{(1)}, Q_{aj}^{(n)}\} + \sum_{p=0}^{n-1}\operatorname{tr}(Q^{(0)}Q^{(p)})Q_{ij}^{(n-1-p)}$$
$$- \sum_{p=0}^{n-1}([Q^{(0)}Q^{(p)}, Q^{(n-1-p)}]_+)_{ij}\,. \tag{28}$$

The algebra (27) has been introduced as an Ansatz. This guess-work was based on preliminary and rather tedious calculations (in fact we have checked the validity of (27) up to the 5th order). In proposing the cubic form of the Ansatz to all orders we should also verify that the Jacobi identity is indeed satisfied. This aim is achieved observing that the saturated piece of the charges gives rise, on its own, to an isomorphic algebra.

Consider the improved basis: from the examples listed in Appendix B we see that each one of them has a higher-order piece, containing the maximum number of current components (the component j_0) in the integrand. Inspired also by the saturated character of the algebra (27) we propose the definition of the *saturated* charges

$$\bar{Q}^{(n)} = \bar{Q}_a^{(n)} t^a , \tag{29}$$

$$\bar{Q}_a^{(n)} = -\frac{1}{2} \operatorname{tr}(t^a t^{a_0} \cdots t^{a_n}) \int \prod_{i=0}^{n} dx_i \mathscr{J}^{a_0 \quad a_n}(x_0, \ldots, x_n) , \tag{30}$$

where the non-local densities

$$\mathscr{J}^{a_0 \quad a_n}(x_0, \ldots, x_n) \equiv j_0^{a_0}(x_0)\varepsilon(x_0 - x_1) j_0^{a_1}(x_1) \cdots \varepsilon(x_{n-1} - x_n) j_0^{a_n}(x_n) \tag{31}$$

can be seen as linear *chains* of current components $j_0^a(x_i)$ in a given basis $\{t^a\}$ for the $O(N)$ algebra, connected by non-local ε functions. We emphasize that the saturated charges $\bar{Q}^{(n)}$ are not conserved quantities. Nevertheless we can prove that they realize the algebra (27) and use this fact to verify that the Ansatz satisfies the Jacobi identity.

We first define the space of all possible chains and derive their algebra from Eq. (8). Then, from the definition (30) we project the results into an algebra of saturated charges. After some arid computations and with the help of the identity (A.16) we obtain the following result:

$$\{\bar{Q}_a^{(m)}, \bar{Q}_b^{(n)}\} = \operatorname{tr}(t^a t^b \bar{Q}^{(m+n)}) - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \operatorname{tr}(t^a \bar{Q}^{(i)} \bar{Q}^{(j)} t^b \bar{Q}^{(m+n-i-j-2)}) . \tag{32}$$

By means of formula (A.8) one recognizes that it is isomorphic to the Ansatz (27). Although this algebra has been derived for the $O(N)$ model, one can rewrite the traces appearing on the r.h.s. of (32) in terms of the structure constants of the group and therefore generalize that algebra for other groups.

Concerning the Jacobi identity, we begin by stressing that the above realization of the Ansatz was built up from the elementary current component j_0: as the Dirac brackets $\{j_0^a(x), j_0^b(y)\}$, given in Eq. (8), obey the Jacobi identity by hypothesis, and since the chains are defined in Eq. (31) as products of j_0-components, it follows that the algebra of chains also satisfies the Jacobi identity. On the other hand, the saturated charges are constructed by simple integrations and linear combinations of chains, therefore implying that the algebra (32) obeys the Jacobi identity as well.

If we had considered the algebra of all chains, including those having the component j_1, the corresponding integrations and trace-projections would lead us to the algebra of improved charges. The Jacobi identity of the algebra obtained in this way would follow from the algebraic properties of (8) as well. The role of the intertwiner field is marginal due to its character as a projector.

From the relation between chains and saturated charges, we also understand how the linear and cubic parts of the algebra (27) are constrained: both are constructed from the same chains, with the same number of current components.

4. Yangian Algebra

Here we comment on the connection between the algebra (27) and Yangian structures. As was discussed by Drinfeld in Ref. [19], Hopf algebras play a very important role in the framework of the quantum inverse problem [20]. There are two classes of Hopf algebras that can be specially recognized as central issues in conformal field theories and integrable models, namely affine Lie algebras [20, 21] and Yangians [21, 22]. In the last case, it has been shown that the problem of finding rational r-matrices boils down to the determination of irreducible representations of Yangians.

