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Internal Lifschitz Singularities for One Dimensional Schrödinger Operators

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Abstract. The integrated density of states of the periodic plus random one-dimensional Schrödinger operator $H_\omega = -\Delta + V_{\text{per}} + \sum_i q_i(\omega)f(\circ - i)$; $f \geq 0$, $q_i(\omega) \geq 0$, has Lifschitz singularities at the edges of the gaps in $Sp(H_\omega)$. We use Dirichlet-Neumann bracketing based on a specifically one-dimensional construction of bracketing operators without eigenvalues in a given gap of the periodic ones.

1. Introduction

In this paper we will consider the behavior of the integrated density of states (IDS) for the one-dimensional random Schrödinger operator.

$$\begin{aligned} H_\omega(g) &= -\Delta + V_{\text{per}} + gV_\omega \\ &= T + gV_\omega , \end{aligned} \tag{1.1}$$

where

$$V_{\text{per}}(x+1) = V_{\text{per}}(x) \tag{1.2}$$

is a periodic, piecewise continuous function, $g > 0$,

$$V_\omega(x) = \sum_{n \in \mathbb{Z}} q_n(\omega) f(x-n) , \tag{1.3}$$

with piecewise continuous $f \geq 0$, $\operatorname{supp} f \subset \left(-\frac{1}{2}, \frac{1}{2}\right)$, and $q_n(\omega)$ are independent, identically distributed (iid) random variables. Their distribution function μ is assumed to have compact support

$$\operatorname{supp} \mu \subset [0, 1] \tag{1.4}$$

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$$\mu[0,x) = O(x^{\delta_-}),\\ \mu(1-x,1] = O(x^{\delta_+}) \tag{1.5}$$

for some $\delta_-, \delta_+ \geq 0$.

The integrated density of states is defined by

$$\sigma(E, H_\omega) = \sigma(E) = \lim_{L\uparrow\infty} \frac{1}{|\Lambda_L|} \#(H^{b,c}_{\omega,\Lambda_L} - E), \tag{1.6}$$

where $\Lambda_L = (-L, L)$, $H^{b,c}_{\omega,\Lambda_L}$ is some restriction defined by boundary conditions (b.c.) of H_ω to $L^2(\Lambda_L)$ and $\#(H)$ is the number of eigenvalues of the operator H which are ≤ 0. The limit in Eq. (1.6) exists under much weaker assumptions on the stochastic potential V_ω implied by than those in Eq. (3)–(5) [15].

An intuitive physical argument led I. M. Lifschitz to predict that the density of states, $\varrho(E) = d\sigma/dE$, has an universal type asymptotic behavior

$$\ln \varrho(E) \sim -\text{const}\,|E - E_c|^{-\frac{d}{2}}, \tag{1.7}$$

for $E \in SpH_\omega$ near the fluctuative spectral edges E_c of H_ω (here d is the space dimension; $d = 1$ in this paper). There are many rigorous proofs of somewhat weaker statements,

$$\lim_{Sp(H_\omega)\ni E\to E_c} \frac{\ln|\ln|\sigma E) - \sigma(E_c)||}{\ln|E - E_c|} = -\frac{d}{2}, \tag{1.8}$$

or

$$\ln|\sigma(E) - \sigma(E_c)| = -\frac{\Phi_c(E)}{|E - E_c|^{d/2}}, \tag{1.9}$$

with Φ_c having sometimes a weak singularity ($\sim \ln|E - E_c|$) at E_c [1, 3, 5, 7, 9–14, 17–19]. But nearly all have dealt only with the lowest spectral edges of H_ω.

For the finite-difference analogue of H_ω, a theorem of type Eq. (1.9) was proven [11] for all spectral edges, while a simpler proof of Eq. (1.8) may be found in [18]. Kirsch and Nitzschner [6] have considered a disordered one-dimensional Kronig-Penney model (with point interactions) which has an infinite number of gaps in its spectrum [4]. The upper spectral edges (lowest edges of the gaps) in this model are nonfluctuative and $\lim_{Sp(H_\omega)\ni E\to E_c} \frac{\ln|\sigma(E) - \sigma(E_c^-)|}{\ln|E - E_c|} = \frac{1}{2}$. Near one-half of the fluctuative spectral edges they have proven that

$$\lim_{SpH_\omega\ni E\to E_c} \sup \frac{\ln|\ln|\sigma(E) - \sigma(E_c)||}{\ln|E - E_c|} \leq -\frac{1}{2}. \tag{1.10}$$

