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Cyclic Monodromy Matrices for $sl(n)$ Trigonometric R-Matrices

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Abstract: The algebra of monodromy matrices for $sl(n)$ trigonometric R-matrix is studied. It is shown that a generic finite-dimensional polynomial irreducible representation of this algebra is equivalent to a tensor product of L-operators. Cocommutativity of representations is discussed and intertwiners for factorizable representations are written through the Boltzmann weights of the $sl(n)$ chiral Potts model.

Introduction

Let us consider an algebra generated by noncommutative entries of the matrix $T(u)$ satisfying the famous bilinear relation originated from the quantum inverse scattering method [13, 20]

$$R(\lambda-\mu)T(\lambda)T(\mu)=T(\mu)T(\lambda)R(\lambda-\mu)\,,$$

where $R(\lambda)$ is R-matrix – a solution of the Yang–Baxter equation. For historical reasons this algebra is called the algebra of monodromy matrices. It possesses a natural bialgebra structure with the coproduct (1.5). If $\mathfrak{g}$ is a simple finite-dimensional Lie algebra and $R(\lambda)$ is a $\mathfrak{g}$-invariant R-matrix the algebra of monodromy matrices after a proper specialization gives the Yangian $Y(\mathfrak{g})$ introduced by Drinfeld [11]. If $R(\lambda)$ is the corresponding trigonometric R-matrix [2, 14] (see (1.1) for $sl(n)$ case) this algebra is closely connected with $U_q(\mathfrak{g})$ and $U_q(\hat{\mathfrak{g}})$ at zero level [11, 14, 15, 22, 23]. In the last case it is convenient to use a new variable $u=\exp\lambda$ rather than λ. If $R(\lambda)$ is $sl(2)$ elliptic R-matrix [1, 5] the algebra of monodromy matrices gives rise to Sklyanin's algebra [24].

In this paper we shall study algebras of monodromy matrices for $sl(n)$ trigonometric R-matrices [6, 19, 21]. In the framework of the quantum inverse scattering method finite-dimensional irreducible representations of these algebras which depend polynomially on the spectral parameter u are of special interest. They correspond to integrable models on a finite lattice. L-operators are irreducible representations with linear dependence on the spectral parameter, and usually we get a polynomial representation as a tensor product of L-operators. The question is to examine whether all finite-dimensional polynomial irreducible representations can be obtained in this way. For the $sl(2)$ case corresponding to the R-matrix of the six-vertex model the answer is known. If ω is generic then each wanted representation is equivalent to a tensor product of L-operators [27, 28]. If ω is a root of 1 the situation is more complicated. In this case only generic representations are equivalent to tensor products of L-operators, but there also exist representations, which are not of this form [28]. For generic ω in the $sl(n)$ case finite-dimensional irreducible representations were described in [7, 12], but to obtain all of them from L-operators the notion of an L-operator should be generalized. Here we study the $sl(n)$ case for ω being a root of 1 and obtain the same results as for the $sl(2)$ case [28].

As is well known, the deformation parameter being a root of 1 is a peculiar case for quantum groups [8]. It is the same for algebras of monodromy matrices under consideration if $\omega^N = 1$. In this case a generic polynomial finite-dimensional irreducible representation is cyclic (without highest and lowest vectors). Moreover, as usual irreducible representations do not cocommute; their tensor products in direct and inverse orders are not equivalent in contrast to what takes place for generic ω. The whole set of irreducible representations exfoliate to varieties of cocommuting representations. For a couple of cocommuting representations one can define an intertwiner realizing an equivalence of two tensor products. Intertwiners give us solutions of the Yang–Baxter equation, representations playing a role of spectral parameters. In the $sl(2)$ case an intertwiner for L-operators can be written as a product of four factors and each of them can be expressed explicitly through the Boltzmann weights of the chiral Potts model [4, 28]. A direct generalization of this construction for the $sl(n)$ case leads to the $sl(n)$ chiral Potts model [3] and minimal representations of $U_q(\widehat{gl}(n))$ [9]. Unfortunately, minimal L-operators from [3] (which correspond to minimal representations of $U_q(\widehat{gl}(n))$ [9]) are not generic from the point of view of this paper. For a generic L-operator if the necessary factorization exists it contains n factors instead of two factors for a minimal one, so an intertwiner is a product of n^2 factors. But explicit expressions for these factors can be written through the same Boltzmann weight of the $sl(n)$ chiral Potts model. Recently, another factorization for a generic L-operator was obtained and the corresponding formula for an intertwiner was written by use of the same Boltzmann weight [16].

The paper is organized as follows. In the first section we give definitions and formulate results without proofs. The next two sections contain proofs of Theorems 1, 2. In the fourth section we introduce factorized L-operators and build their intertwiners; the connection with the $sl(n)$ chiral Potts model is also discussed. In the last sections we give technical details and necessary proofs. Some proofs which can be done by explicit calculation are omitted.

1. The Algebra of Monodromy Matrices

Let us define an algebra of monodromy matrices for the $sl(n)$ trigonometric R-matrix. Denote for short $\mathcal{M} = \operatorname{End}\mathbb{C}^n$. The R-matrix $R(u)$ is considered as an element of $\mathcal{M}^{\otimes 2}$ and has the following nonzero entries:

$$R^{ii}_{ii}(u) = 1 - u\omega\,,$$

$$R^{ij}_{ij}(u) = \omega_{ij}(1-u), \quad R^{ij}_{ji}(u) = u^{\theta_{ij}}(1-\omega), \quad i \neq j\,, \tag{1.1}$$

where $\theta_{ij} = \begin{cases} 1, & i<j \\ 0, & i \geqq j \end{cases}$, $\omega_{ij}\omega_{ji} = \omega^{1+\delta_{ij}}$ and δ_{ij} is the Kronecker symbol. We also introduce a tensor ε such that $\omega_{ij} = \omega^{\varepsilon_{ij}}$. This definition of $R(u)$ differs slightly from the original one [6, 21]. A variable u is called the spectral parameter. $R(u)$ satisfies the Yang–Baxter equation:

$$\overset{12}{R}(u)\overset{13}{R}(uv)\overset{23}{R}(v) = \overset{23}{R}(v)\overset{13}{R}(uv)\overset{12}{R}(u)\,.$$

Here we use the standard matrix notations, the superscripts indicating the way of embedding $\mathcal{M} \subset \mathcal{M}^{\otimes 3}$ as corresponding factors.

Definition 1.1. *The algebra of monodromy matrices $\mathcal{A}$ is an associative algebra defined by generators $T_{ij}(u)$, H_i, $i,j = 1, \ldots, n$ and relations*

$$R(u)\overset{1}{T}(uv)\overset{2}{T}(v) = \overset{2}{T}(v)\overset{1}{T}(uv)R(u)\,, \tag{1.2}$$

$$[\hat{\omega}_l \otimes H_l, T(u)] = 0, \quad \hat{\omega}_l = \mathrm{diag}(1, \ldots, \underset{l\text{-th}}{\omega}, \ldots, 1)\,,$$

$$H_iH_j = H_jH_i, \quad \prod_l H_l = 1\,, \tag{1.3}$$

where $T(u) \in \mathcal{M} \otimes \mathcal{A}$ with entries $T_{ij}(u) \in \mathcal{A}$.

Here and later $\prod_l \equiv \prod_{l=1}^n$ and the same convention is implied for sums. A more explicit form of Eq. (1.3) is

$$H_lT_{ij}(u) = T_{ij}(u)H_l\omega^{\delta_{lj}-\delta_{li}}\,. \tag{1.4}$$

One can introduce the natural coproduct $\Delta\colon \mathcal{A} \to \mathcal{A}^{\otimes 2}$:

$$\Delta(T(u)) = T_1(u)T_2(u) \in \mathcal{M} \otimes \mathcal{A}^{\otimes 2}\,,$$

$$\Delta(H_l) = H_l \otimes H_l \tag{1.5}$$

(subscripts indicate the way of embedding $\mathcal{A} \subset \mathcal{A}^{\otimes 2}$) and counit $\varepsilon\colon \mathcal{A} \to \mathbb{C}$:

$$\varepsilon(T(u)) = I, \quad \varepsilon(H_l) = 1$$

making $\mathcal{A}$ a bialgebra, hence a tensor product of $\mathcal{A}$-modules is also $\mathcal{A}$-module. The algebra $\mathcal{A}$ is closely connected with the algebra $U_q(\widehat{gl}(n))$, but does not exactly coincide with it In Sect. 8 we shall discuss the structure of the algebra $\mathcal{A}$ in more detail.

We are interested in a special class of representation of the algebra $\mathcal{A}$. Often the representation will be indicated by a superscript.

Definition 1.2. *A representation π of the algebra $\mathcal{A}$ is called a polynomial representation if* $\dim \pi < \infty$, *$T^\pi(u)$ is polynomial on u and $T^\pi_{ij}(0) = 0$ for $i<j$.* $\deg \pi \equiv \deg T^\pi = \max_{ij}(\theta_{ij} + \deg T^\pi_{ji})$ *is called a degree of the representation π.*

The algebra $\mathcal{A}$ has the well known element $\det_q T(u)$ which is called the quantum determinant (the exact definition of $\det_q T(u)$ is given in Sect. 6). Henceforward we assume that all ε_{ij} are integer.

Lemma 1.1. *$Q(u) = \det_q T(u)\prod_{il} H_l^{\varepsilon_{li}}$ is a central element.*

Proof. In Sect. 6. $\square$

Lemma 1.2. $\Delta(\det_q T(u)) = \det_q T(u) \otimes \det_q T(u)$.

Proof. In Sect. 7. □

For a polynomial representation π, $\deg \pi = M$, $T(u) \equiv T^\pi(u)$ we define

$$T_{ii}(u) = T_{ii}^\infty(-u)^M + \cdots + T_{ii}^0 ,$$
$$T_{ij}(u) = (-u)^{\theta_{ij}}(T_{ij}^\infty(-u)^{M-1} + \cdots + T_{ij}^0), \quad i \neq j ,$$
$$Q(u) = Q^\infty(-u)^{nM} + \cdots + Q^0 . \tag{1.6}$$

Lemma 1.3. *Let π be a polynomial representation, $T(u) \equiv T^\pi(u)$, $H_i \equiv H_i^\pi$. Operators $t_i^\infty = T_{ii}^\infty \cdot \prod_l H_l^{-\varepsilon_{il}}$ and $t_i^0 = T_{ii}^0 \cdot \prod_l H_l^{\varepsilon_{li}}$ commute with $T(u)$, $H_1, \ldots, H_n$.*

It is obvious that $Q^\infty = \prod_i t_i^\infty$, $Q^0 = \prod_i t_i^0$.

Henceforth throughout the paper we take ω being a primitive N^{th} root of 1. In this case the algebra $\mathscr{A}$ has an additional large set of central elements. To describe them explicitly we introduce an operation $\langle\cdot\rangle$ as follows: $\langle \mathcal{O} \rangle (u^N) = \prod_{k=1}^N \mathcal{O}(u\omega^k)$.

Lemma 1.4. *$\langle T_{ij} \rangle(v)$, $H_1^N, \ldots, H_n^N$ are central elements.*

Proof. In Sect. 7. □

Define the element $\langle T \rangle(v) \in \mathscr{M} \otimes \mathscr{A}$ such that $\langle T \rangle_{ij}(v) = \langle T_{ij} \rangle(v)$.

Lemma 1.5. $\Delta(\langle T \rangle(v)) = \langle T_1 \rangle(v) \langle T_2 \rangle(v)$.

For any $\mathscr{T} \in \mathscr{M}$ let $A_k^{\mathscr{T}}$, $B_k^{\mathscr{T}}$, $C_k^{\mathscr{T}}$ be the following minors:

$A_k^{\mathscr{T}}$ is the principal minor generated by the first k rows and columns.
$B_k^{\mathscr{T}}$ is generated by the first k rows and $k+1$ columns (except the k^{th} column).
$C_k^{\mathscr{T}}$ is generated by the first $k+1$ rows and k columns (except the k^{th} row).

Definition 1.3. *$\mathscr{T}(v) \in \mathscr{M}[v]$, $\deg \mathscr{T} = M$ is called an A-polynomial if it enjoys the properties*

(1) $\mathscr{T}_{ij}(0) = 0$ if $i < j$.
(2) $\deg \mathscr{T}_{ij} < M$ *if* $i > j$.
(3) *For any $k < n$ $A_k^{\mathscr{T}}(v)$ has exactly kM nonzero simple zeros.*
(4) *If $A_k^{\mathscr{T}}(v_0) = 0$ then $B_k^{\mathscr{T}}(v_0) \neq 0$ and $C_k^{\mathscr{T}}(v_0) \neq 0$.*

$A\mathscr{M}[v]$ denotes the set of all A-polynomials.

It is evident that $\deg A_k^{\mathscr{T}} = kM$, $\deg B_k^{\mathscr{T}} \leqq kM$, $\deg C_k^{\mathscr{T}} < kM$ and $A_k^{\mathscr{T}}(0) \neq 0$, $B_k^{\mathscr{T}}(0) = 0$.