The Yangian $Y(A)$ is defined in terms of a Lie algebra A by a linear mapping

$$J : A \to Y(A) .$$

Taking $J(I_a) = J_a$, $I_a \in A$, we have the following commutation relations:

$$[I_a, I_b] = f_{abc} I_c , \tag{33a}$$

$$[I_a, J_b] = f_{abc} J_c , \tag{33b}$$

$$[J_a, [J_b, I_c]] - [I_a, [J_b, J_c]] = a_{abcdef}\{I_d, I_e, I_f\} , \tag{33c}$$

$$[[J_a, J_b], [I_r, J_s]] + [[J_r, J_s], [I_a, J_b] = (a_{abcdef} f_{rsc} + a_{rscdef} f_{abc})\{I_d, I_e, I_f\} , \tag{33d}$$

where f_{abc} are the structure constants of the algebra A, $a_{abcdef} = \frac{1}{24} f_{adi} f_{bej} f_{cfk} f_{ijk}$ and $\{X_1, X_2, X_3\} = \sum X_i X_j X_k$ corresponds to a total symmetrization.

It follows from Eq. (27) that the charges $Q^{(0)}$ and $Q^{(1)}$ generate, via Dirac brackets, the Yangian of the $O(N)$ algebra. This classical manifestation of the Yangian structure was studied in Ref. [23] and the algebra of non-local charges identified as a Poisson-Hopf algebra.

We therefore conclude that the general properties of Hopf algebras underlie the Ansatz (27). Indeed non-local conserved currents in two dimensions and their algebraic structure have been studied in [24] and the fact that Yangians are realised in quantum field theories by the non-local currents was proved in [25] and further analyzed in [26]. Consequences for lattice models have been discussed recently [27].

However we notice that the algebraic relations (33) do not define uniquely the remaining charges $\{Q^{(n)}; n \geqq 2\}$. In Sect. 3 we have compared the standard and improved sets and verified that they are related by (non-linear) combinations. They do obey different algebras, although sharing the generators $Q^{(0)}$, $Q^{(1)}$.

The algebra of standard charges is usually summarized by the algebra of their generating functional, the monodromy matrix $T(\lambda)$. The problem can be formulated in terms of a Lax formalism: one defines the Lax pair

$$L_\mu(x; \lambda) = \frac{2\lambda}{\lambda^2 - 1}(\lambda \eta_{\mu\nu} - \varepsilon_{\mu\nu}) j^\nu , \tag{34}$$

and the transfer matrix $T(x, y; \lambda)$ via the equations

$$(\partial_\mu + L_\mu) T(x, y; \lambda) = 0, \quad \mu = 0, 1 , \tag{35}$$

for which the integrability condition $[\partial_0 + L_0, \partial_1 + L_1] = 0$ is equivalent to Eqs. (35). The solution [28] of Eq. (35) is then given by a path-ordered exponential,

$$T(x, y; \lambda) = P \exp \int_y^x L(\xi; \lambda) \cdot d\xi \tag{36}$$

and the monodromy matrix $T(\lambda) = T(\infty, -\infty; \lambda)$ generates the *standard* non-local charges $\hat{Q}^{(n)}$ through

$$T(\lambda) = \exp \sum_{n \geqq 0} \lambda^{n+1} \hat{Q}^{(n)} . \tag{37}$$

Its Poisson bracket relations are given in terms of a classical r-matrix:

$$\{T(\lambda), T(\mu)\} = [r(\lambda, \mu), T(\lambda) \otimes T(\mu)] ,$$

$$r(\lambda, \mu) = \frac{I_a \otimes I_a}{\lambda^{-1} - \mu^{-1}} . \tag{38}$$

In principle one can extract the algebra $\{\hat{Q}^{(m)}, \hat{Q}^{(n)}\}$ from Eq. (38); for instance, the brackets involving $Q^{(0)}$ and $Q^{(1)}$ follow easily.

Finding the complete algebra is cumbersome, though. However, as concerns the on-shell dynamics, this is not a real problem because conservation of the first pair of charges is sufficient. Indeed, one can prove that there is no particle production [10] and only the $2 \to 2$ scattering has to be computed, since in that case the S-matrix factorizes [29]. Moreover, Poincaré transformation properties of the fields and non-local charges can be used to obtain the action of the latter on asymptotic states. Indeed, for the commutation relation of the charge $Q^{(1)}$ with the generator τ of Lorentz transformations, one generally finds

$$[\tau, Q_{ij}^{(1)}] = \gamma Q_{ij}^{(0)} , \tag{39}$$

where γ is a normalization constant, which depends on the group $\left(\text{for the } O(N) \text{ case } \gamma = -\frac{N-2}{2\pi N}\right)$. One finds [5] for the action of $Q_{ij}^{(1)}$ on asymptotic one-particle states

$$Q_{ij}^{(1)} |0k\rangle = i\gamma 0 \{\delta_{jk} |0i\rangle - \delta_{ik} |0j\rangle\} , \tag{40a}$$

while for a two-particle state, it is computed from [30]

$$\begin{aligned}\langle \phi_1 \phi_2 | Q^{(1)} | \phi_1 \phi_2 \rangle = \lim_{t \to \pm \infty} & \int dx dy \varepsilon(x-y) \langle \phi_1 | j_0(t,x) | \phi_1 \rangle \langle \phi_2 | j_0(t,y) | \phi_2 \rangle \\ & + \langle \phi_1 | Q^{(1)} | \phi_1 \rangle + \langle \phi_2 | Q^{(1)} | \phi_2 \rangle ,\end{aligned} \tag{40b}$$

almost completely fixing the on shell dynamics.