The model considered in [6] has an infinite number of gaps for any (positive) value of the coupling constant g due to the zero-range potential. In our case, since for $g = 0$ the one-dimensional Schrödinger operator with periodic potential T has generically an infinite number of gaps, it seems reasonable to assume that, for sufficiently small g, H_ω will have the same property. Indeed, Kirsch and Martinelli [4] have proved:

Theorem 1. *Let H_ω be given by Eqs.* (1)–(5) *and*

$$T(g) = T + g\sum_{n\in\mathbb{Z}} f(x - n). \tag{1.11}$$

Then, the set

$$\{x \in \mathbb{R} | x \notin SpT(0) \cup SpT(g), \sigma(x;T(0)) = \sigma(x;T(g))\} \in \mathrm{Res}(H_\omega)\,. \tag{1.12}$$

Proof. By ergodicity $SpH_\omega(g)$ is a.s. nonrandom and if $x \in Sp(H_\omega)$ then x is a growth point for $\sigma(x; H_\omega)$. But

$$T = T(0) \le H_\omega(g) \le T(g)\,, \tag{1.13}$$

so that

$$\sigma(E;T(g)) \le \sigma(E;H_\omega(g)) \le \sigma(E;T(0))\,. \quad \square \tag{1.14}$$

Remark 1. If $\operatorname{supp}\mu = [0,1]$, then Eq. (1.12) yields all the gaps in $Sp(H_\omega)$ [4].

Since the spectral edges of $T(g)$ are, for small enough g, analytic functions of g, it is obvious that for small enough $g, g < g_n^c$, the n^{th} gap [1] of $T = T(0)$ is not closed. In the following we shall assume that $g < g_n^c$ for the particular gap we are studying and, by redefining $g_n^c f = f$, we may assume $g \le 1$.

Let $T_1^\zeta(g)$, $\zeta = e^{i\theta} \in U(1)$ be the quasiperiodic restriction of $T(g)$ to $L^2(0,1)$, $\mathscr{D}(T_1^\zeta) = \{\varphi \in C^1[0,1] | \varphi'' \in L^2,\ \varphi(1) = \zeta\varphi(0),\ \varphi'(1) = \zeta\varphi'(0)\}$. Let $\lambda_n(\theta, g) = \lambda_n[T_1^\zeta(g)]$ be its n^{th} eigenvalue (in nondecreasing order).

Define

$$E_{n-1}^+ = \lambda_n((n-1)\pi, 0)\,; \qquad E_n^- = \lambda_n(n\pi, 1)\,, \qquad n = 1, 2, \ldots. \tag{1.15}$$

By Theorem (1.1) and Theorem XIII.90 of [16] the set

$$\mathscr{E} = \{E_0^+\} \bigcup_{\substack{n=1 \\ E_n^- < E_n^+}} \{E_n^-, E_n^+\} \subset Sp[H_\omega]\,, \tag{1.16}$$

is a set of finite spectral edges of H_ω, and, by the previous Remark, if $\operatorname{supp}\mu = [0,1]$ there are no other (finite) edges.

Now we can state our main result:

Theorem 2. *Let H_ω be given by Eqs.* (1.1)–(1.3) *with μ satisfying* (1.4) *and* (1.5) *Then, for any edge $E^c \in \mathscr{E}$:*

$$\lim_{Sp(H_\omega) \ni E \to E_c} \frac{\ln|\ln|\sigma(E) - \sigma(E_c)||}{\ln|E - E_c|} = -\frac{1}{2}\,. \tag{1.17}$$

Remark 2. Inspection of the proof will show that the result may be extended to f with larger support than $[0,1]$ *but with* $f \ge 0$. In particular the result of Kirsch and Simon [7] for E_0^+: the limit in Eq. (1.18) is equal to $-1/\min(\alpha, 2)$, if $f = O(|x|^{-\alpha-1})$, $\alpha > 0$, as $|x| \to \infty$, extends to all $\mathscr{E}$.

Thus, the result known for the lowest edge [7] is valid for all the other edges. We will prove Theorem 2 by a combination of standard techniques: Dirichlet-Neumann bracketing and large deviation estimates [5, 6, 10–12, 17–19].