Let Υ_M be a variety of sets $\Sigma = \{\mathscr{T}(v) \in A\mathscr{M}[v], \mathscr{Q}(u) \in \mathbb{C}[u], h_i, z_i^\infty, z_i^0\}_{i=1}^n$ such that $\deg \mathscr{T} = M$ and

$$\mathscr{T}_{ii}(v) = ((-v)^M (z_i^\infty)^N + \cdots + (z_i^0)^N h_i^{-1}) \prod_l h_l^{\varepsilon_{il}} ,$$
$$\mathscr{Q}(u) = (-u)^{nM} \prod_i z_i^\infty + \cdots + \prod_i z_i^0 ,$$
$$\det \mathscr{T}(v) = \langle \mathscr{Q} \rangle(v) \prod_{il} h_l^{\varepsilon_{il}}, \quad \prod_l h_l = 1 . \tag{1.7}$$

Lemma 1.6. *Υ_M is diffeomorphic to a dense open set in $\mathbb{C}^{n^2M+2n-1}$.*

Proof. In Sect. 2. □

Definition 1.4. *The polynomial representation π is called an A-representation if $\langle T\rangle^\pi(v) \in A\mathscr{M}[v]$ and $\deg\langle T\rangle^\pi = \deg\pi$. An irreducible A-representation of degree 1 is called an elementary representation (L-operator).*

For any irreducible A-representation π we put

$$\Sigma^\pi = \{\langle T\rangle^\pi(v), Q^\pi(u), (H_i^N)^\pi, (t_i^\infty)^\pi, (t_i^0)^\pi\} .$$

Lemma 1.7. *$\Sigma^\pi \in \Upsilon_M$, $M = \deg\pi$.*

Proof. In Sect. 7. □

Theorem 1. *For any set $\Sigma \in \Upsilon_M$ there exists a unique irreducible A-representation π such that $\Sigma^\pi = \Sigma$. Moreover, $\deg\pi = M$ and $\dim\pi = N^{(n-1)nM/2}$.*

Remark. Minimal L-operators from [3] do not fall into the set of A-representations. It is *a posteriori* obvious, since their dimension is equal to N^{n-1} which is less than it should be for irreducible A-representations according to Theorem 1. But one can also see *a priori* that in the case of a minimal L-operator the conditions (3) and (4) of Definition 1.3, which have to be checked for the corresponding matrix consisting of central elements, fail for $k > 2$ and $k > 1$ respectively.

Theorem 2. *A generic irreducible A-representation of degree $M \geqq 1$ is equivalent to a tensor product of M elementary representations.*

Remark. One can check if a representation π is equivalent to a tensor product of elementary representations using only $\langle T\rangle^\pi(v)$.

2. The Proof of Theorem 1. Uniqueness

In order to prove Theorem 1 we shall describe the construction of an irreducible A-representation inspired by Drinfeld's new realization of Yangians [12] and the ideas of the functional Bethe ansatz [26]. Let us introduce the special elements of the algebra $\mathscr{A}$-quantum minors of $T(u)$; the exact definition and the calculation of commutation relations for quantum minors is given in Sect. 6. The following quantum minors will play an important role:

$\hat{A}_k(u)$ is a principal minor generated by the first k rows and columns;

$\hat{B}_k(u)$ is generated by the first k rows and $k+1$ columns (except the k^{th} column);

$\hat{C}_k(u)$ is generated by the first $k+1$ rows and k columns (except the k^{th} row);

$\hat{D}_k(u)$ is generated by the first $k+1$ rows and columns (except the k^{th} row and column);

It is also convenient to introduce improved minors whose commutation relations are simpler than for original ones:

$$\begin{gathered}A_k(u) = \hat{A}_k(u)\hat{H}_k, \quad B_k(u) = \hat{B}_k(u)\hat{H}_k ,\\ C_k(u) = \hat{C}_k(u)\hat{H}_k, \quad D_k(u) = \hat{D}_k(u)\hat{H}_{k-1}\prod_l H^{-\varepsilon_{k+1,l}} ,\\ \hat{H}_k = \prod_{i=1}^{k}\prod_l H_l^{-\varepsilon_{il}} .\end{gathered} \tag{2.1}$$

Main commutation relations read as follows:

$$[A_i(u), A_j(v)] = [A_i(u), H_l] = 0\,,$$
$$[A_i(u), B_i(v)] = [A_i(u), C_j(v)] = [B_i(u), C_j(v)] = 0, \quad i \neq j\,,$$
$$[B_i(u), B_i(v)] = [C_i(u), C_i(v)] = 0\,, \tag{2.2}$$
$$H_l B_i(u) = \omega^{\delta_{i+1,l} - \delta_{il}} B_i(u) H_j, \quad H_l C_i(u) = \omega^{\delta_{il} - \delta_{i+1,l}} C_i(u) H_l\,,$$
$$\begin{matrix} B_i(u)B_j(v) = \omega^{\eta_{ji}} B_j(v) B_i(u) \\ C_i(u)C_j(v) = \omega^{\eta_{ij}} C_j(v) C_i(u) \end{matrix}, \quad |i-j|>1\,,$$
$$\eta_{ij} = \varepsilon_{i,j+1} + \varepsilon_{i+1,j} - \varepsilon_{ij} - \varepsilon_{i+1,j+1}\,, \tag{2.3}$$
$$(u-v)A_i(u)B_i(v) = (u - v\omega)B_i(v)A_i(u) - v(1-\omega)B_i(u)A_i(v)\,,$$
$$\omega(u-v)A_i(u)C_i(v) = (u\omega - v)C_i(v)A_i(u) + u(1-\omega)C_i(u)A_i(v)\,, \tag{2.4}$$
$$D_i(u)A_i(u\omega) - \omega B_i(u)C_i(u\omega)H^{(i)} = A_{i+1}(u\omega)A_{i-1}(u)\,, \tag{2.5}$$
$$H^{(i)} = \prod_l H^{\varepsilon_{il} - \varepsilon_{i+1,l}}\,, \tag{2.6}$$

where $A_0(u) = 1$, $A_n(u) = Q(u)$. Note that

$$H^{(i)}B_j(u) = \omega^{\eta_{ij}} B_j(u) H^{(i)}, \quad H^{(i)}C_j(u) = \omega^{-\eta_{ij}} C_j(u) H^{(i)}\,.$$

Let us also define improved minors of $\langle T\rangle(v)$:

$$A_k^{\langle\rangle}(v) = A_k^{\langle T\rangle}(v)\hat{H}_k^N, \quad B_k^{\langle\rangle}(v) = B_k^{\langle T\rangle}(v)\hat{H}_k^N\,,$$
$$C_k^{\langle\rangle} = C_k^{\langle T\rangle}(v)\hat{H}_k^N\,, \tag{2.7}$$

where minors $A_k^{\langle T\rangle}(v)$, $B_k^{\langle T\rangle}(v)$, $C_k^{\langle T\rangle}(v)$ were defined above.

Lemma 2.1. $\langle A_i\rangle(v) = A_i^{\langle\rangle}(v)$, $\langle B_i\rangle(v) = B_i^{\langle\rangle}(v)$, $\langle C_i\rangle(v) = C_i^{\langle\rangle}(v)$.

Proof. In Sect. 7. □

Denote by $\mathring{\mathscr{A}}$ the subalgebra generated by $\{\hat{A}_k(u), \hat{B}_k(u), \hat{C}_k(u), H_k\}_{k=1}^{n-1}$. Certainly, $\mathring{\mathscr{A}}$ is also generated by $\{A_k(u), B_k(u), C_k(u), H_k\}_{k=1}^{n-1}$.

Now let us fix throughout this section in irreducible A-representation π of degree M and take all elements of the algebra $\mathscr{A}$ in this representation. (The explicit indication of π will be omitted.) Let $\{\zeta_{ij}\}$ be the set of all zeros of the polynomial $A_i^{\langle\rangle}(v)$. Because π is an A-representation, all these zeros are nonzero and simple. Introduce operators α_{kj}, β_{kj}, γ_{kj} as follows:

$$A_k(u) = A_k^\infty \prod_{j=1}^{kM} (\alpha_{kj} - u), \quad \alpha_{kj}^N = \zeta_{kj}, \quad A_k^\infty = \prod_{i=1}^{k} t_i^\infty\,, \tag{2.8}$$
$$\beta_{ij} = B_i(\alpha_{ij}), \quad \gamma_{ij} = C_i(\alpha_{ij})\,. \tag{2.9}$$

When substituting α_{ij} instead of the spectral parameter the ordering of non-commuting factors has to be chosen. We prefer to put all α's to the right, but one can choose another ordering and all the following remains correct. Equations (2.2)–(2.6) and Lemma 2.1 lead to the following relations for these operators:

$$[\alpha_{ik}, \alpha_{jl}] = [\alpha_{ik}, H_l] = [H_i, H_l] = 0\,,$$

$$\alpha_{ik}\beta_{jl} = \beta_{jl}\alpha_{ij}\omega^{\delta_{ij}\delta_{kl}}, \quad \alpha_{ik}\gamma_{jl} = \gamma_{jl}\alpha_{ij}\omega^{-\delta_{ij}\delta_{kl}}\,,$$

$$H_i\beta_{jl} = \omega^{\delta_{i,j+1}-\delta_{ij}}\beta_{jl}H_i, \quad H_i\gamma_{jl} = \omega^{\delta_{ij}-\delta_{i,j+1}}\gamma_{jl}H_i\,, \tag{2.10}$$

$$[\beta_{ik}, \beta_{il}] = [\beta_{ik}, \gamma_{jl}] = [\gamma_{ik}, \gamma_{il}] = 0, \quad i \neq j\,,$$

$$\beta_{ik}\beta_{jl} = \beta_{jl}\beta_{ik}\omega^{\eta_{ij}}, \quad \gamma_{ik}\gamma_{jl} = \gamma_{jl}\gamma_{ik}\omega^{-\eta_{ij}}, \quad |i-j| > 1\,, \tag{2.11}$$

$$\omega\beta_{ik}\gamma_{ik}H^{(i)} = -A_{i+1}(\alpha_{ik})A_{i-1}(\alpha_{ik}\omega^{-1})\,,$$

$$\gamma_{ik}\beta_{ik}H^{(i)} = -A_{i+1}(\alpha_{ik}\omega)A_{i-1}(\alpha_{ik})\,, \tag{2.12}$$

$$\beta_{ij}^N = B_i^{\langle\rangle}(\zeta_{ij}), \quad \gamma_{ij}^N = C_i^{\langle\rangle}(\zeta_{ij})\,, \tag{2.13}$$

$$A_k^\infty \prod_{j=1}^{kM} \alpha_{kj} = \prod_{i=1}^{k} t_i^0 H_i^{-1}\,. \tag{2.14}$$

Since π is an A-representation β_{ij} and γ_{ij} are invertible (see (2.13)). For present the definition (2.8) of operators α_{ij} is formal. To make it sensible we introduce a vector $\mathbf{v}$ – a common eigenvector of $A_i(u)$, $i = 1, \ldots, n-1$ and the subspace $V = \pi(\mathring{\mathscr{A}})\mathbf{v}$.

Lemma 2.2.
1. *V is spanned by common eigenvectors of $A_i(u)$ with different eigenvalues.*
2. *α_{ij}, β_{ij}, γ_{ij} can be well defined on V as operators satisfying relations* (2.10)–(2.14).
3. $\dim V = N^{(n-1)nM/2}$.

Proof. Evidently we can define α_{ij} and $\mathbf{v}$ claiming $\mathbf{v}$ to be its eigenvector with the appropriate eigenvalue. Then the subspace V can be set up step by step starting from $\mathbf{v}$ by use of β_{kl} and γ_{kl}. At every step the definition of α_{ij} can be naturally extended to fulfill relations (2.10). It is easy to check that this construction can be realized selfconsistently giving the subspace V of the required dimension and operators α_{ij}, β_{ij}, γ_{ij} on it satisfying the relations (2.10)–(2.14). And for the operators $B_k(u)$ and $C_k(u)$ we have the interpolating formulae:

$$B_k(u) = u\sum_{i=1}^{kM} \beta_{ki}\alpha_{ki}^{-1}P_{ki}(u), \quad C_k(u) = \sum_{i=1}^{M} \gamma_{ki}P_{ki}(u)\,,$$

where
$$P_{ki}(u) = \prod_{\substack{j=1\\ j\neq i}}^{kM} \frac{u-\alpha_{kj}}{\alpha_{ki}-\alpha_{kj}}\,. \tag{2.15}$$

□

Remark. By the definition of α's one can retell the first point saying that V is spanned by common eigenvectors of α's with different eigenvalues.

One can also see that for $\mathbf{v}'$ – another common eigenvector of $A_i(u)$ V and $V' = \pi(\mathring{\mathscr{A}})\mathbf{v}'$ are isomorphic as $\pi(\mathring{\mathscr{A}})$-orbits.