The above procedure gives results equivalent to the solution of Yang-Baxter equations. Following [31] one can consider the particle multiplets as fundamental representations of the Yangian $Y(A)$. In particular, the well-known fusion procedure is equivalent to the decomposition of a product state into $Y(A)$ irreducible components [32]. For theories with purely elastic scattering a closed bootstrap program may be fulfilled [32, 33].

Concerning symmetry transformations, it is also known [35] that Yangians correspond to (quantum group) symmetries of many integrable models. These symmetry transformations are generated by the non-local charges through a Lie-Poisson action [35] (as opposed to the more familiar Hamiltonian action). Here we exemplify the transformations generated by the first few improved charges of the $O(N)$ sigma model,

$$
\begin{aligned}
\delta_{ij}^{(0)}\Phi &= \{Q_{ij}^{(0)}, \Phi\}\,, \\
\delta_{ij}^{(1)}\Phi &= \{Q_{ij}^{(1)}, \Phi\} + (Q_{ia}^{(0)}\delta_{aj}^{(0)} - Q_{ja}^{(0)}\delta_{ai}^{(0)})\Phi\,, \\
\delta_{ij}^{(2)}\Phi &= \{Q_{ij}^{(2)}, \Phi\} + \{Q_{ia}^{(0)}\delta_{aj}^{(0)} - Q_{ja}^{(0)}\delta_{ai}^{(1)})\Phi \\
&\quad + (Q_{ia}^{(1)}\delta_{aj}^{(0)} - Q_{ja}^{(1)}\delta_{ai}^{(0)})\Phi + Q_{ia}^{(0)}(\delta_{ab}^{(0)}\Phi)Q_{bj}^{(0)}\,,
\end{aligned}
\tag{41}
$$

for which we have verified the following commutation relations

$$
\begin{aligned}
[\delta_{ij}^{(0)}, \delta_{kl}^{(0)}] &= (\delta\circ\delta^{(0)})_{ij,kl}\,, \\
[\delta_{ij}^{(0)}, \delta_{kl}^{(1)}] &= (\delta\circ\delta^{(1)})_{ij,kl}\,, \\
[\delta_{ij}^{(1)}, \delta_{kl}^{(1)}] &= (\delta\circ\delta^{(2)})_{ij,kl}\,,
\end{aligned}
\tag{42}
$$

agreeing with the result $[\delta_{ij}^{(m)}, \delta_{kl}^{(n)}] = (\delta\circ\delta^{(m+n)})_{ij,kl}$ found in [11]. Notice that one cannot find a "Hamiltonian" generator $G^{(1)}$ such that $\delta^{(1)}\Phi = \{G^{(1)}, \Phi\}$, which is at present understood as the root of preliminary misunderstandings about the algebra of charges.

It is also worth mentioning that a number of technical difficulties may arise when the theory does not possess the ultra-locality property, that is when the algebra of the space component of the Lax pair – in our case the time component of the current $j_0(x)$ – contains terms other than $\delta(x)$ distributions. The appearance of $\delta'(x)$ terms (as in the WZNW model of Sect. 5) means that one should also modify the $\delta(z)$ part with s-terms in order that the Jacobi identity be satisfied. This problem has been discussed at length by Maillet [34] (see also [28]).

On the other hand, the purely algebraic construction that we have used circumvents the non-ultralocality problems and provides a concrete classical realization of the Yangian algebra, obeyed by the generators $Q^{(0)}$, $Q^{(1)}$ and the remaining charges.

From the studies of the quantum case, it is known that the standard non-local charges satisfy commutation relations of Yangians and one would expect the same algebra at the classical level. In this sense one could interpret the algebra (27) as another presentation of the Yangian, in terms of a particular set of conserved charges whose immediate virtue is to provide a concise, transparent and explicit form for the complete algebra.

However we do expect that the improved charges will surpass this initial advantage and become an useful tool in off-shell scattering calculations. Indeed it is not clear that the simple outcome of the on-shell picture will persist and one would need charges of higher genera in order to obtain constraints strong enough to determine the correlation functions. This more difficult problem is currently under investigation by the authors.

5. Algebra of Non-Local Charges in WZNW Model

We first re-analyze the current algebra for the principal chiral model with a Wess–Zumino term. This model [36] contains a free coupling constant λ and, for special values of λ, is equivalent to the conformally invariant WZNW model, while the ordinary chiral model is taken from the limit $\lambda\to 0$. Therefore, the current algebra derived below is a generalization of the current algebras for these two special cases. For the WZNW model, at the critical point the current algebra is known to consist of two commuting Kac–Moody algebras, while for the ordinary chiral model, it has been presented previously.