The bracketing operators for an arbitrary partition have eigenvalues inside the gaps of $Sp(H_\omega)$. The one-dimensional case discussed in this paper is distinguished by the fact that the bracketing restrictions of the periodic operators $T(g)$ to an interval have exactly one eigenvalue in each of the gaps of $Sp[T(g)]$. This is the content

[1] Generically $Sp(T(g))$ has an infinite number of gaps. Only for some rather special V_{per} (elliptic functions) there is a finite number

of Theorem 3 in the next section,[2] which also contains some known facts on one-dimensional Schrödinger operators with various boundary conditions on an interval. In Sect. 3 we will show that, by an adequate choice of the partition, the eigenvalues of the bracketing operators in a given gap may be pushed to a predetermined edge. Using these operators, the proof of Theorem 2 becomes a rather standard undertaking and will be sketched in the last section.

2. Some Facts on $-\Delta + V$ on an Interval

Let the real function V be piecewise continuous one some finite interval $J = [a, b]$ and define the operator T_J by

$$(T_J^{\text{b.c.}} f)(x) = -\frac{d^2 f}{dx^2} + V(x) f(x) \tag{2.1}$$

on $L^2(J)$ with

$$\mathscr{D}(T_y^{\text{b.c.}}) = \{f \in C^1(J) | f'' \in L^2(J),\ f \text{ satisfies boundary conditions}\}.$$

We will consider the following types of boundary conditions which lead to selfadjoint operators bounded from below and having compact resolvents:
a) N – *Neumann:* $f'(a) = f'(b) = 0$;
b) D – *Dirichlet:* $f(a) = f(b) = 0$;
c) ζ – *quasiperiodic:* $f'(b) = \zeta f(a)$, $f'(b) = \zeta f'(a)$, $\zeta \in U(1)$.

Whenever it is unambiguous we will write $T^{\text{b.c.}}$ for $T_J^{\text{b.c.}}$. The following proposition summarizes known facts on the eigenvalues and eigenfunctions of $T^{\text{b.c.}}$ [2, 16].

Proposition 1. *Let $T^{\text{b.c.}}$ be defined as above, ε_n^N, ε_n^D, $\varepsilon_n(\zeta)$, $n = 1, 2, \ldots$, be their eigenvalues arranged in a nondecreasing sequence and $u_n^{\text{b.c.}}$ – the corresponding eigenfunctions. Then*
1) *$\varepsilon_n^N, \varepsilon_n^D$ and $\varepsilon_n(\zeta), \zeta^2 \neq 1$ are simple,*

$$\varepsilon_n(\zeta) = \varepsilon_n(\zeta^{-1}), \tag{2.2}$$

$$\begin{aligned} \varepsilon_{2m-1}(1) &< \varepsilon_{2m}(1), \\ \varepsilon_{2m}(-1) &< \varepsilon_{2m+1}(-1), \qquad m = 1, 2, \ldots . \end{aligned} \tag{2.3}$$

2) *u_1^{+1} and u_n^ζ, $\zeta^2 \neq 1$, have no zeros on J; u_n^N and u_n^D have exactly $n-1$ zeros in (a, b); u_{2m}^{+1} and u_{2m+1}^{+1} have exactly $2m$ zeros and u_{2m-1}^{-1} have exactly $2m-1$ zeros in $[a, b]$ regarded as a cricle, if the respective eigenvalues are nondegenerate. In the case of degeneracy, the statement remains true if the functions are chosen to be real.*
3) *$\varepsilon_n(\zeta)$ are analytic in a neighborhood of $\mathscr{C} = U(1) \backslash \{-1, 1\}$ and continuous at $\zeta = \pm 1$; if $\varepsilon_n(\zeta_0)$, $\zeta_0^2 = 1$ is nondegenerate, then ε_n is analytic at ζ_0. When ζ goes from -1 to $+1$ on the unit circle $(-1)^n \varepsilon_n(\zeta)$ increases monotonically.*

We refer the reader to Eastham [3], where the proof of most of the assertions may be found.

