

On the Completeness of the Set of Classical \mathcal{W} -Algebras Obtained from DS Reductions

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Abstract: We clarify the notion of the DS – generalized Drinfeld-Sokolov – reduction approach to classical \mathcal{W} -algebras. We first strengthen an earlier theorem which showed that an $sl(2)$ embedding $\mathcal{S} \subset \mathcal{G}$ can be associated to every DS reduction. We then use the fact that a \mathcal{W} -algebra must have a quasi-primary basis to derive severe restrictions on the possible reductions corresponding to a given $sl(2)$ embedding. In the known DS reductions found to date, for which the \mathcal{W} -algebras are denoted by $\mathcal{W}_{\mathcal{G}}^{\mathcal{S}}$ -algebras and are called canonical, the quasi-primary basis corresponds to the highest weights of the $sl(2)$. Here we find some examples of noncanonical DS reductions leading to \mathcal{W} -algebras which are direct products of $\mathcal{W}_{\mathcal{G}}^{\mathcal{S}}$ -algebras and “free field” algebras with conformal weights $\Delta \in \{0, \frac{1}{2}, 1\}$. We also show that if the conformal weights of the generators of a \mathcal{W} -algebra obtained from DS reduction are nonnegative $\Delta \geq 0$ (which is the case for all DS reductions known to date), then the $\Delta \geq \frac{3}{2}$ subsectors of the weights are necessarily the same as in the corresponding $\mathcal{W}_{\mathcal{G}}^{\mathcal{S}}$ -algebra. These results are consistent with an earlier result by Bowcock and Watts on the spectra of \mathcal{W} -algebras derived by different means. We are led to the conjecture that, up to free fields, the set of \mathcal{W} -algebras with nonnegative spectra $\Delta \geq 0$ that may be obtained from DS reduction is exhausted by the canonical ones.

1. Introduction

The study of nonlinear extensions of the Virasoro algebra by conformal primary fields was initiated by A.B. Zamolodchikov in the pioneering paper [1]. Such algebras, known as \mathcal{W} -algebras, play an important rôle in two dimensional conformal field theories, gravity models and integrable systems. (For detailed reviews, see e.g. [2, 3].) At least three distinct methods are used in the literature for constructing \mathcal{W} -algebras. These can be labelled as direct constructions [1, 4, 5], the methods of extracting \mathcal{W} -algebras from conformal field theories (the most important of which is the coset

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construction) [6–10], and Hamiltonian reductions of affine Kac-Moody (KM) algebras to \mathcal{W} -algebras [11–13]. The Hamiltonian KM reduction method has been intensively pursued recently both in the classical [14–17] and in the quantum framework [18–22], and proved to be the most productive source of \mathcal{W} -algebras so far. As reviewed in [23], the \mathcal{W} -algebras obtained by this method are symmetry algebras of Toda type field theories, first studied in [24]. (See also [25, 26].)

In constructing a reduction of the KM Poisson bracket algebra to a classical \mathcal{W} -algebra we start by imposing certain *first class* constraints on the KM current, and consider the ring \mathcal{R} of differential polynomials in the current which are invariant under the gauge transformations generated by the first class constraints. The crucial questions are:

- (A) *free generation*: whether the ring \mathcal{R} of differential polynomial invariants is freely generated.
- (B) *conformal property*: whether
 - (b1) \mathcal{R} contains a gauge invariant Virasoro density.
 - (b2) \mathcal{R} has a \mathcal{W} -basis.

Here by \mathcal{W} -basis is meant a basis which consists of a Virasoro density and fields which are primary with respect to this Virasoro. See also Sect. 2 for the notion of *classical \mathcal{W} -algebra* used throughout the paper.

These are *quite separate* issues and it is easy to construct examples for which (A) holds but not (B) and (b1) holds but not (A). (They are of course interrelated since (b2) obviously requires (A) and (b1).) Naturally, the answers to *both* (A) and (B) must be positive for a KM reduction to produce a classical \mathcal{W} -algebra.

All Hamiltonian KM reductions that are, to the date of writing, known to produce a (classical) \mathcal{W} -algebra are so called DS – generalized Drinfeld-Sokolov – reductions. In a DS reduction one makes the technical assumption that a certain mechanism is applicable, whose essence is that a freely generating basis (not necessarily the \mathcal{W} -basis) of \mathcal{R} can be constructed by a gauge fixing procedure, and the Virasoro element of the \mathcal{W} -basis is obtained by improving the Sugawara formula by adding the derivative of a current component. This mechanism is termed the *DS mechanism* in this paper and will be described in some detail. However, for a generic KM reduction by first class constraints one cannot expect the invariant ring \mathcal{R} to admit a free generating set; in fact, the special gauge fixing procedure involved in the DS mechanism is the only known method whereby the existence of such a basis set can be guaranteed. The distinguished position of DS reductions among all Hamiltonian KM reductions derives from the applicability of this gauge fixing procedure, which places a stringent restriction on the nature of the constraints.

An important recent development concerning the Hamiltonian KM reduction method has been the realization [15, 16] that a DS reduction can be defined to every embedding of the Lie algebra $sl(2)$ into the simple Lie algebras. The construction generalizes the standard case appearing in the construction of KdV type hierarchies by Drinfeld and Sokolov [11], which corresponds to the principal $sl(2)$ embedding [14]. We call these DS reductions manifestly based on the $sl(2)$ embeddings the *canonical DS reductions* and the resultant \mathcal{W} -algebras the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebras, where \mathcal{G} is the simple Lie algebra and $\mathcal{S} \subset \mathcal{G}$ is the $sl(2)$ subalgebra. One of the salient features of the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebras is that the conformal weights of the elements in the \mathcal{W} -basis are determined by the $sl(2)$ spins in the decomposition of the adjoint representation of \mathcal{G} under \mathcal{S} , and the basis elements are naturally associated to the highest weight vectors in this decomposition. Motivated by their natural, group theoretic definition,

and by the fact that at present the canonical DS reductions are the only KM reductions known to produce \mathcal{W} -algebras, it is expected that the $\mathcal{W}_{\mathcal{G}}^{\mathcal{G}}$ -algebras should have an important rôle to play in the classification of \mathcal{W} -algebras.

Our main purpose in this paper is to show that the possible *noncanonical* DS reductions are *severely* restricted. We do manage to construct some noncanonical DS reductions, but their \mathcal{W} -algebras turn out to be direct products of $\mathcal{W}_{\mathcal{G}}^{\mathcal{G}}$ -algebras and “free field” algebras. Another purpose is to clarify the notion of DS reductions, and thus furnish a framework which could be used in further study of KM reductions.

We shall present here a stronger version of our earlier theorem given in [23] which shows that an $sl(2)$ embedding can be associated to every DS type KM reduction. Most considerations in this paper on general DS reductions will be based on this crucial structural result. The source of the inevitable $sl(2)$ structure given by the theorem is that the existence of a \mathcal{W} -basis in \mathcal{R} (more precisely, we shall only need the existence of a quasi-primary basis for this) requires the element of \mathcal{G} defining the improvement term of the Virasoro density to be the semisimple element (or “defining vector” in the terminology of [27]) of an $sl(2)$ subalgebra. An immediate consequence of the $sl(2)$ structure is that the conformal weights Δ of the elements in the \mathcal{W} -basis must necessarily be integral or half-integral. More importantly, since the classification of $sl(2)$ embeddings is known, the $sl(2)$ theorem reduces the problem of listing all DS reductions to the problem of finding the possible different DS reductions that may belong to a given $sl(2)$ embedding. The new results obtained in this paper indicate that the possible DS reductions corresponding to a given $sl(2)$ embedding are extremely restricted. We shall prove that, due again to the existence of a \mathcal{W} -basis (or quasi-primary basis), the dimension of the gauge subalgebra defining the constraints must be at least half the maximal dimension allowed by first classness, which is attained in the canonical DS reduction, and give restrictions on the position of the gauge subalgebra inside \mathcal{G} with respect to the $sl(2)$ embedding. We then show that if the conformal weights of the \mathcal{W} -basis are nonnegative $\Delta \geq 0$, which is the case for all \mathcal{W} -algebras known to date, then the sectors $\Delta \geq \frac{3}{2}$ must be the same as those in the corresponding $\mathcal{W}_{\mathcal{G}}^{\mathcal{G}}$ -algebra. This result may be thought of as complementary (and consistent) to a result in [31] on the possible conformal spectra of \mathcal{W} -algebras, since our assumptions are different. (A more detailed comparison between the results of [31] and our results can be found in the Discussion.) Another important result of this paper is that we shall prove, by providing examples, the existence of noncanonical DS reductions to \mathcal{W} -algebras where there occur extra weights $\Delta = 0, \frac{1}{2}, 1$ in addition to the canonical spectrum. However, in all those noncanonical examples the \mathcal{W} -algebra turns out to be a direct product of the $\mathcal{W}_{\mathcal{G}}^{\mathcal{G}}$ -algebra with trivial “free fields” carrying the extra weights, and thus it is essentially equivalent to the $\mathcal{W}_{\mathcal{G}}^{\mathcal{G}}$ -algebra.

It is clear that the DS reductions form only a special subset of the possible conformally invariant Hamiltonian KM reductions, and it is natural to inquire about the situation in the general case. This question appears largely unexplored at present, but the series of examples considered in the Appendix of this paper gives support to the expectation that in the general case the ring \mathcal{R} is *not freely generated*. Consider, for example, the “ W_n^l -algebras” proposed by Polyakov and Bershadsky [28, 29] using KM reductions with a mixed (first class and second class) system of constraints for $\mathcal{G} = sl(n)$. An investigation [30] showed that the invariant ring \mathcal{R} can be defined similarly to the case of the DS reductions, but there is no guarantee that \mathcal{R} is freely generated, since DS gauge fixing is not applicable, apart from the cases W_n^2 with n odd which are in fact equivalent to particular $\mathcal{W}_{\mathcal{G}}^{\mathcal{G}}$ -algebras. In other words, in

general the familiar Bershadsky-Polyakov reductions cannot be expected to yield \mathscr{W} -algebras in the usual sense of the word, since they fail on requirement (A). Focusing on the particular cases of the W_{2n}^2 -algebras, we shall prove that \mathscr{R} is indeed not freely generated. On this basis we believe that the structure of the invariant ring \mathscr{R} is in general much more complicated than in the case of DS type reductions, which provides the justification for adopting the applicability of DS gauge fixing as one of the main assumptions underlying our present study.

This paper is organized as follows. To clarify the notion of the DS approach to classical \mathscr{W} -algebras in the more general framework of Hamiltonian KM reductions, we provide in Sect. 2 a detailed account of the DS approach and in particular of the canonical DS reductions leading to the $\mathscr{W}_{\mathscr{L}}^{\mathscr{S}}$ -algebras. Section 3 contains the $sl(2)$ theorem, which associates an $sl(2)$ embedding to every DS reduction. Section 4 deals with the restrictions on the possible DS reductions belonging to the same $sl(2)$ embedding. Section 5 gives the argument on the spectrum of the conformal weights, and the new examples of noncanonical DS reductions. In Sect. 6 we give our conclusions, discuss the relationship of our results with those in [31], and point out some open problems. We conclude with the statement of the conjecture mentioned in the Abstract and the discussion of some open questions. There is also an appendix containing as illustration the W_{2n}^2 -reductions which lead to nonfreely generated invariant rings \mathscr{R} .

2. Classical \mathscr{W} -Algebras

By definition, a *classical \mathscr{W} -algebra* is a Poisson bracket algebra built on a finite number of independent fields $W_a(z)$, $a = 1, \dots, N$, defined on the circle S^1 , according to the following requirements. First, the defining Poisson bracket relations are of the form¹

$$\{W_b(z), W_c(w)\} = \sum_i P_{bc}^i(W_1(w), \dots, W_N(w)) \delta^{(i)}(z - w), \quad (2.1)$$

where the P_{bc}^i appearing in the *finite* sum ($i = 0, 1, 2, \dots$) on the right hand side are *differential polynomials* in the generator fields $\{W_a\}_{a=1}^N$, with constant terms allowed. Second, W_1 satisfies the Virasoro Poisson bracket algebra,

$$\{W_1(z), W_1(w)\} = -W_1'(w)\delta(z - w) + 2W_1(w)\delta'(z - w) + \frac{c}{24\pi}\delta'''(z - w). \quad (2.2)$$

Third, the rest of the generators W_a , $a = 2, \dots, N$, are conformal primary fields with respect to W_1 ,

$$\{W_1(z), W_a(w)\} = -W_a'(w)\delta(z - w) + \Delta_a W_a(w)\delta'(z - w), \quad a = 2, \dots, N. \quad (2.3)$$

The classical Virasoro centre c and the conformal weights Δ_a , $a = 2, \dots, N$, are (in general complex) numbers. These constant parameters and the “structure polynomials” P_{bc}^i are restricted by the antisymmetry and the Jacobi identity of the

¹ Conventions: $\delta(z - w) := \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} z^{n-1} w^{-n}$ is the delta-function on S^1 ($|z| = |w| = 1$) for which $\oint dz f(z)\delta(z - w) = f(w)$; we use $\delta^{(i)}(z - w) = \left(\frac{d}{dz}\right)^i \delta(z - w)$. The Virasoro $W_1(z)$ in (2.2) and the KM current $J(z)$ in (2.4) have their Laurent modes, $L_n := i \oint dz W_1(z) z^{n+1}$ and $J_n := -i \oint dz J(z) z^n$, which fulfil the standard Virasoro and affine KM algebras with centre c and k , respectively (up to an overall factor $(-i)$ to be replaced by 1 upon quantization)

Poisson bracket. Two classical \mathscr{W} -algebras are regarded to be equivalent if their defining relations can be brought to the same form by a differential polynomial change of basis, $W_a \rightarrow \tilde{W}_a = \tilde{W}_a(W_1, \dots, W_N)$, such that the inverse transformation, $\tilde{W}_a \rightarrow W_a = W_a(\tilde{W}_1, \dots, \tilde{W}_N)$ is also given by differential polynomials.

