Commun. Math. Phys. 158, 135–153 (1993)

Communications in
Mathematical
Physics

Quantum Grassmann Manifolds

P. Šťovíček

Department of Mathematics, Faculty of Nuclear Science, CTU, Trojanova 13,
12000 Prague, Czech Republic

Received: 22 July 1992/in revised form: 16 April 1993

Abstract. Orbits of the quantum dressing transformation for $SU_q(N)$ acting on its solvable dual are introduced. The case is considered when the corresponding classical orbits coincide with Grassmann manifolds. Quantization of the Poisson bracket on a Zariski open subset of the Grassmann manifold yields a $*$-algebra generated by the quantum coordinate functions. The commutation relations are written in a compact form with the help of the R-matrix. Finite-dimensional irreducible representations of $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$ are derived from the $*$-algebra structure.

1. Introduction

A method of orbits (geometric quantization) due to Kirillov-Kostant-Souriau revealed a remarkable relationship between the geometry and the representation theory for classical groups. Important sources of this method are induced representations and the Borel-Weil theory. It is of interest to establish an analogous approach also for quantum groups [1]. One of the most interesting among expected results would be a production of examples of quantum manifolds. In this direction a serious progress has been made. This is true first of all for the representation theory of quantum groups [2–4]. Moreover, the method of induced representations is well developed [5] and a deformation of Schubert cells has been described [6, 7]. To complete this picture one has to recall an important notion of quantum dressing transformation. No doubt that its role is crucial as it substitutes the classical coadjoint action. The dressing transformation is of importance already for classical groups [8], has interesting applications in physics [9] and is closely related to the notions of the generalized Pontryagin dual and the Iwasawa decomposition [10]. A quantum generalization was discussed in [11]. The quantum dual was also derived in the paper [12] where knowledge of the representation theory for quantum compact groups was the starting point. This is in some sense the opposite direction to that we are going to stress in this paper. The geometry of the dressing orbit should be the primary object and a construction of representations is expected to result from it.

The presented paper follows the Faddeev-Reshetikhin-Takhtajan approach to quantum groups [13]. It prolongs some ideas from the paper [14] where the quantum dressing transformation was described for a solvable group acting on its compact dual. Here the opposite case is considered when the group $SU_q(N)$ acts on its solvable dual. The leading idea is that all the rich structure presented in the classical case is now concentrated in the non-commutative multiplication law. A special type of orbits corresponding to integer multiples of fundamental weights are Grassmann manifolds. Their quantization is done here by introducing quantum local coordinates. This assumes a construction of a quantum "restriction" homomorphism from the solvable group onto a "subset" of the dressing orbit (classically this is a Zariski open subset of the Grassmann manifold). Its existence allows a straightforward construction of the finite-dimensional irreducible $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$ module. The general case of arbitrary weight is also available for the corresponding irreducible module is a submodule in a tensor product of $N-1$ modules of the above special type. The obtained examples of quantized Grassmann manifolds generalize the quantum sphere due to Podles' [15]. It should be pointed out that recently there were published also some other papers on quantum flag manifolds, though based on different approaches. Namely, they insist on quantization of the flag algebra and the Plücker relations [16, 17]. In the present paper, the interpretation of elements of the obtained finite-dimensional module as holomorphic sections in some quantum line bundle continues to be open. The special case of quantum sphere S^2 has been treated in [18].

The paper is organized as follows. Section 2 has a preliminary character. Some basic notions are recalled, particularly those related to the quantum dressing transformation of $SU_q(N)$ on its quantum dual AN_q. The quantum dressing orbit is defined as a factor algebra of $\mathscr{A}_q(AN)$. The factorization means that generators of the centre of $\mathscr{A}_q(AN)$ are replaced by some constant parameters specifying the orbit. The main results are presented in the following two sections. Section 3 starts with the Poisson bracket on the classical Grassmann manifold explicitly expressed in conveniently chosen local holomorphic coordinates z_{uv}. Quantization of this bracket yields a $*$-algebra $\mathscr{C}$ generated by z_{uv}, z^*_{st}. The defining commutation relations are written in a compact form with the help of the R-matrix. The quantum "local coordinates" are constructed in terms of a $*$-algebra morphism $\psi: \mathscr{A}_q(AN) \to \mathscr{C}$. The parameters characterizing the quantum dressing orbit are calculated explicitly. Besides, a commutation relation is investigated between polynomials quadratic in the generators of the algebra $\mathscr{A}_q(AN)$. It is shown to be equivalent to the original defining relations for $\mathscr{A}_q(AN)$. Section 4 is devoted to a construction of finite-dimensional irreducible representations of $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$. Owing to the morphism ψ, the algebra $\mathscr{C}$ is a $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$ module. We note that $\mathscr{A}_q(AN)$ and $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$ are isomorphic as $*$-algebras. Afterwards the generators z_{uv} are factorized off and only the quantum "antiholomorphic" coordinate functions z^*_{st} are retained. In this way we get a reduced module. This factorization should be imagined as a quantum analogue to furnishing the classical orbit with a polarization which happens to be a complex structure. The cyclic submodule generated by the unit is investigated in detail. It is shown to be finite-dimensional for a proper choice of the parameters characterizing the quantum dressing orbit. The proof is based on some identity valid for the R-matrix. The invariant scalar product is introduced with the help of the vacuum-value functional on $\mathscr{C}$.

2. Quantum Dressing Transformation

Let us first recall some basic definitions and notations (it coincides with that having been used in [14]). In what follows we assume that the quantum parameter $q = e^{-h} \in (0, 1)$. The quantum integers are defined by $[k] = (q^{-k} - q^k)/(q^{-1} - q)$. As already mentioned in the introduction we are going to restrict ourselves to the case of the quantum group $SU_q(N)$. The corresponding fundamental (vector) representation is denoted by $U = (u_{jk})$. The $*$-algebra of quantum functions $\mathscr{A}_q(SU(N))$ is defined by the relations [13]

$$RU_1U_2 = U_2U_1R, \qquad U^* = U^{-1}, \qquad \det_q U = 1\,.$$

The underlying $N^2 \times N^2$ R-matrix fulfilling the Yang-Baxter equation is given by

$$R_{jk,st} = \delta_{js}\delta_{kt} + (q - q^{sgn(k-j)})\delta_{jt}\delta_{ks}\,. \tag{1}$$

The indices on the LHS should not be confused with the leg indices; $R = R_{12}$. The leg notation surely requires no clarification. Concerning the usual indices, throughout the paper we assume the lexicographical ordering. With this assumption, R is lower-triangular. Replacing q by q^{-1} on the RHS of (1) we get R^{-1}. Furthermore,

$$R_{12}^* = R_{21}\,.$$

Whenever a specification of the dimension is reasonable we shall write $R^{[N]}$ instead of R; $R^{[1]} = q$. The symbol P stands for a flip morphism permuting two factors in a tensor product. Provided the product $\mathbb{C}^N \otimes \mathbb{C}^N$ is concerned, $P_{jk,st} = \delta_{jt}\delta_{ks}$. Another very useful relation valid for the R-matrix is

$$(q^{-1} - q)P = R_{21}^{-1} - R_{12} = R_{12}^{-1} - R_{21}\,. \tag{2}$$

The generalized Pontryagin dual to $SU(N)$ is the solvable group AN. Classically, AN is formed by unimodular upper-triangular matrices with positive elements on the diagonal. The fundamental representation of AN_q is an upper-triangular matrix $\Lambda = (\alpha_{jk})$ with entries from $\mathscr{A}_q(AN)$ fulfilling [14]

$$R\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1R, \qquad \Lambda_1^*R^{-1}\Lambda_2 = \Lambda_2R^{-1}\Lambda_1^*\,; \tag{3}$$

$$\prod \alpha_{jj} = 1\,. \tag{4}$$

An additional requirement $\alpha_{jj} > 0$ means that the elements α_{jj} are supposed to be represented by positive matrices or, in a weakened and more preferable formulation, by diagonalisable matrices with positive eigenvalues. Both $\mathscr{A}_q(SU(N))$ and $\mathscr{A}_q(AN)$ can be turned into $*$-Hopf algebras in a standard manner.