To complete this part of the proof of Theorem 1 it is enough to show that V is invariant with respect to $\pi(\mathscr{A})$. To have more compact notations we shall show that $\pi(\mathscr{A}) \subset \pi(\mathring{\mathscr{A}})$ using α_{ij}, β_{ij}, γ_{ij}. The way of doing this is the following recursive process. The first step is trivial:

$$T_{11}(u) = \hat{A}_1(u), \quad T_{12}(u) = \hat{B}_1(u), \quad T_{21}(u) = \hat{C}_1(u)$$

(see (2.1), (2.6)). $T_{22}(u)$ can be tested by means of the relation

$$\hat{A}_2(u\omega) = T_{22}(u\omega)T_{11}(u) - \omega_{21}T_{21}(u\omega)T_{12}(u) .$$

To pass to the 3 by 3 principal submatrix one has to use relations

$$\hat{B}_2(u\omega) = \omega_{21}(\omega_{31}^{-1}T_{23}(u\omega)T_{11}(u) - T_{21}(u\omega)T_{13}(u)) ,$$
$$\hat{C}_2(u\omega) = \omega_{31}(\omega_{21}^{-1}T_{32}(u\omega)T_{11}(u) - T_{31}(u\omega)T_{12}(u)) . \tag{2.16}$$

Substituting here $u = \alpha_{1i}$ we obtain the interpolating formulae for $T_{13}(u)$, $T_{31}(u)$:

$$T_{13}(u) = -u\omega_{21}^{-1}\hat{H}_1 \sum_{i=1}^{M} \gamma_{1i}^{-1}\hat{B}_2(\alpha_{1i}\omega)\alpha_{1i}^{-1}P_{1i}(u) ,$$

$$T_{31}(u) = -\omega_{32}^{-1}\hat{H}_1 \sum_{i=1}^{M} \hat{C}_2(\alpha_{1i}\omega)\beta_{1i}^{-1}P_{1i}(u) .$$

Now $T_{23}(u)$, $T_{32}(u) \in \pi(\mathring{\mathscr{A}})$ due to (2.16) and to test $T_{33}(u)$ we recall that

$$\hat{A}_3(u\omega) = T_{33}(u\omega)\hat{A}_2(u) + \text{known terms} .$$

For further steps we have to introduce additional quantum minors:

$\hat{B}_{kl}(u)$ is generated by the first k rows and $k-1$ columns together with $(k+l)^{\text{th}}$ column;
$\hat{C}_{kl}(u)$ is generated by the first $k-1$ rows and k columns together with $(k+l)^{\text{th}}$ row;
$\hat{D}^B_{kl}(u)$ is generated by the first $k-1$ rows and columns together with $(k+l)^{\text{th}}$ row and $(k+1)^{\text{th}}$ column;
$\hat{D}^C_{kl}(u)$ is generated by the first $k-1$ rows and columns together with $(k+1)^{\text{th}}$ row and $(k+l)^{\text{th}}$ column;

We also define the corresponding improved minors:

$$B_{kl}(u) = \hat{B}_{kl}(u)\hat{H}_k, \quad D^B_{kl}(u) = \hat{D}^B_{kl}(u)\hat{H}_{k-1}\prod_l H_i^{-\varepsilon_{k+1,i}}$$

$$C_{kl}(u) = \hat{C}_{kl}(u)\hat{H}_k, \quad D^C_{kl}(u) = \hat{D}^C_{kl}(u)\hat{H}_{k-1}\prod_l H_i^{-\varepsilon_{k+1,i}}$$

(cf. (2.1)) and use the relations

$$D^B_{kl}(u)A_k(u\omega) - \omega B_{kl}(u)C_k(u\omega)H^{(k)} = \frac{\omega_{k+l,k}}{\omega_{k+1,k}} B_{k+1,l-1}(u\omega)A_{k-1}(u) ,$$

$$D^C_{kl}(u)A_k(u\omega) - \omega B_k(u)C_{kl}(u\omega)H^{(k)} = \frac{\omega_{k+1,k}}{\omega_{k+l,k}} C_{k+1,l-1}(u\omega)A_{k-1}(u) , \tag{2.17}$$

which look similar to (2.5). To check $T_{i4}(u) \in \pi(\mathring{\mathscr{A}})$, $i = 1, 2, 3$, the following formulae have to be written:

$$H^{(2)}B_{22}(u) = -u\omega_{34}^{-1}\sum_{i=1}^{2M}\gamma_{2i}^{-1}B_3(\alpha_{2i}\omega)A_1(\alpha_{2i})\alpha_{2i}^{-1}P_{2i}(u) , \tag{2.18}$$

$$\hat{B}_{22}(u\omega) = \omega_{21}(\omega_{41}^{-1}T_{24}(u\omega)T_{11}(u) - T_{21}(u\omega)T_{14}(u)) , \tag{2.19}$$

$$T_{14}(u) = -u\omega_{21}^{-1}\hat{H}_1^{-1}\sum_{i=1}^{M}\gamma_{1i}^{-1}\hat{B}_{22}(\alpha_{1i}\omega)\alpha_{1i}^{-1}P_{1i}(u) . \tag{2.20}$$

Equations (2.18), (2.20) are obtained from the first of Eq. (2.17) for $k = 2$ and Eq. (2.19) respectively after the following substitutions: $u = \alpha_{2i}\omega^{-1}$ and $u = \alpha_{1i}$. Now $T_{24}(u) \in \pi(\mathring{\mathscr{A}})$ due to (2.19) and to test $T_{34}(u)$ we use

$$\omega_{41}\omega_{42}\hat{B}_3(u\omega) = \omega_{31}\omega_{32}T_{34}(u\omega)\hat{A}_2(u) + \text{known terms} .$$

In the same manner we can show that $T_{4i}(u) \in \pi(\mathring{\mathscr{A}})$, $i = 1, 2, 3$. In order to test $T_{44}(u)$ and thus to complete this step of the process we look to

$$\hat{A}_4(u\omega) = T_{44}(u\omega)\hat{A}_3(u) + \text{known terms} .$$

It is quite evident how to do the next steps by means of relations (2.17) and interpolating formulae. As a result of this recursive process we can express all $T_{kl}(u)$ through operators α_{ij}, β_{ij}, γ_{ij}. Justifying this formal calculations like in Lemma 2.2 we convince ourselves that $\pi(\mathscr{A})V \subset V$.

Proof of Lemma 1.6. The recursive process described above certainly has the "classical limit" – a very similar one for usual matrix polynomials. It shows that the variety Υ_M can be parametrized by $\mathscr{Q}(u)$, minors $A_i^{\mathscr{T}}(v)$, $B_i^{\mathscr{T}}(v)$, $C_i^{\mathscr{T}}(v)$, $i = 1, \ldots, n-1$ and h_i, z_i^∞, z_i^0, $i = 1, \ldots, n$. Now it is very easy to find independent parameters in which the identity mapping is the required diffeomorphism. □

3. The Proof of Theorem 1. Existence

Let a set $\Sigma \in \Upsilon_M$ be given. We have to find an irreducible A-representation π such that $\Sigma = \Sigma^\pi$. Define the algebra $\mathring{\mathscr{A}}_\Sigma$ by generators $\{\alpha_{ik}, \beta_{ik}, \gamma_{ik}, H_i\}_{i=1}^{M}{}_{k=1}^{iM}$ and relations (cf. (2.7)–(2.14)):

$$[\alpha_{ik}, \alpha_{jl}] = [\alpha_{ik}, H_l] = [H_i, H_l] = 0 ,$$

$$\alpha_{ik}\beta_{jl} = \beta_{jl}\alpha_{ij}\omega^{\delta_{ij}\delta_{kl}}, \quad \alpha_{ik}\gamma_{jl} = \gamma_{jl}\alpha_{ij}\omega^{-\delta_{ij}\delta_{kl}} ,$$

$$H_i\beta_{jl} = \omega^{\delta_{i,j+1}-\delta_{ij}}\beta_{jl}H_i, \quad H_i\gamma_{jl} = \omega^{\delta_{ij}-\delta_{i,j+1}}\gamma_{jl}H_i ,$$

$$[\beta_{ik}, \beta_{il}] = [\beta_{ik}, \gamma_{jl}] = [\gamma_{ik}, \gamma_{il}] = 0, \quad i \neq j ,$$

$$\beta_{ik}\beta_{jl} = \beta_{jl}\beta_{ik}\omega^{\eta_{ij}}, \quad \gamma_{ik}\gamma_{jl} = \gamma_{jl}\gamma_{ik}\omega^{-\eta_{ij}}, \quad |i-j| > 1 ,$$

$$\omega\beta_{ik}\gamma_{ik}H^{(i)} = -A_{i+1}(\alpha_{ik})A_{i-1}(\alpha_{ik}\omega^{-1}) ,$$

$$\gamma_{ik}\beta_{ik}H^{(i)} = -A_{i+1}(\alpha_{ik}\omega)A_{i-1}(\alpha_{ik}) ,$$

$$\alpha_{ij}^N = \zeta_{ij}, \quad \beta_{ij}^N = \hat{h}_iB_i^{\mathscr{T}}(\zeta_{ij}), \quad \gamma_{ij}^N = \hat{h}_iC_i^{\mathscr{T}}(\zeta_{ij}) ,$$

$$\prod_{i=1}^{k} z_i^\infty \prod_{j=1}^{kM} \alpha_{kj} = \prod_{i=1}^{k} z_i^0 H_i^{-1} ,$$

$$A_k(u) = \prod_{i=1}^{k} z_i^\infty \prod_{j=1}^{kM} (\alpha_{kj} - u), \quad H^{(i)} = \prod_l H^{\varepsilon_{il}-\varepsilon_{i+1,l}} ,$$

$$\hat{h}_k = \prod_{i=1}^{k}\prod_l h_l^{-\varepsilon_{il}} .$$

It is easy to see that $\mathring{\mathscr{A}}_\Sigma$ is a simple algebra isomorphic to End $\mathbb{C}^{N(n-1)nM/2}$ so it has a unique irreducible representation and any its representation is faithful. Before we have shown that an irreducible A-representation π generates the irreducible representation of the algebra $\mathring{\mathscr{A}}_{\Sigma^\pi}$. Now we would like to reverse a logic. Let $B(u)$, $C(u)$ be defined by Eq. (2.15) and $\hat{A}(u)$, $\hat{B}(u)$, $\hat{C}(u)$ by Eq. (2.1). Define the homomorphism φ: $\mathscr{A} \to \mathring{\mathscr{A}}_\Sigma$ on generators as follows: $\varphi(H_i) = H_i$ and $\varphi(T_{ij}(u))$ is given by the recursive process described in the previous section. For the definition of φ to be correct all the relations (1.1) have to be preserved by φ. To verify this is to check some polynomial identities on Υ_M. So they have to be checked only for generic Σ and it certainly will be done if an irreducible A-representation π such that $\Sigma^\pi = \Sigma$ will be shown. Though we return almost to the starting point of the consideration we have a profit to solve the problem only for generic Σ. In this case the required irreducible A-representation can be built from some simple primitives.

Later we shall treat $\mathbb{C}^n$-coordinate indices modulo n, excepting the cases when they appear in inequalities. Introduce the algebra $\mathscr{W}$ generated by F_i, G_i, H_i, $i = 1, \ldots, n$ and relations

$$F_iF_j = F_jF_i, \quad F_iH_j = H_jF_i, \quad H_iH_j = H_jH_i\,,$$
$$\omega_{ij}F_i\,G_j = G_jF_i\omega_{i,j+1}, \quad H_iG_j = G_jH_i\omega^{\delta_{i,j+1}-\delta_{ij}}\,,$$
$$\omega_{ij}G_iG_j = G_jG_i\omega_{i+1,j+1}, \quad \prod_l H_l = 1\,. \tag{3.1}$$

Let $f_i = F_i\prod_l H_l^{-\varepsilon_{il}}$, $\mathbf{F} = F_1 \ldots F_n$, and $\mathbf{G} = G_1 \ldots G_n$. Elements f_i, F_i^N, G_i^N, H_i^N, $i = 1, \ldots, n$ and $\mathbf{FG}^{-1}$ clearly generate the center of $\mathscr{W}$. The mapping ϕ: $\mathscr{A} \to \mathscr{W}$:

$$T_{ij}(u) \xrightarrow{\phi} -uF_i\delta_{ij} + (-u)^{\theta_{ij}}G_i\delta_{i+1,j}\,,$$

$$H_l \xrightarrow{\phi} H_l \tag{3.2}$$

is a homomorphism of algebras. It is easy to calculate that

$$Q(u) \xrightarrow{\phi} (-u)^{n-1}\omega^{(1-n)n/2}\left((-1)^n\mathbf{G}\prod_{i=2}^{n}\omega_{1i} - u\mathbf{F}\right),$$

$$\langle T_{ij}\rangle(v) \xrightarrow{\phi} -vF_i^N\delta_{ij} + (-v)^{\theta_{ij}}G_i^N\delta_{i+1,j}\,.$$

For any representation ξ of the algebra $\mathscr{W}$ the representation $\xi \circ \phi$ of the algebra $\mathscr{A}$ will be called a simplest representation.

Let $\mathscr{V} = \operatorname{End}\mathbb{C}^N$ and $X, Z \in \mathscr{V}$ be the following matrices: $X_{ij} = \delta_{i,\,j+1\,(\mathrm{mod}\,N)}$, $Z_{ij} = \omega^i\delta_{ij}$. Define naturally operators X_i, $Z_i \in \mathscr{V}^{\otimes n}$:

$$X_i = I^{\otimes(i-1)} \otimes X \otimes I^{\otimes(n-i-1)}, \quad Z_i = I^{\otimes(i-1)} \otimes Z \otimes I^{\otimes(n-i-1)}$$

and introduce the subspace $\mathscr{H} \subset (\mathbb{C}^N)^{\otimes n}$ as the eigenspace $Z^{\otimes n} = 1$.

Lemma 3.1. *Let a_i, b_i, c_i, $i = 1, \ldots, n$ be arbitrary numbers such that $\prod_i c_i = 1$ and m_{ij}, $i, j = 1, \ldots, n$ be integers such that $m_{i,l+1} - m_{il} - m_{l,i+1} - m_{li} = \varepsilon_{i+1,l+1} - \varepsilon_{il}$. The mapping ξ: $\mathscr{W} \to \operatorname{End}\mathscr{H}$:*

$$F_i = a_i\prod_l Z_l^{\varepsilon_{il}}, \quad G_i = b_iX_{i+1}X_i^{-1}\prod_l Z_l^{m_{il}}, \quad H_i = c_iZ_i$$

is a representation of the algebra $\mathscr{W}$.