In principle, one can construct classical \mathscr{W} -algebras by determining the constant parameters and the structure polynomials directly from the requirements of antisymmetry and Jacobi identity. However, in practice this is hard to carry out systematically, and for this reason in this paper we are interested in the DS reduction approach where these requirements are guaranteed by construction. Moreover, this approach enables us to quantize the resulting algebras directly through the BRST procedure.

2.1. The DS Reduction Approach to Classical \mathscr{W} -Algebras

The general strategy of the DS reduction approach to constructing classical \mathscr{W} -algebras may be formulated as follows. Consider a finite dimensional complex simple Lie algebra \mathscr{G} with the ad-invariant, nondegenerate scalar product $\langle \cdot, \cdot \rangle$. Denote by \mathscr{K} the space of \mathscr{G} -valued smooth functions on the circle, $\mathscr{K} := \{J(z) \mid J(z) \in \mathscr{G}\}$, and let \mathscr{K} carry the “KM Poisson bracket algebra” given by

$$\begin{aligned} \{\langle \alpha, J(z) \rangle, \langle \beta, J(w) \rangle\} &= \langle [\alpha, \beta], J(z) \rangle \delta(z - w) \\ &\quad + K \langle \alpha, \beta \rangle \delta'(z - w), \quad \forall \alpha, \beta \in \mathscr{G}, \end{aligned} \quad (2.4)$$

where $k = -2\pi K \neq 0$ is the KM level. (In other words, the space \mathscr{K} is the fixed level Poisson subspace of the dual of the affine KM Lie algebra carrying the Lie-Poisson bracket.) We henceforth set the constant K to 1 for notational simplicity. Let us choose a subalgebra $\Gamma \subset \mathscr{G}$, with a basis $\{\gamma_i\}$ and an element $M \in \mathscr{G}$ in such a way that the following constraints:

$$\phi_i(z) = 0, \quad \text{where} \quad \phi_i(z) := \langle \gamma_i, J(z) - M \rangle, \quad (2.5)$$

are *first class*. This means that the scalar product $\langle \cdot, \cdot \rangle$ and the antisymmetric 2-form ω_M on \mathscr{G} defined by

$$\omega_M(\alpha, \beta) := \langle M, [\alpha, \beta] \rangle, \quad \forall \alpha, \beta \in \mathscr{G}, \quad (2.6)$$

vanish when restricted to Γ . The constraint surface, $\mathscr{K}_\Gamma \subset \mathscr{K}$, defined by (2.5) consists of currents of the form

$$J(z) = M + j(z), \quad j(z) \in \Gamma^\perp, \quad (2.7)$$

and the first class constraints ϕ_i generate gauge transformations on it,

$$j \longrightarrow \text{Ad}_{e^f}(j) := e^f(j + M)e^{-f} - M + (e^f)'e^{-f}, \quad f(z) \in \Gamma. \quad (2.8)$$

We are interested in the *gauge invariant differential polynomials* in $j(z)$ since, as we shall see below, under certain conditions they furnish a classical \mathscr{W} -algebra.

Let \mathscr{R} be the set of gauge invariant differential polynomials in the components of $j(z)$ (with constant terms allowed). This set is obviously closed with respect to linear combination, ordinary multiplication and application of ∂ . We express this by saying that \mathscr{R} is a *differential ring*. On the other hand, the induced Poisson bracket carried

by the gauge invariant functions on \mathcal{K}_Γ (inherited from the Poisson bracket on \mathcal{K}) also closes on \mathcal{R} . Namely, if $T, U \in \mathcal{R}$ one has

$$\{T(j(z)), U(j(w))\} = \sum_i P_{TU}^i(j(w)) \partial^i \delta(z - w), \quad (2.9)$$

where the sum is finite, and $P_{TU}^i \in \mathcal{R}$ because of the gauge invariance. This implies that if \mathcal{R} is a *freely generated* differential ring, i.e., if there exists a basis $\{W_a\}_{a=1}^N \subset \mathcal{R}$ such that any element of \mathcal{R} can be expressed in a unique way as a differential polynomial in the W_a ’s, then the KM Poisson brackets of the basis elements give an algebra of the form (2.1). In particular, when it is possible to find a \mathcal{W} -basis of \mathcal{R} – by which we mean such a free generating set for which (2.2) and (2.3) also hold – then we have a classical \mathcal{W} -algebra because the Jacobi identity and antisymmetry are guaranteed by construction. Thus, within this approach, our purpose should just be to classify the KM reductions for which the invariant ring \mathcal{R} is freely generated and admits a \mathcal{W} -basis.

2.1.1. Freely Generated Ring Due to DS Gauge. It is rather obvious that \mathcal{R} is not freely generated for a generic first class reduction. For instance, the reductions proposed by Polyakov and Bershadsky [28, 29] aimed at constructing the “ W_n^l -algebras” lead in general to a nonfreely generated ring \mathcal{R} (see the Appendix). To our knowledge, the only systematic method by which one can produce free generators for \mathcal{R} relies on the so called DS gauges, the existence of which places a strong restriction on the reductions. These gauges may be defined as follows.

Definition (DS gauge). *Given a set of first class constraints of type (2.5), we have a DS gauge if the following conditions i) - iii) are met:*

(i) *There exists a diagonalizable element² $H \in \mathcal{S}$ such that*

$$[H, \Gamma] \subset \Gamma, \quad [H, M] = -M. \quad (2.10)$$

(ii) *With a graded linear space V (i.e. $[H, V] \subset V$) defining a direct sum decomposition,*

$$\Gamma^\perp = [M, \Gamma] + V, \quad \text{with } V \cap [M, \Gamma] = \{0\}, \quad (2.11)$$

one can gauge-fix the constrained current (2.7) into the form belonging to the subspace $\mathcal{E}_V \subset \mathcal{K}_\Gamma$ given by

$$\mathcal{E}_V := \{J \mid J(z) = M + j_{\text{DS}}(z), \quad j_{\text{DS}}(z) \in V\}. \quad (2.12)$$

(iii) *The resultant gauge fixed current $j_{\text{DS}}(z) = j_{\text{DS}}(j(z))$, in which the gauge orbit passing through $j(z) \in \mathcal{K}_\Gamma$ meets the global gauge section \mathcal{E}_V , is given by a differential polynomial in the original current $j(z)$.*

Condition i) requires a special element H whose adjoint action ad_H maps Γ into itself and with respect to which M is an eigenvector with nonzero eigenvalue, $[H, M] = \lambda M$ (for later convenience we have scaled H so that $\lambda = -1$). Note that it is not always possible to find such an H for a given pair (Γ, M) , even if we take into account that M can be redefined by $M \rightarrow M + m$, $m \in \Gamma^\perp$, which does not affect the constraints (2.5). The main requirement given by ii), iii) is that \mathcal{E}_V is a global gauge section of (2.8) such that the components of the gauge fixed current j_{DS} ,

² A diagonalizable element defines a grading of \mathcal{S} by means of its eigenvalues in the adjoint representation, and is often referred to as a grading operator of \mathcal{S}

when considered as functions on \mathcal{K}_Γ , are elements of \mathcal{R} . In particular, if Γ consists of nilpotent elements of \mathcal{G} then iii) is implied by the stronger and more practical requirement

(iii') The gauge-fixing equation corresponding to the gauge section \mathcal{E}_V ,

$$j \rightarrow \text{Ad}_{e^f} j = e^f j e^{-f} + (e^f M e^{-f} - M) + (e^f)' e^{-f} = j_{\text{DS}}, \quad (2.13a)$$

where

$$j(z) \in \Gamma^\perp, \quad f(z) \in \Gamma, \quad j_{\text{DS}}(z) \in V, \quad (2.13b)$$

has a differential polynomial solution $f(z) = f(j(z))$.

In all known examples for which a DS gauge exists, Γ actually consists of nilpotent elements and one has property iii)'. This will include all the examples given in this paper.

When a DS gauge is available, we call the procedure by which the general first class constrained current is transformed to such a gauge, i.e., whereby Eq. (2.13) is solved, the *DS gauge fixing procedure* [11, 23]. Note that in principle we need not require that the solution be unique for the gauge transformation e^f though it is actually unique in all known examples. Like for any gauge invariant function, for any $P(j) \in \mathcal{R}$ we have

$$P(j) = P(\text{Ad}_{e^{-f}} j_{\text{DS}}) = P(j_{\text{DS}}(j)) \quad (2.14)$$

by inverting (2.13a). The point is that by using iii) this equation now implies that the components of $j_{\text{DS}}(j)$ form a generating set for \mathcal{R} . Furthermore, these generators of \mathcal{R} are independent since they reduce to independent current components in the DS gauge, i.e., we have

$$j_{\text{DS}}(j(z)) = j(z) \quad \text{on } \mathcal{E}_V, \quad (2.15)$$

which follows directly from the notion of gauge fixing. In conclusion, we see that if a DS gauge exists then \mathcal{R} is freely generated, and a basis is given by the components of $j_{\text{DS}}(j)$.

Clearly, the number of components of the gauge fixed current j_{DS} should be $\dim \mathcal{G} - 2 \dim \Gamma$, and this implies by (2.11-12) that the *nondegeneracy condition*,

$$\text{Ker} (\text{ad}_M) \cap \Gamma = \{0\}, \quad (2.16)$$

is a *necessary* condition for DS gauge fixing. On the other hand, we can provide a reasonably simple *sufficient* condition for DS gauge fixing as follows [23]. Choose a graded subspace $\Theta \subset \mathcal{G}$ which is dual to Γ with respect to the 2-form ω_M , and define V in (2.11) to be the space orthogonal to both Γ and Θ ,

$$V := \Theta^\perp \cap \Gamma^\perp. \quad (2.17)$$

In other words, add the gauge fixing conditions

$$\chi_k(z) := \langle \theta_k, J(z) - M \rangle = 0, \quad \theta_k \in \Theta, \quad (2.18)$$

to the first class constraints (2.5). If, in addition to the nondegeneracy condition (2.16), one has

$$[\Theta, \Gamma]_{\geq 1} \subset \Gamma, \quad (2.19)$$

where the subscript refers to the grading defined by H , and if Γ consists of nilpotent elements of \mathcal{G} , then by using V in (2.17) one indeed obtains a DS gauge. (We refer the reader to [23] for a detailed description of the recursive DS gauge fixing procedure

based on this sufficient condition.) A somewhat stronger sufficient condition for DS gauge fixing is furnished by complementing (2.16) with the condition

$$\Gamma^\perp \subset \mathcal{G}_{>-1}. \quad (2.20)$$

Equations (2.16) and (2.20) together imply

$$\mathcal{G}_{\geq 1} \subset \Gamma \subset \mathcal{G}_{>0}, \quad (2.21)$$

which ensures (2.19) and that Γ consists of nilpotent elements.

We also note the following further consequences of the definition of a DS gauge. First, because of (2.15), the components of $j_{\text{DS}}(j(z))$, defined by means of a basis of V , contain the corresponding components of $j(z)$ in their linear terms. Second, for the very same reason, the induced KM Poisson bracket algebra of the components of $j_{\text{DS}}(j(z))$ can be identified with the Dirac bracket algebra carried by the components of the gauge fixed current, where the second class constraints defining the Dirac bracket are given by combining (2.5) and (2.18) together [14].

2.1.2. The Form of the Virasoro Density and the DS Mechanism. Having assumed the existence of a DS gauge using the grading operator $H \in \mathcal{G}$, next we have to ensure that the polynomial Poisson bracket algebra carried by \mathcal{R} contains the Virasoro subalgebra. For this we shall consider the following density,

$$L_H := \frac{1}{2} \langle J, J \rangle - \langle H, J' \rangle. \quad (2.22)$$

Indeed, one can easily check that this defines a *gauge invariant* Virasoro density, i.e., it not only fulfils the Virasoro algebra but also is an element of \mathcal{R} , provided that in addition to (2.10) one has

$$H \in \Gamma^\perp. \quad (2.23)$$

Of course, the relations (2.10) and (2.23) also imply that the conformal action generated (for $\delta_f z = -f(z)$) by the charge $Q_f = \oint dz f(z) L_H(z)$ on \mathcal{K} ,

$$\delta_f J := -\{Q_f, J\} = f J' + f'(J + [H, J]) + f'' H, \quad (2.24)$$

induces a conformal action on the space of gauge orbits in \mathcal{K}_Γ . We note that the coefficient of the term $\langle H, J' \rangle$ in L_H rescales according to the choice of λ in $[H, M] = \lambda M$; the value -1 in (2.22) is adjusted for our choice $\lambda = -1$. Note also that (2.23) is a rather mild additional assumption to the existence of a DS gauge, since in the examples when DS gauges exist Γ is usually a strictly triangular subalgebra of \mathcal{G} and (2.23) is automatic for the Cartan element H .

Based on the construction we described above, and motivated by the canonical DS reductions which we will recall in the next section, we are interested in reductions of KM Poisson bracket algebras to classical \mathcal{W} -algebras through the following mechanism:

- (i) *The first class constraints (2.5) admit a DS gauge with respect to a grading operator H .*
- (ii) *There exists a \mathcal{W} -basis in the invariant ring \mathcal{R} with respect to the Virasoro density $W_1 := L_H$.*

These assumptions imply the existence of a basis of \mathcal{R} yielding a classical \mathcal{W} -algebra in the sense of Eqs. (2.1-3), and we believe that they are not much stronger than requiring just this to be the case. (To the date of writing, we have no counterexample.)

By definition, in this paper we call a KM reduction defined by first class constraints of type (2.5) a *DS reduction* if the above *DS mechanism* i), ii) is applicable.

Before describing the canonical DS reductions where this mechanism is at work, and whose uniqueness is the main question addressed later, we wish to mention another consequence of the assumptions. Namely, we observe from (2.24) that if $f'' = 0$ then the infinitesimal conformal transformation generated by Q_f leaves the DS gauge fixed current form invariant, and we have

$$\delta_f j_{\text{DS}} = f j'_{\text{DS}} + f'(j_{\text{DS}} + [H, j_{\text{DS}}]), \quad \text{for} \quad f'' = 0. \quad (2.25)$$

Since $f'' = 0$ holds for the infinitesimal scale transformation for which $f(z) \sim z$, we see from (2.25) that the components of $j_{\text{DS}}(j)$ have definite *scale dimensions* given by shifting the grades of the corresponding basis elements of V in (2.11) by +1.