The Chevalley generators of the deformed enveloping algebra $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$ ($\mathfrak{sl}(N,\mathbb{C}) =$ *complexification of* $\mathfrak{su}(N)$) are denoted traditionally by H_j and $X_j^\pm$, $j = 1, 2, \ldots, N-1$. They obey the deformed commutation relations including the Serre relations [1, 2]. An involution on $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$ is defined by $H_j^* = H_j$, $(X_j^\pm)^* = X_j^\mp$. $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$ and $\mathscr{A}_q(SU(N))$ are dual $*$-Hopf algebras provided a pairing between them is chosen as

$$\langle H_j \mid U\rangle = E_{jj} - E_{j+1,j+1}\,, \qquad \langle X_j^+ \mid U\rangle = E_{j,j+1}\,, \qquad \langle X_j^- \mid U\rangle = E_{j+1,j}\,. \tag{5}$$

Moreover, $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$ and $\mathscr{A}_q(AN)$ are isomorphic $*$-algebras. The isomorphism is given by ($j = 1, 2, \ldots, N-1$)

$$\begin{aligned} \exp(hH_j) &= \alpha_{jj}\alpha_{j+1,j+1}^{-1}, \\ (q^{-1}-q)X_j^+ &= q^{-1/2}(\alpha_{jj}\alpha_{j+1,j+1})^{-1/2}\alpha_{j,j+1}^*, \\ (q^{-1}-q)X_j^- &= q^{-1/2}(\alpha_{jj}\alpha_{j+1,j+1})^{-1/2}\alpha_{j,j+1}. \end{aligned} \tag{6}$$

This is why one can concentrate on representations of the algebra $\mathscr{A}_q(AN)$ instead of $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$.

The classical dressing transformation is a Poisson action

$$\mathscr{R}_{\mathrm{cl}}: AN \times SU(N) \to AN$$

induced by the Iwasawa decomposition

$$SL(N,\mathbb{C}) = SU(N) \times AN.$$

$\mathscr{R}_{\mathrm{cl}}(\Lambda, U)$ is the upper-triangular part $\tilde{\Lambda}$ of the matrix $\Lambda U = \tilde{U}\tilde{\Lambda}$. It is a useful fact that every unimodular positive matrix X can be decomposed as $X = \Lambda^*\Lambda$, $\Lambda \in AN$, and this relation is one-to-one. The dressing transformation of positive matrices then reads

$$\mathscr{R}_{\mathrm{cl}}: \Lambda^*\Lambda \to U^*\Lambda^*\Lambda U. \tag{7}$$

The quantum case is formally similar. The quantum dressing transformation is a coaction

$$\mathscr{R}: \mathscr{A}_q(AN) \to \mathscr{A}_q(AN) \otimes \mathscr{A}_q(SU(N))$$

defined by [14]

$$\mathscr{R}(x) = \varrho(x \otimes 1)\varrho^*,$$

where $\varrho \in \mathscr{A}_q(AN) \otimes \mathscr{A}_q(SU(N))$ is the canonical element, $\varrho^* = \varrho^{-1}$. This definition is possible since $\mathscr{A}_q(AN)$ and $\mathscr{A}_q(SU(N))$ are dual as vector spaces. Let us simplify the notation by writing x instead of $x \otimes 1$ and a instead of $1 \otimes a$. Then decomposing the matrix ΛU into the unitary and upper-triangular parts [14], $\Lambda U = \tilde{U}\tilde{\Lambda}$, where entries of both $\tilde{U}$ and $\tilde{\Lambda}$ belong to $\mathscr{A}_q(AN) \otimes \mathscr{A}_q(SU(N))$, we get $\mathscr{R}(\Lambda) = \tilde{\Lambda}$. It follows that

$$\mathscr{R}(\Lambda^*\Lambda) = U^*\Lambda^*\Lambda U. \tag{8}$$

It is also important to note that if the right coaction $\mathscr{R}$ is combined with the pairing (5) the algebra $\mathscr{A}_q(AN)$ becomes a left $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$ module. For $\xi \in \mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$, $f \in \mathscr{A}_q(AN)$, we define

$$\xi \cdot f = (\mathrm{id} \otimes \langle \xi, \cdot \rangle)\mathscr{R}(f) \in \mathscr{A}_q(AN). \tag{9}$$

It holds

$$1 \cdot f = f, \qquad (\xi_1\xi_2) \cdot f = \xi_1 \cdot (\xi_2 \cdot f),$$

and

$$\xi \cdot (fg) = \sum_k (\xi_k^1 \cdot f)(\xi_k^2 \cdot g) \tag{10}$$

where $\Delta\xi = \sum_k \xi_k^1 \otimes \xi_k^2$, Δ is the comultiplication.

The following proposition characterizes the centre of $\mathscr{A}_q(AN)$.

Proposition 2.1. *An element c belongs to the centre of $\mathscr{A}_q(AN)$ if and only if $\mathscr{R}(c) = c \otimes 1$.*

Proof. The condition $\mathscr{R}(c) = c \otimes 1$ means that ϱ commutes with $c \otimes 1$. This is clearly true whenever c belongs to the centre. On the other hand, writing $\varrho = \sum_s x_s \otimes a_s$ with $\{a_s\}$ and $\{x_s\}$ being mutually dual bases, one observes that the equality $\sum x_s c \otimes a_s = \sum c x_s \otimes a_s$ implies that $x_s c = c x_s$ for all s. $\square$

It was proven in [13] that

$$S^2(u_{jk}) = q^{2j-2k} u_{jk},$$

where S designates the antipode on $\mathscr{A}_q(SU(N))$. As U is unitary and $S(U) = U^*$, it follows that

$$\sum_k u_{lk}^* u_{jk} q^{N-2k+1} = q^{N-2j+1} \delta_{jl}.$$

Consequently,

$$\mathscr{R}(\mathrm{tr}(\mathscr{D}(\Lambda^* \Lambda)^k)) = \mathrm{tr}(\mathscr{D}(\Lambda^* \Lambda)^k) \otimes 1,$$

where $\mathscr{D} = \mathrm{diag}(q^{N-1}, q^{N-3}, \ldots, q^{-N+1})$. But it is even known [13] that the elements $\mathrm{tr}(\mathscr{D}(\Lambda^* \Lambda)^k)$, $k = 1, 2, \ldots, N-1$, generate the centre of $\mathscr{A}_q(AN)$. This means that the coaction $\mathscr{R}$ admits factorization. It is natural to define the quantum dressing orbit as the $*$-algebra $\mathscr{A}_q(AN)$ factorized by the relations

$$\mathrm{tr}(\mathscr{D}(\Lambda^* \Lambda)^k) - \gamma_k = 0, \qquad k = 1, 2, \ldots, N-1, \tag{11}$$

with γ_k's being some positive constants.

3. Quantized Grassmann Manifold

We start our discussion from the classical dressing orbit which is a Poisson manifold. According to (7), it is determined unambiguously by the set of eigenvalues (unordered but including the multiplicities) of the matrix $\Lambda^* \Lambda$. We are going to consider a special case when $\Lambda^* \Lambda$ has exactly two different eigenvalues: λ_1 with the multiplicity n and λ_2 with the multiplicity $m, n + m = N$. The orbit is then the Grassmann manifold $\mathbb{G}_m$ whose points are m-dimensional subspaces in $\mathbb{C}^N$, $\dim_C \mathbb{G}_m = mn$. One can write

$$\Lambda^* \Lambda = \lambda_1 \mathbf{I} + (\lambda_2 - \lambda_1) Q, \tag{12}$$

where Q is a projector of rank m and $\mathbf{I}$ stands for a unit matrix. A parameterization of the orbit is given by the parameterization of the projector

$$Q = \begin{pmatrix} \mathbf{I} \\ Z^* \end{pmatrix} (\mathbf{I} + ZZ^*)^{-1} (\mathbf{I}, Z), \tag{13}$$

where $Z = (z_{jk}) \in \mathbb{C}^{m,n}$.

Proposition 3.1. *The Poisson bracket on $\mathbb{G}_m$ is expressed in the above introduced coordinates z_{jk} as follows:*

$$-i\{z_{st}, z_{uv}\} = (\operatorname{sgn}(v-t) - \operatorname{sgn}(u-s))\, z_{ut} z_{sv}\,, \tag{14}$$

$$\begin{aligned}
-i(\lambda_1 - \lambda_2)\,\{z^*_{st}, z_{uv}\} &= 2\lambda_2 \delta_{su}\delta_{tv} + 2\lambda_1 \sum_j z_{uj} z^*_{sj} \sum_k z^*_{kt} z_{kv} \\
&\quad + \delta_{su} \sum_k (\lambda_1 + \lambda_2 + \operatorname{sgn}(s-k)\,(\lambda_1 - \lambda_2))\, z^*_{kt} z_{kv} \\
&\quad + \delta_{tv} \sum_j (\lambda_1 + \lambda_2 - \operatorname{sgn}(t-j)\,(\lambda_1 - \lambda_2))\, z_{uj} z^*_{sj}\,.
\end{aligned} \tag{15}$$

Proof. It is worth recalling that both $SU(N)$ and AN are Poisson subgroups of $SL(N,\mathbb{C})$ and the Poisson bracket on $SL(N,\mathbb{C})$ is given by

$$\begin{aligned}
\{t_{jk}, t_{uv}\} &= i(\operatorname{sgn}(u-j) + \operatorname{sgn}(v-k))\, t_{vj} t_{uk}\,, \\
\{t^*_{jk}, t_{uv}\} &= i\delta_{ju} \sum_\sigma (1 - \operatorname{sgn}(j-\sigma))\, t^*_{\sigma k} t_{\sigma v} \\
&\quad - i\delta_{kv} \sum_\nu (1 + \operatorname{sgn}(k-\nu))\, t_{u\nu} t^*_{j\nu}\,,
\end{aligned} \tag{16}$$

where $T = (t_{jk})$ is the vector representation of $SL(N,\mathbb{C})$.