Now we have got a lot of simplest representations to extract the required irreducible A-representation from a tensor product of simplest representations. Let κ_i, $i = 1, \ldots, nM$ be zeros of $\mathscr{Q}(u)$ (simple for generic case), and let us take nonzero vectors $\Psi_i \in \ker \mathscr{T}(\kappa_i^N)$ which are unique up to scale factor due to (1.7). Define step by step a sequence of simplest representations $\sigma_i = \xi_i \circ \phi$ such that

$$Q^{\sigma_i}(\kappa_i) = 0, \quad \langle T \rangle^{\sigma_i}(\kappa_i^N)\Psi_{ii} = 0\,,$$

$$\prod_{j=1}^{nM}{}^{\otimes} \xi_j(f_l) = z_l^{\infty}, \quad \prod_{j=1}^{nM}{}^{\otimes} \sigma_j(H_l^N) = h_l\,,$$

where
$$\Psi_{1i} = \Psi_i, \quad \Psi_{i+1,j} = \langle T \rangle^{\sigma_i}(\kappa_j^N)\Psi_{ij} \tag{3.3}$$

and take the representation $\pi = \pi(\sigma_1, \ldots, \sigma_{nM})$ such that

$$T^{\pi}(u) = (-u)^{(1-n)M} T_{nM}(u; \sigma_{nM}) \cdot \ldots \cdot T_1(u; \sigma_1)\,,$$

$$H_i^{\pi} = \prod_{j=1}^{nM}{}^{\otimes} \sigma_j(H_i), \quad T_i(u; \sigma_i) = T_i^{\sigma_i}(u)\,. \tag{3.4}$$

Lemma 3.2. $\langle T \rangle^{\pi}(v) = \mathscr{T}(v)$.

Proof. Consider the ratio $\tau(v) = \langle T \rangle^{\pi}(v)\mathscr{T}^{-1}(v)$. This is a meromorphic function having poles only at points κ_i^N. But Eq. (3.3), (3.4) show that for any i $\mathrm{res}_{n=\kappa_i^N}\tau(u) = 0$. Hence, $\tau(v)$ does not depend on v. Taking limits $v \to 0$ and $v \to \infty$ we see that $\tau(v)$ is both an upper triangular matrix with unit diagonal and a lower triangular one. Then $\tau(v)$ is the unit matrix. □

One can easily check that the representation π is a polynomial A-representation of degree M and $Q^{\pi}(u) = \mathscr{Q}(u)$, $t_i^{\infty} = z_i^{\infty}$. As a corollary of Lemma 3.2 we have got that $(t_i^0)^N = \prod_{j=1}^{nM}{}^{\otimes}\sigma_j(G_{i-j}^N) = (z_i^0)^N$. According to Lemma 1.3 the representation π can be restricted to a maximal common eigenspace of operators $t_i^0 = \prod_{j=1}^{nM}{}^{\otimes} \xi_j\left(G_{i-j}\prod_l H_l^{\varepsilon_{li}}\right)$, $i = 1, \ldots, n$. It is obvious that we can choose this eigenspace $\mathscr{H}^0$ such that $t_i^0|_{\mathscr{H}^0} = z_i^0$. So an irreducible component $\pi^0 \subset \pi|_{\mathscr{H}^0}$ is an irreducible A-representation such that: $\Sigma^{\pi^0} = \Sigma$. □

Proof of Theorem 2. This theorem simply follows from formula (3.11) and Theorem 1. Let π_0 be an irreducible A-representation, $\Sigma = \Sigma^{\pi_0}$ and the representation $\pi = \pi(\sigma_1, \ldots, \sigma_{nM})$ is built as described above. One can see that operators t_i^0 are organized as products of commuting factors $t_{ik} = \prod_{j=kn+1}^{(k+1)n}{}^{\otimes} \xi_j\left(G_{i-j}\prod_l H_l^{\varepsilon_{li}}\right)$. Let $\mathscr{H}^k$ be a maximal common eigenspace of t_{ik}, $i = 1, \ldots, n$ and $\bigotimes_{k=1}^{n}\mathscr{H}^k \subset \mathscr{H}^0$. Taking π^k as an irreducible component of $\pi(\sigma_{kn+1}, \ldots, \sigma_{(k+1)n})|_{\mathscr{H}^k}$ it is easy to see that π^k is an elementary representation. The representation

$$\pi^0 = \pi^M \otimes \ldots \otimes \pi^1 \tag{3.5}$$

is an A-representation of degree M, $\dim \pi^0 = N^{(n-1)nM/2}$ and $\Sigma^{\pi^0} = \Sigma$. Therefore it should be irreducible, equivalent to π_0 and (3.5) is its decomposition to a tensor product of elementary representations. □

4. Cocommuting Representations and Intertwiners

Definition4.1. *Representations π_1, π_2 of the algebra $\mathscr{A}$ are called cocommuting representations if the representations $\pi_1 \otimes \pi_2$ and $\pi_2 \otimes \pi_1$ are equivalent. A linear invertible operator* **R** *such that*

$$\mathbf{R}\pi_1 \otimes \pi_2(\Delta(\mathcal{O})) = \pi_2 \otimes \pi_1(\Delta(\mathcal{O}))\mathbf{R} \tag{4.1}$$

for any $\mathcal{O} \in \mathscr{A}$ is called their intertwiner.

Lemma 4.1. *Let π_1, π_2 be cocommuting representations and all central elements are represented in $\pi_1 \otimes \pi_2$ by scalars. Then*

$$[\langle T\rangle^{\pi_1}(v), \langle T\rangle^{\pi_2}(v)] = 0 \,. \tag{4.2}$$

Proof. The statement follows from Lemmas 1.4, 1.5. □

Lemma 4.2. *Let π_1, π_2 be irreducible A-representations and both $\pi_1 \otimes \pi_2$ and $\pi_2 \otimes \pi_1$ be A-representations. Then π_1 and π_2 cocommute if and only if Eq. (4.2) is satisfied and their intertwiner is unique modulo a scalar factor.*

Proof. Due to Theorem 1 both $\pi_1 \otimes \pi_2$ and $\pi_2 \otimes \pi_1$ are irreducible A-representations because of their dimensions. So the part "only if" follows from the previous lemma. On the other hand if Eq. (4.2) is satisfied it follows from Eq. (1.5), (1.6) and Lemmas 1.2, 1.5 that $\Sigma^{\pi_1 \otimes \pi_2} = \Sigma^{\pi_2 \otimes \pi_1}$. Hence returning to Theorem 1 we obtain that they are equivalent irreducible representations. □

So we reduce the problem to consideration of matrix A-polynomials instead of irreducible A-representations. For $\mathscr{T}(v) \in A\mathscr{M}[v]$ let $\mathscr{M}_{\mathscr{T}}[v] \subset \mathscr{M}[v]$ be spanned by $v^k \mathscr{T}^l(v)$, $k, l \geqq 0$.

Lemma 4.3. *Let $\mathscr{P}(v)$, $\mathscr{T}(v) \in A\mathscr{M}[v]$ and $[\mathscr{P}(v), \mathscr{T}(v)] = 0$. Then for generic $\mathscr{T}(v)$ $\mathscr{P}(v) \in \mathscr{M}_{\mathscr{T}}[v]$.*

Lemma 4.4. *Let $\mathscr{T}(v) \in A\mathscr{M}[v]$ and $\mathscr{T}_\lambda(v) = \mathscr{T}(v) - \lambda I$. Then for generic $\mathscr{T}(v)$* corank $\mathscr{T}_\lambda(v) \leqq 1$ *for all λ, u.*

Proof. If λ_0, v_0 such that corank $\mathscr{T}_{\lambda_0}(v_0) > 1$ exist then λ_0 is a common zero of $\det \mathscr{T}_\lambda(v_0)$, $A_{n-1}^{\mathscr{T}_\lambda}(v_0)$ and $B_{n-1}^{\mathscr{T}_\lambda}(v_0)$ as polynomials on λ. Therefore, v_0 is a common zero of three their mutual resultants as polynomials on v. But it is impossible for generic $\mathscr{T}(v)$. □

Proof of Lemma 4.3. Let us recall that if $\chi \in \mathscr{M}$ has a "simple" spectrum in a sense that corank $(\mathscr{X} - \lambda I) \leqq 1$ for all λ then the set $\{\mathscr{X}^k\}_{k=0}^{n-1}$ is a basis of its commutant. A generic $\mathscr{T}(v)$ has a "simple" spectrum for all v, so $\mathscr{P}(v) = \sum_{k=0}^{n-1} P_k(v)\mathscr{T}^k(v)$. Treating this equality as a system of linear equations for functions $P_k(v)$ we see that it has a unique solution for any finite v. Taking into account Cramer's formulae one can see that $P_k(v)$ must be whole rational functions, i.e. polynomials. The same idea applied to the highest order terms (infinite v) gives the equality for degrees: $\deg \mathscr{P} = \max_k(\deg P_k, k \deg \mathscr{T})$. □

Certainly, if $\mathscr{P}_1(v)$, $\mathscr{P}_2(v) \in \mathscr{M}_{\mathscr{T}}[v]$ then $[\mathscr{P}_1(v), \mathscr{P}_2(v)] = 0$. And vice versa, one can say that if $[\mathscr{P}_1(v), \mathscr{P}_2(v)] = 0$ then generically $\mathscr{P}_1(v), \mathscr{P}_2(v) \in \mathscr{M}_{\mathscr{T}}[v]$ for some $\mathscr{T}(v)$.

Later we shall use the following trivial idea: *A nonzero meromorphic function is not zero at a generic point.*

Lemma 4.5. *For a generic A-polynomial $\mathcal{P}(v)$ its power $\mathcal{P}^m(v)$ is also an A-polynomial.*

Corollary. *For generic $\mathcal{T}(v) \in A\mathcal{M}[v]$ and $\mathcal{P}_1(v), \mathcal{P}_2(v) \in \mathcal{M}_{\mathcal{T}}[v]$ $\mathcal{P}_1(v), \mathcal{P}_2(v) \in A\mathcal{M}[v]$.*

Now let us return to the intertwiners. Due to Theorem 1 the space of irreducible A-representations of degree M is Υ_M and all of them can be realized in the same space $V^M = \mathbb{C}^{N(n-1)nM/2}$. Define $\mathcal{R}_{\mathcal{T}}$ as a set of irreducible representations π such that $\langle T\rangle^\pi(v) \in \mathcal{M}_{\mathcal{T}}[v]$. We want to treat an intertwiner as a function of the intertwining representations and it can be done. According to Lemmas 4.3, 4.5 intertwiners for commuting irreducible A-representations of degrees M, M' define modulo a scalar factor a locally holomorphic $\mathrm{Hom}(V^M, V^{M'})$-valued function on $\bigcup_{\mathcal{T}} \mathcal{R}_{\mathcal{T}}^{\times 2} \cap (\Upsilon_M \times \Upsilon_{M'})$. Moreover this function evidently is a nearly meromorphic function, only a common scalar factor can be multivalued. Later we imply an intertwiner to be considered as a function of representations in the sense described above.

Lemma 4.6. *Let $\mathbf{R}(\pi_1, \pi_2)$ be an intertwiner for cocommuting irreducible representations π_1, π_2. Then generally $\operatorname{tr}\mathbf{R}(\pi_1, \pi_2) \neq 0$.*

Proof. It is sufficient to take $\pi_1 = \pi^{\otimes l}$ and $\pi_2 = \pi^{\otimes m}$ for some irreducible A-representation π and integers l, m. Generally $\pi^{\otimes 2}$ is also an irreducible A-representation and $\mathbf{R}(\pi, \pi)$ is proportional to the permutation operator. Now one can give the explicit expression for the intertwiner $\mathbf{R}(\pi_1, \pi_2)$ and show that $\operatorname{tr}\mathbf{R}(\pi_1, \pi_2) \propto N^k$, where k is the maximal common factor of l and m. □

This lemma shows that $\operatorname{tr}\mathbf{R}(\pi_1, \pi_2) = 1$ is a good normalization condition making an intertwiner a pure meromorphic function.