2.2. The Canonical DS Reductions and the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -Algebras

The DS mechanism works in the *canonical DS reductions* which are associated to the $sl(2)$ embeddings in \mathcal{S} in the following way. Let $\mathcal{S} = \{M_-, M_0, M_+\} \subset \mathcal{S}$ be an $sl(2)$ subalgebra with standard commutation relations

$$[M_0, M_{\pm}] = \pm M_{\pm}, \quad [M_+, M_-] = 2M_0. \quad (2.26)$$

Consider the grading of \mathcal{S} defined by the eigenvalues of $\text{ad}_{M_0} = [M_0, \cdot]$,

$$\mathcal{S} = \sum_m \mathcal{S}_m, \quad \text{where} \quad [M_0, X] = mX, \quad \forall X \in \mathcal{S}_m. \quad (2.27)$$

Choose a subspace $\mathcal{P}_{\frac{1}{2}} \subset \mathcal{S}_{\frac{1}{2}}$ for which

$$\omega_{M_-}(\mathcal{P}_{\frac{1}{2}}, \mathcal{P}_{\frac{1}{2}}) = \{0\}, \quad \dim \mathcal{P}_{\frac{1}{2}} = \frac{1}{2} \dim \mathcal{S}_{\frac{1}{2}}, \quad (2.28a)$$

and define the *canonical subalgebra* Γ_c by

$$\Gamma_c := \mathcal{P}_{\frac{1}{2}} + \mathcal{S}_{\geq 1}. \quad (2.28b)$$

The canonical first class constraints are obtained from (2.5) by taking $\Gamma := \Gamma_c$ and $M := M_-$, and thus the constrained current takes the form

$$J(z) = M_- + j_c(z), \quad j_c(z) \in \Gamma_c^{\perp} \quad \text{with} \quad \Gamma_c^{\perp} = [M_-, \mathcal{P}_{\frac{1}{2}}] + \mathcal{S}_{\geq 0}. \quad (2.29)$$

Note that with respect to $M := M_-$ the dimension of $\Gamma = \Gamma_c$ is the maximal one allowed by the first classness of the constraints and the nondegeneracy condition (2.16). It is also easy to check that DS gauges are available by using $H := M_0$ as the grading operator and that $L_{M_0} \in \mathcal{R}$.

The \mathcal{W} -basis of \mathcal{R} is constructed by means of the “highest weight gauge” [14], which is the particular DS gauge obtained by taking

$$V := \text{Ker}(\text{ad}_{M_+}) \quad (2.30)$$

in the direct sum decomposition of type (2.11). For this, we first fix a basis $\{Y_{l,n}\} \subset \text{Ker}(\text{ad}_{M_+})$ of highest weight vectors,

$$[M_0, Y_{l,n}] = lY_{l,n}, \quad Y_{1,1} := M_+ / \langle M_-, M_+ \rangle, \quad (2.31)$$

where n is a multiplicity index and $\langle M_-, Y_{l,n} \rangle = 0$ for $Y_{l,n} \neq Y_{1,1}$. We then write the current resulting from the gauge fixing, $j_{\text{hw}}(j_c(z))$, in the form

$$j_{\text{hw}}(j_c(z)) = \sum_{l,n} W_{l,n}(j_c(z)) Y_{l,n}, \quad (2.32)$$

and $\{W_{l,n}(j_c)\} \subset \mathcal{R}$ is a basis of the invariant ring. It turns out that, except $W_{1,1}$, $W_{l,n}$ is a primary field of weight $(l+1)$ with respect to L_{M_0} [15, 23]. The Virasoro density $W_1 := L_{M_0}$ given by

$$W_1(j_c) = \frac{1}{2} \langle M_- + j_c, M_- + j_c \rangle - \langle M_0, j'_c \rangle, \quad (2.33a)$$

can be rewritten as

$$W_1(j_c) = \frac{1}{2} \langle j_{\text{hw}}^{\text{sing}}(j_c), j_{\text{hw}}^{\text{sing}}(j_c) \rangle + W_{1,1}(j_c), \quad (2.33b)$$

where

$$j_{\text{hw}}^{\text{sing}}(j_c) := \sum_n W_{0,n}(j_c) Y_{0,n} \quad (2.33c)$$

is the $sl(2)$ singlet part of $j_{\text{hw}}(j_c)$. The singlet components $W_{0,n}(j_c)$ generate a KM algebra under the induced Poisson bracket, and the first term in (2.33b) is just the corresponding Sugawara formula. The second term $W_{1,1}(j_c)$ in (2.33b) is another Virasoro density, that commutes with the singlet Sugawara density. We see from the above that the required \mathcal{W} -basis, $\{W_a\}_{a=1}^N \subset \mathcal{R}$, $N = \dim \text{Ker}(\text{ad}_{M_+})$, can be obtained from the basis $\{W_{l,n}\} \subset \mathcal{R}$ by exchanging $W_{1,1}$ with W_1 , and calling the rest of the basis elements W_2, \dots, W_N . The resulting classical \mathcal{W} -algebra is called the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra. By the remark given at the end of Subsect. 2.1.1., the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra can be interpreted also as the Dirac bracket algebra carried by the components of the current in the highest weight gauge (after the aforementioned change of basis is made).

It is worth stressing that, apart from those $sl(2)$ embeddings for which there are no singlets in the adjoint of \mathcal{G} , the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra contains the singlet KM subalgebra generating the group of canonical transformations:

$$j_{\text{hw}}(z) \longrightarrow e^{\alpha^i(z)Y_{0,i}} j_{\text{hw}}(z) e^{-\alpha^i(z)Y_{0,i}} + (e^{\alpha^i(z)Y_{0,i}})' e^{-\alpha^i(z)Y_{0,i}}. \quad (2.34)$$

It follows that the generators of the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra given by the nonsinglet components of j_{hw} fall into representations of the Lie algebra of the singlets, and that one can further reduce the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra by using this sub-KM symmetry, i.e., by putting constraints on the singlet components of j_{hw} . However, such “secondary reductions” do not in general lead to new \mathcal{W} -algebras based on independent fields according to the requirements (2.1-3) (see also the Appendix).

Having our hands on the above rather nice examples, it appears natural to ask how close they are to an exhaustive set of \mathcal{W} -algebras that can be obtained through the DS mechanism. This question will be even more natural after establishing in the next section that in a certain sense there is indeed an $sl(2)$ embedding behind any \mathcal{W} -algebra obtained in this way.

3. The Existence of an $sl(2)$ Structure

In this section we prove a theorem, which allows one to associate an $sl(2)$ embedding to every reduction yielding a \mathcal{W} -algebra by means of the DS mechanism reviewed in the previous section. More precisely, our assumption will be that \mathcal{R} is freely generated due to the existence of a DS gauge and possesses a quasi-primary basis with respect to L_H . We shall then conclude that $H \in \mathcal{S}$ must belong to an $sl(2)$ subalgebra. This immediately implies that *the conformal weights must be either integral or half-integral in every \mathcal{W} -algebra arising in this way*. This theorem is a stronger version of the previous result in [23], and the rest of the paper is devoted to uncovering its implications. To make the proof as clear as possible we shall proceed through two preliminary lemmas.

Consider conformally invariant first class constraints described by a triple (Γ, M, H) . The form of the constrained current is given by Eq. (2.7) and L_H (2.22) defines an element of the invariant ring \mathcal{R} . For any vector field $f(z) \frac{d}{dz} \in \text{diff } S^1$, the infinitesimal conformal transformation $\delta_f J$ is generated by the charge Q_f according to (2.24). This conformal action preserves the constraint surface $\mathcal{K}_\Gamma \subset \mathcal{K}$, and we have

$$\delta_f j = f j' + f'(j + [H, j]) + f'' H. \quad (3.1)$$

Consider now the subspace of special configurations, $\mathcal{E}_0 \subset \mathcal{K}_\Gamma$, given by

$$\mathcal{E}_0 := \{ J \mid J(z) = M + h(z)H \}. \quad (3.2)$$

This subspace \mathcal{E}_0 is invariant under conformal transformations, and the field $h(z)$ transforms according to

$$\delta_f h = f h' + f' h + f''. \quad (3.3)$$

By definition, $U(z)$ is called a *quasi-primary* field of scale dimension Δ (which is the conformal weight if $U(z)$ is primary) if it transforms as

$$\delta_f U = f U' + \Delta f' U \quad (3.4)$$

under the *Möbius subgroup* of the conformal group generated by the vector fields with $f''' = 0$. As far as scale dimension Δ is concerned, it can be defined even for a non-quasi-primary field $U(z)$ if it satisfies (3.4) for $f'' = 0$. For example, the field $h(z)$ is not quasi-primary but has scale dimension 1. We then have the following statement.

Lemma 1. *There is no quasi-primary differential polynomial $p(j)$ on \mathcal{K}_Γ whose restriction to \mathcal{E}_0 (3.2) satisfies*

$$p(j)|_{\mathcal{E}_0} = Ah \quad (3.5)$$

with a nonzero constant A .

Proof. Since \mathcal{E}_0 is invariant under conformal transformations and Ah is not quasi-primary for $A \neq 0$, we see that a differential polynomial $p(j)$ satisfying (3.5) cannot be quasi-primary. Q.E.D.

On the other hand, we also have the following statement.

Lemma 2. *Suppose the constraints admit a DS gauge for which the complementary space V in (2.11) is graded by H , and*

$$H \notin [M, \Gamma]. \quad (3.6)$$

Then there exists a gauge invariant differential polynomial $P^H(j) \in \mathcal{R}$ whose restriction to \mathcal{E}_0 (3.2) is proportional to the field h ,

$$P^H(j)|_{\mathcal{E}_0} = Ah, \quad (3.7)$$

where A is a nonzero constant.

Proof. Let $j_{\text{DS}}(j) \in V$ be the gauge transform of the general current $j \in \mathcal{K}_\Gamma$ to the DS gauge. Recall that the components of $j_{\text{DS}}(j)$ generate \mathcal{R} and have scale dimensions given by shifting the grades of the corresponding basis elements of V by 1. Recall also that the components of $j_{\text{DS}}(j)$ contain the corresponding components of j . From these facts and (3.6) we see that the restriction of $\langle H, j_{\text{DS}}(j) \rangle$ to \mathcal{E}_0 (3.2) contains a term proportional to h and has scale dimension 1. Clearly, we can thus take $P^H(j) := \langle H, j_{\text{DS}}(j) \rangle$ to be the required element of \mathcal{R} . Q.E.D.

The $sl(2)$ theorem will result by combining the statements of the two lemmas. The theorem uses the notion of a quasi-primary basis. By definition, the basis $\{W_a\}_{a=1}^N \subset \mathcal{R}$ is a *quasi-primary basis* if the basis elements are quasi-primary fields with respect to the given Virasoro density. Clearly, a \mathcal{W} -basis is a quasi-primary basis.

Theorem. Suppose that the conformally invariant first class constraints described by (Γ, M, H) admit a DS gauge with respect to the grading operator H . Suppose furthermore that there exists a quasi-primary basis of gauge invariant differential polynomials $\{W_a\}_{a=1}^N \subset \mathcal{R}$ ($N = \dim \mathcal{S} - 2\dim \Gamma$) with respect to L_H . Then there exists an element $M_+ \in \Gamma$ such that the standard $sl(2)$ commutation relations (2.26) hold with $M_- := M$ and $M_0 := H$.

Proof. Suppose that we have (3.6). Then by Lemma 2 we have an element $P^H(j) \in \mathcal{R}$ whose restriction to \mathcal{E}_0 has the property (3.7). On the other hand, since we assumed a quasi-primary basis in \mathcal{R} we can express $P^H(j)$ as a differential polynomial in the basis,

$$P^H(j) = P(W_1(j), W_2(j), \dots, W_N(j)). \quad (3.8)$$

When restricted to \mathcal{E}_0 the $W_a(j)$ ’s in the r.h.s. of (3.8) either vanish or become quasi-primary differential polynomials in h . However, due to Lemma 1, none of the nonvanishing ones can contain a term proportional to h and hence the r.h.s. of (3.8) does not reduce to the expression Ah . Since this contradicts (3.7), we conclude that (3.6) cannot hold. Thus there must exist an element $\gamma \in \Gamma$ such that $H = [M, \gamma]$. Decomposing γ into a grade 1 part and the rest, $\gamma = \gamma_1 + \gamma_{\neq 1}$, we obtain $[M, \gamma_{\neq 1}] = 0$ on account of the grading. The nondegeneracy condition (2.16) then implies $\gamma_{\neq 1} = 0$. (The element $\gamma_{\neq 1}$ must be in Γ , since Γ is assumed to be graded (2.10).) Thus γ has grade 1 and we have

$$H = [M, \gamma], \quad \text{and} \quad [H, \gamma] = \gamma. \quad (3.9)$$

Combining (3.9) with $[H, M] = -M$ in (2.10), we find that the set $\{M, H, \gamma\}$ forms the required $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\}$ of (2.26). Q.E.D.

Since a \mathcal{W} -basis is necessarily a quasi-primary basis, the theorem says that one can always find an $sl(2)$ subalgebra by the set given above for any \mathcal{W} -algebra obtained from DS reduction. This suggests that the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebras, which are manifestly based on the $sl(2)$ subalgebras of \mathcal{S} , are in fact “natural” in the context of the DS reduction approach. This in turn leads us to the question as to whether the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebras are

the only classical \mathcal{W} -algebras that may be obtained from DS reduction. We shall try to answer this question in the rest of the paper.