Let us first evaluate the Poisson bracket on $\mathbb{G}_m$ at the point $Z = 0$. Expressing Λ from (12), (13) as a power series in the entries of Z and retaining the terms up to the first order,

$$\Lambda = \begin{pmatrix} \lambda_2 \mathbf{I} & (1 - (\lambda_1/\lambda_2))\, Z \\ 0 & \lambda_1 \mathbf{I} \end{pmatrix} + O(Z^2)\,,$$

and substituting Λ into (16) instead of T we get

$$\begin{aligned}
\{z_{st}, z_{uv}\}_{Z=0} &= 0\,, \\
\{z^*_{st}, z_{uv}\}_{Z=0} &= 2i\lambda_2(\lambda_1 - \lambda_2)^{-1} \delta_{su}\delta_{tv}\,.
\end{aligned}$$

To get the complete bracket one can employ the fact that $\mathscr{R}_{\mathrm{cl}}$ is a Poisson action and so

$$\{\mathscr{R}^*_{\mathrm{cl}} f, \mathscr{R}^*_{\mathrm{cl}} g\} = \mathscr{R}^*_{\mathrm{cl}} \{f, g\}\,. \tag{17}$$

Regard the fundamental representation U as a matrix of functions living on $SU(N)$ and split it into the blocks

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{18}$$

where $A = (a_{jk})$ is an $m \times m$ matrix, $B = (b_{jk})$ is an $m \times n$ matrix, etc. It holds

$$\begin{aligned}
\mathscr{R}^*_{\mathrm{cl}} Z &= (A + ZC)^{-1}(B + ZD) \\
&= A^{-1}B + A^{-1}ZD - A^{-1}ZCA^{-1}B + O(Z^2)\,.
\end{aligned}$$

Substitute z_{uv} for g and z_{st} respectively z^*_{st} for f in (17) and evaluate both sides at the point (Z_0, U_0), $Z_0 = 0$ and

$$U_0 = \begin{pmatrix} (\mathbf{I} + ZZ^*)^{-1/2} & Z(\mathbf{I} + Z^*Z)^{-1/2} \\ -Z^*(\mathbf{I} + ZZ^*)^{-1/2} & (\mathbf{I} + Z^*Z)^{-1/2} \end{pmatrix}.$$

Straightforward calculations then lead to (14), (15). Let us note only that if $A^{-1} = (\tilde{a}_{jk})$ then

$$\{\tilde{a}_{st}, b_{uv}\} = -\sum_{j,k} \tilde{a}_{sj} \{a_{jk}, b_{uv}\} \tilde{a}_{kt} \,. \qquad \square$$

To quantize the Poisson bracket means to introduce a $*$-algebra $\mathscr{C}$ generated by the elements z_{st}, z^*_{uv} in such a way that the obligatory correspondence rule

$$[f, g] = ih\{f, g\} + O(h^2)$$

is obeyed. Let us define $\mathscr{C}$ by the relations

$$[z_{st}, z_{uv}] = (q^{\operatorname{sgn}(s-u)} - q^{\operatorname{sgn}(t-v)}) z_{ut} z_{sv} \,, \tag{19}$$

$$\begin{aligned}(\lambda_1 - \lambda_2)[z^*_{st}, z_{uv}] = {} & (q - q^{-1})\lambda_2 \delta_{su}\delta_{tv} \\ & + (q - q^{-1})\lambda_1 \sum_j z_{uj} z^*_{sj} \sum_k z^*_{kt} z_{kv} \\ & + \delta_{su} \sum_k ((q^{\operatorname{sgn}(s-k)} - q^{-1})\lambda_1 + (q - q^{\operatorname{sgn}(s-k)})\lambda_2) z^*_{kt} z_{kv} \\ & + \delta_{tv} \sum_j ((q - q^{\operatorname{sgn}(t-j)})\lambda_1 + (q^{\operatorname{sgn}(t-j)})\lambda_2) z_{uj} z^*_{sj} \,. \end{aligned} \tag{20}$$

To support this definition we note that, first, the correspondence rule is actually fulfilled. Second, it is not the most important though useful fact that the commutation relations (19), (20) can be rewritten in a compact form with the help of the R-matrix,

$$R^{[m]}_{21} Z_1 Z_2 = Z_2 Z_1 R^{[n]}_{12} \,, \tag{21}$$

$$\begin{aligned}\lambda_2 \mathbf{I} + \lambda_1 Z_1 Z^*_1 Z^*_2 Z_2 + \frac{1}{q^{-1} - q} \, Z^*_2 P(\lambda_1 R^{[m]-1}_{21} - \lambda_2 R^{[m]}_{12}) Z_2 & \\ - \frac{1}{q^{-1} - q} \, Z_1 P(\lambda_1 R^{[n]}_{21} - \lambda_2 R^{[n]-1}_{12}) Z^*_1 = 0 \,. & \end{aligned} \tag{22}$$

But the third reason is decisive, especially in view of expected applications to the representation theory. The commutation relations (21), (22) allow to introduce quantum local coordinates on the dressing orbit in terms of a $*$-algebra morphism ψ.

Observe that the relations (3) imply that the matrix $\Lambda_2 R^{-1}_{21}$ commutes with $\Lambda^*_1 \Lambda_1$ and hence the same is true for the matrix $R^{-1}_{21} \Lambda^*_2 \Lambda_2 R^{-1}_{12}$. Thus

$$\Lambda^*_1 \Lambda_1 R^{-1}_{21} \Lambda^*_2 \Lambda_2 R^{-1}_{12} = R^{-1}_{21} \Lambda^*_2 \Lambda_2 R^{-1}_{12} \Lambda^*_1 \Lambda_1 \,. \tag{23}$$

An equivalent equation is obtained provided the legs 1 and 2 are interchanged,

$$\Lambda^*_1 \Lambda_1 R^{-1}_{21} \Lambda^*_2 \Lambda_2 R_{21} = R_{12} \Lambda^*_2 \Lambda_2 R^{-1}_{12} \Lambda^*_1 \Lambda_1 \,. \tag{24}$$

It is desirable to reverse this procedure. This is actually the case provided one can assume that the following implication holds in the $*$-algebra generated by the entries of the matrix Λ:

$$f > 0 \,, \qquad f^2 g = q^{2\sigma} g f^2 \Rightarrow fg = q^\sigma g f \,, \tag{25}$$

where $\sigma \in \mathbb{R}$ is arbitrary. Clearly, (25) is fulfilled for finite-dimensional representatives.

Proposition 3.2. *Let $\Lambda = (\alpha_{jk})$ be an upper-triangular matrix with entries from some $*$-algebra and with positive elements on the diagonal. Assume that the condition* (25) *is fulfilled in the $*$-subalgebra generated by the entries of the matrix Λ. Then the relations* (3) *and* (24) *are equivalent.*

Proof. Let us begin the proof with a remark concerning the notation. To avoid indices in the exponent we shall write, if necessary, $\delta(j,k)$ instead of δ_{jk}. We have to prove the implication only in one direction: (24) $\Rightarrow$ (3). Decompose

$$\Lambda_1 R_{21}^{-1} \Lambda_2^* = \tilde{X}\tilde{Y}\,, \tag{26}$$

where the matrix $\tilde{X}$ is lower-triangular and $\tilde{Y}$ is upper-triangular with units on the diagonal. Since

$$(\Lambda_1 R_{21}^{-1} \Lambda_2^*)_{jk,st} = q^{-\delta(k,s)} \alpha_{js} \alpha_{tk}^* + (q^{-1} - q)\,\delta_{ks} \sum_{\sigma<k} \alpha_{j\sigma} \alpha_{t\sigma}^*\,,$$

and Λ is upper-triangular, it holds

$$(\Lambda_1 R_{21}^{-1} \Lambda_2^*)_{jk,st} = 0 \quad \text{for} \quad j > s\,,$$
$$(\Lambda_1 R_{21}^{-1} \Lambda_2^*)_{jk,jt} = q^{-\delta(j,k)} \alpha_{jj} \alpha_{tk}^*\,.$$

Consequently,

$$\tilde{X}_{jk,st} = q^{-\delta(j,k)} \delta_{js} \alpha_{jj} \alpha_{tk}^*\,, \tag{27}$$

$$\tilde{Y}_{jk,jt} = \delta_{kt}\,. \tag{28}$$

After the substitution of (26) and its adjoint into (24) we get

$$\Lambda_1^* \tilde{X}\tilde{Y} \Lambda_2 R_{21} = R_{12} \Lambda_2^* \tilde{Y}^* \tilde{X}^* \Lambda_1\,.$$