Lemma 4.7. *Let $\pi_a \in \mathcal{R}_{\mathcal{T}}$, $a = 1, 2, 3$ be irreducible A-representations such that all $\pi_a \otimes \pi_b$ $(a \neq b)$ are A-representations. Then intertwiners $\mathbf{R}(\pi_a, \pi_b)$ satisfy the Yang–Baxter equation*

$$\mathbf{R}_{12}(\pi_1, \pi_2)\mathbf{R}_{13}(\pi_1, \pi_3)\mathbf{R}_{23}(\pi_2, \pi_3) = \mathbf{R}_{23}(\pi_2, \pi_3)\mathbf{R}_{13}(\pi_1, \pi_3)\mathbf{R}_{12}(\pi_1, \pi_2) .$$

Proof. We consider both sides of this equality as functions on $\bigcup_{\mathcal{T}} \mathcal{R}_{\mathcal{T}}^{\times 3}$. Put

$$\mathfrak{R} = (\mathbf{R}_{23}(\pi_2, \pi_3)\mathbf{R}_{13}(\pi_1, \pi_3)\mathbf{R}_{12}(\pi_1, \pi_2))^{-1}\mathbf{R}_{12}(\pi_1, \pi_2)\mathbf{R}_{13}(\pi_1, \pi_3)\mathbf{R}_{23}(\pi_2, \pi_3) .$$

$\mathfrak{R}$ commutes with all operators of the representation $\pi_1 \otimes \pi_2 \otimes \pi_3$ which is generically an A-representation, hence $\mathfrak{R}$ is a scalar. Moreover, from

$$\mathbf{R}_{12}(\pi_1, \pi_2)\mathbf{R}_{13}(\pi_1, \pi_3)\mathbf{R}_{23}(\pi_2, \pi_3)\mathbf{R}_{12}(\pi_1, \pi_2)^{-1} = \mathfrak{R}\mathbf{R}_{23}(\pi_2, \pi_3)\mathbf{R}_{13}(\pi_1, \pi_3)$$

we see that $\operatorname{tr}(\mathbf{R}_{13}(\pi_1, \pi_3)\mathbf{R}_{23}(\pi_2, \pi_3)) = \mathfrak{R}\operatorname{tr}(\mathbf{R}_{23}(\pi_2, \pi_3)\mathbf{R}_{13}(\pi_1, \pi_3))$. So $\mathfrak{R} = 1$. □

Proof of Lemma 4.5. It is enough for any degree l and power m to give an example of a polynomial $\mathcal{P}(v)$, $\deg \mathcal{P} = l$ satisfying items 1, 2 of Definition 1.3 such that $A_k^{\mathcal{P}^m}(v)$ has simple zeros and to give an example of a similar polynomial $S(v)$, $\deg \mathcal{S} = l$ such that $A_k^{\mathcal{S}^m}(v)$ and $B_k^{\mathcal{S}^m}(v)$ have no common zeros. We shall take $\mathcal{P}(v)$

as follows:

$$\begin{array}{ll} \mathscr{P}_{ii}(v) = (v - w_i)^l\,, \quad w_i \neq w_j \quad \text{if } i \neq j & \\ \mathscr{P}_{i,k+1}(v) = v\,, \quad \mathscr{P}_{k+1,i}(v) = \varepsilon & i = 1, \ldots, k \end{array}$$

$$\mathscr{P}_{ii}(v) = 1, \quad i = k+1, \ldots, n, \quad \mathscr{P}_{ij}(v) = 0 \quad \text{otherwise}\,.$$

One can calculate that for $\varepsilon \to 0$,

$$A_k^{\mathscr{P}^m}(v) = \prod_{i=1}^{k} (v - w_i)^{lm} + \varepsilon v \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j \neq i}}^{k} (u - w_j)^{lm} \sum_{s=0}^{m-2} (v - w_i)^s + o(\varepsilon)$$

so $A_k^{\mathscr{P}^m}(v)$ have simple zeros for small enough ε.

We shall seek for a polynomial $\mathscr{S}(v)$ of the following type:

$$\mathscr{S}(v) = \begin{pmatrix} \mathbf{a}(v) & v\mathbf{b} & 0 \\ 0 & (w - v)^l & 0 \\ 0 & 0 & (w - v)^l I \end{pmatrix},$$

where $\mathbf{a}(v)$ is a k by k block, $\mathbf{b}$ is a k-column and I is the $(n - k - 1)$ dimensional unit matrix. Let $\mathbf{a}(v)$ be a k-dimensional A-polynomial of degree l, $\det \mathbf{a}(v)$ has simple zeros, $\det \mathbf{a}(\omega) \neq 0$ and the principal $(k-1)^{\text{th}}$ minor of $\mathbf{a}(v)$ is not zero at zeros of $\det \mathbf{a}(v)$. One can build such a matrix $\mathbf{a}(v)$ in a way similar to the formulae (3.3), (3.4). Let us also take $\mathbf{b} \notin \operatorname{im} \mathbf{a}(v)$ at zeros of $\det \mathbf{a}(v)$. The technical exercise is to show that $B_k^{\mathscr{S}^m}(v)$ is not zero at zeros of $A_k^{\mathscr{S}^m}(v)$. □

5. $sl(n)$ Chiral Potts Model

Unfortunately, no reasonable explicit expression for intertwiners of generic A-representations can be obtained directly, even for the $sl(2)$ case. The way to obtain such an expression in this case is to use the factorization of A-representations to simplest representations. As a result formulae for intertwiners through the Boltzmann weights of the chiral Potts model can be got [4, 28]. The first generalization of the chiral Potts model to the $sl(n)$ case was proposed in [3, 9] and corresponding formulae for intertwiners of minimal cyclic representations were written.

In this section we will introduce a special class of elementary A-representations – factorizable representations. For intertwiners of cocommuting factorizable representations explicit formulae will be given. Although minimal representations are not A-representations the same factors as in Boltzmann weights of the $sl(n)$ chiral Potts model [3, 9] happen to be employed here (cf. [16]).

Let us take a two-dimensional subspace $\Pi \subset \mathbb{C}^{2n}$ and introduce a couple (Γ, Φ), where Γ is a variety:

$$\Gamma = \{p \in \mathbb{C}^{2n} \mid \langle p \rangle \in \Pi\}\,,$$

$$p = \begin{pmatrix} a_1 & \ldots & a_n \\ b_1 & \ldots & b_n \end{pmatrix}, \quad \langle p \rangle = \begin{pmatrix} a_1^N & \ldots & a_n^N \\ b_1^N & \ldots & b_n^N \end{pmatrix},$$

and Φ is an n by n matrix such that $\Phi_{ij}^N \Phi_{kl}^{-N} = \dfrac{\partial(a_i^N, b_j^N)}{\partial(a_k^N, b_l^N)}$. Here the right-hand side is a Jacobian calculated on the subspace Π. Always later we shall refer only to Γ implying the couple (Γ, Φ). Let $\mathring{\mathscr{W}}$ be the quotient of the algebra $\mathscr{W}$ modulo relations $\mathbf{F}\mathbf{G}^{-1} = 1$,

$$F_i^N = 1, \quad G_i^N = \omega_{1i}\omega_{i1}^{-1}, \quad H_i^N = 1, \quad i = 1, \ldots, n$$

and $\mathscr{Z}$ be the center of $\mathring{\mathscr{W}}$. We shall retain the same notations for generators in case of $\mathring{\mathscr{W}}$ keeping in mind new extra relations. One can see that $\mathscr{Z}$ is generated by $f_i = F_i \prod_l H_l^{-\varepsilon_{il}}$, $i = 1, \ldots, n$.

Define the simplest L-operator $L(u, p) \in \mathscr{M} \otimes \mathring{\mathscr{W}}$ as follows:

$$L_{ij}(u, p) = \Phi_{ii}^{-1}(-ua_i F_i \delta_{ij} + (-u)^{\theta_{ij}} b_i G_i \delta_{i+1,j}) . \tag{5.1}$$

Try to find a solution of the "skew intertwining" relation

$$\begin{gathered} S(p, \tilde{p}) L_2(u, \tilde{p}^1) L_1(u, p) = L_2(u, p^1) L_1(u, \tilde{p}) S(p, \tilde{p}) , \\ [S(p, \tilde{p}), H_i \otimes H_i] = 0 , \end{gathered} \tag{5.2}$$

where $S(p, \tilde{p}) \in \mathring{\mathscr{W}}^{\otimes 2}$,

$$p^1 = \begin{pmatrix} a_1 & \ldots & a_{n-1} & a_n \\ b_2 & \ldots & b_n & b_1 \end{pmatrix}, \quad \tilde{p}^1 = \begin{pmatrix} \tilde{a}_1 & \ldots & \tilde{a}_{n-1} & \tilde{a}_n \\ \tilde{b}_2 & \ldots & \tilde{b}_n & \tilde{b}_1 \end{pmatrix}$$

and subscripts indicate the way of embedding $\mathring{\mathscr{W}} \subset \mathring{\mathscr{W}}^{\otimes 2}$ as corresponding factor. Introduce elements

$$J_i = F_{i+1}^{-1} G_i \otimes G_i^{-1} F_i, \quad K_i = (H_{i+1}^{-1} \otimes H_i) J^i$$

such that $J_i^N = K_i^N = (-1)^{N-1}$ and define the subalgebra $\mathscr{T}_m \subset \mathring{\mathscr{W}}^{\otimes m}$ generated by

$$J_i(k) = 1^{\otimes(k-1)} \otimes J_i \otimes 1^{\otimes(m-k-1)} \quad \begin{matrix} i = 1, \ldots, n \\ k = 1, \ldots, m-1 \end{matrix} .$$

Define also the subalgebra $\mathscr{K}_m \subset \mathring{\mathscr{W}}^{\otimes m}$ generated by $\mathscr{Z}^{\otimes m}$ and

$$K_i(k) = 1^{\otimes(k-1)} \otimes K_i \otimes 1^{\otimes(m-k-1)} \quad \begin{matrix} i = 1, \ldots, n \\ k = 1, \ldots, m-1 \end{matrix} .$$

Lemma 5.1. *Let $p, \tilde{p}$ belong to the same variety Γ and $\langle p \rangle \neq \langle \tilde{p} \rangle$. Then there exists generically a unique modulo $\mathscr{K}_2$ solution $S(p, \tilde{p})$ of Eq.* (5.2):

$$S(p, \tilde{p}) = \sum_{s \in \mathbb{Z}_N^n} W_{p\tilde{p}}(\mathbf{s}) \omega^{s_1 s_n} \prod_i \omega^{(1-s_i)s_i/2} J_1^{s_1} \ldots J_n^{s_n} , \tag{5.3}$$

where

$$\begin{gathered} W_{p\tilde{p}}(\mathbf{s}) = \left(\frac{\Phi_{ii}^N}{b_i^N \tilde{a}_i^N - a_i^N \tilde{b}_i^N} \right)^{\frac{s_n - s_0}{N}} \prod_i \prod_{j=1}^{s_i - s_{i-1}} \frac{b_i \tilde{a}_i \omega - a_i \tilde{b}_i \omega^j}{\Phi_{ii}} , \\ s_{i-1} \leqq s_i, \quad i = 1, \ldots, n, \quad s_0 = s_n (\mathrm{mod}\ N) . \end{gathered} \tag{5.4}$$

(Cf. (0.5), (0.6) from [3].)

Remark. The first ratio in the r.h.s. of Eq. (5.4) actually does not depend on i. Inequalities there describe a convenient choice of the representative for $\mathbf{s}$.

Lemma 5.2. *$S(p, \tilde{p})$ satisfies the inversion relation*

$$S(p, \tilde{p})S(\tilde{p}, p) = \mathfrak{J}(p, \tilde{p})$$

and the skew Yang–Baxter equation:

$$\begin{aligned}&(S(\tilde{p}, \hat{p}) \otimes 1)(1 \otimes S(p^1, \hat{p}^1))(S(p, \tilde{p}) \otimes 1)\\&= \varrho(p, \tilde{p}, \hat{p})(1 \otimes S(p, \tilde{p}))(S(p^1, \hat{p}^1) \otimes 1)(1 \otimes S(\tilde{p}, \hat{p}))\end{aligned}$$

where $\varrho(p, \tilde{p}, \hat{p})$ is a nonzero scalar and

$$\mathfrak{J}(p, \tilde{p}) = N^{n+1} \prod_i \frac{b_i \tilde{a}_i - a_i \tilde{b}_i}{b_i^N \tilde{a}_i^N - a_i^N \tilde{b}_i^N} \cdot \frac{\prod_i b_i^N \tilde{a}_i^N - \prod_i a_i^N \tilde{b}_i^N}{\prod_i b_i \tilde{a}_i - \prod_i b_i \tilde{a}_i} .$$

This lemma corresponds to Theorem 4.1 from [10] and the inversion relation (0.8) from [3] or (A.1) from [17]. (It should be noted that for $p \in \Gamma$ we suppose that $p^1 \in \Gamma^1$ with $\Phi_{ij}^1 = \Phi_{i, j+1}$.)

Introducing the products $\mathrm{L}^m(u, \mathbf{p})$ and $\mathrm{S}(\mathbf{p}, \tilde{\mathbf{p}})$:

$$\mathrm{L}^m(u, \mathbf{p}) = L_m(u, p_m^m) \cdot \ldots \cdot L_1(u, p_1^1) \in \mathscr{M} \otimes \mathring{\mathscr{W}}^{\otimes m} ,$$

$$\mathbf{p} = (p_1 \ldots p_m) \in \Gamma^{\times m} ,$$

$$\mathrm{S}(\mathbf{p}, \tilde{\mathbf{p}}) = \prod_i \prod_{j=i+1}^{i+n} S(p_i^i, \tilde{p}_{j-i}^{j-i}) \in \mathring{\mathscr{W}}^{\otimes 2m}$$

(i is increasing and j is decreasing from left to right in this product), we get usual intertwining relation

$$\mathrm{S}(\mathbf{p}_1, \mathbf{p}_2)\mathrm{L}_2^n(u, \mathbf{p}_2)\mathrm{L}_1^n(u, \mathbf{p}_1) = \mathrm{L}_1^n(u, \mathbf{p}_1)\mathrm{L}_2^n(u, \mathbf{p}_2)\mathrm{S}(\mathbf{p}_1, \mathbf{p}_2) , \tag{5.5}$$

where subscripts indicate embeddings $\mathring{\mathscr{W}}^{\otimes n} \subset \mathring{\mathscr{W}}^{\otimes n} \otimes \mathring{\mathscr{W}}^{\otimes n}$.

Lemma 5.3. $\mathrm{S}(\mathbf{p}_1, \mathbf{p}_2)$ *satisfies the Yang–Baxter equation*

$$\begin{aligned}&(\mathrm{S}(\mathbf{p}_2, \mathbf{p}_3) \otimes \mathbf{1})(\mathbf{1} \otimes \mathrm{S}(\mathbf{p}_1, \mathbf{p}_3))(\mathrm{S}(\mathbf{p}_1, \mathbf{p}_2) \otimes \mathbf{1})\\&= (\mathrm{S}(\mathbf{p}_1, \mathbf{p}_2) \otimes \mathbf{1})(\mathbf{1} \otimes \mathrm{S}(\mathbf{p}_1, \mathbf{p}_3))(\mathrm{S}(\mathbf{p}_2, \mathbf{p}_3) \otimes \mathbf{1})\end{aligned}$$

where $\mathbf{1} = 1^{\otimes n}$.

To prove the announced lemmas we have to study some extra subalgebras.