Theorem 3. *(Bracketing of Neumann and Dirichlet eigenvalues). Let $T^{\text{b.c.}} = T_{(a,b)}^{\text{b.c.}}$ and $\varepsilon_n^{N(D)}$, $\varepsilon_n(\zeta)$, $n = 1, 2, \ldots$ be respectively the eigenvalues of $T^{N(D)}$ and T^ζ*

[2] I am indebted to the anonymous referee of this paper who suggested the straightforward proof of Theorem 3 which is given here

respectively, ordered in increasing sequence (if necessary by continuity for $\zeta^2 \to 1$). Then:

$$\begin{aligned} &\varepsilon_n((-1)^n) \le \varepsilon_{n+1}^N \le \varepsilon_{n+1}((-1)^n); \\ &\varepsilon_n((-1)^n) \le \varepsilon_n^D \le \varepsilon_{n+1}((-1)^n), \qquad n = 1, 2, \ldots; \end{aligned} \tag{2.4}$$

and all the bounds are attainable.

Proof. Let us first show that neither the Dirichlet nor the Neumann eigenvalues can coincide with any $\varepsilon_n(\zeta)$, $\zeta^2 \neq 1$, i.e. that the Dirichlet/Neumann eigenvalues are either in the gaps or at the band edges.

Assuming the contrary, let $\varepsilon = \varepsilon_n(\zeta)$, for some $n \in \mathbb{N}$, $\zeta^2 \neq 1$, be a Neumann eigenvalue. By Proposition 1, $\varepsilon = \varepsilon_n(\zeta^{-1})$, so that u_n^ζ and $u_n^{\zeta^{-1}}$ are linearly independent (and complex conjugate) solutions of the equation:

$$\frac{-d^2u}{dx^2} + Vu = \varepsilon u. \tag{2.5}$$

Let

$$u_n^\zeta(x) = M(x)e^{i\varphi(x)}, \tag{2.6}$$

with $M > 0$ by Proposition 1. By adding a suitable constant phase to φ, the (real) Neumann eigenfunction may be written as:

$$u^N(x) = M(x)\cos(\varphi(x)), \tag{2.7}$$

while the boundary conditions satisfied by M and φ are

$$\begin{aligned} M(1) = M(0), &\qquad M'(1) = M'(0), \\ \varphi(1) = \varphi(0) + \arg(\zeta) + 2k\pi, &\qquad \varphi'(1) = \varphi'(0), \end{aligned} \tag{2.8}$$

for some integer k. By assumption, u^N satisfies the Neumann conditions, which yield for M and φ:

$$\begin{aligned} M'(0)\cos(\varphi(0)) - \varphi'(0)M(0)\sin(\varphi(0)) = 0, \\ M'(1)\cos(\varphi(1)) - \varphi'(1)M(1)\sin(\varphi(1)) = 0. \end{aligned} \tag{2.9}$$

The compatibility condition of Eqs. (2.8) and (2.9) is

$$\tan(\varphi(0)) = \tan(\varphi(0)) + \arg(\zeta)),$$

which implies $\zeta^2 = 1$, contradicting the assumption.

The reasoning in the Dirichlet case is quite similar.

If $V = 0$, then

$$\begin{aligned} \varepsilon_n(\zeta) = [(n-1)\pi + (-1)^{n+1}|\arg(\zeta)|]^2, \\ \varepsilon_n^D = (n\pi)^2, \qquad \varepsilon_n^N = [(n-1)\pi)^2, \qquad n = 1, 2, \ldots. \end{aligned}$$

The Dirichlet and Neumann eigenvalues satisfy Eq. (2.4) with all the $\le$ signs replaced by $=$.

Now, for piecewise continuous V, $T^{\text{b.c.}} + gV$ is an entire analytic family (see e.g. [16]). Since all the eigenvalues of the Dirichlet, Neumann and (for $\zeta^2 = 1$) ζ-operators are nondegenerate and for all real g, $\varepsilon_k^{N/D}(g) \neq \varepsilon_k(\zeta, g)$, $\zeta^2 \neq 1$, we have

$$\varepsilon_n(g)\zeta) \le \varepsilon_n^D(g) \le \varepsilon_n^D(g) \le \varepsilon_n(\zeta, g), \qquad n = 1, 2, \ldots,$$

wherefrom Eq. (2.4) follows by the monotonicity in ζ (Proposition 1, 3).