4. Restrictions on Γ for a Given $sl(2)$ Embedding

In Sect. 3 we found that there exists an $sl(2)$ embedding, $\mathcal{S} = \{M_- = M, M_0 = H, M_+ \in \Gamma\}$, to any system of conformally invariant first class constraints given by a triple (Γ, M, H) for which \mathcal{R} is freely generated due to the existence of a DS gauge and possesses a quasi-primary basis with respect to L_H . This result reduces the problem of listing all DS reductions to the problem of finding all allowed Γ 's for given $sl(2)$ embeddings in \mathcal{G} (whose classification is known), such that the triple (Γ, M_-, M_0) leads to a \mathcal{W} -algebra. In this section we shall see that the same requirement used earlier to uncover the $sl(2)$ structure, i.e., that there exists a quasi-primary basis in \mathcal{R} , also gives considerably strong restrictions on the allowed Γ . (As in the previous section we only need to require the existence of a quasi-primary basis in \mathcal{R} rather than a \mathcal{W} -basis.) In particular, we shall prove that Γ must satisfy a certain number of inequalities on the dimensions of its graded subspaces. The simplest ones among them are

$$\dim \Gamma_q \geq \frac{1}{2} \dim \mathcal{G}_q, \quad \forall q \geq 1, \quad (4.1)$$

which imply that $\Gamma_{\geq 1}$ must be at least half as large as $(\Gamma_c)_{\geq 1}$. To derive (4.1), we shall use a two-step DS gauge fixing procedure based on a semi-direct sum structure of the gauge subalgebra Γ . We shall examine the existence of a quasi-primary basis by asking if the DS gauge fixed current can be expressed as a differential polynomial in such a basis. For this purpose it will be convenient to expand every element of \mathcal{R} as a differential polynomial in the partially gauge fixed current provided by the first step of the gauge fixing, described in Sect. 4.1. By inspecting the linear term in the expansion of the fully DS gauge fixed current and requiring that it be compatible with the existence of a quasi-primary basis, in Sect. 4.2 we shall prove a proposition from which the inequalities in (4.1) follow.

4.1. Gauge Fixing with Respect to M_+

In order to implement the first step of the two-step gauge fixing for the system given by the triple (Γ, M_-, M_0) , let us write the first class constrained current $J(z) \in \mathcal{K}_\Gamma$ in the form:

$$J(z) = M_- + h(z)M_0 + j_+(z)M_+ + t(z), \quad \text{with } t(z) \in \Gamma^\perp \cap \mathcal{S}^\perp. \quad (4.2)$$

That this is possible follows from $M_0 \in \Gamma^\perp$ (2.23), and from $M_+ \in \Gamma$ (and hence $M_+ \in \Gamma^\perp$) which is required by the Theorem in Sect. 3. The conformal transformation (2.24) then reads

$$\delta_f h = fh' + f'h + f'', \quad \delta_f j_+ = fj'_+ + 2f'j_+, \quad \delta_f t = ft' + f'(t + [M_0, t]). \quad (4.3)$$

It will be useful to consider the subgroup of the gauge group generated by M_+ , which acts on $J(z)$ according to

$$h \rightarrow h + 2\alpha, \quad j_+ \rightarrow j_+ + \alpha' - \alpha h - \alpha^2, \quad t \rightarrow e^{\alpha M_+} t e^{-\alpha M_+}. \quad (4.4)$$

A complete, polynomial gauge fixing of this gauge freedom is obtained by restricting the current to the form

$$I(z) = M_- + \omega(z)M_+ + u(z), \quad \text{with } u(z) \in \Gamma^\perp \cap \mathcal{S}^\perp. \quad (4.5)$$

The general current (4.2) is transformed to this M_+ -gauge section by choosing the parameter $\alpha(z)$ to be $\alpha(z) = -\frac{1}{2}h(z)$. As a result, the differential polynomials given by

$$u = e^{-\frac{1}{2}hM_+}te^{\frac{1}{2}hM_+}, \quad \omega = j_+ + \frac{1}{4}h^2 - \frac{1}{2}h', \quad (4.6)$$

are invariant under the M_+ -transformations (4.4), and freely generate the ring of the M_+ -gauge invariant differential polynomials. Since every Γ -gauge invariant differential polynomial is necessarily M_+ -gauge invariant (since $M_+ \in \Gamma$), we see that every $P \in \mathcal{R}$ can be expressed in terms of the M_+ -gauge invariants, u and ω . In more detail, we can write

$$P = P(u, \omega) = \sum_i P_i(u, \omega), \quad (4.7)$$

where the P_i are uniquely determined differential polynomials that are *homogeneous* of degree i in their arguments u, ω . This expansion is very convenient for investigating the transformation properties of differential polynomials under infinitesimal Möbius transformations ($f''' = 0$), because the expansion is covariant under such transformations. Indeed, from (4.3) and (4.6) we find that u and ω transform in a *linear, homogeneous* way under Möbius transformations:

$$\delta_f u = fu' + f'(u + [M_0, u]) - \frac{1}{2}f''[M_+, u], \quad \delta_f \omega = f\omega' + 2f'\omega. \quad (4.8)$$

(For completeness, we note that under a general conformal transformation $\delta_f \omega$ picks up also the usual f''' term.) Since the conformal transformation of a differential polynomial is determined through the transformation of its arguments, we obtain

$$(\delta_f P)_i(u, \omega) = (\delta_f P_i)(u, \omega). \quad (4.9)$$

As a consequence, we find that a differential polynomial $P(u, \omega) = \sum_i P_i(u, \omega)$ is *quasi-primary of scale dimension Δ if and only if $P_i(u, \omega)$ is quasi-primary of scale dimension Δ for all i* . On account of this, we have the following general idea for deriving restrictions on Γ from requiring the existence of a quasi-primary basis in \mathcal{R} : We should look at the linear, quadratic etc. terms in the expansion of the components of the DS gauge fixed current $j_{\text{DS}}(u, \omega)$ that generate \mathcal{R} , and inspect the conditions under which they can be expressed as differential polynomials in homogeneous, quasi-primary differential polynomials in u and ω , since such differential polynomials enter the expansion of the quasi-primary basis.

We shall see shortly how the above idea works in the simplest linear case, but before that we wish to mention some further features of the gauge fixing with respect to M_+ . First, this partial gauge fixing is stable under the subgroup of the gauge group generated by

$$\tilde{\Gamma} := \Gamma \cap \mathcal{S}^\perp, \quad (4.10)$$

i.e., by the subalgebra $\tilde{\Gamma} \subset \Gamma$ defined by removing M_+ from Γ so that the rest is orthogonal to M_- . The stability of the M_+ -gauge section (4.5) under the $\tilde{\Gamma}$ -gauge

transformations can be seen explicitly by observing the $\tilde{\Gamma}$ -invariance of the partial gauge fixing condition,

$$\langle M_0, J(z) \rangle = 0, \quad (4.11)$$

that restricts the current to the form (4.5). Second, Γ has the following semi-direct sum structure:

$$\Gamma = \text{span}\{M_+\} \oplus_s \tilde{\Gamma}, \quad (\text{i.e., } [M_+, \tilde{\Gamma}] \subset \tilde{\Gamma}). \quad (4.12)$$

Accordingly, one can write the element $g(z) = e^{\gamma(z)}$, $\gamma(z) \in \Gamma$, of the gauge group in the product form

$$g(z) = e^{\tilde{\gamma}(z)} \cdot e^{\alpha(z)M_+} \quad \text{with} \quad \tilde{\gamma}(z) \in \tilde{\Gamma}, \quad (4.13)$$

and thereby fix the M_+ -gauge-freedom first in the way given above, and fix the $\tilde{\Gamma}$ -gauge-freedom subsequently. Having performed the first step, from now on we regard $I(z)$ in (4.5) as our new variable, whose components have the transformation rule (4.8) under the Möbius group and upon which the further $\tilde{\Gamma}$ gauge fixings are to be performed. The variables u, ω are simpler to deal with than the original variables t, j_+, h , since the Möbius group acts homogeneously on the former (4.8) whereas it acts inhomogeneously on the latter (4.3).

4.2. Half-Maximality of Γ from the Linear Terms

Below we prove a proposition from which the dimensional estimate (4.1) will follow as a corollary. The proof will be based on a preliminary lemma, which is an analogue of Lemma 1 of Sect. 3.

Let $q \in \{1, \frac{3}{2}, 2, \frac{5}{2}, \dots\}$ be fixed and (if exists) choose a nonzero element $T_{-q} \in (\Gamma^\perp)_{-q}$. As can be readily seen, we have

$$(\text{ad}_{M_+})^{2q}(T_{-q}) \neq 0. \quad (4.14)$$

Define $\mathcal{E}[T_{-q}]$ to be the following subspace of the space of M_+ -gauge fixed currents given by (4.5):

$$\mathcal{E}[T_{-q}] := \{ I \mid I(z) = M_- + \sum_{i=0}^{i_{\max}} v_{i-q}(z) (\text{ad}_{M_+})^i(T_{-q}) \}, \quad (4.15)$$

where i_{\max} is the largest natural number for which $(\text{ad}_{M_+})^{i_{\max}}(T_{-q}) \neq 0$ (from (4.14) we have $i_{\max} \geq 2q$), and the current components $v_{i-q}(z)$ are arbitrary. In other words, the special configurations $\mathcal{E}[T_{-q}]$ are given by the M_+ -gauge-fixed current (4.5), where all the components including ω vanish, except for a single “ M_+ -string” of u -fields, namely, the $v_{i-q}(z)$ ’s. The point is that the subspace $\mathcal{E}[T_{-q}]$ is invariant under the Möbius transformations (4.8), which act on $I(z) \in \mathcal{E}[T_{-q}]$ as

$$\begin{aligned} \delta_f v_{i-q} &= f v'_{i-q} + (1 + i - q) f' v_{i-q} - \frac{1}{2} f'' v_{i-q-1}, \\ \forall i &= 0, \dots, i_{\max}, \quad (v_{-q-1} = 0). \end{aligned} \quad (4.16)$$

This means that, under this transformation, the notion of quasi-primary differential polynomials is well-defined even when restricted to the subspace $\mathcal{E}[T_{-q}]$.

Let us set $b_q = 0$ or $\frac{1}{2}$ for q integral or half-integral, respectively. Then for any integer $0 \leq k \leq q - b_q - 1$, the most general linear differential expression of scale dimension $k + b_q + 1$ that can be formed from $I(z) \in \mathcal{E}[T_{-q}]$ is given by

$$p_k = A_0 v_{k+b_q} + \sum_{i=1}^{q+k+b_q} A_i \partial^i v_{k+b_q-i}, \quad (4.17)$$

where the A_i are arbitrary constants. We then have the following auxiliary statement.

Lemma. *If the linear differential polynomial p_k in (4.17) is quasi-primary on $\mathcal{E}[T_{-q}]$, then $A_0 = 0$ for $0 \leq k \leq q - b_q - 1$.*

Proof. By computing the Möbius transformation of p_k in (4.17) through (4.16), we obtain

$$\delta_f p_k = f p'_k + (k + b_q + 1) f' p_k + f'' U(p_k), \quad (4.18)$$

where

$$U(p_k) = \frac{1}{2} \sum_{i=1}^{q+k+b_q} [i(2k + 2b_q + 1 - i) A_i - A_{i-1}] \partial^{i-1} v_{k+b_q-i}. \quad (4.19)$$

For p_k to be quasi-primary on $\mathcal{E}[T_{-q}]$ one must have $U(p_k) = 0$ for any current $I(z) \in \mathcal{E}[T_{-q}]$. The observation that the coefficient of A_i in (4.19) vanishes for $i = 2k + 2b_q + 1$ (which occurs for $0 \leq k \leq q - b_q - 1$) leads at once to $A_{2k+2b_q} = A_{2k+2b_q-1} = \dots = A_1 = A_0 = 0$. Q.E.D.

We now prove the main result of the section.

Proposition. *Suppose that there exist a DS gauge and a quasi-primary basis in \mathcal{R} (at the linear level). Then for $q = 1, \frac{3}{2}, 2, \frac{5}{2}, \dots$, Γ must satisfy the following relations:*

$$(\text{ad}_{M_+})^{q+k+b_q} ((\Gamma^\perp)_{-q}) \subset [M_-, \Gamma_{k+b_q+1}], \quad \forall k = 0, 1, \dots, q - b_q - 1, \quad (4.20)$$

where $b_q = 0$ or $\frac{1}{2}$ depending on whether q is integral or half-integral.

Proof. We can transform the M_+ -gauge fixed current $I(z)$ in (4.5) to the fully DS gauge fixed current by a $\tilde{\Gamma}$ -gauge-transformation, and the components of the resulting $j_{\text{DS}}(u, \omega) \in V$ freely generate \mathcal{R} , where V is given in (2.11) which defines the DS gauge. It follows from the differential polynomial nature of the DS gauge fixing that, when decomposed according to (4.7), the linear terms of the components of $j_{\text{DS}}(u, \omega)$, defined by using some graded basis of V , contain the corresponding components of $I(z)$. We also know (cf. Section 2) that the components of $j_{\text{DS}}(u, \omega)$ have definite scale dimensions. From these facts it follows that if (4.20) did not hold for some q and some k , then we could find a component $P(u, \omega) \in \mathcal{R}$ of $j_{\text{DS}}(u, \omega)$ whose linear term $P_1(u, \omega)$ reduces to an expression of the form (4.17) with $A_0 \neq 0$ when restricted to a subspace $\mathcal{E}[T_{-q}]$ defined for a T_{-q} with $(\text{ad}_{M_+})^{q+k+b_q}(T_{-q}) \notin [M_-, \Gamma_{k+b_q+1}]$. On the other hand, if there exists a quasi-primary basis in \mathcal{R} , then this $P(u, \omega)$ can be expressed as a differential polynomial in the basis, and thus $P_1(u, \omega)$ must be a differential linear combination of the quasi-primary linear terms of the basis elements, that is,

$$P_1(u, \omega) = Q_{k+b_q+1}(u, \omega) + Q'_{k+b_q}(u, \omega) + Q''_{k+b_q-1}(u, \omega) + \dots, \quad (4.21)$$

where $Q_i(u, \omega)$ is a linear quasi-primary differential polynomial of scale dimension i . Since $P_1(u, \omega)$ contains the term v_{k+b_q} , $Q_{k+b_q+1}(u, \omega)$ must contain it as well. Clearly, this is a contradiction, because due to the lemma there is no such quasi-primary differential polynomial of the form (4.7) whose linear term contains a nonzero multiple of v_{k+b_q} when restricted to $\mathcal{E}[T_{-q}]$. We therefore conclude that (4.20) must hold. Q.E.D.³

From the above proposition, we easily obtain the following dimensional bounds.