We begin by fixing our conventions. The target space for the chiral models to be considered here will be a simple Lie group G (which is usually, although not necessarily, assumed to be compact) with Lie algebra $\mathbf{g}$, and we use the trace in some irreducible representation to define the invariant scalar product $(\cdot,\cdot)$ on $\mathbf{g}$, normalized so that the long roots have length $\sqrt{2}$, as well as the invariant closed three-form ω on $\mathbf{g}$ giving rise to the Wess-Zumino term. Explicitly, for $X, Y, Z \in \mathbf{g}$, we have,

$$(X, Y) = -\operatorname{tr}(XY)\,, \tag{43}$$

while

$$\omega(X, Y, Z) = \frac{1}{4\pi}\operatorname{tr}(X[Y, Z])\,. \tag{44}$$

Obviously, $(\cdot,\cdot)$ and ω extend to a bi-invariant metric $(\cdot,\cdot)$ on G and to a bi-invariant three-form ω on G, respectively: the latter can alternatively be represented in terms of the left-invariant Maurer–Cartan form $g^{-1}dg$ or right-invariant Maurer–Cartan form dgg^{-1} on G, as follows:

$$\omega = \frac{1}{12\pi}\operatorname{tr}(g^{-1}dg)^3 = \frac{1}{12\pi}\operatorname{tr}(dgg^{-1})^3\,. \tag{45}$$

[Due to the Maurer–Cartan structure equation, this representation implies that ω is indeed a closed three-form on G, and the normalization in Eqs. (44) and (45) is chosen so that $\omega/2\pi$ generates the third de Rham cohomology group $H^3(G, \mathbb{Z})$ of G over the integers, at least when G is simply connected; cf. Ref. [37].]

In part of what follows, we work in terms of (arbitrary) local coordinates u^i on G, representing the metric $(\cdot,\cdot)$ by its components g_{ij} and the three-form ω by its components ω_{ijk}. Then the total action of the so-called Wess–Zumino–Novikov–Witten (WZNW) theory is the sum

$$S = S_{CH} + nS_{WZ}\,, \tag{46}$$

where the action for the ordinary chiral model, S_{CH} is given by (1), and the Wess–Zumino term is

$$S_{WZ} = \frac{1}{6}\int_B d^3x\,\varepsilon^{\kappa\lambda\mu}\omega_{ijk}(\tilde{\varphi})\,\partial_\kappa\tilde{\varphi}^i\,\partial_\lambda\tilde{\varphi}^j\,\partial_\mu\tilde{\varphi}^k = \int_B \tilde{\varphi}^\star\omega\,. \tag{47}$$

Here, φ and $\tilde{\varphi}$ are the basic field and the extended field of the theory, respectively, i.e. φ is a (smooth) map from a fixed two-dimensional Lorentz manifold Σ to G and $\tilde{\varphi}$ is a (smooth) map from an appropriate three-dimensional manifold B to G, chosen such that Σ is the boundary of B and φ is the restriction of $\tilde{\varphi}$ to that boundary. The conformally invariant WZNW model is obtained at $\lambda = \sqrt{4\pi/|n|}$, while the ordinary chiral model can be recovered in the limit $\lambda \to 0$.

Before proceeding further, we find it convenient to pass to a more standard notation, writing g and $\tilde{g}$, rather than φ and $\tilde{\varphi}$, for the basic field and the extended field of the theory, respectively, and using the explicit definitions (43) of the metric $(\cdot,\cdot)$ on G and (44) of the three-form ω on G. Then

$$S_{CH} = -\frac{1}{2\lambda^2}\int d^2x\,\eta^{\mu\nu}\,\mathrm{tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g)\,, \tag{48}$$

while

$$S_{WZ} = \frac{1}{4\pi}\int_0^1 dr \int d^2x\,\varepsilon^{\mu\nu}\,\mathrm{tr}(\tilde{g}^{-1}\partial_r\tilde{g}\tilde{g}^{-1}\partial_\mu\tilde{g}\tilde{g}^{-1}\partial_\nu\tilde{g})\,. \tag{49}$$

(Here, the extended field $\tilde{g}$ is assumed to be constant outside a tubular neighbourhood $\Sigma\times[0,1]$ of the boundary Σ of B, and r is the coordinate normal to the boundary.) Next, we decompose the currents j_μ and J_μ, both of which take values in $\mathbf{g}_L\otimes\mathbf{g}_R$, into left and right currents, all of which take values in $\mathbf{g}$: $j_\mu=(j^L_\mu, j^R_\mu)$, $J_\mu=(J^L_\mu, J^R_\mu)$. Explicitly,

$$j^L_\mu = -\frac{1}{\lambda^2}\partial_\mu g g^{-1},\quad j^R_\mu = +\frac{1}{\lambda^2}g^{-1}\partial_\mu g\,, \tag{50}$$