Comparing the decomposition of both sides into a product of the lower-triangular and upper-triangular parts we conclude that there exists an invertible diagonal matrix Ω, $\Omega_{jk,st} = \omega_{jk} \delta_{js} \delta_{kt}$, such that

$$\Lambda_1^* \tilde{X} = R_{12} \Lambda_2^* \tilde{Y}^* \Omega\,, \qquad \tilde{Y} \Lambda_2 R_{21} = \Omega^{-1} \tilde{X}^* \Lambda_1\,.$$

Comparing these equations with their adjoints one finds that Ω is self-adjoint, $\Omega^* = \Omega$ and hence $\omega_{jk}^* = \omega_{jk}$. So it is enough to consider only one equation written in the form

$$\tilde{X}^* \Lambda_1 R_{21}^{-1} = \Omega \tilde{Y} \Lambda_2\,. \tag{29}$$

According to (27), (28) it holds

$$(\tilde{X}^* \Lambda_1 R_{21}^{-1})_{jk,jk} = q^{-2\delta(j,k)} \alpha_{kk} \alpha_{jj}^2\,, \qquad (\tilde{Y} \Lambda_2)_{jk,jk} = \alpha_{kk}\,.$$

It follows that $\omega_{jk} = q^{-2\delta(j,k)} \alpha_{kk} \alpha_{jj}^2 \alpha_{kk}^{-1}$. As $\omega_{jk}^* = \omega_{jk}$, we have $\alpha_{jj}^2 \alpha_{kk}^2 = \alpha_{kk}^2 \alpha_{jj}^2$ and owing to the assumption (25), $\alpha_{jj}\alpha_{kk} = \alpha_{kk}\alpha_{jj}$. Consequently there exists the square root

$$\omega_{jk}^{1/2} = q^{-\delta(j,k)} \alpha_{jj}\,. \tag{30}$$

Set

$$X = \tilde{X} \Omega^{-1/2}\,, \qquad Y = \Omega^{1/2} \tilde{Y}\,. \tag{31}$$

Then (26) and (29) mean that

$$\Lambda_1 R_{21}^{-1} \Lambda_2^* = XY\,, \qquad X^* \Lambda_1 R_{21}^{-1} = Y \Lambda_2\,. \tag{32}$$

Further it holds

$$(\tilde{X}^* \Lambda_1 R_{21}^{-1})_{jk,jt} = q^{-2\delta(j,t)} \alpha_{kt} \alpha_{jj}^2\,, \qquad (\tilde{Y} \Lambda_2)_{jk,jt} = \alpha_{kt}\,.$$

Employing once more the assumption (25) one gets

$$q^{-\delta(j,k)} \alpha_{jj} \alpha_{kt} = q^{-\delta(j,t)} \alpha_{kt} \alpha_{jj}\,.$$

As an immediate consequence we have $X = \Lambda_2^*$ [cf. (27)]. Use now the substitution $Y = K\Lambda_1$. K is again upper-triangular. The Eqs. (31) then read

$$\Lambda_1 R_{21}^{-1} \Lambda_2^* = \Lambda_2^* K \Lambda_1\,, \tag{33}$$

$$\Lambda_2 \Lambda_1 R_{21}^{-1} = K \Lambda_1 \Lambda_2\,. \tag{34}$$

Adjoint Eq. (33), interchange the legs 1 and 2 and compare with the original equation. It follows that $K = PK^*P$, i.e., $K_{jk,st} = K_{ts,kj}^*$. Consequently,

$$K_{jk,st} = 0 \quad \text{for} \quad (j,k) > (s,t) \quad \text{or} \quad (k,j) < (t,s)\,. \tag{35}$$

Besides, owing to (28), (30) and (31) it holds

$$K_{jk,jt} = q^{-\delta(j,k)} \delta_{kt}\,. \tag{36}$$

As a final step we shall show that

$$K_{jk,st} = (q^{-1} - q) \delta_{jt} \delta_{ks} \quad \text{for} \quad j < s,\ k > t\,. \tag{37}$$

We proceed by induction in (s,t). Equation (34) means that

$$\sum_{\sigma,\nu} K_{jk,\sigma\nu} \alpha_{\sigma s} \alpha_{\nu t} = \alpha_{kt} \alpha_{js} + (q^{-1} - q^{\operatorname{sgn}(s-t)}) \alpha_{ks} \alpha_{jt}\,. \tag{38}$$

Suppose we are given a double index (s,t) such that $K_{jk,\sigma\nu} = (q^{-1} - q)\delta_{j\nu}\delta_{k\sigma}$ for all couples (σ,ν), $\sigma \le s$, $\nu \le t$ and $(\sigma,\nu) \ne (s,t)$ and whenever $j < \sigma$, $k > \nu$. This assumption should be regarded as being fulfilled even if no such couple (σ,ν) exists. Suppose further that $j < s$, $k > t$. Owing to (35) and (36), $K_{jk,\sigma\nu}$ is nonzero only if $j = \sigma$, $k = \nu$ or $j < \sigma$, $k > \nu$. Bearing in mind that Λ is upper-triangular we get

$$\begin{aligned} \text{LHS of (38)} &= K_{jk,st} \alpha_{ss} \alpha_{tt} + \vartheta (q^{-1} - q) \sum_{\sigma > j, \nu < k} \delta_{j\nu} \delta_{k\sigma} \alpha_{\sigma s} \alpha_{\nu t} \\ &= K_{jk,st} \alpha_{ss} \alpha_{tt} + \vartheta (q^{-1} - q) \alpha_{ks} \alpha_{jt}\,, \end{aligned}$$

where $\vartheta = 0$ if $(k,j) = (s,t)$ and $\vartheta = 1$ otherwise,

$$\text{RHS of (38)} = (q^{-1} - q) \alpha_{ks} \alpha_{jt}\,.$$

Comparing both expressions we conclude that $K_{jk,st} = 0$ if $(k,j) \ne (s,t)$ and $K_{ts,st} = q^{-1} - q$.

The equalities (35), (36) and (37) altogether mean that $K = R_{21}^{-1}$. But then (33) and (34) immediately give (3). □

Let $\psi(\Lambda)$ be the upper-triangular matrix with entries from the algebra $\mathscr{C}$ such that

$$\begin{aligned}\psi(\Lambda^*\Lambda) &= \psi(\Lambda)^*\,\psi(\Lambda)\\ &= \lambda_1\mathbf{I} + (\lambda_2-\lambda_1)\begin{pmatrix}\mathbf{I}\\ Z^*\end{pmatrix}(\mathbf{I}+ZZ^*)^{-1}(\mathbf{I},Z)\,.\end{aligned} \tag{39}$$

The inversion $(\mathbf{I}+ZZ^*)^{-1}$ should be considered as a formal power series.

Proposition 3.3. *The matrix* $\psi(\Lambda)$ *obeys the relation* (23).

Proof. Let us retain the notation (13) also in the quantum case. It holds again $Q^2=Q$, $Q^*=Q$. Rewrite Eq. (23) as

$$R_{21}\psi(\Lambda_1^*\Lambda_1)R_{21}^{-1}\psi(\Lambda_2^*\Lambda_2) = \psi(\Lambda_2^*\Lambda_2)R_{12}^{-1}\psi(\Lambda_1^*\Lambda_1)R_{12}\,, \tag{40}$$

and substitute (39) to get

$$\begin{aligned}&\lambda_1R_{21}Q_1R_{21}^{-1} + (\lambda_2-\lambda_1)R_{21}Q_1R_{21}^{-1}Q_2\\ &\quad= \lambda_1R_{12}^{-1}Q_1R_{12} + (\lambda_2-\lambda_1)Q_2R_{12}^{-1}Q_1R_{12}\,.\end{aligned} \tag{41}$$

Set

$$S_{12} = (q^{-1}-q)^{-1}(\lambda_1R_{12}-\lambda_2R_{21}^{-1})\,. \tag{42}$$

The relation (2) combined with (42) yields

$$\begin{aligned}(\lambda_1-\lambda_2)R_{12} &= (q^{-1}-q)(S_{12}+\lambda_2P)\,,\\ (\lambda_1-\lambda_2)R_{21}^{-1} &= (q^{-1}-q)(S_{12}+\lambda_1P)\,.\end{aligned}$$