Corollary. *For all $q \geq 1$ and $0 \leq k \leq q - b_q - 1$, we have*

$$\dim \Gamma_q + \dim \Gamma_{k+b_q+1} \geq \dim \mathcal{S}_q. \quad (4.22)$$

Proof. From (4.14) we obtain

$$\begin{aligned} \dim \left[(\text{ad}_{M_+})^{q+k+b_q} ((\Gamma^\perp)_{-q}) \right] &= \dim (\Gamma^\perp)_{-q} = \dim \mathcal{S}_{-q} - \dim \Gamma_q \\ &= \dim \mathcal{S}_q - \dim \Gamma_q. \end{aligned} \quad (4.23)$$

On the other hand, from (4.20) and the nondegeneracy condition (2.16) we have

$$\dim \left[(\text{ad}_{M_+})^{q+k+b_q} ((\Gamma^\perp)_{-q}) \right] \leq \dim [M_-, \Gamma_{k+b_q+1}] = \dim \Gamma_{k+b_q+1}. \quad (4.24)$$

Combining (4.23) and (4.24) we get (4.22). The inequalities (4.1) are recovered upon choosing $k = q - b_q - 1$. Q.E.D.

The relations (4.20) restrict the size of Γ considerably as well as its position in \mathcal{S} with respect to the $sl(2)$ subalgebra \mathcal{S} . To derive (4.20) we only used the requirement that the *linear* terms of the DS gauge fixed current should be expressible in terms of the linear terms of a quasi-primary basis of \mathcal{B} . It is plausible that by carrying on the analysis to the quadratic and higher levels one should obtain further restrictions on Γ from the requirement of the existence of a quasi-primary basis. Unfortunately, it appears at the moment that such an analysis does not yield a clearcut condition on Γ , and for this reason this issue will not be pursued further in this paper.

5. Conformal Spectrum and Decoupling in Noncanonical DS Reductions

Suppose that we construct a \mathscr{W} -algebra by using the DS mechanism but *not* by a canonical DS reduction described in Sect. 2. Suppose also that in the \mathscr{W} -algebra *no negative conformal weight occurs* (in fact, so far we have no example with a negative weight) with respect to L_{M_0} . We shall then show in Sect. 5.1 that the $\Delta \geq \frac{3}{2}$ part of the conformal spectrum, given by the weights of the basis elements in the \mathscr{W} -basis (or quasi-primary basis), is *completely fixed* by the $sl(2)$ subalgebra $\mathcal{S} \subset \mathcal{G}$ associated to the reduction by the theorem of Sect. 3. In other words, the $\Delta \geq \frac{3}{2}$ part of the conformal spectrum is the *same* as for the corresponding $\mathscr{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra obtained by canonical DS reduction. In the subsequent Sects. 5.2 and 5.3 we show by examples that *there do exist noncanonical DS reductions*, where the resultant \mathscr{W} -algebras possess extra “low-lying” weights $\Delta \in \{0, \frac{1}{2}, 1\}$ in addition

³ If Γ is known to be graded by the $sl(2)$ Casimir, then for fixed q the conditions in (4.20) for $k > 0$ all follow from that for $k = 0$

to the canonical conformal spectrum of $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$. However, we find that these \mathcal{W} -algebras are not essentially different from the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebras, since they decouple into the direct product of a $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -subalgebra and a system of free fields. It would be interesting to know whether the decoupling mechanism we exhibit here in specific examples works in other noncanonical DS reductions too. As far as the decoupling of weight $\frac{1}{2}$ fields is concerned, one may expect this to be a general phenomenon in DS reductions by analogy with the general decoupling theorem established in the context of meromorphic conformal field theory by Goddard and Schwimmer [32].

5.1. Conformal Spectrum from the $sl(2)$ Embedding

Consider a \mathcal{W} -algebra resulting from DS reduction. Let V be the graded complementary space defining the DS gauge, given in (2.11). By the theorem of Sect. 3, we can assume that the grading is by the $sl(2)$ generator M_0 . We noted in Sect. 2 (see (2.25)) that the generators of \mathcal{R} provided by the components of $j_{\text{DS}}(j)$ have definite scale dimensions obtained from the grades of the basis of V by a shift by $+1$. This clearly implies that the spectrum of conformal weights in any \mathcal{W} -basis (or quasi-primary basis) of \mathcal{R} , with respect to L_{M_0} , is determined by the spectrum of M_0 -grades in V in the same way.

Let us now consider the case where *no negative conformal weight occurs* in our \mathcal{W} -algebra. Note that we have the equality

$$\dim V_m = \dim \mathcal{S}_m - \dim \Gamma_{-m} - \dim \Gamma_{m+1}, \quad \forall m, \quad (5.1)$$

on account of the decomposition (2.11) and the nondegeneracy condition (2.16). If we combine this equality with the *nonnegativity* assumption,

$$\dim V_m = 0 \quad \text{for} \quad m \leq -\frac{3}{2}, \quad (5.2)$$

then we get the formula

$$\dim V_m = \dim \mathcal{S}_m - \dim \mathcal{S}_{m+1} \quad \text{for} \quad m \geq \frac{1}{2}. \quad (5.3)$$

This tells us that the $\Delta \geq \frac{3}{2}$ sectors of the conformal weights of the generators of our \mathcal{W} -algebra are necessarily *the same* as for the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra, where \mathcal{S} is the $sl(2)$ containing M_0 . Thus the conformal spectrum can be different only for the weight 0 and $\frac{1}{2}$ sectors (which do not exist in the canonical case), and the weight 1 sector. By summing over all the grades in (5.1) and comparing it with the corresponding sum taken for the canonical DS reduction, we derive the formula for the dimension of these sectors,

$$\dim V_{-1} + \dim V_{-\frac{1}{2}} + (\dim V_0 - \dim (V_c)_0) = 2(\dim \Gamma_c - \dim \Gamma). \quad (5.4)$$

Note that the dimension of Γ_c is the maximal one allowed by the first-classness of the constraints and the nondegeneracy condition (2.16), and that $(V_c)_0$ is the space of $sl(2)$ singlets in \mathcal{S} . It is also useful to spell out from (5.1) the dimensions of the extra sectors more explicitly,

$$\begin{aligned} \dim V_0 - \dim (V_c)_0 &= \dim V_{-1} = \dim \mathcal{S}_1 - \dim \Gamma_1 - \dim \Gamma_0, \\ \dim V_{-\frac{1}{2}} &= \dim \mathcal{S}_{\frac{1}{2}} - 2\dim \Gamma_{\frac{1}{2}}. \end{aligned} \quad (5.5)$$

This means that we must have at least as many conformal vectors as in the canonical case, the number of extra conformal vectors equals that of the conformal scalars, and conformal spinors occur whenever $\dim \Gamma_{\frac{1}{2}}$ is smaller than in the canonical case. We next present examples where such extra low-lying fields indeed occur, and shall see that, in those examples, the \mathcal{W} -algebra decouples into the direct product of a subalgebra isomorphic to $\mathcal{W}_{\mathcal{F}}^{\mathcal{G}}$ and extra “free fields” of weight 0, $\frac{1}{2}$ and 1.

5.2. Decoupling of Weight $\frac{1}{2}$ Fields

Consider a *half-integral* $sl(2)$ embedding $\mathcal{S} = \{M_-, M_0, M_+\} \subset \mathcal{G}$. Recall that the canonical first class constraints are defined by Γ_c in (2.28) and restrict the current to the form given in (2.29). In this section we are interested in noncanonical DS reductions that are “marginal modifications” of the canonical DS reduction obtained by removing some of the canonical constraints belonging to grade $\frac{1}{2}$ elements of Γ_c . This means that our modified gauge subalgebra Γ is of the type

$$\mathcal{G}_{\geq 1} \subset \Gamma \subset \Gamma_c, \quad (5.6)$$

and the constraint surface \mathcal{K}_Γ consists of currents of the form

$$J(z) = M_- + j(z), \quad j(z) \in \Gamma^\perp, \quad \text{with} \quad \Gamma^\perp = (\Gamma^\perp)_{-\frac{1}{2}} + \mathcal{G}_{\geq 0}. \quad (5.7)$$

From the sufficient condition (2.21), the gauge group admits the DS gauge fixing (with the grading defined by M_0) and hence the corresponding ring \mathcal{R} is freely generated. It is also clear that $L_{M_0} \in \mathcal{R}$, but it is not obvious whether there exists a \mathcal{W} -basis in \mathcal{R} . However, one sees from (5.6) that if there is a \mathcal{W} -basis in \mathcal{R} then it must contain a subset of generators whose conformal weights coincide with those of the $\mathcal{W}_{\mathcal{F}}^{\mathcal{G}}$ -algebra and $2(\dim \Gamma_c - \dim \Gamma)$ additional conformal spinors (see (5.5)). In fact, below we shall exhibit two subrings, $\mathcal{R}_{\frac{1}{2}}$ and $\widehat{\mathcal{R}}$, in \mathcal{R} , and the section is devoted to proving the following statements:

- (i) The subrings $\mathcal{R}_{\frac{1}{2}}$ and $\widehat{\mathcal{R}}$ are closed (in the usual local sense given in (2.9)) with respect to the induced Poisson bracket carried by \mathcal{R} , and commute with each other under the Poisson bracket.
- (ii) The subring $\mathcal{R}_{\frac{1}{2}} \subset \mathcal{R}$ is freely generated by a basis consisting of weight $\frac{1}{2}$ bosonic free fields.
- (iii) The subring $\widehat{\mathcal{R}}$ is freely generated by a basis subject to the $\mathcal{W}_{\mathcal{F}}^{\mathcal{G}}$ -algebra under the Poisson bracket.
- (iv) The union of the bases of $\mathcal{R}_{\frac{1}{2}}$ and $\widehat{\mathcal{R}}$ gives a basis of \mathcal{R} .
- (v) The Virasoro generator $L_{M_0} \in \mathcal{R}$ is the sum of the Virasoro generators of the subrings $\mathcal{R}_{\frac{1}{2}}$ and $\widehat{\mathcal{R}}$, $L_{M_0} = \mathcal{L}_{\frac{1}{2}} + \widehat{\mathcal{L}}$.
- (vi) The \mathcal{W} -basis of \mathcal{R} is obtained from the decoupled basis in iv) by replacing the Virasoro generator $\widehat{\mathcal{L}}$ of the $\mathcal{W}_{\mathcal{F}}^{\mathcal{G}}$ -algebra carried by $\widehat{\mathcal{R}}$ by $L_{M_0} \in \mathcal{R}$.

Let us begin by considering the subalgebra $\widehat{\Gamma} \subset \mathcal{G}$ given by

$$\widehat{\Gamma} := \widehat{\Gamma}_{\frac{1}{2}} + \mathcal{G}_{\geq 1}, \quad (5.8)$$

where the subspace $\widehat{\Gamma}_{\frac{1}{2}} \subset \mathcal{G}_{\frac{1}{2}}$ is defined by

$$[M_-, \widehat{\Gamma}_{\frac{1}{2}}] = (\Gamma^\perp)_{-\frac{1}{2}}. \quad (5.9)$$

One easily verifies the following relations satisfied by the subalgebras introduced above:

$$[\widehat{\Gamma}, \Gamma] \subset \Gamma, \quad [M_-, \widehat{\Gamma}] \subset \Gamma^\perp, \quad \widehat{\Gamma} \subset \Gamma^\perp, \quad (5.10a)$$

$$\text{Ker} \left(\text{ad}_{M_-} \right) \cap \widehat{\Gamma} = \{0\}, \quad (5.10b)$$

$$\Gamma^\perp = [M_-, \widehat{\Gamma}] + \text{Ker} \left(\text{ad}_{M_+} \right), \quad (5.10c)$$

$$\Gamma \subset \Gamma_c \subset \widehat{\Gamma}. \quad (5.10d)$$

As we shall see shortly, the construction will mainly depend on these relations. Defining

$$\phi_\alpha(z) := \langle \alpha, J(z) \rangle - \langle \alpha, M_- \rangle, \quad \alpha \in \mathcal{G}, \quad (5.11)$$

we see that (5.10a) is equivalent to the equation,

$$\{\phi_{\hat{\gamma}}(z), \phi_{\gamma}(w)\}|_{\mathcal{K}_\Gamma} = 0, \quad \hat{\gamma} \in \widehat{\Gamma}, \quad \gamma \in \Gamma. \quad (5.12)$$

This implies that the KM transformations generated by $\widehat{\Gamma}$,

$$J \longrightarrow \text{Ad}_{e^F} J := e^F J e^{-F} + (e^F)' e^{-F}, \quad F(z) \in \widehat{\Gamma}, \quad (5.13)$$

which contain the gauge transformations corresponding to $F(z) \in \Gamma$, are well-defined on the constraint surface \mathcal{K}_Γ (i.e., preserve the form (5.7)). Therefore we can define $\widehat{\mathcal{R}} \subset \mathcal{R}$ to be the subring consisting of the $\widehat{\Gamma}$ -invariant (invariant under (5.13)) differential polynomials on \mathcal{K}_Γ . It also follows from (5.12) that $\widehat{\mathcal{R}}$ is closed with respect to the induced Poisson bracket carried by \mathcal{R} , i.e., if $T, U \in \widehat{\mathcal{R}}$, then P_{TU}^n in (2.9) belongs to $\widehat{\mathcal{R}}$. Furthermore, by writing $\widehat{\Gamma}$ in the form,

$$\widehat{\Gamma} = \Gamma + \quad, \quad \text{with} \quad \cap \Gamma = \{0\}, \quad (5.14)$$

we obtain from (5.12) that the current components $\phi_\sigma(z)$, $\sigma \in \quad$ are Γ -invariant on \mathcal{K}_Γ and hence belong to \mathcal{R} . We define $\mathcal{R}_{\frac{1}{2}}$ to be the subring of \mathcal{R} generated by these current components. It is easy to see that the induced Poisson bracket closes on $\mathcal{R}_{\frac{1}{2}}$ too in the usual local sense. To finish the proof of statement i), we just note that $\widehat{\mathcal{R}}$ and $\mathcal{R}_{\frac{1}{2}}$ commute with each other under the Poisson bracket since $\widehat{\mathcal{R}}$ consists of $\widehat{\Gamma}$ -invariants, and the current components ϕ_σ , that generate the differential ring $\mathcal{R}_{\frac{1}{2}}$ by definition, generate infinitesimal $\widehat{\Gamma}$ -transformations through the Poisson bracket.