It remains to show the attainability of the bounds in Eq. (3.1). Let V_{per} be the continuation of V to $\mathbb{R}$ by periodicity and define V_y on (a, b) by

$$V_y(x) = V_{\text{per}}(x + y), \qquad x \in (a, b). \tag{2.10}$$

Let $T_y^{\text{b.c.}}(y) = -\Delta^{\text{b.c.}} + V_y$ and $\varepsilon_n^{\text{b.c.}}(y)$ – its eigenvalues. T_y^ζ is unitarily equivalent to $T^\zeta = T_0^\zeta$ by the cyclic translation operator. Since the eigenfunctions of $T^{\pm 1}$ are real and C^1, there are points $y_{n,\alpha}^{n,\pm}$ at which $\frac{d}{dx} u_n^{\pm 1}(y_{n,\alpha}^{N,\pm}) = 0$. Remember that $u_n^{\pm 1}$ is (anti) periodic for any $n \in \mathbb{N}$. With the exception of u_1^{+1} the eigenfunctions also have zeros: $y_{\eta,\beta}^{D,\pm}$. Thus, the $(n+1)^{\text{st}}$ eigenvalue of T_y^N attains the lower bound (3.1) for $y = y_{n,\alpha}^{N,(-1)^n}$ and the upper one for $y = y_{n+1,\alpha}^{N,(-1)^n}$. The Dirichlet case is similar. □

Let us now consider $T_{(0,L)}^{\text{b.c.}}$, $2 \le L \in \mathbb{N}$, with a periodic potential, $V(x+1) = V(x)$. It is obvious that the eigenvalues of $T_{(0,L)}^\zeta$ may be obtained from those of $T_{(0,1)}^\zeta$. By Theorem 3 we may also locate $L - 1$ eigenvalues of $T_{(0,L)}^{D(N)}$ in each of the bands of T and bracket the remaining one eigenvalue per gap of T. Thus:

Proposition 2. *Let $T_L^{\text{b.c.}} = T_{(0,L)}^{\text{b.c.}}$, with V having unit period. Let $\varepsilon_k(\zeta)$, $u_k^\zeta(x)$ be the eigenvalues and eigenfunctions of T_1^ζ and $\varepsilon_k(\zeta, L)$ and $u_k^\zeta(x, L)$ those for T_L^ζ, $L \in \mathbb{N}$, arranged in nondecreasing order. Then*

1) $$\varepsilon_{(k-1)L+m}(\zeta, L) = \varepsilon_k(\zeta_m^{(L,k)}), \qquad m = 1, 2, \ldots, L,\ k = 1, 2, \ldots, \tag{2.11}$$

where $\zeta_m^{(L,k)}$ are the L roots of the equation

$$\eta^L = \zeta. \tag{2.12}$$

2) *The eigenvalues of T satisfy*

$$\begin{gathered} \varepsilon_{(k-1)L+m}^D(L) = \varepsilon_{(k-1)L+m+1}^N(L) = \varepsilon_k((-1)^{k+1} e^{im\pi/L}), \\ \varepsilon_k((-1)^k) \le \varepsilon_{kL}^D(L), \qquad \varepsilon_{kL+1}^N(L) \le \varepsilon_{k+1}((-1)^{k+1}), \end{gathered} \tag{2.13}$$

where $m = 1, 2, \ldots, L-1$, $k = 1, 2, \ldots$.

Remark 4. Let $T_{\text{per}} = -\Delta + V_{\text{per}}$ on $L^2(\mathbb{R})$. Then, (see e.g. [1])

$$Sp(T_{\text{per}}) = \bigcup_{\substack{n \in \mathbb{N} \\ \zeta \in U(1)}} \varepsilon_n(\zeta) = \bigcup_{n \in \mathbb{N}} [a_n, b_n], \tag{2.14}$$

where $a_n = \varepsilon((-1)^{n+1}) < b_n = \varepsilon_n((-1)^n)$.

We have shown that for any $n \in \mathbb{N}$,

$$\varepsilon_{n+1}^N(y), \varepsilon_n^D(y) \in [b_n, a_{n+1}]. \tag{2.15}$$

The periodic functions $\varepsilon_{n+1}^N(y)$ and $\varepsilon_n^D(y)$ oscillate in the interval (2.15) attaining its edges at least once in each period. If the n^{th} gap is closed, $b_n = a_{n+1}$, they are pinned (constant).

3. Bracketing Operators without Eigenvalues in a Gap

As we have seen the Dirichlet and Neumann operators on an interval have generically eigenvalues in the gaps of the ζ (quasiperiodic) operators. Nevertheless we may use the method of proof of Theorem 3 to construct approximating operators bracketing H_ω which have no eigenvalues in a given gap of $Sp(H_\omega)$. By using these we will achieve the proof of Theorem 2 in the next section.