Using this substitution in (41) we get after some straightforward manipulations an equivalent form of (40),

$$S_{21}Q_1S_{12}Q_2 = Q_2S_{21}Q_1S_{12}\,.$$

Set

$$V(\mathbf{I},Z)\,,\qquad W=\begin{pmatrix}-Z\\ \mathbf{I}\end{pmatrix}.$$

In view of the form of the matrix Q (13) one easily finds that it is enough to verify the equality

$$V_2S_{21}Q_1S_{12}W_2 = 0\,.$$

This equation if expressed in terms of the matrix Z and provided the relation (21) is employed amounts to two equations:

$$-(S_{21}^{[m]}-\lambda_1PZ_1Z_1^*)(\mathbf{I}+Z_1Z_1^*)^{-1}R_{21}^{-1}Z_2 + \frac{\lambda_1-\lambda_2}{q^{-1}-q}Z_2(\mathbf{I}+Z_1Z_1^*)^{-1} = 0\,,$$

$$-\frac{\lambda_1-\lambda_2}{q^{-1}-q}Z_1^*(\mathbf{I}+Z_1Z_1^*)^{-1}R_{21}^{-1}Z_2 + (Z_2S_{21}^{[n]}Z_1^*-\lambda_2P)(\mathbf{I}+Z_1Z_1^*)^{-1} = 0\,.$$

But again after some straightforward manipulations both of these two equations can be shown to follow from (22). □

Denote by $\tilde{\mathscr{A}}_q(AN)$ the $*$-algebra determined by the relations (3) but with the conditon (4) being temporarily abandoned. Propositions 3.2 and 3.3 guarantee the existence of a $*$-algebra morphism

$$\psi\colon\tilde{\mathscr{A}}_q(AN)\to\mathscr{C}\,.$$

It plays the role of the desired quantum local coordinates on the dressing orbit. It remains to determine the constants γ_k in (11) in order to specify the orbit.

Proposition 3.4. *It holds*

$$(\lambda_1 - \lambda_2)\,\mathrm{tr}(\mathscr{D}\psi(\Lambda^*\Lambda)^k) = [n]\,\lambda_1^k(q^{-m}\lambda_1 - q^m\lambda_2) + [m]\,\lambda_2^k(q^n\lambda_1 - q^{-n}\lambda_2)\,. \quad (43)$$

Proof. First note that

$$\psi(\Lambda^*\Lambda)^k = \lambda_1^k\mathbf{I} + (\lambda_2^k - \lambda_1^k)Q\,.$$

Let $a, b \in \mathbb{R}$ be some parameters. Then

$$\begin{aligned}&a\mathbf{I} + (b-a)Q\\ &= \begin{pmatrix} b(\mathbf{I}+ZZ^*)^{-1} + a(\mathbf{I}+ZZ^*)^{-1}ZZ^* & * \\ * & b(\mathbf{I}+Z^*Z)^{-1}Z^*Z + a(\mathbf{I}+Z^*Z)^{-1}\end{pmatrix}.\end{aligned}$$

The expression $\mathrm{tr}(\mathscr{D}(a\mathbf{I} + (b-a)Q))$ is linear in a and b. The coefficient standing at a is

$$\sum_{j=1}^{m} q^{m+n-2j+1}(Z(\mathbf{I}+Z^*Z)^{-1}Z^*)_{jj} + \sum_{k=1}^{n} q^{-m+n-2k+1}((\mathbf{I}+Z^*Z)^{-1})_{kk}\,. \quad (44)$$

Next we are going to simplify this expression. Equation (22) can be rewritten with the help of (2) as

$$\begin{aligned}(\lambda_2 - \lambda_1)P &- \frac{\lambda_1 - \lambda_2}{q^{-1} - q}\, Z_2 R_{12}^{-1} Z_1^*\\ &+ \frac{\lambda_1 - \lambda_2}{q^{-1} - q}\, Z_1^* R_{12} Z_2 + \lambda_1(\mathbf{I} + Z_1^*Z_1)P(\mathbf{I} + Z_1Z_1^*) = 0\,.\end{aligned}$$

Further manipulations based on the relations (21) and (2) yield

$$\begin{aligned}&(\mathbf{I} + Z_1^*Z_1)^{-1}Z_1^*R_{12}Z_2\\ &= Z_2R_{21}(\mathbf{I} + Z_1^*Z_1)^{-1}Z_1^* - \frac{q^{-1}-q}{\lambda_1 - \lambda_2}(\lambda_2\mathbf{I} + \lambda_1 Z_1^*Z_1)(\mathbf{I} + Z_1^*Z_1)^{-1}P\,.\end{aligned}$$

Comparing the matrix elements of both sides with the indices (st, ts) we get

$$\begin{aligned}&q((\mathbf{I}+Z^*Z)^{-1}Z^*)_{st}Z_{ts} - (q^{-1}-q)\sum_{\sigma>t}((\mathbf{I}+Z^*Z)^{-1}Z^*)_{s\sigma}Z_{\sigma s}\\ &= qZ_{ts}((\mathbf{I}+Z^*Z)^{-1}Z^*)_{st} - (q^{-1}-q)\sum_{\nu>s} Z_{t\nu}((\mathbf{I}+Z^*Z)^{-1}Z^*)_{\nu t}\\ &\quad - (q^{-1}-q)(\lambda_1-\lambda_2)^{-1}((\lambda_2\mathbf{I} + \lambda_1 Z^*Z)(\mathbf{I}+Z^*Z)^{-1})_{ss}\,. \quad (45)\end{aligned}$$

Regard s as being fixed, t as varying from 1 to m and $((\mathbf{I}+Z^*Z)^{-1}Z^*)_{st}Z_{ts}$ as an unknown which is to be obtained from (45). More precisely, we are interested only in the sum $\sum_t((\mathbf{I}+ZZ)^{-1}Z^*)_{st}Z_{ts}$. As

$$(1, 1, \ldots, 1)\begin{pmatrix} q & (q-q^{-1)} & (q-q^{-1)} & \ldots \\ 0 & q & (q-q^{-1}) & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix}^{-1} = (q^{-1}, q^{-3}, \ldots, q^{-2m+1})\,,$$

we get

$$((\mathbf{I}+Z^*Z)^{-1}Z^*Z)_{ss} = -\frac{q^{-2m}-1}{\lambda_1-\lambda_2}((\lambda_2\mathbf{I}+\lambda_1 Z^*Z)(\mathbf{I}+Z^*Z)^{-1})_{ss}$$
$$+q\sum_{j=1}^{m} q^{-2j+1} Z_{js}((\mathbf{I}+Z^*Z)^{-1}Z^*)_{sj}$$
$$-(q^{-1}-q)\sum_{\nu>s}\sum_{j=1}^{m} q^{-2j+1} Z_{j\nu}((\mathbf{I}+Z^*Z)^{-1}Z^*)_{\nu j}\,.$$

Now we consider the element $x_\nu = \sum_j q^{-2j+1} Z_{j\nu}((\mathbf{I}+Z^*Z)^{-1}Z^*)_{\nu j}$ as an unknown and are again interested only in the sum $\sum x_\nu$. The result is

$$\sum_{j=1}^{m} q^{-2j+1}(Z(\mathbf{I}+Z^*Z)^{-1}Z^*)_{jj}$$
$$=\sum_{k=1}^{n} q^{-2k+1}\left(\left(Z^*Z+\frac{q^{-2m}-1}{\lambda_1-\lambda_2}(\lambda_2\mathbf{I}+\lambda_1 Z^*Z)\right)(\mathbf{I}+Z^*Z)^{-1}\right)_{kk}.$$

With this identity one easily finds that (44) is equal to

$$[n]\,(q^{-m}\lambda_1 - q^m\lambda_2)(\lambda_1-\lambda_2)^{-1}\,.$$

The coefficient standing at b can be obtained similarly. □

4. Construction of Representations

In the classical case, the vector space of functions living on the orbit of the coadjoint action becomes naturally an $SU(N)$ module. But this structure is too rough to construct irreducible representations. One has to employ the symplectic structure or the descent Poisson structure, impose the quantization condition and pass to the vector space of holomorphic (or antiholomorphic) sections in a line-bundle based on the orbit. In the quantum case, the Poisson bracket is already concealed in the non-commutative multiplication law. As we shall see, the relations (21), (22) allow a straightforward construction of irreducible representations. Nevertheless, it is of interest to express in local coordinates also the quantum dressing transformation which replaces the classical coadjoint action.