In order to establish ii), we make a concrete choice for the space \quad in (5.14) (the subring $\mathcal{R}_{\frac{1}{2}}$ is easily seen to be independent of the choice). We do this by first choosing a subspace $\mathcal{Q}_{\frac{1}{2}} \subset \mathcal{G}_{\frac{1}{2}}$ on which the 2-form ω_{M_-} vanishes and for which $\mathcal{G}_{\frac{1}{2}} = \mathcal{P}_{\frac{1}{2}} + \mathcal{Q}_{\frac{1}{2}}$, with $\mathcal{P}_{\frac{1}{2}}$ appearing in the definition of Γ_c (2.28). It follows that if we define the subspaces $\mathcal{P}, \mathcal{Q} \subset \widehat{\Gamma}_{\frac{1}{2}}$ by requiring

$$\mathcal{P}_{\frac{1}{2}} = \Gamma_{\frac{1}{2}} + \mathcal{P}, \quad \text{and} \quad \mathcal{Q} := \mathcal{Q}_{\frac{1}{2}} \cap [M_-, \Gamma_{\frac{1}{2}}]^\perp, \quad (5.15a)$$

then we can take

$$:= \mathcal{P} + \mathcal{Q}. \quad (5.15b)$$

These definitions guarantee that we can choose bases $\{X_i\} \subset \mathcal{P}$, $\{Y_i\} \subset \mathcal{Q}$ so that we have

$$\omega_{M_-}(X_i, Y_k) = \delta_{ik}, \quad \omega_{M_-}(X_i, X_k) = \omega_{M_-}(Y_i, Y_k) = 0. \quad (5.16)$$

The corresponding basis of $\mathcal{R}_{\frac{1}{2}}$ is given by the current components

$$p_i(z) := \phi_{X_i}(z) \quad \text{and} \quad q_i(z) := \phi_{Y_i}(z), \quad (5.17)$$

whose Poisson brackets read

$$\begin{aligned} \{p_i(z), q_k(w)\}|_{\mathcal{H}_\Gamma} &= \delta_{ik} \delta(z-w), \\ \{p_i(z), p_k(w)\}|_{\mathcal{H}_\Gamma} &= \{q_i(z), q_k(w)\}|_{\mathcal{H}_\Gamma} = 0. \end{aligned} \quad (5.18)$$

One readily checks that these elements of \mathcal{R} are weight $\frac{1}{2}$ conformal primary fields both with respect to L_{M_0} and with respect to their own free field Virasoro density $\mathcal{L}_{\frac{1}{2}}$ given by

$$\mathcal{L}_{\frac{1}{2}} := \frac{1}{2} \sum_i (p'_i q_i - p_i q'_i). \quad (5.19)$$

Thus we have exhibited the basis of $\mathcal{R}_{\frac{1}{2}}$ claimed in statement ii).

To prove the main statement iii), observe that (5.10b) is the analogue of the earlier nondegeneracy condition (2.16) and (5.10c) is similar to decomposition (2.11) used to define a DS gauge. This suggests that the subspace of currents \mathcal{E}_{hw} given by

$$\mathcal{E}_{\text{hw}} := \left\{ J \mid J(z) = M_- + j_{\text{hw}}(z), \quad j_{\text{hw}}(z) \in \text{Ker} \left(\text{ad}_{M_+} \right) \right\}, \quad (5.20)$$

that defined the highest weight gauge for the canonical DS reduction, is a global, polynomial section of the \widehat{F} -action (5.13) on \mathcal{H}_Γ . This follows if we show that the equation (i.e., the analogue of (2.13)),

$$j \longrightarrow \text{Ad}_{e^F} j := e^F(j + M_-)e^{-F} - M_- + (e^F)'e^{-F} = j_{\text{hw}}, \quad (5.21a)$$

with

$$j(z) \in \Gamma^\perp, \quad F(z) \in \widehat{\Gamma}, \quad j_{\text{hw}}(z) \in \text{Ker} \left(\text{ad}_{M_+} \right), \quad (5.21b)$$

has a unique, differential polynomial solution $F(z) = F(j(z))$. Indeed, if this is so then the resultant $j_{\text{hw}}(j(z))$ is also a differential polynomial in j on account of $\widehat{\Gamma} \subset \mathcal{G}_{>0}$, which implies that $\text{Ad}_{e^F} j$ is a finite differential polynomial in F . The construction then guarantees that the components of $j_{\text{hw}}(j)$ are $\widehat{\Gamma}$ -invariants and freely generate $\widehat{\mathcal{R}}$, in analogy with the way one constructs a basis of gauge invariants through DS gauge fixing. Although we could verify the unique, polynomial solubility of (5.21) directly by a recursive procedure based on the grading similarly as for DS gauge fixing [23], it will be advantageous to solve (5.21) by a two-step procedure utilizing that \mathcal{E}_{hw} (5.20) is a gauge section in the canonical case. In the two-step procedure first we reduce the current $j \in \Gamma^\perp$ to the canonical form $j_c \in \Gamma_c^\perp$ and then employ

the usual DS procedure to the highest weight gauge fixing available for the canonical DS reduction. To implement this, we write the current $j(z) \in \Gamma^\perp$ in the form

$$j(z) = \sum_i p_i(z)[Y_i, M_-] + r(z), \quad r(z) \in \Gamma_c^\perp, \quad (5.22)$$

where $\{Y_i\} \subset \mathcal{Q}$ is the basis introduced earlier, p_i is defined in (5.17), and we used that $\Gamma^\perp = [M_-, \mathcal{Q}] + \Gamma_c^\perp$. Then we see that the first step is implemented by the KM transformation

$$j \longrightarrow \text{Ad}_{e^{-p \cdot Y}} j := j_c(j). \quad (5.23a)$$

In the second step the resultant current $j_c(j) \in \Gamma_c^\perp$ can be brought to the subspace \mathcal{E}_{hw} (5.20) by a unique Γ_c -transformation (which is a particular $\widehat{\Gamma}$ -transformation on account of (5.10d)) since \mathcal{E}_{hw} is known to represent a global gauge section for the canonical DS reduction,

$$j_c \longrightarrow \text{Ad}_{e^{f_c}} j_c := j_{\text{hw}}(j_c), \quad \text{with} \quad f_c \in \Gamma_c. \quad (5.23b)$$

After this two step process, the group element e^F in (5.21) turns out to be

$$e^F = e^{f_c} e^{-p \cdot Y}, \quad (5.24)$$

where $f_c = f_c(j_c(j))$ is a differential polynomial in its argument. This implies the unique solubility of (5.21) for F since the group parameters F and $(f_c, p \cdot Y)$ are related to each other in a one-to-one, polynomial manner on account of the grading. From the unique, polynomial solubility of (5.21) we conclude that the ring $\widehat{\mathcal{R}}$ is freely generated by the components of $j_{\text{hw}}(j) = j_{\text{hw}}(j_c(j))$, whose number is $\dim \mathcal{S} - 2 \dim \Gamma_c$ (notice that the function $j_{\text{hw}}(j_c)$ appearing here is the same as that occurring in the canonical case).

It is now not difficult to see that the Poisson bracket algebra formed by the basis $j_{\text{hw}}(j)$ of the subring $\widehat{\mathcal{R}}$ is isomorphic to the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra. This follows from the very fact that the components of $j_{\text{hw}}(j)$ are $\widehat{\Gamma}$ -invariant and hence commute with the canonical constraints, i.e., $\{\phi_{\gamma_c}(z), j_{\text{hw}}(j(w))\}|_{\mathcal{H}_\Gamma} = 0$ for $\forall \gamma_c \in \Gamma_c$, and from the fact that on the subspace \mathcal{E}_{hw} , $j_{\text{hw}}(j(z))$ reduces to the highest weight gauge current $j_{\text{hw}}(z)$ defined in (5.20). More explicitly, from the first fact we observe that for $j_{\text{hw}}(j(z))$ the Dirac bracket defined for the set of canonical second class constraints specifying the constraint surface $\mathcal{E}_{\text{hw}} \subset \mathcal{K}$ (5.20) is identical to the Poisson bracket,

$$\{j_{\text{hw}}(j(z)), j_{\text{hw}}(j(w))\} = \{j_{\text{hw}}(j(z)), j_{\text{hw}}(j(w))\}^* \quad \text{on} \quad \mathcal{E}_{\text{hw}}. \quad (5.25)$$

Then from the second fact we see that the r.h.s. of (5.25) is equivalent to the Dirac bracket of the current components $j_{\text{hw}}(z)$ entering the definition (5.20),

$$\{j_{\text{hw}}(j(z)), j_{\text{hw}}(j(w))\}^* = \{j_{\text{hw}}(z), j_{\text{hw}}(w)\}^* \quad \text{on} \quad \mathcal{E}_{\text{hw}}. \quad (5.26)$$

As mentioned in Sect. 2, the r.h.s. of (5.26) forms the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra after the change of the basis in which the M_+ -component of $j_{\text{hw}}(z)$ is replaced by the Virasoro density $L_{M_0}(j_{\text{hw}}(z))$. (Here $j_{\text{hw}}(z)$ simply means the current defined on the subspace \mathcal{E}_{hw} (5.20) and is to be distinguished from the function $j_{\text{hw}}(j(z))$ defined on \mathcal{H}_Γ .) By combining the last two equations, and noting that $\{j_{\text{hw}}(j(z)), j_{\text{hw}}(j(w))\}$ is $\widehat{\Gamma}$ -invariant and thus determined by its restriction to the section \mathcal{E}_{hw} , we obtain the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -basis of \mathcal{R} required by statement iii) similarly as in the canonical case. Namely,

we modify the basis provided by the components of $j_{\text{hw}}(j)$ by replacing the M_+ -component of $j_{\text{hw}}(j)$ with the Virasoro density $\widehat{\mathcal{L}}(j) \in \widehat{\mathcal{R}}$ given by

$$\widehat{\mathcal{L}}(j) := L_{M_0}(j_{\text{hw}}(j)) = \frac{1}{2} \langle M_- + j_{\text{hw}}(j), M_- + j_{\text{hw}}(j) \rangle, \quad (5.27)$$

where we observed that $\langle M_0, j_{\text{hw}} \rangle = 0$.

To demonstrate statement iv), we show that $\{p_i, q_i, j_{\text{hw}}(j)\} \subset \mathcal{R}$ is a basis of \mathcal{R} . We do this by showing that any Γ -invariant differential polynomial $P(j) \in \mathcal{R}$ can be expressed as a differential polynomial in this set. For this purpose, it will be useful to decompose the unique solution $F(j) \in \widehat{\Gamma}$ of (5.21) into a sum according to (5.14),

$$F = \epsilon + f, \quad \text{with} \quad \epsilon \in \Gamma, \quad f \in \Gamma. \quad (5.28)$$

By substituting this into (5.21) and inspecting the lowest grade part of this equation, we obtain

$$\epsilon = \sum_i q_i X_i - \sum_i p_i Y_i, \quad (5.29)$$

where p_i, q_i are the gauge invariant components of j defined by (5.17). On account of these equations and $[\Gamma, \Gamma] \subset \Gamma$ which holds for grading reasons, we can write

$$e^F = e^{\epsilon+f} = e^{q \cdot X - p \cdot Y} e^{\tilde{f}}, \quad \text{with} \quad \tilde{f} \in \Gamma, \quad (5.30)$$

where $\tilde{f} = \tilde{f}(\epsilon, f)$ is determined by the Baker-Campbell-Hausdorff formula. We then see by inverting (5.21) using (5.30) that $j \in \Gamma^\perp$ can be written in the form

$$j = \text{Ad}_{e^{-f}}(\text{Ad}_{e^{p \cdot Y - q \cdot X}} j_{\text{hw}}), \quad (5.31)$$

where \tilde{f}, j_{hw} are uniquely determined differential polynomials in j . If now $P(j) \in \mathcal{R}$ is an arbitrary Γ -invariant, then we have

$$P(j) = P(\text{Ad}_{e^{-f}}(\text{Ad}_{e^{p \cdot Y - q \cdot X}} j_{\text{hw}})) = P(\text{Ad}_{e^{p \cdot Y - q \cdot X}} j_{\text{hw}}). \quad (5.32)$$

This implies that the ring \mathcal{R} is indeed generated by the set $\{p_i, q_i, j_{\text{hw}}(j)\} \subset \mathcal{R}$. Of course, the number of the elements in the basis set is

$$\dim \Gamma + \dim \mathcal{S} - 2 \dim \Gamma_c = \dim \mathcal{S} - 2 \dim \Gamma, \quad (5.33)$$

as required. Having proved statement iv) for specific bases of the subrings $\mathcal{R}_{\frac{1}{2}}, \widehat{\mathcal{R}}$ by the above, the statement obviously holds for any two such bases as well.