and, by definition,

$$\begin{aligned} J^L_\mu &= (\eta_{\mu\nu}+\alpha\varepsilon_{\mu\nu})j^{L\nu} = -\frac{1}{\lambda^2}(\eta_{\mu\nu}+\alpha\varepsilon_{\mu\nu})\partial^\nu g g^{-1}\,,\\ J^R_\mu &= (\eta_{\mu\nu}-\alpha\varepsilon_{\mu\nu})j^{R\nu} = +\frac{1}{\lambda^2}(\eta_{\mu\nu}-\alpha\varepsilon_{\mu\nu})g^{-1}\partial^\nu g\,, \end{aligned} \tag{51}$$

where $\alpha=\dfrac{n\lambda^2}{4\pi}$. The scalar field j, when viewed as taking values in the space of endomorphisms of $\mathbf{g}_L\oplus\mathbf{g}_R$, is given by the (2×2)-block matrix

$$j=\frac{1}{\lambda^2}\begin{pmatrix} 1 & -\mathrm{Ad}(g)\\ -\mathrm{Ad}(g)^{-1} & 1\end{pmatrix}. \tag{52}$$

In other words, for $X=(X_L,X_R)$ in $\mathbf{g}_L\oplus\mathbf{g}_R$,

$$j(X)=\frac{1}{\lambda^2}(X_L-\mathrm{Ad}(g)X_R,\,X_R-\mathrm{Ad}(g)^{-1}X_L)\,. \tag{53}$$

It can be shown that the covariant currents J_μ defined by Eqs. (51) differ from the Noether currents $\hat{j}_\mu$ for the chiral model with a Wess–Zumino term by a total curl, and that current conservation (which for both types of currents has the same physical content, because a total curl is automatically conserved) is equivalent to the equations of motion.

Now in terms of an arbitrary basis $\{t_a\}$ of $\mathbf{g}$, with structure constants f^{abc} defined by $[t^a,t^b]=f^{abc}t^c$, the various currents are represented by their components

$$\begin{aligned} j^{La}_\mu &= (j_\mu, t^{La}) = -\mathrm{tr}(j^L_\mu t^a), \qquad & j^{Ra}_\mu &= (j_\mu, t^{Ra}) = -\mathrm{tr}(j^R_\mu t^a)\,,\\ J^{La}_\mu &= (J_\mu, t^{La}) = -\mathrm{tr}(J^L_\mu t^a), \qquad & J^{Ra}_\mu &= (J_\mu, t^{Ra}) = -\mathrm{tr}(J^R_\mu t^a)\,, \end{aligned} \tag{54}$$

and the scalar field j by its components

$$\eta_{ab} = (j, t^{La} \otimes t^{Lb}) = (j, t^{Ra} \otimes t^{Rb}) = -\frac{1}{\lambda^2} \operatorname{tr}(t^a t^b), \tag{55}$$

$$j^{ab} = (j, t^{La} \otimes t^{Rb}) = \frac{1}{\lambda^2} \operatorname{tr}(g^{-1} t^a g t^b), \tag{56}$$

where

$$t^{La} = (t^a, 0), \quad t^{Ra} = (0, t^a). \tag{57}$$

With this notation, we see that the current Dirac brackets imply the following brackets relations for the components of the currents j^a_μ:

$$\begin{aligned}
\{j_0^{La}(x), j_0^{Lb}(y)\} &= -f^{abc} j_0^{Lc}(x)\delta(x-y) + \alpha f^{abc} j_1^{Lc}(x)\delta(x-y), \\
\{j_0^{La}(x), j_1^{Lb}(y)\} &= -f^{abc} j_1^{Lc}(x)\delta(x-y) + \eta_{ab}\delta'(x-y), \\
\{j_1^{La}(x), j_1^{Lb}(y)\} &= 0, \\
\{j_0^{Ra}(x), j_0^{Rb}(y)\} &= -f^{abc} j_0^{Rc}(x)\delta(x-y) - \alpha f^{abc} j_1^{Rc}(x)\delta(x-y), \\
\{j_0^{Ra}(x), j_1^{Rb}(y)\} &= -f^{abc} j_1^{Rc}(x)\delta(x-y) + \eta_{ab}\delta'(x-y), \\
\{j_1^{Ra}(x), j_1^{Rb}(y)\} &= 0, \\
\{j_0^{La}(x), j_0^{Rb}(y)\} &= \alpha j'^{ab}(x)\delta(x-y), \\
\{j_0^{La}(x), j_1^{Rb}(y)\} &= j^{ba}(y)\delta'(x-y), \\
\{j_0^{Ra}(x), j_1^{Lb}(y)\} &= j^{ab}(y)\delta'(x-y), \\
\{j^L_{1,a}(x), j^R_{1,b}(y)\} &= 0.
\end{aligned} \tag{58}$$

They must be supplemented by the commutation relations between the components of the currents j_μ and those of the field j:

$$\begin{aligned}
\{j_0^{La}(x), j^{bc}(y)\} &= -f^{abd} j^{dc}(x)\delta(x-y), \\
\{j_0^{Ra}(x), j^{bc}(y)\} &= -f^{acd} j^{bd}(x)\delta(x-y), \\
\{j_1^{La}(x), j^{bc}(y)\} &= 0, \\
\{j_1^{Ra}(x), j^{bc}(y)\} &= 0.
\end{aligned} \tag{59}$$

Finally, the components of the field j commute among themselves: $\{j^{ab}(x), j^{cd}(y)\} = 0$.