Lemma 5.4. *Let us consider the subalgebra $\mathscr{S} \subset \mathring{\mathscr{W}}$ generated by $F_{i+1}^{-1} G_i$, $i = 1, \ldots, n$. Then $\mathscr{S}'$ – the commutant of $\mathscr{S}$ is generated by $F_i^{-1} G_i$, $i = 1, \ldots, n$ and $\mathscr{Z}$.*

Proof. Commutation relations in $\mathring{\mathscr{W}}$ are homogeneous, so modulo factors belonging to $\mathscr{Z}$ we have to test only monomials of H_i's and G_i's. But $E = \prod_i H_i^{\mu_i} G_i^{\nu_i} \in \mathscr{S}'$ if and only if $\mu_{i+1} - \mu_i = \sum_j \nu_j(\varepsilon_{ij} - \varepsilon_{i+1, j})$, so $E \in \prod_i (F_i^{-1} G_i)^{\nu_i} \mathscr{Z}$. □

Lemma 5.5. *The commutant of the subalgebra $\mathscr{L}_m^\circ \subset \mathring{\mathscr{W}}^{\otimes m}$ generated by $\{H_i^{\otimes m}, F_i^{\otimes (m-k)} \otimes G_{i-1} \otimes \ldots \otimes G_{i-k}\}_{i=1}^{n}{}_{k=1}^{m}$ is equal to $\mathscr{K}_m$.*

Proof. Denote the commutant of $\mathscr{L}_m^\circ$ by $\mathscr{L}_m'$. One can check that $\mathscr{K}_m \subset \mathscr{L}_m'$. Obviously, $1^{\otimes (m-1)} \otimes F_{i+1}^{-1} G_i \in \mathscr{L}_m^\circ$ so $\mathscr{L}_m' \subset \mathring{\mathscr{W}}^{\otimes (m-1)} \otimes \mathscr{S}'$. This imply that $\mathscr{L}_m'$ is generated by $\mathscr{L}_{m-1}' \otimes 1$ and $1^{\otimes (m-2)} \otimes \mathscr{K}_2$. Step by step we can reduce the problem to $m = 1$ and show that $\mathscr{L}'_m$ is generated by $\mathscr{L}'_1 \otimes 1^{\otimes(m-1)}$ and $\mathscr{K}_m$. But $\mathscr{L}_1^\circ = \mathring{\mathscr{W}}$ and $\mathscr{L}'_1 = \mathscr{Z}$. □

Lemma 5.6. *Let $\mathscr{L}_m(\mathbf{p}) \subset \mathring{\mathscr{W}}^{\otimes m}$ be the subalgebra generated by $H_i^{\otimes m}$, $i = 1, \ldots, n$ and all entries of $\mathrm{L}^m(u, \mathbf{p})$. The commutant of $\mathscr{L}_m(\mathbf{p})$ is equal to $\mathscr{K}_m$ for generic $\mathbf{p}$.*

Proof. One can check that $\mathscr{K}_m$ commute with $\mathscr{L}_m(\mathbf{p})$. So it is enough to prove the statement only for one variety Γ and one point $\mathbf{p} \in \Gamma^{\times m}$. We shall use the trick of the "trigonometric limit" [10]. Let us take Γ containing $p^\circ = \begin{pmatrix} 1 \ldots 1 \\ 0 \ldots 0 \end{pmatrix}$ and tend $p_i \to p^\circ$, $i = 1, \ldots, m$ one after another. In this limit $\mathscr{L}_m(\mathbf{p})$ goes to $\mathscr{L}_m^\circ$ which commutant is equal to $\mathscr{K}_m$ according to the previous lemma. □

Proof of Lemma 5.1. Substituting the expressions (5.3) into Eq. (5.2) we get identities

$$[S(p, \tilde{p}), G_{i+1} \otimes G_i] = [S(p, \tilde{p}), H_i \otimes H_i] = 0$$

and equations

$$S(p, \tilde{p}) F_{i+1} \otimes G_i \left(\frac{b_i \tilde{a}_i}{\Phi_{ii}} J_i + \frac{a_{i+1} \tilde{b}_{i+1}}{\Phi_{i+1\,i+1}} \right)$$
$$= F_{i+1} \otimes G_i \left(\frac{a_i \tilde{b}_i}{\Phi_{ii}} J_i + \frac{b_{i+1} \tilde{a}_{i+1}}{\Phi_{i+1\,i+1}} \right) S(p, \tilde{p})$$

which together with commutation relations

$$J_i J_j = J_j J_i \omega^{\delta_{i+1,j} - \delta_{i,j+1}}, \quad J_i (F_{j+1} \otimes G_j) = (F_{j+1} \otimes G_j) J_i \omega^{\delta_{i,j+1} - \delta_{ij}}$$

lead to functional equations for $W_{p\tilde{p}}(\mathbf{s})$:

$$\frac{W_{p\tilde{p}}(\mathbf{s})}{W_{p\tilde{p}}(\mathbf{s} - \mathbf{e}_i)} = \frac{\Phi_{i+1,i+1}(\omega b_i \tilde{a}_i - a_i \tilde{b}_i \omega^{s_i - s_{i-1}})}{\Phi_{ii}(\omega b_{i+1} \tilde{a}_{i+1} - a_{i+1} \tilde{b}_{i+1} \omega^{s_{i+1} - s_i + 1})},$$
$$\mathbf{e}_i = (0, \ldots, \underset{i^{\text{th}}}{1}, \ldots, 0).$$

The formula (5.4) gives a solution of these equations. Clearly $S(p, p) = 1 \otimes 1$ so $S(p, \tilde{p})$ is generically invertible. If $\hat{S}(p, \tilde{p})$ is another solution of Eq. (5.2) then the ratio $S^{-1}(p, \tilde{p}) \hat{S}(p, \tilde{p})$ commutes with $\mathscr{L}_2((p, \tilde{p}))$ and, hence, generically belongs to $\mathscr{K}_2$. □

Lemma 5.7. *The intersection $\mathscr{T}_m \cap \mathscr{K}_m$ is generated by scalars.*

Proof. It is easy to see that $\mathscr{T}_m \cap \mathscr{K}_m \subset \mathscr{Z}^{\otimes m}$, but it is also clear that $\mathscr{T}_m \cap \mathscr{Z}^{\otimes m}$ is generated by scalars. □

Proof of Lemma 5.2. $\mathfrak{J}(p, \tilde{p})$ commutes with $\mathscr{L}_2(p, \tilde{p})$, so generically $\mathfrak{J}(p, \tilde{p}) \in \mathscr{T}_2 \cap \mathscr{K}_2$ and hence is a scalar. Therefore $S^{-1}(p, \tilde{p}) \in \mathscr{T}_2$ and we see that $\varrho(p, \tilde{p}, \hat{p})$ commutes with $\mathscr{L}_3(p, \tilde{p}, \hat{p})$ which follows to $\varrho(p, \tilde{p}, \hat{p}) \in \mathscr{T}_3 \cap \mathscr{K}_3$. The explicit formula for $\mathfrak{J}(p, \tilde{p})$ can be obtained in the same way as the inversion relation (A.1) from [17]. □

Proof of Lemma 5.3. Consider the ratio

$$\mathfrak{R}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = ((\mathrm{S}(\mathbf{p}_2, \mathbf{p}_3) \otimes \mathbf{1})(\mathbf{1} \otimes \mathrm{S}(\mathbf{p}_1, \mathbf{p}_3))(\mathrm{S}(\mathbf{p}_1, \mathbf{p}_2) \otimes \mathbf{1}))^{-1}$$
$$\times (\mathrm{S}(\mathbf{p}_1, \mathbf{p}_2) \otimes \mathbf{1})(\mathbf{1} \otimes \mathrm{S}(\mathbf{p}_1, \mathbf{p}_3))(\mathrm{S}(\mathbf{p}_2, \mathbf{p}_3) \otimes \mathbf{1}).$$

Similar to the previous proof $\mathfrak{R}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \in \mathcal{T}_{3m} \cap \mathcal{K}_{3m}$ and is a scalar. So it is represented by the same scalar in any representation of $\mathring{\mathcal{W}}^{\otimes 3m}$. Let σ be a nonzero representation of $\mathring{\mathcal{W}}$. Taking the representation $\sigma^{\otimes 3m}$ of $\mathring{\mathcal{W}}^{\otimes 3m}$ and computing $\det \sigma^{\otimes 3m}(\mathcal{R}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)) = 1$ we see that $\mathfrak{R}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ is a root of 1. Hence it is constant. In conclusion, it is clear that $\mathfrak{R}(\mathbf{p}, \mathbf{p}, \mathbf{p}) = 1$ if $\mathbf{p} = (p, \ldots, p)$. □

Similar to (3.2) the mapping $\phi_m(\mathbf{p})\colon \mathcal{A} \to \mathring{\mathcal{W}}^{\otimes m}$:

$$T(u) \to (-u)^{1-m}\mathrm{L}^m(u, \mathbf{p}), \quad H_l \to H_l^{\otimes m}$$

is a homomorphism of algebras. Let $\mathcal{W}^\circ$ be the quotient of algebra $\mathring{\mathcal{W}}$ over relations $f_i = 1, i = 1, \ldots, n$ and $\iota\colon \mathring{\mathcal{W}} \to \mathcal{W}^\circ$ be the canonical projection. One can check that $\mathcal{W}^\circ$ is a simple algebra isomorphic to $(\operatorname{End} \mathbb{C}^N)^{\otimes(n-1)}$. Let σ° be the irreducible representation of $\mathcal{W}^\circ$, $\sigma = \sigma^\circ \circ \iota$ and consider the representation $\pi_m(\mathbf{p}) = \sigma^{\otimes m} \circ \phi_m(\mathbf{p})$ of the algebra $\mathcal{A}$.

Lemma 5.8. *$\pi_m(\mathbf{p})$ is completely reducible for generic* $\mathbf{p}$.

Proof. It is clear that any irreducible representation of $\mathcal{W}^\circ$ can be obtained from the construction of the Lemma 3.1 by proper choosing of parameters. In particular it means that all generators of $\mathcal{W}^\circ$ are represented in σ° by unitary operators and the same is the fact for generators of $\mathcal{L}_m^\circ$ in the representation $\sigma^{\otimes m}$ modulo scalar factors. Hence $\sigma^{\otimes m}$ is completely reducible with respect to $\mathcal{L}_m^\circ$ and generically with respect to $\mathcal{L}_m(\mathbf{p})$. (Use "trigonometric limit.") Since $\operatorname{im} \phi_m(\mathbf{p}) = \mathcal{L}_m(\mathbf{p})$ the statement is proved. □

Lemma 5.9. *Invariant subspaces of $\pi_m(\mathbf{p})$ are invariant with respect to $\sigma^{\otimes m}(\mathcal{T}_m)$.*

Proof. It suffices to prove the statement only for generic $\mathbf{p}$, where $\pi_m(\mathbf{p})$ is completely reducible. Moreover, we can look to only irreducible subspaces. Let P be projector onto such subspace along all others. As $(\sigma^\circ)^{\otimes m}$ is the faithful irreducible representation of $(\mathcal{W}^\circ)^{\otimes m}$ we can write $P = \sigma^{\otimes m}(\mathcal{O})$ with some $\mathcal{O}$ belonging to the commutant of $\mathcal{L}_m(\mathbf{p})$ which is equal to $\mathcal{K}_m$ for generic $\mathbf{p}$. Therefore $\mathcal{O}$ commute with $\mathcal{T}_m$ and $\operatorname{im} P$, $\ker P$ are invariant with respect to $\sigma^{\otimes m}(\mathcal{T}_m)$. □

Corollary. Invariant subspaces of $\pi_m(\mathbf{p})$ do not generically depend on $\mathbf{p}$.

Proof. Let the subalgebra $\mathcal{L}_m^\dagger \subset \mathring{\mathcal{W}}^{\otimes m}$ be generated by $\mathcal{L}_m^\circ$ and $\mathcal{T}_m$. Clearly for any $\mathbf{p}$ $\mathcal{L}_m(\mathbf{p}) \subset \mathcal{L}_m^\dagger$. Together with the lemma it means that invariant subspaces of $\sigma^{\otimes m}$ with respect to $\mathcal{L}_m^\dagger$ are also invariant subspaces of $\pi_m(\mathbf{p})$ for generic $\mathbf{p}$ and vice versa. □

Lemma 5.10. *Irreducible parts of $\pi_n(\mathbf{p})$ are irreducible A-representations for generic* $\mathbf{p}$.