Concerning statement v), observe first the following chain of the equalities:

$$\begin{aligned} \widehat{\mathcal{L}}(j) &= L_{M_0}(j_{\text{hw}}(j_c(j))) = L_{M_0}(j_c(j)) \\ &= \frac{1}{2} \langle M_- + j_c(j), M_- + j_c(j) \rangle - \langle M_0, j'_c(j) \rangle, \end{aligned} \quad (5.34)$$

where all equalities are due to definitions except the second one, which is due to the Γ_c -gauge-invariance of L_{M_0} on the constraint surface \mathcal{R}_{Γ_c} of the canonical DS reduction. Then, by using (5.19) and (5.23a), it is a matter of direct verification to derive

$$L_{M_0}(j) := \frac{1}{2} \langle M_- + j, M_- + j \rangle - \langle M_0, j' \rangle = \mathcal{L}_{\frac{1}{2}}(j) + \widehat{\mathcal{L}}(j), \quad (5.35)$$

as claimed in statement v).

Finally, since $L_{M_0}(j) \in \mathcal{R}$ is linear in $\widehat{\mathcal{B}} \in \widehat{\mathcal{R}}$, statement vi) is now obvious from the above.

In summary, in this section we have shown that the reduction belonging to Γ (5.6) leads to a \mathcal{W} -algebra that is isomorphic to the direct product of the $\mathcal{W}_{\mathcal{G}}^{\mathcal{G}}$ -algebra with a system of weight $\frac{1}{2}$ bosonic free fields. The number of the (p, q) pairs is $\frac{1}{2} \dim(\mathcal{G}_{\frac{1}{2}})$ in the extreme case when $\Gamma = \mathcal{G}_{\geq 1}$, and 0 in the other extreme case $\Gamma = \Gamma_c$. Obviously, the systems obtained by adding such free fields to the $\mathcal{W}_{\mathcal{G}}^{\mathcal{G}}$ -algebra cannot be considered genuinely new \mathcal{W} -algebras. The above construction whereby we have seen the decoupling mainly depended on the properties collected under (5.10), but at some points also on the specific grading structure of our example. In particular, the fact that Γ in (5.6) differs from Γ_c only by elements in $\mathcal{G}_{\frac{1}{2}}$ is a sufficient condition for the construction to work in general. Nevertheless, this construction could perhaps serve as a “prototype” in a more general study of noncanonical DS reductions of $\Gamma \subset \Gamma_c$ type (we have no other kind of noncanonical example). Although the range of validity of this type of construction is not clear, it is certainly not restricted to the above family of examples, as is illustrated by a new example in the next section.

5.3. Decoupling of Weight (0, 1) Fields

The modifications of the canonical DS reductions described in the previous section were obtained by removing some of the canonical constraints belonging to lowest grade elements of Γ_c in the case of a half-integral $sl(2)$ embedding. In the case of an integral $sl(2)$ embedding the same idea cannot be applied *in general*, since the DS gauge fixing would not be applicable for the modified system of constraints. There are however particular cases where the idea works, and we here present a simple example based on the $sl(2)$ subalgebra of the Lie algebra B_2 belonging to a short root. We shall see that the modified reduction leads to a \mathcal{W} -algebra that decouples into the direct product of the corresponding $\mathcal{W}_{\mathcal{G}}^{\mathcal{G}}$ -algebra and a (p, q) pair of free fields with conformal weights (0, 1), quite analogously to what we have seen in Sect. 5.2.

The root diagram of the Lie algebra B_2 consists of the vectors

$$\pm e_1, \quad \pm e_2, \quad \pm(e_1 \pm e_2). \quad (5.36)$$

The algebra is spanned by the step operators and the Cartan elements,

$$E_{\pm e_1}, \quad E_{\pm e_2}, \quad E_{\pm(e_1 \pm e_2)}, \quad H_{e_1}, \quad H_{e_2}, \quad (5.37)$$

which we normalize by $[H_{e_i}, E_{e_i}] = E_{e_i}$. We consider the $sl(2)$ subalgebra belonging to the short root e_1 ,

$$M_{\pm} := E_{\pm e_1}, \quad M_0 := H_{e_1}. \quad (5.38)$$

For the corresponding canonical DS reduction we have

$$\Gamma_c = \text{span} \{E_{e_1+e_2}, E_{e_1}, E_{e_1-e_2}\}, \quad (5.39)$$

and

$$\text{Ker} \left(\text{ad}_{M_+} \right) = \text{span} \{E_{e_1+e_2}, E_{e_1}, E_{e_1-e_2}, H_{e_2}\}. \quad (5.40)$$

The adjoint representation decomposes under the $sl(2)$ according to $10 = 3 \times 3 + 1$. The first class constraints of the modified reduction are determined by the pair (Γ, M) where we define $M := M_-$ and

$$\Gamma := \text{span} \{E_{e_1+e_2}, E_{e_1}\}. \quad (5.41)$$

One can directly check that the DS gauge fixing is applicable in this case. To see the structure of the reduced system we proceed analogously as in Sect. 5.2. We define

$$\widehat{\Gamma} := \Gamma + \quad , \quad \text{with} \quad := \text{span} \{E_{e_1-e_2}, E_{e_2}\}, \quad (5.42)$$

and then the analogues of the relations in (5.10) are satisfied. By using these relations we can verify also in this case that the Poisson bracket algebra carried by the ring, \mathcal{R} , of Γ -invariant differential polynomials decouples into the direct product of the $\mathcal{W}_{\mathcal{G}}^{\mathcal{S}}$ -subalgebra carried by the subring, $\widehat{\mathcal{R}}$, of $\widehat{\Gamma}$ -invariants, and the Γ -invariant current components

$$p(z) := \frac{1}{\sqrt{2}} \langle E_{e_1-e_2}, J(z) \rangle \quad \text{and} \quad q(z) := \frac{1}{\sqrt{2}} \langle E_{e_2}, J(z) \rangle \quad (5.43)$$

generating another subring, $\mathcal{R}_{(0,1)}$. On the constraint surface defined by Γ , these current components satisfy the analogue of (5.18) (since $\langle M_-, [E_{e_1-e_2}, E_{e_2}] \rangle = 2$ in our convention). The notation $\mathcal{R}_{(0,1)}$ reflects the fact that in this case p is a conformal scalar and q is a conformal vector with respect to $L_{M_0} \in \mathcal{R}$. These conformal weights are assigned to the pair (p, q) by the quadratic Virasoro density given by

$$\mathcal{L}_{(0,1)} := p'q, \quad (5.44)$$

and $L_{M_0} \in \mathcal{R}$ decomposes into the sum of this Virasoro density and that of the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -subalgebra, similarly as for the weight $\frac{1}{2}$ fields in Sect. 5.2.

6. Discussion

The purpose of the present paper was to gain a better understanding of the DS reduction approach to classical \mathcal{W} -algebras in general, and in particular to investigate the completeness of the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebras in the set of \mathcal{W} -algebras that may be obtained from DS reductions. On the basis of the definition given in Sect. 2, we proved in Sect. 3 that all DS reductions can be determined by triples of the form $(\Gamma, M = M_-, H = M_0)$, where $\mathcal{S} = \{M_-, M_0, M_+\}$ is an $sl(2)$ subalgebra of the underlying simple Lie algebra \mathcal{G} and $M_+ \in \Gamma$. This way we reduced the problem of listing all DS reductions to the problem of finding all possible Γ 's for given $sl(2)$ embeddings (whose classification is known). Then we went on to exhibit restrictions on the allowed Γ 's in Sect. 4. The basic idea there was that by inspecting the expansion of the DS gauge fixed current and requiring term by term that it be compatible with the existence of a quasi-primary basis in \mathcal{R} one obtains conditions on Γ . We completed the analysis only at the linear level, but it should be possible to pin down Γ more closely by analysing the quadratic and higher order terms of the expansion. We also wish to emphasize at this point that the linear conditions on Γ given by the proposition in Sect. 4.2 are to be combined with the requirements imposed on Γ by the first classness of the constraints together with the severe restriction for the existence of a DS gauge. All in all, we think Γ is already very much constrained by these conditions, but further study would be needed to have the allowed Γ 's under complete control, ideally by deriving their list.

We left the previous train of thoughts in Sect. 5 to some extent. We there first showed that if the conformal weight spectrum resulting from a DS reduction is nonnegative then its $\Delta \geq \frac{3}{2}$ subsector is necessarily the same as that of the

corresponding $\mathcal{W}_\mathcal{G}$ -algebra. We then found examples of new, noncanonical DS reductions, which in principle yield new \mathcal{W} -algebras for which extra low-lying weights $\Delta \in \{0, \frac{1}{2}, 1\}$ do occur. However, in the examples we also found a mechanism whereby the resulting \mathcal{W} -algebras were identified as direct products of $\mathcal{W}_\mathcal{G}$ -algebras and systems of free fields, i.e., they turned out to be not essentially new.

Our theorem on an $sl(2)$ embedding being associated to every DS reduction and our result on the conformal weights are consistent with the more abstract results in [31] where an embedding of the Möbius $sl(2)$ into a finite Lie algebra was associated to every classical \mathcal{W} -algebra with positive, half-integral conformal spectrum by using completely different methods. More precisely, in [31] the classical \mathcal{W} -algebra was viewed as the limit of a quantum one. This led to some unnecessarily restrictive assumptions, which we removed in a recent paper [33]. But, even taking this into account, the assumptions in [31] and in the present work are different. For instance, refs. [31, 33] exclude conformal scalars (and spinors), which are some of the free fields that occur in our examples. It is known that our “DS $sl(2)$ embedding” and the “Möbius $sl(2)$ embedding” of [31] are isomorphic for the *canonical* DS reductions [31, 33], but it is not clear that they are isomorphic for *all* possible noncanonical DS reductions which are the cases in which we are interested here. The exact relationship between the results in [31, 33] and the present paper will be clear when a more complete classification of \mathcal{W} -algebras and DS reductions becomes available.

Pending such a complete classification, the results derived in this paper give a strong support to the *conjecture that the set of \mathcal{W} -algebras with nonnegative spectra $\Delta \geq 0$ that may be obtained from DS reductions is exhausted by the $\mathcal{W}_\mathcal{G}$ -algebras and decoupled systems consisting of $\mathcal{W}_\mathcal{G}$ -algebras and systems of free fields.* On the basis of the results in [31, 33], it is also natural to ask whether the $\mathcal{W}_\mathcal{G}$ -algebras are exhaustive even outside the DS approach.

We wish close this paper by mentioning some other open questions related to DS reductions, and to the above conjecture. First, let us recall that the definition of the classical \mathcal{W} -algebra (and that of the DS reduction) assumes a preferred Virasoro density. In view of the notion of isomorphism between classical \mathcal{W} -algebras, we are naturally led to the following basic questions:

1. Are there nontrivial possibilities for finding two \mathcal{W} -bases, $\{W_a\}$ and $\{\tilde{W}_a\}$, both freely generating an invariant ring \mathcal{R} , such that the weights Δ_a , $\tilde{\Delta}_a$, and the centres c , \tilde{c} , relative to W_1 and \tilde{W}_1 respectively, are not identical?
2. Are there “accidental isomorphisms” between $\mathcal{W}_\mathcal{G}$ -algebras belonging to group theoretically *inequivalent* $sl(2)$ embeddings?

We note here that the conformal structure is not unique in a rather trivial way in the cases where the \mathcal{W} -basis contains a (p, q) pair, $\{p(z), q(w)\} = \delta(z - w)$, decoupled from the rest, since one can assign conformal weights $(h, 1 - h)$ to the pair with any h by building an appropriate quadratic Virasoro density out of p, q .

Second, we used in Sects. 3 and 4 the notion of a quasi-primary basis, which is *in principle weaker* than the notion of \mathcal{W} -basis, and derived conditions from its existence by looking only at the *linear* part of the ring \mathcal{R} . Then in Sect. 5 we derived results from the assumption that the conformal spectrum is nonnegative, and constructed noncanonical DS reductions which were all found to lead to decoupled systems containing a $\mathcal{W}_\mathcal{G}$ -subalgebra, but we made no attempt to establish these results more generally. In fact, these questions are open:

3. What is the full set of conditions implied by the existence of a \mathcal{W} -basis in the invariant ring \mathcal{R} ?
4. Does every \mathcal{W} -algebra obtained from a noncanonical DS reduction contain a $\mathcal{W}_{\mathcal{F}}^{\mathcal{G}}$ -subalgebra? If it is so, is such an algebra always “completely reducible”?
5. Do DS reductions exist with *negative* conformal weights occurring in the \mathcal{W} -basis of \mathcal{R} with respect to L_{M_0} ?

Third, the existence of a DS gauge is the only *sufficient* condition we are aware of whereby one can guarantee the invariant ring \mathcal{R} to be freely generated. In fact, we have no nontrivial example for \mathcal{R} being freely generated without the applicability of DS gauge fixing. Hence we should ask the following question:

6. Are there other sufficient conditions than the existence of a DS gauge for ensuring that the invariant ring \mathcal{R} is freely generated?

We also wish to note that in most KM reductions by first class constraints \mathcal{R} may not be freely generated, simply by a genericity argument. (We explicitly show the non-existence of a free generating set for the examples in the Appendix.) Moreover, if the reduction is by conformally invariant first class constraints then \mathcal{R} may in general be generated by invariants that include a Virasoro density and are subject to differential polynomial relations. Thus there is a large set of extended conformal algebras built on generating fields obeying differential–algebraic constraints that one may derive from KM reduction, and it is an open question whether one can or cannot make sense of quantum versions of such algebras.

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Note added. After the first version of this paper was submitted there appeared a preprint [35] containing a decoupling algorithm and ref. [36] dealing with an example of classical coset construction where the analogue of the ring \mathcal{R} is *infinitely* generated. With regard to an aspect of the $sl(2)$ structure, we also wish to mention refs. [37, 38].

Appendix: W_{2n}^2 Examples of Nonfreely Generated Rings

In this appendix we consider the W_{2n}^2 -algebras of [29] and show by inspection that the corresponding ring \mathcal{R} is *not freely generated*. These examples illustrate the difficulties one has to face in general if one wants to describe the structure of the invariant ring for KM reductions for which DS gauges do not exist. (These difficulties appear similar to the ones encountered in the general case of the GKO coset construction [3]).