By Dirichlet-Neumann bracketing ([5, 7, 10, 12], σ is bracketed by the expectation values of the integrated density of states for the restrictions of H_ω,

$$L^{-1}\mathbb{E}\{\#(H^D_{\omega,L} - E)\} \le \sigma(E) \le L^{-1}\mathbb{E}\{\#(H^N_{\omega,L} - E)\}\,, \tag{3.1}$$

for any $L \in \mathbb{N}$. As we have seen, for $T^N_L(g)$, $L-1$ eigenvalues per band are in $Sp(T(g))$ and, generically, there is one eigenvalue in each gap of $SpT(g)$. But

Lemma 1. *Let H_ω be bounded from above (below) by*

$$H^{\mathrm{b.c.}}_\omega(L,y) = \bigoplus_{m\in\mathbb{Z}} H^{\mathrm{b.c.}}_{\omega,(mL+y,(m+1)L+y)}\,, \tag{3.2}$$

$L \in \mathbb{N}$, $y \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, *where $H^{\mathrm{b.c.}}_{\omega,(a,b)}$ is the restriction of H_ω to $L^2(a,b)$ with boundary conditions* b.c. $= D$(b.c. $= N$). *Let (E_n^-, E_n^+) be the n^{th} gap of H_ω. Then, one may choose $y = y_n^D(y_n^N)$ such that the* IDS *of $H^{\mathrm{b.c.}}_\omega(L, y_n^{\mathrm{b.c.}})$*

$$\sigma(E; L, y) = \sigma(E) = n,\ \forall E \in (E_n^-, E_n^+)\,. \tag{3.3}$$

Proof. By Proposition 1 there are $L-1$ eigenvalues of $T^{D(N)}_{\alpha,\alpha+L}(g)$ of $T(g)$ in each band of $Sp(T(g))$. By Theorem 3 and Remark 4 the eigenvalues that lie generically in the gap are periodic functions of α, attaining the spectral edges at least once per period.

Let us consider b.c. $= N$ and choose a $y = y_n^N \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ for which the $(n+1)^{\mathrm{st}}$ eigenfunction of $T_1^{(-1)^n}(0)$ has zero derivative. Then, the $(nL+1)^{\mathrm{st}}$ eigenvalue of $T^N_{(y_n^N, y_n^N+L)}(0)$

$$\lambda_{nL+1}[T^N_{y_n^N, y_n^N+L)}(0)] = \varepsilon_{n+1}((-1)^n) = E_n^+\,. \tag{3.4}$$

By Proposition 2, for $g=1$

$$\lambda_{nL}[T^N_{(y_n^N, y_n^N+L)}(1)] = \varepsilon_n((-1)^n e^{\frac{i\pi}{L}}, 1) < \varepsilon_n((-1)^n, 1) = E_n^-\,. \tag{3.5}$$

Since

$$T^N_{(y_n^N, y_n^N+L)}(0) \le H^N_{\omega,(y_n^N, y_n^N+L)} \le T^N_{(y_n^N, y_n^N+L)}(1)\,, \tag{3.6}$$

we obtain

$$\lambda_{nL}[H^N_{\omega,(y_n^N, y_n^N+1)}] < E_n^- \tag{3.7}$$

and

$$\lambda_{nL+1}[H^N_{\omega,(y_n^N, y_n^N+1)}] \ge E_n^+\,. \tag{3.8}$$

For the Dirichlet case we choose $y = y_n^D \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ for which the n^{th} eigenfunction of $T_1^D(1)$ has a zero. Then, the nL^{th} eigenvalue of $T^D_{(y_n^D, y_n^D+1)}(1)$,

$$\lambda_{nL}[T^D_{(y_n^D, y_n^D+L)}] = \varepsilon_n((-1)^n, 1) = E_n^-\,, \tag{3.9}$$

and using Eq. (3.6) with N replaced by D, we obtain

$$\lambda_{nL}[H^D_{\omega,(y^D_n,y^D_n+L)}] \leq E^-_n \,, \tag{3.10}$$

$$\lambda_{nL+1}[H^D_{\omega,(y^D_n,y^D_n+L)}] > E^+_n \,. \tag{3.11}$$

It remains to note that Eqs. (3.7), (3.8) and (3.10), (3.11) remain obviously valid if we add an integer to y. □