Denote by $\mathscr{C}_{ah}$ the subalgebra of $\mathscr{C}$ generated by the "antiholomorphic" quantum coordinate functions z^*_{st}. Hence the defining relation for $\mathscr{C}_{ah}$ is

$$R^{[n]}_{21} Z^*_1 Z^*_2 = Z^*_2 Z^*_1 R^{[m]}_{12}\,. \tag{46}$$

The quantum dressing transformation $\mathscr{R}$ (8) if expressed in the local coordinates reads

$$\mathscr{R}(Z^*) = (B^* + D^*Z^*)(A^* + C^*Z^*)^{-1}\,, \tag{47}$$

where we assume that the fundamental representation U of $SU_q(N)$ splits into blocks as in (18). According to the property (10) it is enough to specify the action only on the generators z^*_{kl}. Recalling the pairing (5) we arrive after some straightforward calculations at the formulas

$$
\begin{aligned}
H_j \cdot z^*_{kl} &= (\delta_{kj} - \delta_{k,j+1}) z^*_{kl} && \text{for } j = 1, \ldots, m-1, \\
&= (\delta_{l1} + \delta_{km}) z^*_{kl} && j = m, \\
&= (-\delta_{l,j-m} + \delta_{l,j-m+1}) z^*_{kl} && j = m+1, \ldots, N-1, \\
X^+_j \cdot z^*_{kl} &= \delta_{k,j+1} z^*_{jl} && \text{for } j = 1, \ldots, m-1, \\
&= q^{-\tau/2} z^*_{ml} z^*_{k1}, \ \tau = 1 - \delta_{i1} + \delta_{km} && j = m, \\
&= -q^{-1} \delta_{l,j-m} z^*_{k,j-m+1} && j = m+1, \ldots, N-1, \\
X^-_j \cdot z^*_{kl} &= \delta_{kj} z^*_{j+1,l} && \text{for } j = 1, \ldots, m-1, \\
&= -q^{1/2} \delta_{km} \delta_{l1} && j = m, \\
&= -q \delta_{l,j-m+1} z^*_{k,j-m} && j = m+1, \ldots, N-1.
\end{aligned}
\tag{48}
$$

Let us now proceed to the construction of irreducible representations. Denote by $\mathscr{I}$ the left ideal in $\mathscr{C}$ generated by the elements z_{st}. Then $\mathscr{C}/\mathscr{I}$ is a left $\tilde{\mathscr{A}}_q(AN)$ module. The factor morphism $\mathscr{C} \to \mathscr{C}/\mathscr{I}$ if restricted to the subalgebra $\mathscr{C}_{ah}$ becomes a linear isomorphism and so one can identify $\mathscr{C}/\mathscr{I}$ with $\mathscr{C}_{ah}$. Denote by $\mathscr{M}$ the cyclic submodule in $\mathscr{C}/\mathscr{I}$ with the cyclic vector 1.

Proposition 4.1. *$\mathscr{M}$ is spanned by* 1 *and by entries of the matrices ($R = R^{[m]}$, $r = 1, 2, \ldots$)*

$$
\begin{aligned}
&Z^*_1 Z^*_2 \ldots Z^*_r (q^n \lambda_1 R_{21} R_{31} \ldots R_{r1} - q^{-n} \lambda_2 R^{-1}_{12} R^{-1}_{13} \ldots R^{-1}_{1r}) \\
&\quad \times (q^n \lambda_1 R_{32} \ldots R_{r2} - q^{-n} \lambda_2 R^{-1}_{23} \ldots R^{-1}_{2r}) \\
&\quad \times \ldots \times (q^n \lambda_1 R_{r,r-1} - q^{-n} \lambda_2 R^{-1}_{r-1,r}) \times (q^n \lambda_1 - q^{-n} \lambda_2).
\end{aligned}
\tag{49}
$$

Proof. The submodule $\mathscr{M}$ is spanned by 1 and by entries of the matricies $Q_1 Q_2 \ldots Q_r$. 1. Q can be written in a block-like form

$$
Q = \begin{pmatrix} (\mathbf{I} + ZZ^*)^{-1} & Z(\mathbf{I} + Z^*Z)^{-1} \\ Z^*(\mathbf{I} + ZZ^*)^{-1} & Z^*Z(\mathbf{I} + Z^*Z)^{-1} \end{pmatrix}.
$$

Denote by $\mathscr{V}$ the subspace in $\mathscr{C}/\mathscr{I}$ spanned by 1 and by entries of the matrices (49). We have to show that, first, $\mathscr{V}$ is Q-invariant, second, $\mathscr{V} \subset \mathscr{M}$. The verification will be done in several steps.

(i) *The solution to the system of algebraic equations ($R = R^{[n]}$)*

$$
\frac{1}{q^{-1} - q} (X_1 (\lambda_1 R_{12} - \lambda_2 R^{-1}_{21}) Y_2)_{st,uv} = F_{st,uv}
$$

with the unknown quantities $X_{su} Y_{tv} = (X_1 Y_2)_{st,uv}$ is

$$
\begin{aligned}
X_{su} Y_{tv} = &\frac{q^{-1} - q}{(\lambda_1 - \lambda_2)(q^n \lambda_1 - q^{-n} \lambda_2)} \\
&\times \sum_{\sigma\nu} (q^n \lambda_1 R^{-1}_{12} - q^{-n} \lambda_2 R_{21})_{\sigma t, u\nu} q^{2n-2\sigma} F_{s\nu,\sigma v}.
\end{aligned}
$$

To see this note that $X_{su} Y_{tv} = (q^{-1} - q)(\lambda_1 - \lambda_2)^{-1} F_{st,uv}$ for $u \neq t$. For $u = t$, we fix s and v and solve the system of algebraic equations with the vector of unknown quantities $(X_{s1}Y_{1v}, X_{s2}Y_{2v}, \ldots, X_{sn}Y_{nv})$, with the RHS $(F_{s1,1v}, F_{s2,2v}, \ldots, F_{sn,nv})$ and with the matrix $\mathbb{M}$,

$$\begin{aligned}\mathbb{M}_{jk} &= (q\lambda_1 - q^{-1}\lambda_2)/(q^{-1}-q) \quad \text{for} \quad j=k\,,\\ &= -\lambda_1 \qquad\qquad\qquad\qquad\qquad\quad j<k\,,\\ &= -\lambda_2 \qquad\qquad\qquad\qquad\qquad\quad j>k\,.\end{aligned}$$

$\mathbb{M}$ can be inverted,

$$\begin{aligned}M^{-1}_{jk} = \frac{(q^{-1}-q)^2}{(\lambda_1-\lambda_1)(q^n\lambda_1 - q^{-n}\lambda_2)} &\times \frac{q^{n-1}\lambda_1 - q^{-n+1}\lambda_2}{q^{-1}-q} \quad \text{for} \quad j=k\,,\\ &\times q^{n+2j-2k}\lambda_1 \qquad\qquad\qquad j<k\,,\\ &\times q^{-n+2j-2k}\lambda_2 \qquad\qquad\quad\; j>k\,.\end{aligned}$$

The result easily follows.

(ii) *It holds*

$$(\mathbf{I}+ZZ^*)^{-1}\cdot 1 = (\lambda_1 - q^{-2n}\lambda_2)(\lambda_1-\lambda_2)^{-1}\mathbf{I}\,. \tag{50}$$

To see (50) note that the relation (22) implies

$$(q^{-1}-q)^{-1}Z_1(\lambda_1 R_{12} - \lambda_2 R_{21}^{-1})Z_2^*\cdot 1 = \lambda_2 P\,.$$

According to (i), we can solve the last equation to get

$$z_{uv}z^*_{st}\cdot 1 = \frac{q^{-1}-q}{q^n\lambda_1 - q^{-n}\lambda_2}\, q^{-n+2t-1}\lambda_2\delta_{su}\delta_{tv}\,.$$

The result easily follows.

(iii) *It holds*

$$\begin{aligned}&(\mathbf{I}+Z_1Z_1^*)^{-1}\cdot Z_2^*Z_3^*\ldots Z_r^*\\ &\quad= q^{-n}(\lambda_1-\lambda_2)^{-1}Z_2^*Z_3^*\ldots Z_r^*(q^n\lambda_1 R_{21}R_{31}\ldots R_{r1}\\ &\qquad - q^{-n}\lambda_2 R_{12}^{-1}R_{13}^{-1}\ldots R_{1r}^{-1})R_{r1}^{-1}\ldots R_{31}^{-1}R_{21}^{-1}\,.\end{aligned} \tag{51}$$

As an immediate consequence one finds that, first, $\mathscr{V}$ is $Z^*(\mathbf{I}+ZZ^*)^{-1}$-invariant, second, $\mathscr{V}\subset\mathscr{M}$.

To check (51) note that from (22) one can derive

$$\begin{aligned}&(\mathbf{I}+Z_1Z_1^*)^{-1}Z_2^*\\ &\quad= Z_2^*R_{12}^{-1}Z_1^*(\mathbf{I}+Z_1Z_1^*)^{-1}R_{21}^{-1} - (q^{-1}-q)(\lambda_1-\lambda_2)^{-1}\lambda_1 Z_2^*R_{12}^{-1}P\,.\end{aligned}$$

Proceeding by induction one can prove that

$$\begin{aligned}&(\mathbf{I}+Z_1Z_1^*)^{-1}Z_2^*Z_3^*\ldots Z_r^*\\ &\quad= Z_2^*Z_3^*\ldots Z_r^*R_{12}^{-1}R_{13}^{-1}\ldots R_{1r}^{-1}(\mathbf{I}+Z_1Z_1^*)^{-1}R_{r1}^{-1}\ldots R_{31}^{-1}R_{21}^{-1}\\ &\qquad - \frac{\lambda_1}{\lambda_1-\lambda_2}\, Z_2^*Z_3^*\ldots Z_r^*R_{12}^{-1}R_{13}^{-1}\ldots R_{1r}^{-1}\\ &\qquad \times (R_{r1}^{-1}\ldots R_{31}^{-1}R_{21}^{-1} - R_{1r}\ldots R_{13}R_{12})\,.\end{aligned}$$

Now it is enough to apply (50).