Using the explicit representation of the theory in terms of group-valued fields, it is very simple to check the results using the decomposition of the momentum in terms of a local and a non-local piece as was done in Ref. [5].

We are now in a position to generalize the previous results for the WZNW model. Classically, the equations of motion are given by the conservation laws

$$\begin{aligned}
\partial_\mu(j^{R\mu} - \alpha\varepsilon^{\mu\nu} j^R_\nu) &= 0, \\
\partial_\mu(j^{L\mu} + \alpha\varepsilon^{\mu\nu} j^L_\nu) &= 0.
\end{aligned} \tag{60}$$

The currents $j^{R,L}_\mu$ satisfy the zero-curvature condition

$$\partial_\mu j^{R,L}_\nu - \partial_\nu j^{R,L}_\mu + \lambda^2 [j^{R,L}_\mu, j^{R,L}_\nu] = 0. \tag{61}$$

Concerning the covariant currents $J_\mu^{R,L}$, the above equations imply

$$\partial^\mu J_\mu^{R,L} = 0 \,,$$
$$\partial_\mu J_\nu^{R,L} - \partial_\nu J_\mu^{R,L} + \lambda^2 [J_\mu^{R,L}, J_\nu^{R,L}] = 0 \,, \tag{62}$$

so that one coud follow the algorithm described by Eqs. (18)–(21) to build up new conserved non-local charges. In particular, the first one reads

$$Q^{L(1)} = \int dx (J_1^L + \lambda^2 J_0^L \partial^{-1} J_0^L - \alpha J_0^L) \,. \tag{63}$$

On the other hand, the algebraic construction of improved charges described in Sect. 3 can also be performed for the WZ case with few modifications. The chain algebra construction is extended as well as definition of saturated charges. In this case we need to use the following Dirac bracket for the current J_0^L (written in matrix components)

$$\{(J_0^L)_{ij}(x), (J_0^L)_{kl}(y)\} = (\delta \circ J_0^L)_{ij,kl}(x)\delta(x-y) + \alpha(\delta\circ\delta)_{ij,kl}\delta'(x-y) \,, \tag{64}$$

and we are led to the following Dirac brackets (we suppose $m \geqq n$ with no loss of generality):

$$\begin{aligned}\{Q_{ij}^{(m)}, Q_{kl}^{(n)}\} &= (\delta\circ Q^{(n+m)})_{ij,kl} - \sum_{p=0}^{m-1}\sum_{q=0}^{n-1} (Q^{(p)}Q^{(q)}\circ Q^{(m+n-p-q-2)})_{ij,kl} \\ &+ 4\alpha\left((\delta\circ Q^{(n+m-1)})_{ij,kl} - \sum_{p=0}^{m-2}\sum_{q=0}^{n-1}(Q^{(p)}Q^{(q)}\circ Q^{(m+n-p-q-3)})_{ij,kl}\right)\end{aligned} \tag{65}$$

or equivalently, denoting by $\{,\}_{WZ}$ the bracket for the Wess–Zumino model and $\{,\}_{CH}$ for previous brackets of the chiral model, we summarize the results by $(n \geqq m)$

$$\{Q^{(m)}, Q^{(n)}\}_{WZ} = \{Q^{(m)}, Q^{(n)}\}_{CH} + 4\alpha\{Q^{(m-1)}, Q^{(n)}\}_{CH} \,. \tag{66}$$

Some remarks are in order now. First, concerning the chain algebra, it clearly goes through the Wess–Zumino case. Therefore, the Jacobi identities are valid here as well. The algebra for the right sector follows directly from (66) through $\alpha \to -\alpha$. Also the mixed brackets $\{Q^{L(m)}, Q^{R(n)}\}$ vanish since $\{(J_0^L)_{ij}(x), (J_0^R)_{kl}(y)\} = 0$. We observe that, due to the non-ultra-local contribution in the bracket (64) the Yangian generated by $Q^{R,L(0)}$, $Q^{R,L(1)}$ acquires an extra term as compared with Eq. (33b). Such extension is parametrized by the coupling α and the complete dependence on α can be summarized by the result (66).

6. Conclusions

We have computed the classical algebra of conserved non-local charges of the so-called improved basis. The result is characterized by the order n of the non-local charges $Q^{(n)}$, which in fact can be defined in terms of its genus [30], as computed from scattering theory. Therefore, classifying the genus, one verifies that in the right-hand side of the Dirac algebra, only the possible highest genus contributes with a non-vanishing coefficient. The algebra obtained is a saturated cubic deformation of a Kac–Moody algebra and consistent with the Yangian algebraic structure found in the literature by alternative methods.