Proof. It is sufficient to consider only one variety Γ. Let us take it such that $\begin{pmatrix} a, \ldots, a \\ b, \ldots, b \end{pmatrix} \in \Gamma$ for any a, b. One can easily reduce the problem to the following one: To prove that generically $\mathcal{U}(v) = (-v)^{-1}\prod_k U^{(k)}(v) \in A\mathcal{M}[v]$, where $U_{ij}^{(k)} = -v\delta_{ij} + (-v)^{\theta_{ij}} b_k \delta_{i+1,j}$. Computing $\mathcal{U}(v)$ explicitly we can see that $\mathcal{U}_{ij}(v) = (d_n - v)\delta_{ij} + (-v)^{\theta_{ij}} d_l \delta_{i+1,j}$, where $\prod_k (b_k - v) = \sum_l d_l v^{n-l}$. Taking d_1, d_{n-1}, $d_n \neq 0$ and $d_l = 0$ otherwise we obtain that $A_1(v) = d_n - v$, $B_1(v) = -vd_1$ if $l = 0$ and $A_{l+1} = (d_n - v)^{l+1} + v^l d_1^l d_{n-1}$, $B_{l+1} = -vd_1(d_n - v)^l$, $C_{l+1} = v^{l-1} d_1^{l-1} d_{n-1}(v - d_n)$ if $l > 0$. Therefore generically $\mathcal{U}(v) \in A\mathcal{M}[v]$. □

Definition 5.1. *Irreducible parts of* $\pi_n(\mathbf{p})$ *are called factorizable representations.*

Finally, we have got the following picture. Let V be an irreducible subspace of $\sigma^{\otimes n}$ with respect to $\mathscr{L}_n^\dagger$ and $\pi(\mathbf{p}, V) = \pi_n(\mathbf{p})|_V$. One can see that $\dim V = N^{n(n-1)/2}$. The subspace V suffices to collect all factorizable representations because for any $\mathbf{p}$ and irreducible subspace V' one can find $\mathbf{p}'$ such that $\pi(\mathbf{p}, V') = \pi(\mathbf{p}', V)$, $\mathbf{p}, \mathbf{p}' \in \Gamma^{\times n}$. Let $\mathbf{P}$ be the permutation operator corresponding to $\sigma^{\otimes n} \otimes \sigma^{\otimes n}$. Then by virtue of (5.5) the representations $\pi_n(\mathbf{p})$ and $\pi_n(\mathbf{p}')$ are cocommuting if $\mathbf{p}, \mathbf{p}'$ are in the same variety $\Gamma^{\times n}$ and $\mathbf{R}(\mathbf{p}, \mathbf{p}') = \mathbf{P}\sigma^{\otimes 2n}(\mathrm{S}(\mathbf{p}, \mathbf{p}'))$ is their intertwiner in the sense of Eq. (4.1). $\mathbf{R}(\mathbf{p}, \mathbf{p}')$ can be restricted to $V \otimes V$ giving the intertwiner for cocommuting factorizable representations $\pi(\mathbf{p}, V)$, $\pi(\mathbf{p}', V)$. So we have got an explicit formula for an intertwiner of special elementary representations – factorizable representations. Unfortunately, counting of parameters shows that factorizable representations do not cover the total set of elementary representations. On the other hand it is not surprising because we can see from Lemma 4.3 that a generic variety of cocommuting elementary representations is 3-dimensional but a variety of cocommuting factorizable representations is at least $(n + 1)$-dimensional, which is larger for $n > 2$.

Remark. It is well known that any solution of the Yang–Baxter equation can be considered as a matrix of Boltzmann weights (maybe complex) for some solvable lattice vertex model with states on edges. In particular, $\mathrm{S}(\mathbf{p}, \mathbf{p}')$ and $\mathbf{R}(\mathbf{p}, \mathbf{p}')$ also define some $sl(n)$ generalizations of the chiral Potts model with N^{n^2} and $N^{(n-1)n/2}$ local states per edge respectively. The first obtained model is reducible and contains the second one as an irreducible part. The second model is equivalent to the model considered in [16]. A discussion of these models in more details will be done in the forthcoming paper.

6. Quantum Minors and Quantum Determinant

Now we want to discuss some technical problems skipped before. In this section it is not necessary to suppose that ε_{ij} are integers and ω is a root of 1. Only the condition $\omega_{ij}\omega_{ji} = \omega^{1+\delta_{ij}}$ is assumed. It is more convenient to study a little bit more general situation. We introduce a new R-matrix $\bar{R}(u)$ replacing in Eq. (1.1) a tensor ε by a similar tensor $\bar{\varepsilon}$ and change the definition of the algebra $\mathscr{A}$ substituting $\bar{R}(u)$ instead of $R(u)$ in the left-hand side of the relation (1.2):

$$\bar{R}(u)\overset{1}{T}(uv)\overset{2}{T}(v) = \overset{2}{T}(v)\overset{1}{T}(uv)R(u)\,. \tag{6.1}$$

Let $V = \mathbb{C}^n$ and $e_1, \ldots, e_n$ be the canonical basis of V. Later we regard monodromy matrices as matrices over $\mathscr{A}$, naturally acting in the $\mathscr{A}$-bimodule $V_{\mathscr{A}} = \mathscr{A} \otimes_{\mathbb{C}} V$. We assume the embedding $1 \otimes \mathrm{id}$: $V \to V_{\mathscr{A}}$ taking place. Let us introduce the $\mathscr{A}$-bimodules $V^{\otimes m} = \underbrace{V_{\mathscr{A}} \otimes_{\mathscr{A}} \ldots \otimes_{\mathscr{A}} V_{\mathscr{A}}}_{m} = \mathscr{A} \otimes_{\mathbb{C}} V^{\otimes m}$ and their submodules $V_{\mathscr{A}}^{\wedge m} = \mathscr{A} \otimes_{\mathbb{C}} V^{\wedge m}$, $V^{\wedge m}$ being spanned by completely antisymmetric tensors. Define $b_m \in \operatorname{End} V^{\otimes m}$ as follows:

$$b_m e_{i_1} \otimes \ldots \otimes e_{i_m} = \prod_{l=1}^{m} \prod_{k=1}^{l-1} \omega_{i_k i_l}^{\theta_{i_l i_k}} e_{i_1} \otimes \ldots \otimes e_{i_m}\,.$$

$\bar{b}_m$ is defined by the similar formula with $\bar{\omega}_{ij} = \omega^{\bar{\varepsilon}_{ij}}$. One can check that

$$V^{\wedge m} = b_m\left(\bigcap_{k=1}^{m-1} \ker \overset{k,k+1}{R(\omega)}\right) = \bar{b}_m\left(\bigcap_{k=1}^{m-1} \ker \overset{k,k+1}{\bar{R}(\omega)}\right).$$

As usual the definition of quantum minors is based on the fusion procedure [18]. By virtue of the relation

$$\overset{k,k+1}{\bar{R}(\omega)} \overset{k}{T}(u\omega^{1-k}) \overset{k-1}{T}(u\omega^{-k}) = \overset{k-1}{T}(u\omega^{-k}) \overset{k}{T}(u\omega^{1-k}) \overset{k,k+1}{R(\omega)} \tag{6.2}$$

following from (6.1) $V_{\mathscr{A}}^{\wedge m}$ is an invariant submodule for $T^{\otimes_q m}(u)$:

$$T^{\otimes_q m}(u) = \bar{b}_m \overset{1}{T}(u) \cdot \ldots \cdot \overset{m}{T}(u\omega^{1-m}) b_m^{-1}.$$

Definition 6.1. $T^{\wedge m}(u) = T^{\otimes_q m}(u)|_{V_{\mathscr{A}}^{\wedge m}}$, $\det_q T(u) = T^{\wedge n}(u)$. *Entries of* $T^{\wedge m}(u)$ *are called quantum minors and* $\det_q T(u)$ *is called the quantum determinant.*

Proof of Lemma 1.2. Equation (1.5) gives the correct coproduct only in the original case: $R(u) = \bar{R}(u)$, $b_m = \bar{b}_m$. In this case it is obvious from the definition that $\Delta(T^{\wedge m}(u)) = T_1^{\wedge m}(u) T_2^{\wedge m}(u)$. □

Proof of Lemma 1.1. For a moment we have to indicate explicitly R-matrices taking part in the relations defining the algebra of monodromy matrices. Three such algebras are necessary: $\mathscr{A} = \mathscr{A}_{\bar{R}R}$, $\mathscr{A}_{RR}$ and $\mathscr{A}_{\bar{R}\bar{R}}$. The Yang–Baxter equation shows that R-matrices $R(u)$, $\bar{R}(u)$ generate some representations χ, $\bar{\chi}$ of the algebras $\mathscr{A}_{RR}$, $\mathscr{A}_{\bar{R}\bar{R}}$ in $\mathbb{C}^n$ respectively. Taking the m^{th} tensor power of Eq. (6.1) and using the definition of the quantum determinant we have got

$$\det_q T(u) \bar{\rho}(u/v) T(v) = T(v) \rho(u/v) \det_q T(u),$$

where $$\rho(u) = f(u)(\det_q T_{RR})^{\chi}(u), \quad \bar{\rho}(u) = f(u)(\det_q T_{\bar{R}\bar{R}})^{\bar{\chi}}(u), \tag{6.3}$$

and $f(u)$ is an arbitrary scalar factor. The easiest way to calculate $\rho(u)$, $\bar{\rho}(u)$ explicitly is to use different expressions for $\det_q T(u)$ for calculating different entries of ρ, $\bar{\rho}$. For each entry the most convenient expression has only one nonzero term. As a result the matrices ρ, $\bar{\rho}$ can be written as follows:

$$\rho(u) = \prod_{k,l} \hat{\omega}_l^{\varepsilon_{kl}}, \quad \bar{\rho}(u) = \prod_{k,l} \hat{\omega}_l^{\bar{\varepsilon}_{kl}},$$

and using Eq. (1.4) in case of $R(u) = \bar{R}(u)$ completes the proof. □

Corollary. $[\det_q T(u), \det_q T(v)] = 0$.

Proof. It follows from (6.3) since $\det \rho = \det \bar{\rho}$. □

Let us identify V with $V^{\wedge(m-1)}$ and $V^{\wedge 2}$ with $V^{\wedge(m-2)}$ as follows:

$$e_i \leftrightarrow (-1)^{n-i} e_i \wedge \ldots \wedge e_{i-1} \wedge e_{i+1} \wedge \ldots \wedge e_n$$

$$e_i \wedge e_j \leftrightarrow (-1)^{i+j} e_1 \wedge \ldots \wedge e_{i-1} \wedge e_{i+1} \wedge \ldots \wedge e_{j-1} \wedge e_{j+1} \wedge \ldots \wedge e_n$$

$$i < j \tag{6.4}$$

and take the elements written above as standard basic elements for these spaces.

Lemma 6.1.

$$T(u)d^{-1}(T^{\wedge(n-1)}(u\omega^{-1}))^t\bar{d} = \det_q T(u)\,, \tag{6.5}$$

$$T^{\wedge 2}(u)\ell^{-1}(d^{\wedge 2})^{-1}(T^{\wedge(n-2)}(u\omega^{-2}))^t\bar{d}^{\wedge 2}\bar{\ell} = \det_q T(u)\,,$$

$$d = \prod_l \prod_{k=1}^{l-1} \hat{\omega}_l^{\varepsilon_{lk}}, \quad \bar{d} = \prod_l \prod_{k=1}^{l-1} \hat{\omega}_l^{\bar{\varepsilon}_{lk}}\,,$$

$$\ell e_i \wedge e_j = \omega_{ij} e_i \wedge e_j, \quad \bar{\ell} e_i \wedge e_j = \bar{\omega}_{ij} e_i \wedge e_j, \quad \ell, \bar{\ell} \in \operatorname{End} V^{\wedge 2}\,. \tag{6.6}$$

Proof. One has the natural embeddings $V_{\mathscr{A}}^{\wedge n} \subset V_{\mathscr{A}} \otimes_{\mathscr{A}} V_{\mathscr{A}}^{\wedge(n-1)} \subset V^{\otimes m}$ and $V_{\mathscr{A}}^{\wedge n} \subset V_{\mathscr{A}}^{\wedge 2} \otimes_{\mathscr{A}} V_{\mathscr{A}}^{\wedge(n-2)} \subset V^{\otimes m}$, so $\det_q T(u)$ can be calculated in two steps. At first $T^{\otimes_q m}(u)$ is restricted to the tensor product $V_{\mathscr{A}} \otimes_{\mathscr{A}} V_{\mathscr{A}}^{\wedge(n-1)}$ or $V_{\mathscr{A}}^{\wedge 2} \otimes_{\mathscr{A}} V_{\mathscr{A}}^{\wedge(n-2)}$ and then to $V_{\mathscr{A}}^{\wedge n}$. Taking into account relations (6.4) in this calculation we obtain the statement. □

Corollary.

$$\check{R}(u)\overset{1}{T}{}^{\wedge(n-1)}(uv)\overset{2}{T}{}^{\wedge(n-1)}(v) = \overset{2}{T}{}^{\wedge(n-1)}(v)\overset{1}{T}{}^{\wedge(n-1)}(uv)\hat{R}(u)\,,$$

$$\hat{R}(u) = (\rho \otimes I)(R(u))^t(I \otimes \rho)^{-1}, \quad \check{R}(u) = (\bar{\rho} \otimes I)(\bar{R}(u))^t(I \otimes \bar{\rho})^{-1}\,. \tag{6.7}$$

Proof. One can transform Eq. (6.1) to this formulae using Lemma 1.1 and Eq. (6.5). □

Let us also introduce $\tilde{T}(u)$ as follows:

$$\tilde{T}(u) = \ell b_2 \overset{2}{\rho}{}^{-1}(\overset{2}{T}{}^{\wedge(n-1)}(u\omega^{-1}))^t(\overset{1}{T}{}^{\wedge(n-1)}(u))^t \frac{2}{\rho}\bar{b}_2^{-1}\bar{\ell}^{-1}\,.$$

Equation (6.7) shows that $V_{\mathscr{A}}^{\wedge 2}$ is an invariant submodule for $\tilde{T}(u)$ and one can put $\hat{T}(u) = \tilde{T}(u)|_{V_{\mathscr{A}}^{\wedge 2}}$. Using Eqs. (6.3)–(6.6) one can show that

$$(\hat{T}(u))^t = \det_q T(u) T^{\wedge(n-2)}(u\omega^{-1})\,. \tag{6.8}$$

Due to the structure of the R-matrices (see Eq. (1.1)) we can consider submatrices of $T(u)$ as monodromy matrices of smaller size: commutation relations inside a submatrix are also described by the relation (6.1) if one substitutes there for the original matrices submatrices of $T(u)$, $R(u)$, $\bar{R}(u)$ corresponding each others. And quantum minors of $T(u)$ are quantum determinants of its submatrices treated as smaller monodromy matrices. This is the important thing permitting us to compute commutation relations of quantum minors step by step by means of Eq. (6.5), (6.7), (6.8).