As discussed in [30], the W_{2n}^2 -algebras can be obtained by reducing the KM algebra of $\mathcal{G} = sl(2n)$ by first class constraints of type (2.5) with (Γ, M) being the following. Consider the $sl(2)$ subalgebra $\mathcal{S} = \{M_-, M_0, M_+\} \subset sl(2n)$ under which the $2n$ dimensional representation decomposes into $2n = n + n$, and note that the singlets of \mathcal{S} in the adjoint of $sl(2n)$ form another $sl(2)$ subalgebra $\sigma = \{m_-, m_0, m_+\} \subset sl(2n)$. The gauge algebra Γ of the required first class constraints is given by the semidirect sum

$$\Gamma = \text{span}\{m_+\} \oplus_s \Gamma_c, \quad ([m_+, \Gamma_c] \subset \Gamma_c), \quad (\text{A.1})$$

where $\Gamma_c \subset sl(2n)$ is the canonical subalgebra (2.28) belonging to \mathcal{S} , and $M = M_-$. The DS gauge fixing is not applicable to these first class constraints since, on account

of $[M_-, m_+] = 0$, the nondegeneracy condition (2.16) is not satisfied. More precisely, the $\Gamma_c \subset \Gamma$ part of the gauge freedom can still be fixed in the usual differential polynomial way, and after doing this [30] the independent components of the partially gauge fixed current may be displayed as

$$\begin{pmatrix} e & b \\ 0 & -e \end{pmatrix}, \quad \begin{pmatrix} S_i + E_i & B_i \\ C_i & S_i - E_i \end{pmatrix}, \quad i = 1, \dots, (n-1). \quad (\text{A.2})$$

Except S_1 that enters the Virasoro $L := L_{M_0} = e^2 + S_1$ linearly, all these fields are primary; e, b have conformal weight 1, the fields with index i have conformal weight $(i+1)$. These components are differential polynomials in the original first class constrained current, since they were obtained by applying the standard DS gauge fixing to the Γ_c gauge freedom. It also follows [30] that the components

$$e, \quad S_i, \quad C_i, \quad (\text{A.3})$$

are invariant under the residual gauge transformations generated by $m_+ \in \Gamma$, while the rest transforms according to

$$\begin{aligned} E_i &\longrightarrow E_i + \alpha C_i, \\ B_i &\longrightarrow B_i - 2\alpha E_i - \alpha^2 C_i, \\ b &\longrightarrow b - 2\alpha e + \alpha'. \end{aligned} \quad (\text{A.4})$$

Thus the problem of finding a generating set for \mathcal{R} is reduced to the problem of finding a generating set for the differential polynomial invariants in the components in (A.2) under the very simple gauge transformation rule (A.4). We below investigate this problem by using the following observations. First, notice that \mathcal{R} is *graded by scale dimension*, i.e., the homogeneous pieces with respect to scale dimension belong to \mathcal{R} separately for any element of \mathcal{R} . One sees this for example from the fact that the gauge transformation (A.4) preserves scale dimension for scalar α . (One could identify \mathcal{R} as a certain factor-ring, and see its being graded by scale dimension from that too.) Thus it is natural to look for a *homogeneous* generating set in \mathcal{R} , i.e., one consisting of elements having definite scale dimensions. Second, because the basic ingredients in (A.2) from which the elements of \mathcal{R} are constructed have positive scale dimensions, one sees that \mathcal{R} is *positively graded*, and the subspaces of \mathcal{R} with fixed scale dimension have finite dimension.

This implies that if we want to select a homogeneous generating set, we can proceed by starting from the elements of lowest scale dimension in \mathcal{R} and include at each scale dimension a minimal set of elements in the generating set in such a way that the elements of \mathcal{R} up to that scale dimension are differential polynomials in these elements and the elements of lower scale dimension. We can implement this procedure by inspection up to some finite scale dimension. On the other hand, if there is a basis (free generating set) in \mathcal{R} then the number of basis elements cannot be greater than the number of degrees of freedom in the reduced system (obtained by simple counting). Hence we can conclude the *nonexistence of a homogeneous basis* in \mathcal{R} either (a) if we have collected as many generators as the number of reduced degrees of freedom and then exhibit an element of \mathcal{R} that cannot be expressed as a differential polynomial in these generators, or (b) if we find relations between the selected generators after having completed the selection up to a given scale dimension. By using this reasoning, we shall find that in the cases we consider the ring \mathcal{R} *does not admit a homogeneous basis*. As a consequence, it does not admit a \mathcal{W} -basis, since that would be a particular homogeneous basis. We think the nonexistence of a

homogeneous free generating set implies that \mathcal{R} does not admit *any* free generating set, but this will not be shown here. (We should note that the nature of the generating set of \mathcal{R} has not been investigated so far, although the analogues of the above first class constraints and a differential rational gauge fixing procedure were given in [30] for W_k^l in general.) We first consider the simplest case $n = 2$, i.e., W_4^2 .

i) The case W_4^2 . In this case the reduced system contains 5 degrees of freedom since the number of fields in (A.2) is now 6 and we have the one parameter gauge freedom (A.4). It is clear that the 3 gauge invariant components e , S , C must be included in the generating set of \mathcal{R} we are looking for. (We suppress the index i in (A.2-4), which in the present case takes only the value 1.) The next simplest gauge invariants will be obtained by means of the rational gauge fixing

$$E \longrightarrow E + \alpha C = 0 \quad \Longrightarrow \quad \alpha = -\frac{E}{C}. \quad (\text{A.5})$$

By plugging back this value of the gauge parameter into (A.4), we obtain the following 2 differential rational gauge invariants:

$$\begin{aligned} B &\longrightarrow (E^2 + BC)/C := R_1, \\ b &\longrightarrow (bC^2 + 2eEC + (EC' - E'C))/C^2 := R_2. \end{aligned} \quad (\text{A.6})$$

By a similar argument used for a DS gauge fixed current (see (2.14)), it is easy to see that it is possible to express all differential rational invariants, and thus also the elements of \mathcal{R} since polynomials are special rationals, as differential rational functions in the components of the gauge fixed current resulting from the rational gauge fixing. In particular, observing that the denominators in (A.6) are invariants themselves, we obtain the elements of \mathcal{R} given by the numerators of R_1 , R_2 ,

$$X := E^2 + BC, \quad P := bC^2 + 2eEC + (EC' - E'C). \quad (\text{A.7})$$

By inspection, it is not hard to see that there are in fact no simpler (i.e., ones with lower scale dimension) elements of \mathcal{R} in terms of which we could express X and P , which contain B and b , respectively. From this and the fact that the number of degrees of freedom is 5, we conclude that either the set $\{e, L, C, X, P\}$ is a homogeneous basis for \mathcal{R} , or otherwise there is no such free generating set in \mathcal{R} . (If this was a free generating set then it was also a \mathcal{W} -basis. For this reason we exchanged the generator S for the Virasoro $L = e^2 + S$, which is obviously allowed.)

Let us next observe that the following combination of the rational invariants

$$\begin{aligned} K := C^3 R_2^2 - (C')^2 R_1 &= b^2 C^3 + 4ebEC^2 + 4e^2 E^2 C + 2bC(EC' - E'C) \\ &\quad + 4e(E^2 C' - EE'C) - B(C')^2 \end{aligned} \quad (\text{A.8})$$

is a differential *polynomial* belonging to \mathcal{R} . Since only P contains b in the combination bC^2 , we see that $K \in \mathcal{R}$ cannot be expressed as a differential polynomial in the set $\{e, L, C, X, P\}$. Therefore \mathcal{R} does not admit a (homogeneous) free generating set. Consistently, if we now say that the generating set of \mathcal{R} will be $\{e, L, C, X, P, K, \dots\}$ then we receive 1 differential polynomial relation between the first 6 generators,

$$P^2 - KC - (C')^2 X = 0. \quad (\text{A.9})$$

Observe also that this generating set will not consist of a Virasoro and primary fields, since K in (A.8) is not a primary field. Moreover, it should be stressed that we have no argument even for the existence of a *finite* generating set in \mathcal{R} ! We are not sure

if a finite generating set exists in this case or not, but it is conceivable for example that if we consider the elements of \mathcal{R} only up to some finite scale dimension, then we find a generating set for that part consisting of g elements with r relations in such a way that $g - r$ is always 5, but both g and r tend to ∞ as we increase the scale dimension.

One can actually see already from the W_4^2 case that \mathcal{R} is never freely generated for W_{2n}^2 because a similar argument may be applied to those cases, too. But it is worth also having a closer look at the next case, where this can be seen even without considering such a “tricky object” as K above (the “trick” there being the cancellation of the terms proportional with $1/C$, that are present before the subtraction in (A.8)).

ii) The case W_6^2 . The number of reduced degrees of freedom given by simple counting is in this case 9. We have now 5 linear invariants in (A.3). By using rational gauge fixing or just looking at the transformation rule of the capital letters in (A.4), we obtain the following 4 quadratic invariants:

$$\begin{aligned} X_1 &:= E_1^2 + B_1 C_1, & X_{12} &:= 2E_1 E_2 + B_1 C_2 + C_1 B_2, \\ X_2 &:= E_2^2 + B_2 C_2, & Y &:= C_1 E_2 - C_2 E_1. \end{aligned} \quad (\text{A.10})$$

It is clear that we have to include all the above linear and quadratic invariants in the generating set of \mathcal{R} . This already implies that \mathcal{R} cannot be freely generated since we can verify the relation

$$Y^2 - C_1^2 X_2 - C_2^2 X_1 + C_1 C_2 X_{12} = 0. \quad (\text{A.11})$$

Let us nevertheless continue the selection of the generating set a bit further. So far we have 9 generators and 1 relation and none of the generators we already have contains the component b . The simplest invariants involving b are the following 3 analogues of P in (A.7),

$$\begin{aligned} P_1 &:= bC_1^2 + 2eC_1 E_1 + (C_1' E_1 - C_1 E_1'), \\ P_2 &:= 2bC_1 C_2 + 2e(E_1 C_2 + E_2 C_1) + (C_1' E_2 + C_2' E_1 - C_1 E_2' - C_2 E_1'), \\ P_3 &:= bC_2^2 + 2eC_2 E_2 + (C_2' E_2 - C_2 E_2'). \end{aligned} \quad (\text{A.12})$$

It is easy to see that we also have to include these 3 invariants in the generating set of \mathcal{R} we are looking for. Together with these 3 generating elements we receive also 2 new relations:

$$\begin{aligned} C_1 C_2 P_2 - C_2^2 P_1 - C_1^2 P_3 + (C_1 C_2' - C_1' C_2) Y &= 0, \\ C_1^2 P_3 - C_2^2 P_1 + (C_1 C_2) Y' - (C_1 C_2') Y - 2eC_1 C_2 Y &= 0. \end{aligned} \quad (\text{A.13})$$

Thus the counting of degrees of freedom, $9 = 12 - 3$, is “correct” at this stage, though the set

$$\{e, L, C_1, S_2, C_2, X_1, X_{12}, Y, X_2, P_1, P_2, P_3\}, \quad (\text{A.14})$$

where we exchanged S_1 for L , is not a generating set of \mathcal{R} . Indeed, in addition to invariants like K in (A.8), one may check that for example the following elements of \mathcal{R} cannot be expressed as differential polynomials in this set,

$$\begin{aligned} T_1 &:= C_2 [bC_1^2 + (C_1' E_1 - C_1 E_1')]^2 + 4eC_1^2 E_2 [bC_1^2 + (C_1' E_1 - C_1 E_1')] \\ &\quad - 4e^2 C_1^4 B_2, \\ T_2 &:= C_1 [bC_2^2 + (C_2' E_2 - C_2 E_2')]^2 + 4eC_2^2 E_1 [bC_2^2 + (C_2' E_2 - C_2 E_2')] \\ &\quad - 4e^2 C_2^4 B_1. \end{aligned} \quad (\text{A.15})$$

By adding these 2 invariants to the generating set, we also receive 2 new relations, and it is not clear to us if the procedure would terminate at a certain higher scale dimension or not. The only firm conclusion we can draw from the above is that \mathcal{R} is not freely generated for W_6^2 and that the generating set (whatever it is) is pretty complicated.

iii) Further remarks. Some further remarks are now in order. First, the analysis given above clearly implies that \mathcal{R} is not freely generated also for any W_{2n}^2 . For example, one can see this for any $n \geq 3$ simply by looking only at the corresponding linear and quadratic invariants. The analogous statement is likely to be true for any W_k^l ($1 < l < k$), except the cases W_{2n+1}^2 which coincide with particular $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebras for $\mathcal{S} = sl(2n+1)$. More generally, we may expect the structure of the invariant ring \mathcal{R} to be similarly complicated for a generic KM reduction by first class constraints. (For further study of the structure of these complicated rings, one can find references on the mathematical literature on differential rings in [34].)

Finally, we wish to note that the above considered reduction of the KM algebra by first class constraints can be naturally reinterpreted as the following two-step reduction procedure [30]. The first step consists in reducing the KM algebra to the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra with $\mathcal{S} = sl(2n)$ and \mathcal{S} given at the beginning of the appendix. The second step consists in further reducing the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra by using its sub-KM algebra given by the singlets (see also the remark at the end of Sect. 2.2.). This sub-KM algebra is now just the $sl(2)$ KM algebra of the components belonging to $\sigma = \{m_-, m_0, m_+\} \subset \text{Ker}(\text{ad}_{M_+})$. The “secondary reduction” of the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra (i.e., the second step of the KM reduction) has been defined by putting the m_+ -component – the lower-left entry of the first matrix in Eq. (A.2) – to the degenerate 0 value. The reader might wonder what happens if one puts that component to 1, rather than 0, which means that one would perform a DS reduction on the singlet sub-KM algebra as the secondary reduction of the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra. In fact, one can verify that this “DS reduction after DS reduction” gives no new kind of algebra; it leads just to the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebra, where $\mathcal{S} = \mathcal{S} + \sigma$, with the sum applied to the $sl(2)$ generators. Clearly, this has a natural generalization, that is, DS reductions of $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ -algebras to other canonical algebras as far as there is a semisimple part in the singlet KM to perform a secondary DS reduction. We leave this for a future study.

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