(iv) $\mathscr{V}$ *is* $(\mathbf{I}+ZZ^*)^{-1}$*-invariant.*

This assertion is a consequence of (iii) and of the following relation:

$$\begin{aligned}
&R_{12}^{-1}R_{13}^{-1}\dots R_{1r}^{-1}R_{r1}^{-1}\dots R_{31}^{-1}R_{21}^{-1}(aR_{32}R_{42}\dots R_{r2}-bR_{23}^{-1}R_{24}^{-1}\dots R_{2r}^{-1})\\
&\quad\times\dots\times(aR_{r,r-1}-bR_{r-1,1}^{-1})\\
&\quad=(aR_{32}R_{42}\dots R_{r2}-bR_{23}^{-1}R_{24}^{-1}\dots R_{2r}^{-1})\times\dots\times(aR_{r,r-1}-bR_{r-1,1}^{-1})\\
&\quad\times R_{1r}^{-1}\dots R_{13}^{-1}R_{12}^{-1}R_{21}^{-1}R_{31}^{-1}\dots R_{r1}^{-1}\,,
\end{aligned}$$

which can be proven by applying repeatedly the Yang-Baxter equation.

(v) *It holds*

$$\begin{aligned}
&Z_1(\mathbf{I}+Z_1^*Z_1)^{-1}\cdot Z_2^*Z_3^*\dots Z_r^*\\
&\quad=\frac{q^{-1}-q}{\lambda_1-\lambda_2}\lambda_2\sum_k(q^n\lambda_1-q^{-n}\lambda_2)^{-k+1}P_{1k}\\
&\quad\times(q^n\lambda_1R_{1,k-1}^{-1}-q^{-n}\lambda_2R_{k-1,1})\\
&\quad\times\dots\times(q^n\lambda_1R_{12}^{-1}-q^{-n}\lambda_2R_{21})Z_2^*\dots\hat{Z}_k^*\dots Z_r^*\\
&\quad\times R_{k2}^{-1}\dots R_{k,k-1}^{-1}(q^n\lambda_1R_{k+1,k}\dots R_{rk}\\
&\quad-q^{-n}\lambda_2R_{k,k+1}^{-1}\dots R_{kr}^{-1})R_{rk}^{-1}\dots R_{k+1,k}^{-1}\Gamma_1\,,
\end{aligned}\tag{52}$$

where $\Gamma=\operatorname{diag}(q^{-2n+1},q^{-2n+3},\dots,q^{-1})$ *and the hat indicates that the corresponding factor is omitted.*

The relation (52) can be proven by induction. The equality (22) implies

$$\begin{aligned}
&\frac{1}{q^{-1}-q}Z_1(\mathbf{I}+Z_1^*Z_1)^{-1}(\lambda_1R_{12}-\lambda_2R_{21}^{-1})Z_2^*\\
&\quad=\frac{\lambda_1-\lambda_2}{q^{-1}-q}Z_2^*R_{12}^{-1}Z_1(\mathbf{I}+Z_1^*Z_1)^{-1}+\lambda_2P(\mathbf{I}+Z_2Z_2^*)^{-1}\,.
\end{aligned}$$

Afterwards the result stated in (i) should be applied.

(vi) *It holds*

$$\begin{aligned}
&(aR_{32}\dots\hat{R}_{k2}\dots R_{r2}-bR_{23}^{-1}\dots\hat{R}_{2k}^{-1}\dots R_{2r}^{-1})\\
&\quad\times\dots\times(aR_{k+1,k}\dots R_{rk}\widehat{-bR_{k,k+1}^{-1}}\dots R_{kr}^{-1})\times\dots\times(aR_{r,r-1}-bR_{r-1,r}^{-1})\\
&\quad\times(aR_{rk}\dots\hat{R}_{kk}\dots R_{2k}-bR_{kr}^{-1}\dots\hat{R}_{kk}^{-1}\dots R_{k2}^{-1})R_{2k}^{-1}\dots R_{k-1,k}^{-1}\\
&=R_{k2}^{-1}\dots R_{k,k-1}^{-1}(aR_{32}R_{42}\dots R_{r2}-bR_{23}^{-1}R_{24}^{-1}\dots R_{2r}^{-1})\\
&\quad\times\dots\times(aR_{r,r-1}-bR_{r-1,r}^{-1})\,.
\end{aligned}\tag{53}$$

This identitiy can be proven by induction in r. Let us sketch the induction step $r-1\to r$. Owing to (2) we have $R_{k2}^{-1}R_{2k}^{-1}=\mathbf{I}+(q^{-1}-q)R_{k2}^{-1}P_{2k}$ and so

$$\begin{aligned}
&(aR_{rk}\dots\hat{R}_{kk}\dots R_{2k}-bR_{kr}^{-1}\dots\hat{R}_{kk}^{-1}\dots R_{k2}^{-1})R_{2k}^{-1}\\
&\quad=(aR_{rk}\dots\hat{R}_{kk}\dots R_{3k}-bR_{kr}^{-1}\dots\hat{R}_{kk}^{-1}\dots R_{k3}^{-1})\\
&\quad-(q^{-1}-q)bR_{kr}^{-1}\dots\hat{R}_{kk}^{-1}\dots R_{k2}^{-1}P_{2k}\,.
\end{aligned}$$

After this substitution we get the LHS of (53) written as a sum of two summands. The induction hypothesis is applicable to the first summand and afterwards the identities are used following from the Yang-Baxter equation

$$\begin{aligned}
&R_{32}\ldots\hat{R}_{k2}\ldots R_{r2}R_{k3}^{-1}\ldots R_{k,k-1}^{-1}\\
&\quad= R_{k2}^{-1}R_{k3}^{-1}\ldots R_{k,k-1}^{-1}R_{32}\ldots R_{r2}\,,\\
&R_{23}^{-1}\ldots\hat{R}_{2k}^{-1}\ldots R_{2r}^{-1}R_{k3}^{-1}\ldots R_{k,k-1}^{-1}\\
&\quad= R_{k2}^{-1}R_{k3}^{-1}\ldots R_{k,k-1}^{-1}R_{23}^{-1}\ldots R_{2,k-1}^{-1}R_{k2}R_{2,k+1}^{-1}\ldots R_{2r}^{-1}\,.
\end{aligned}$$

To deal with the second summand note that the Yang-Baxter equation implies

$$\begin{aligned}
&(aR_{32}\ldots\hat{R}_{k2}\ldots R_{r2}-bR_{23}^{-1}\ldots\hat{R}_{2k}^{-1}\ldots R_{2r}^{-1})\\
&\qquad\times\ldots\times(aR_{r,r-1}-bR_{r-1,r}^{-1})R_{kr}^{-1}\ldots\hat{R}_{kk}^{-1}\ldots R_{k2}^{-1}P_{2k}\\
&\quad= R_{k2}^{-1}\ldots\hat{R}_{kk}^{-1}\ldots R_{kr}^{-1}P_{2k}(aR_{43}\ldots\hat{R}_{k3}\ldots R_{r3}\\
&\qquad-bR_{34}^{-1}\ldots\hat{R}_{3k}^{-1}\ldots R_{3r}^{-1})\times\ldots\times(aR_{r,r-1}\ldots bR_{r-1,r}^{-1})\\
&\qquad\times(aR_{rk}\ldots\hat{R}_{kk}\ldots R_{3k}-bR_{kr}^{-1}\ldots\hat{R}_{kk}^{-1}\ldots R_{k3}^{-1})\,.
\end{aligned}$$

Using this identity we get the second summand written in a form which allows the induction hypothesis to be applied and after that both summands can be again recombined to yield the RHS of (53).