This result permits us to try to obtain constraints on the correlation functions of the theory, similarly to the massive perturbation of the $k = 1$ WZW model [9]. This problem evaded solution for several years, but with this approach, one should be able to accomplish such desired constraints, once one knows a realization of charges in terms of integro-differential operators. Indeed, for the asymptotic charges one finds such representations [5, 10].

Further problems related to the role of monodromy matrices are at present under investigation; in particular, it would be interesting to find and interpret a generating functional for the improved non-local charges. Concerning the quantum theory, we recall that, for sigma models with a simple gauge group the quantum non-local charge algebra must be the same as we have computed substituting Dirac brackets by $(-i)$ times commutators [15].

Finally, we remark that the WZNW theory presents an algebra that is analogous to the chiral case. In fact, the WZNW theory has been treated using the Bethe Ansatz [38], with results analogous in some sense to the sigma model case, and one expects many similarities between them. The usefulness of such interpolating cases has been stressed in [39].

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Appendix A

We list here some useful formulae concerning the special $O(N)$ product $A \circ B$ and the constraints involving the currents j and j_μ.

The product $A \circ B$ is defined as follows:

$$(A \circ B)_{ij,kl} = A_{ik}B_{jl} - A_{il}B_{jk} + A_{jl}B_{ik} - A_{jk}B_{il} \,, \tag{A.1}$$

and possesses the properties

$$(A \circ B)_{ij,kl} = (B \circ A)_{ij,kl} = (A^t \circ B^t)_{kl,ij} \,, \tag{A.2}$$

$$(A \circ B)_{ij,ka}C_{al} - (k \leftrightarrow l) = (A \circ BC)_{ij,kl} + (AC \circ B)_{ij,kl} \,, \tag{A.3}$$

$$(A \circ B)_{ia,kl}C_{aj} - (i \leftrightarrow j) = (A \circ C^t B)_{ij,kl} + (C^t A \circ B)_{ij,kl} \,, \tag{A.4}$$

$$C_{ia}(A \circ B)_{aj,kl} - (i \leftrightarrow j) = (CA \circ B)_{ij,kl} + (A \circ CB)_{ij,kl} \,, \tag{A.5}$$

$$A_{ia}(B \circ C)_{ab,kl}D_{bj} - (i \leftrightarrow j) = (AB \circ D^t C)_{ij,kl} + (D^t B \circ AC)_{ij,kl} \,, \tag{A.6}$$

$$A_{ka}(B \circ C)_{ij,ab}D_{bl} - (k \leftrightarrow l) = (BA^t \circ CD)_{ij,kl} + (BD \circ CA^t)_{ij,kl} \,, \tag{A.7}$$

$$\frac{1}{4} t^a_{ji} t^b_{lk} (A \circ B)_{ij,kl} = \operatorname{tr}(t^a A t^b B) \,. \tag{A.8}$$

Now we list the constraints among the currents:

$$(j_\mu \circ j_\nu)_{ij,kl} = (j_\mu)_{ij}(j_\nu)_{kl} + (j_\nu)_{ij}(j_\mu)_{kl} \,, \tag{A.9}$$

$$(j_\mu \circ j)_{ij,kl} = 0 \,, \tag{A.10}$$

$$(j \circ j)_{ij,kl} = 0 \,, \tag{A.11}$$

$$[j_\mu, j]_+ = j_\mu \,, \tag{A.12}$$

$$[j, j]_+ = 2j \,, \tag{A.13}$$

$$[j_\mu, j] = -\partial_\mu j \,, \tag{A.14}$$

$$(j_1 j) = \frac{1}{2} j_1 - \frac{1}{2} \partial j \,. \tag{A.15}$$

The $O(N)$ t-matrices, contracted by a factor $f^{a_i b_j c}$, merge as follows

$$-\frac{1}{4} \operatorname{tr}(t^{a_{i+1}} \cdots t^{a_m} t^a t^{a_0} \cdots t^{a_i}) f^{a_i b_j c} \operatorname{tr}(t^{b_j} \cdots t^{b_n} t^b t^{b_0} \cdots t^{b_{j-1}})$$

$$= \frac{1}{4} \operatorname{tr}(t^c \times (t^{a_{i+1}} \cdots t^{a_m} t^a t^{a_0} \cdots t^{a_{i-1}} + (-)^m t^{a_{i-1}} \cdots t^{a_0} t^a t^{a_m} \cdots t^{a_{i+1}})$$

$$\times (t^{b_{j+1}} \cdots t^{b_n} t^b t^{b_0} \cdots t^{b_{j-1}} + (-)^n t^{b_{j-1}} \cdots t^{b_0} t^b t^{b_n} \cdots t^{b_{j+1}})) \,, \tag{A.16}$$