Lemma 6.2. *Let $T_{\mathbf{ij}}$, $T_{\mathbf{kl}}$ be quantum minors and one of them includes another. Then*

$$T_{\mathbf{ij}}(u)T_{\mathbf{kl}}(u) = T_{\mathbf{kl}}(u)T_{\mathbf{ij}}(u)\Psi_{\mathbf{jl}}\bar{\Psi}_{\mathbf{ik}}^{-1}\,,$$

$$\bar{\Psi}_{\mathbf{ik}} = \prod_{i \in \mathbf{i}} \prod_{k \in \mathbf{k}} \bar{\omega}_{ik}, \quad \Psi_{\mathbf{jl}} = \prod_{j \in \mathbf{j}} \prod_{l \in \mathbf{l}} \omega_{jl}\,,$$

where bold letters are multi-indices.

Proof. If the smaller minor is an entry of $T(u)$ the statement follows from the proof of Lemma 1.1 because the larger minor can be considered as a quantum determinant. The general case can be got simply by multiplications. □

Now we can prove relations (2.2)–(2.5), (2.17) for quantum minors. Some of Eq. (2.2) and (2.3) are evident and others follow from Lemma 6.2. Equation (2.4) can be obtained from the relation (6.7) applied to the principal submatrix generated by the first $(i+1)$ rows and columns (its quantum determinant is the quantum minor $\hat{A}_{i+1}(u)$). The relation (6.8) applied to the same submatrix leads to Eq. (2.5). And the same relation applied to the submatrices generating quantum minors $\hat{B}_{kl}(u)$ and $\hat{C}_{kl}(u)$ gives Eq. (2.17).

7. Comultiplication of Central Elements

The fusion procedure is also very helpful in handling of central elements. Now we again require ω to be a primitive N^{th} root of 1 but ε_{ij} can still be complex. Let W^m be the kernel of the complete symmetrizing projector in $V^{\otimes m}$. It is clear that $\ker \overset{k,k+1}{R}(\omega) \subset b_m W^m$ if $k < m$. Define

$$\mathbf{R}^m = \left(\prod_{j=1}^{m} \prod_{i=1}^{j-1} \overset{ij}{R}(\omega^{j-i}) \right) b_m ,$$

both indices growing from right to left. $\mathbf{R}^m$ will be considered as function of ω_{ij}.

Let $V^{\langle\rangle} \subset V^{\otimes N}$ be the subspace spanned by the elements $e_i^{\otimes N} = e_i \otimes \ldots \otimes e_i$, $i = 1, \ldots, n$ and $V^{\langle\rangle}_{\mathscr{A}} = \mathscr{A} \otimes_{\mathbb{C}} V^{\langle\rangle}$.

Lemma 7.1. *Generically* $\ker \mathbf{R}^N = W^N \oplus V^{\langle\rangle}$.

Proof. Using the Yang–Baxter equation (1.2) one can move any factor $\overset{k,k+1}{R}(\omega)$ in the product for $\mathbf{R}^N$ to the very right and show that $W^N \subset \ker \mathbf{R}^N$. It is also clear that $\overset{ij}{R}(u)|_{V^{\langle\rangle}} = 1 - u\omega$. Evidently $W^N \cap V^{\langle\rangle} = 0$. So $W^N \oplus V^{\langle\rangle} \subset \ker \mathbf{R}^N$ and it remains to prove that generically $\dim \ker R^N = \dim W^N + \dim V^{\langle\rangle}$. Here the right-hand side does not depend on ω_{ij} at all and it is enough to calculate the left-hand side only for one special case. Let us test the limit $\omega_{ij} \to 0$ for $i < j$. In this limit

$$\begin{aligned} R(u)e_i \otimes e_i &= (1 - u\omega)e_i \otimes e_i , \\ R(u)e_i \otimes e_j &= (1 - \omega)e_j \otimes e_i + o(1) \qquad\qquad i<j . \\ R(u)e_j \otimes e_i &= \omega_{ij}((1 - u)e_j \otimes e_i + o(1)) , \end{aligned}$$

From these equalities one can see that $\mathbf{R}_0^N = \lim_{\omega_{ij} \to 0} \mathbf{R}^N$ is finite and $\operatorname{im} \mathbf{R}_0^N$ is spanned by $\{e_{i_1} \otimes \ldots \otimes e_{i_N}\colon i_1 \geqq \ldots \geqq i_N,\, i_1 \neq i_N\}$. Hence $\dim \ker \mathbf{R}_0^N = \dim W^N + \dim V^{\langle\rangle}$. But generically $\dim \ker \mathbf{R}^N \leqq \dim \ker \mathbf{R}_0^N$, so the statement is proved. $\square$

Lemma 7.2. *Let $K \in \mathscr{M}^{\otimes N}$ be a projector such that $W \subset \ker K$ and $V^{\langle\rangle} \subset \operatorname{im} K$. Then $KT^{\otimes_q N}(u)|_{W_{\mathscr{A}}} = 0$ and $KT^{\otimes_q N}(u)|_{V^{\langle\rangle}} = \langle T \rangle (u^N)$.*

Proof. By virtue of Eq. (6.2) $W^N_{\mathscr{A}}$ is an invariant submodule for $T^{\otimes_q N}(u)$. Due to Eq. (6.1) one has the relation

$$\mathbf{R}^m T^{\otimes_q m}(u) = \overset{m}{T}(u\omega^{1-m}) \cdot \ldots \cdot \overset{1}{T}(u) \mathbf{R}^m ,$$

which shows that $\mathscr{A}\otimes_{\mathbb{C}}\ker\mathbf{R}^N$ is also an invariant submodule for $T^{\otimes_q N}(u)$. Therefore according to Lemma 7.1 $W_{\mathscr{A}}\oplus V_{\mathscr{A}}^{\langle\rangle}$ is its invariant submodule too and the statment follows from the straightforward computation. □

Proof of Lemma 1.5. This lemma is a corollary of the definition (1.5) of the coproduct and the previous lemma. □

Proof. of Lemma 1.4. Let $\chi,\bar{\chi}$ be the representations of the algebras $\mathscr{A}_{RR}$, $\mathscr{A}_{\bar{R}\bar{R}}$ generated by R-matrices $R(u)$, $\bar{R}(u)$ in $\mathbb{C}^n$. Equation (6.1) and Lemma 7.2 together give

$$\bar{R}^{\langle\rangle}(v)\langle \overset{1}{T}\rangle(u^N v)\overset{2}{T}(u)=\overset{2}{T}(u)\langle \overset{1}{T}\rangle(u^N v)R^{\langle\rangle}(v)\,,$$
$$R^{\langle\rangle}(v)=\langle T\rangle^{\chi}(v),\quad \bar{R}^{\langle\rangle}(v)=\langle T\rangle^{\bar{\chi}}(v)\,.$$

$R^{\langle\rangle}(v)$, $\bar{R}^{\langle\rangle}(v)$ can be calculated easily and are equal to $(1-v)I\otimes I$ if all ε_{ij} are integers. □

Proof of Lemma 2.1. Since all entries of $\langle T\rangle(v)$ mutually commute its minors can be defined as usual. The slightly more general statment will be proved.
Let $T_{\mathrm{ij}}(u)$ be a quantum minor of $T(u)$ and $\langle T\rangle_{\mathrm{ij}}(v)$ be the corresponding minor of $\langle T\rangle(v)$. Then

$$\langle T_{\mathrm{ij}}\rangle(v)=\langle T\rangle_{\mathrm{ij}}(v)\prod_{\substack{i,k\in\mathrm{i}\\ i>k}}\bar{\tau}_{ik}\prod_{\substack{j,l\in\mathrm{j}\\ j>l}}\tau_{jl}\,,$$
$$\bar{\tau}_{ik}=(-1)^{(N-1)\bar{\varepsilon}_{ik}},\quad \tau_{jl}=(-1)^{(N-1)\varepsilon_{jl}}\,. \tag{7.1}$$

As before we treat quantum minors of $T(u)$ as quantum determinants of its submatrices. So we have to prove this formula only for the quantum determinant supposing that it is proved yet for all proper quantum minors. The complete set of formulas for all quantum minors can be obtained by induction with respect to the minor's size. The base of the induction is the case when a minor is simply an entry; in this case the formula (7.1) is tautological. In order to prove the formula (7.1) for the quantum determinant let us take the N^{th} tensor power of Eq. (6.5). Using the commutation relations (6.3) to carry $\det_q T(u)$ through $T(v)$ we come to

$$T^{\otimes_q N}(u)\,\hat{T}^{\otimes_q N}(u)=\langle \det_q T\rangle(u^N)\,,$$
$$\hat{T}^{\otimes_q N}(u)=b_N(d^{\otimes N})^{-1}\prod_i{}^{\otimes}\overset{i}{\rho}{}^{(N-i)}(\overset{N}{T}{}^{\wedge(n-1)}(u))^t\cdot\ldots\cdot(\overset{1}{T}{}^{\wedge(n-1)}(u\omega^{N-1}))^t$$
$$\times\prod_i{}^{\otimes}\overset{i}{\bar{\rho}}{}^{i}\bar{d}^{\otimes N}\bar{b}_N^{-1}\,. \tag{7.2}$$

Let K be the same projector as in Lemma 7.2. By the straightforward computation taking into account Eq. (7.1) for proper minors one can check that

$$K\hat{T}^{\otimes_q N}(u)|_{V^{\langle\rangle}}=\langle T\rangle^{\wedge(n-1)}(u^N)\prod_i\prod_{j=1}^{i-1}\tau_{ij}\bar{\tau}_{ij}\,.$$

Now Eq. (7.2) multiplied by K from the left side gives the required formula

$$\prod_i\prod_{j=1}^{i-1}\tau_{ij}\bar{\tau}_{ij}\langle \det_q T\rangle(v)=\langle T\rangle(v)\langle T\rangle^{\wedge(n-1)}(v)=\det\langle T\rangle(v)\,. \tag{7.3}$$

□

Proof of Lemma 1.7. The only nontrivial property to be checked is

$$\det\langle T\rangle^{\pi}(v) = \langle Q^{\pi}\rangle(v)\prod_{i,l} h_l^{\varepsilon_{il}} = \langle \det_q T\rangle^{\pi}(v)\,.$$

But it was already proved above (cf. Lemma 1.1 and Eq. (7.3)). □

8. Algebra of Monodromy Matrices and $U_q(\hat{gl}(n))$

Let us make two remarks about the structure of the algebra $\mathscr{A}$. At first there exists an algebra isomorphism between $\mathscr{A}_{RR}$ and $\mathscr{A}_{\overline{RR}}$ if $\bar{\varepsilon}_{ij} = \varepsilon_{ij} + s_{ij} - s_{ji}$ for some integers s_{ij}. It looks as follows:

$$\mathscr{A}_{\overline{RR}} \ni T_{ij}(u) \to \prod_i H_i^{s_{il}} T_{ij}(u) \prod_l H_l^{-s_{lj}} \in \mathscr{A}_{RR}$$

$$H_l \to H_l\,.$$

This mapping does not preserve the coproduct so it is not a bialgebra isomorphism.

Now let us take a polynomial representation π of degree M such that $t_i^0 = t_i^\infty = 1$, $i = 1, \ldots, n$. We put $T(u) \equiv T^{\pi}(u)$, $H_l \equiv H_l^{\pi}$ and introduce operators E_i, F_i, G_i, $i = 1, \ldots, n$ as follows:

$$E_i = (T_{ii}^0)^{-1} T_{i+1,i}^0, \quad F_i = (T_{ii}^\infty)^{-1} T_{i,i+1}^\infty\,,$$

$$G_i = (T_{ii}^\infty)^{-1} T_{i+1,i+1}^0 = \prod_l H^{-(\varepsilon_{il}+\varepsilon_{l,i+1})}\,.$$

For $n > 2$ they satisfy commutation relations

$$H_l E_i = E_i H_l \omega^{\delta_{li}-\delta_{l,i+1}}, \quad H_l F_i = F_i H_l \omega^{\delta_{l,i+1}-\delta_{li}}, \quad \prod_l H_l = 1\,,$$

$$[E_i, F_j] = (\omega - 1) G_i (H_{i+1} - H_i)\delta_{ij}\,,$$

$$E_i E_j = E_j E_i \omega^{\eta_{ij}}, \quad F_i F_j = F_j F_i \omega^{\eta_{ji}}, \quad |i-j| > 1\,,$$

$$\left.\begin{array}{l} \omega^{\eta_{ji}} E_i^2 E_j - (\omega+1) E_i E_j E_i + \omega^{\eta_{ij}} E_j E_i^2 = 0 \\ \omega^{\eta_{ij}} F_i^2 F_j - (\omega+1) F_i F_j F_i + \omega^{\eta_{ji}} F_j F_i^2 = 0 \end{array}\right., \quad |i-j| = 1$$

which look similar to the commutation relations for $U_q(\widehat{gl}(n))$. More precisely, for $\omega^N = 1$, N being odd, $\varepsilon_{ij} = \frac{N+1}{2}(1+\delta_{ij})$ and $q = \omega^{(N+1)/2}$ the operators

$$k_i = H_i^{(N+1)/2}, \quad e_i = \frac{E_i}{(q-q^{-1})}, \quad f_i = \frac{F_i}{(1-\omega)}$$

satisfy the commutation relations for $U_q(\widehat{gl}(n))$ at level 0.

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