4. Proof of Theorem 2

In the previous section we have proven Lemma 1 which gives us bracketing operators for $Sp(H_\omega)$. Now we can return to our primary task. We will proceed by Dirichlet-Neumann bounding taking a single upper/lower operator for both spectral edges bordering a given gap – the one defined in Lemma 2. For the sake of simpler notations, having set on proving the theorem near the edges E^-_n, E^+_n of a given gap, we will omit the n, y, L dependence of the bracketing operators, writing H^D_ω for $H^D_{\omega,(y^D_n,y^D_n+L)}$ and H^N_ω in the Neumann case.

Definition 1. Let $X^{N(D)}_\pm(\omega, L, C)$, $C > 0$, be the events

$$\left\{ H^{N(D)}_\omega \text{ has an eigenvalue in the interval } \left(E^\pm_n - \frac{C}{L^2}, E^\pm_n + \frac{C}{L^2} \right) \right\}.$$

The proof becomes a simple exercise given the following:

Lemma 2. *For sufficiently large L and C^{-1} there are L-independent constants $A, B > 0$ such that*

$$\ln \mathbb{P}[X^N_\pm(\circ, L, C)] \leq -AL \ln L \,, \tag{4.1}$$

$$\ln \mathbb{P}[X^D_\pm(\circ, L, C)] \geq -BL \ln L \,, \tag{4.2}$$

if in Eq. (1.5) $\delta_\pm > 0$. *For* $\delta_\pm = 0$ *the logarithms should be dropped from the* r.h.s. *of Eqs. (1–2). Here $\mathbb{P}[X]$ is the probability of the event X.*

Indeed, $H^{N(D)}_\omega$ has no eigenvalues in (E^-_n, E^+_n). For sufficiently small $C > 0$, let

$$Sp(H_\omega) \ni E = E^\pm_n \pm \frac{C}{L^2} \,. \tag{4.3}$$

Taking into account Eqs. (3.7)–(3.11), only λ_{n_L} may be in $[E, E^-_n]$, respectively only $\lambda_{nL+1} \in [E^+_n, E]$. Thus, by Eq. (4.1), $\sigma(E) - \sigma(E^+_n)$ is bracketed by $f_D(E), f_N(E)$, with

$$f_{D/N}(E) = \frac{1}{L(E)} \mathbb{P}[X^{D/N}_+(\circ, L(E), C)] \,, \tag{4.4}$$

$L(E) = C|E - E^+_n|^{-1/2}$ and a similar pair for $\sigma(E^-_n) - \sigma(E)$. Taking the limit $E \to E^\pm_n$ yields Eq. (1.17). □

Before proceeding further we will state a generalization by Kirsch and Nitzschner [6] of Temple's inequality, which may be proven in the same way as Theorem XIII.5 of [16].

Lemma 3. *Let H be selfadjoint, semibounded and with compact resolvent. Let*

Lemma 3. *Let H be selfadjoint, semibounded and with compact resolvent. Let*

$$\lambda_n(H) \le \nu_n < 0 < \nu_{n+1} \le \lambda_{n+1}(H), \tag{4.5}$$

and $\varphi \in \mathscr{D}(H)$, $\|\varphi\| = 1$, $(\varphi, H\varphi) = 0$. Then:

$$\begin{aligned} \lambda_n(H) &\ge -\frac{\|H\varphi\|^2}{\nu_{n+1}}; \\ \lambda_{n+1}(H) &\le -\frac{\|H\varphi\|^2}{\nu_n}. \end{aligned} \tag{4.6}$$

Proof of Lemma 2. Let us start with Eq. (1) for E_n^-. For $-1 \le i \le L+1$ and some $\xi \in [0,1]$ define

$$q_i(\omega_N) = \begin{cases} 1-\xi, & \text{if } all\ q_j(\omega) \ge 1-\xi,\ -1 \le j \le L+1, \\ 0, & \text{otherwise}. \end{cases} \tag{4.7}$$