Appendix B

We choose the antiderivative operator as

$$\partial^{-1} A(x) = \frac{1}{2} \int dy\, \varepsilon(x-y) A(y), \quad \varepsilon(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \,. \\ +1, & x > 0 \end{cases} \tag{B.1}$$

With this definition we have antisymmetric boundary conditions for the potentials $\chi^{(n)}$,

$$\chi^{(n)}(\pm\infty) = \pm \frac{1}{2} \int dx\, J_0^{(n)}(x) = \pm \frac{1}{2} \hat{Q}^{(n)} \,. \tag{B.2}$$

The first six improved non-local charges read

$$Q^{(0)} = \int dx\, j_0 \,,$$

$$Q^{(1)} = \int dx\, (j_1 + 2 j_0 \partial^{-1} j_0) \,,$$

$$Q^{(2)} = \int dx (2 j_0 + 2 j_1 \partial^{-1} j_0 - 2 \partial^{-1} j_0 j_1 - 4 \partial^{-1} j_0 j_0 \partial^{-1} j_0) \,,$$

$$\begin{aligned} Q^{(3)} = \int dx [& 3 j_1 + 8 j_0 \partial^{-1} j_0 + 2 j_1 \partial^{-1} j_1 \\ & - 4(\partial^{-1} j_0 j_1 \partial^{-1} j_0 + \partial^{-1} j_0 j_0 \partial^{-1} j_1 + \partial^{-1} j_1 j_0 \partial^{-1} j_0) \\ & + 8 \partial^{-1} (\partial^{-1} j_0 j_0) j_0 \partial^{-1} j_0] \,, \end{aligned}$$

$$\begin{aligned} Q^{(4)} = \int dx \{& 6 j_0 + 10 j_1 \partial^{-1} j_0 - 10 \partial^{-1} j_0 j_1 - 24 \partial^{-1} j_0 j_0 \partial^{-1} j_0 \\ & - 4(\partial^{-1} j_1 j_0 \partial^{-1} j_1 + \partial^{-1} j_0 j_1 \partial^{-1} j_1 + \partial^{-1} j_1 j_1 \partial^{-1} j_0) \\ & + 8 [\partial^{-1} (\partial^{-1} j_0 j_0)(j_0 \partial^{-1} j_1 + j_1 \partial^{-1} j_0) \\ & - (\partial^{-1} j_0 j_1 + \partial^{-1} j_1 j_0) \partial^{-1} (j_0 \partial^{-1} j_0)] \\ & + 16 \partial^{-1} (\partial^{-1} j_0 j_0) j_0 \partial^{-1} (j_0 \partial^{-1} j_0)\} \,, \end{aligned}$$

$$
\begin{aligned}
Q^{(5)} = \int dx \{ & 10j_1 + 32j_0\partial^{-1}j_0 + 12j_1\partial^{-1}j_1 \\
& - 28(\partial^{-1}j_0j_1\partial^{-1}j_0 + \partial^{-1}j_0j_0\partial^{-1}j_1 + \partial^{-1}j_1j_0\partial^{-1}j_0) \\
& - 4\partial^{-1}j_1j_1\partial^{-1}j_1 + 64\partial^{-1}(\partial^{-1}j_0j_0)j_0\partial^{-1}j_0 \\
& + 8[\partial^{-1}(\partial^{-1}j_1j_0)j_0\partial^{-1}j_1 + \partial^{-1}(\partial^{-1}j_0j_0)j_1\partial^{-1}j_1 \\
& + \partial^{-1}(\partial^{-1}j_0j_1)j_0\partial^{-1}j_1 + \partial^{-1}(\partial^{-1}j_1j_0)j_1\partial^{-1}j_0 \\
& + \partial^{-1}(\partial^{-1}j_0j_1)j_1\partial^{-1}j_0 + \partial^{-1}(\partial^{-1}j_1j_1)j_0\partial^{-1}j_0] \\
& + 16[\partial^{-1}(\partial^{-1}j_0j_0)j_0\partial^{-1}(j_0\partial^{-1}j_1) \\
& + \partial^{-1}(\partial^{-1}j_0j_0)j_0\partial^{-1}(j_1\partial^{-1}j_0) + \partial^{-1}(\partial^{-1}j_1j_0)j_0\partial^{-1}(j_0\partial^{-1}j_0) \\
& + \partial^{-1}(\partial^{-1}j_0j_1)j_0\partial^{-1}(j_0\partial^{-1}j_0) + \partial^{-1}(\partial^{-1}j_0j_0)j_1\partial^{-1}(j_0\partial^{-1}j_0)] \\
& - 32\partial^{-1}(\partial^{-1}(\partial^{-1}j_0j_0)j_0)j_0\partial^{-1}(j_0\partial^{-1}j_0)\} \,. \qquad \text{(B.3)}
\end{aligned}
$$

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