Obviously,

$$H_{\omega_N}^N \le H_\omega^N, \tag{4.8}$$

so that $X_-^N(\omega_N, L, C) \Rightarrow X_-^N(\omega, L, C)$. But

$$H_{\omega_N}^N = T^N(1-\xi), \tag{4.9}$$

in the first case in Eq. (7) and $H_{\omega_N} = T^N(0)$ in the second. In the latter case there are no eigenvalues in $\left(E_n^- - \dfrac{C}{L^2}, E_n^-\right)$ for sufficiently small CL^{-2}. In the former, by Proposition 2 and Theorem 3 for $g = 1-\xi$,

$$\lambda_{nL}(T^N(1-\xi)) = \varepsilon_n(e^{i\pi(n-L-1)}, 1-\xi). \tag{4.10}$$

Since $\varepsilon_n((-1)^n, g)$ is not degenerate for g in some neighborhood of 1, it will be analytic in ζ near $\zeta_0 = (-1)^n$ and also analytic in g near $g = 1$. For sufficiently small L^{-1} and ζ,

$$\varepsilon_n((-1)^n e^{\frac{i\pi}{L}}, 1-\xi) = \varepsilon_n((-1)^n, 1) - \frac{\alpha}{L^2} - \xi \left.\frac{\partial \varepsilon_n((-1)^n, q)}{\partial g}\right|_{g=1-\xi} + \dots. \tag{4.11}$$

Taking $\xi = \dfrac{\beta}{L^2}$, noting that $\dfrac{\partial \varepsilon_n(\zeta, g)}{\partial g} > 0$, and using Eq. (1.5), we see that, if $\delta_- > 0$, Eq. (1.1) is valid while for $\delta_- = 0$ there is no $\ln L$ term.

To prove (4.2) near E_n^-, let us redefine again $q_i(\omega)$ by

$$q_i(\omega_D) = \begin{cases} 1, & \text{if } q_i(\omega) > 1-\xi, \\ 1-\xi, & \text{if } q_i(\omega) \le 1-\xi, \end{cases} \tag{4.12}$$

for some $\xi \in [0,1]$, $-1 \le i \le L+1$. Since

$$H_{\omega_D}^D \le H_\omega^D,$$

then $X_-^D(\omega, L, C) \Rightarrow X_-^D(\omega_D, L, C)$. Now,

$$H_{\omega_D}^D = T^D(1) - \sum_i [1 - q_i(\omega_D)]\, f(\circ - i). \tag{4.13}$$

Let

$$\varphi = \frac{1}{L} u_n^{(-1)^n}, \qquad ||\varphi|| = 1, \tag{4.14}$$

where $u_n^{(-1)^n}$ is the normalized n^{th} eigenfunction of $T_1^D(1)$ continued periodically and restricted to our interval:

$$T^D(1)\varphi = \lambda_{nL}[T^D(1)]\varphi. \tag{4.15}$$

Since the sum in Eq. (5.13) is nonnegative,

$$\lambda_{nL-1}[H^D_{\omega_D}] \le \varepsilon_n((-1)^n)e^{\frac{i\pi}{L}}, 1) \le E_n^- - \frac{\alpha}{L^2}, \tag{4.16}$$

for L sufficient large, where we used Proposition 2. Now

$$(\varphi, H^D_{\omega_D}\varphi) = E_n^- - \frac{N_+ h_1 \beta}{L^3} = F, \tag{4.17}$$

where N_+ is the number of $q_i(\omega_D)$ which are $= 1-\zeta$, $h_1 = \int_0^1 f(x)\,|u_n^{(-1)^n}(x)|^2 dx > 0$ and we set $\zeta = \beta L^{-2}$. Defining

$$H = H^D_{\omega_D} - F, \tag{4.18}$$

and choosing $\beta < \alpha/h_1$ we may apply Lemma 4 to obtain an upper bound to $\lambda_{n_L}(H^D_{\omega_D})$:

$$\lambda_{nL}(H^D_{\omega_D} \le E_n^- - \frac{N_+\beta h_1}{L^3} + \frac{N_+\beta^2 h_2^2}{L^3}\,\frac{1 - N_+ h_1^2/L h_2^2}{\alpha - N_+ h_1 \beta/L}. \tag{4.19}$$

Here $h_2^2 = \int_0^1 f^2(x)\,|u_n^{(-1)^n}(x)|^2 dx$. Now by standard large derivation arguments (see e.g. Sect. 4 of [11]) we may establish Eq. (4.2) for E_n^-. The case of E_n^+ is essentially the same.

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