(vii) *$\mathscr{V}$ is $Z(\mathbf{I}+Z^*Z)^{-1}$-invariant.*

First apply to (52) the identity

$$\begin{aligned}
&(aR_{k+1,k}\ldots R_{rk}-bR_{k,k+1}^{-1}\ldots R_{kr}^{-1})R_{rk}^{-1}\ldots R_{k+1,k}^{-1}\\
&\quad\times(aR_{32}\ldots R_{r2}-bR_{23}^{-1}\ldots R_{2r}^{-1})\times\ldots\times(aR_{r,r-1}-bR_{r-1,r}^{-1})\\
&= (aR_{32}\ldots R_{r2}-bR_{23}^{-1}\ldots R_{2r}^{-1})\times\ldots\times(aR_{r,r-1}-bR_{r-1,r}^{-1})\\
&\quad\times R_{k+1,k}^{-1}\ldots R_{rk}^{-1}(aR_{rk}\ldots R_{k+1,k}-bR_{kr}^{-1}\ldots R_{k,k+1}^{-1})\,,
\end{aligned}\tag{54}$$

which follows from the repeatedly used Yang-Baxter equation. The result is then obtained with the help of the equality (53).

(viii) *$\mathscr{V}$ is $Z^*Z(\mathbf{I}+Z^*Z)^{-1}$-invariant.*

Multiply the equality (52) by Z_1^* from the left and notice that $Z_1^*P_{1k}=P_{1k}Z_k^*$ and

$$Z_k^*Z_2^*\ldots\hat{Z}_k^*\ldots Z_r^*R_{k,k-1}^{-1}=R_{2k}^{-1}\ldots R_{k-1,k}^{-1}Z_2^*Z_3^*\ldots Z_r^*\,.$$

Applying again the identity (54) we get the result. □

Next an identity is presented valid for the R-matrix (1).

Proposition 4.2. *For $R=R^{[m]}$, $r>m\sigma$, $\sigma=0,1,2,\ldots$, it holds*

$$\begin{aligned}
&(q^{-\sigma}R_{21}R_{31}\ldots R_{r1}-q^{\sigma}R_{12}^{-1}R_{13}^{-1}\ldots R_{1r}^{-1})(q^{-\sigma}R_{32}\ldots R_{r2}-q^{\sigma}R_{23}^{-1}\ldots R_{2r}^{-1})\\
&\quad\times\ldots\times(q^{-\sigma}R_{r,r-1}-q^{\sigma}R_{r-1,r}^{-1})(q^{-\sigma}-q^{\sigma})=0\,.
\end{aligned}\tag{55}$$

Proof. We shall prove the adjoint equality rather than (55). Let us denote by $\{e_1, e_2, \ldots, e_m\}$ the standard basis in $\mathbb{C}^m$. We are going to verify the following assertion ($\sigma = 0, 1, 2, \ldots$):

$$(q^{-\sigma} - q^{\sigma})(q^{-\sigma} R_{r-1,r} - q^{\sigma} R_{r,r-1}^{-1}) \times \ldots \times (q^{-\sigma} R_{1r} \ldots R_{12} - q^{\sigma} R_{r1}^{-1} \ldots R_{21}^{-1}) \\ \times e_{j(1)} \otimes \ldots \otimes e_{j(r)} = 0\,, \tag{56}$$

whenever there are $(\sigma + 1)$ mutually equal indices among $j_1, j_2, \ldots, j_r$.

First note that the relation (1) means

$$R_{12} e_j \otimes e_k = e_j \otimes e_k + (q - q^{\operatorname{sgn}(j-k)}) e_k \otimes e_j\,.$$

Consequently one finds that

$$R_{1r} \ldots R_{13} R_{12} e_{j(1)} \otimes e_{j(2)} \otimes \ldots \otimes e_{j(r)} \\ = \sum c(k_1, k_2, \ldots, k_r)\, e_{k(1)} \otimes e_{k(2)} \otimes \ldots \otimes e_{k(r)}\,,$$

where $c(k_1, k_2, \ldots, k_r)$ is nonzero only in the following two cases:
(a) $(k_1, k_2, \ldots, k_r) = (j_1, j_2, \ldots, j_r)$; then $c(k_1, k_2, \ldots, k_r) = q^{\nu}$ with ν being equal to the number of indices among $j_2, j_3, \ldots, j_r$ coinciding with j_1.
(b) There exist indices $1 < l_2 < \ldots < l_s \le r$, $s \ge 2$, such that $j_1 < j_{l(2)} < \ldots < j_{l(s)}$ and $(k_1, k_2, \ldots, k_r) = (j_{l(s)}, \ldots, j_1, \ldots, j_{l(2)}, \ldots, j_{l(s-1)}, \ldots)$, i.e. $(k_1, k_2, \ldots, k_r)$ is obtained from $(j_1, j_2, \ldots, j_r)$ by the cyclic permutation of the indices $j_1, j_{l(2)}, \ldots, j_{l(s)}$, and all remaining indices keep their position.

An analogous discussion can be done also for $R_{r1}^{-1} \ldots R_{21}^{-1} e_{j(1)} \otimes \ldots \otimes e_{j(r)}$. Now to prove the above assertion we proceed by induction in r. For $r = 1$, the assertion is clearly valid. To perform the induction step $r - 1 \to r$ we distinguish two cases:
(I) The assumed subcollection of $(\sigma + 1)$ equal indices is contained in $(j_2, \ldots, j_r)$. Then the identity following from the Yang-Baxter equation,

$$(q^{-\sigma} R_{r-1,r} - q^{\sigma} R_{r,r-1}^{-1}) \times \ldots \times (q^{-\sigma} R_{2r} \ldots R_{23} - q^{\sigma} R_{r2}^{-1} \ldots R_{32}^{-1}) \\ \times (q^{-\sigma} R_{1r} \ldots R_{12} - q^{\sigma} R_{r1}^{-1} \ldots R_{21}^{-1}) \\ = (q^{-\sigma} R_{12} \ldots R_{1r} - q^{\sigma} R_{21}^{-1} \ldots R_{r1}^{-1}) \\ \times (q^{-\sigma} R_{r-1,r} - q^{\sigma} R_{r,r-1}^{-1}) \times \ldots \times (q^{-\sigma} R_{2r} \ldots R_{23} - q^{\sigma} R_{r2}^{-1} \ldots R_{32}^{-1})\,,$$

combined with the induction hypothesis implies (56).
(II) The index j_1 belongs to the subcollection of $(\sigma+1)$ equal indices. Then according to the above discussion,

$$(q^{-\sigma} R_{1r} \ldots R_{12} - q^{\sigma} R_{r1}^{-1} \ldots R_{21}^{-1}) e_{j(1)} \otimes \ldots \otimes e_{j(r)} \\ = q^{-\sigma}(q^{\sigma} e_{j(1)} \otimes \ldots \otimes e_{j(r)}) - q^{\sigma}(q^{-\sigma} e_{j(1)} \otimes \ldots \otimes e_{j(r)}) \\ + \sum c'(k_1, \ldots, k_r) e_{k(1)} \otimes \ldots \otimes e_{k(r)}\,,$$

and there are $(\sigma + 1)$ mutually equal indices among $k_2, k_3, \ldots, k_r$ whenever the coefficient $c'(k_1, \ldots, k_r)$ is nonzero. In this case the induction hypothesis again implies (56).

To complete the proof it is enough to observe that if $r > m\sigma$ then there are at least $(\sigma + 1)$ equal indices among $j_1, j_2, \ldots, j_r$. $\square$

Combining Propositions 4.1 and 4.2 with a result due to Rosso [3] according to which every finite-dimensional representation of $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$ is completely reducible we get

Corollary 4.3. *For*

$$\lambda_1 = q^{\mu-n-\sigma}\,, \qquad \lambda_2 = q^{\mu+n+\sigma}\,, \qquad \mu = n + \sigma(n-m)/N\,, \tag{57}$$

$\sigma = 0,1,2,3,\ldots$, *$\mathscr{M} = \mathscr{M}_\sigma$ is a finite-dimensional irreducible $\mathscr{A}_q(AN)$ (and hence $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$) module. The unit is the lowest weight vector in $\mathscr{M}_\sigma$,*

$$\begin{aligned}\exp(hH_j)\cdot 1 &= 1 \quad \textit{for} \quad j \neq m\\ &= q^\sigma \quad \textit{for} \quad j = m\,.\end{aligned} \tag{58}$$

This formula assumes that the condition (4) is again restored. But $\prod \alpha_{jj}$ lies in the centre of $\mathscr{A}_q(AN)$ and with the choice (57), $\prod \alpha_{jj}\cdot 1 = 1$. Thus $\prod \alpha_{jj}$ acts on $\mathscr{M}$ as the unit operator.

According to another result stated in [3], the vectors $X^+_{k(1)}\ldots X^+_{k(r)}\cdot 1$ span the module $\mathscr{M}$. As a consequence one easily derives that provided there exists an invariant scalar product $\langle\cdot\mid\cdot\rangle$ on $\mathscr{M}$, i.e.

$$\langle \xi\cdot v_1 \mid v_2\rangle = \langle v_1 \mid \xi^*\cdot v_2\rangle$$

for all $\xi \in \mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$, $v_1, v_2 \in \mathscr{M}$, then it is determined unambiguously up to a positive factor. But the existence of the Haar measure for $SU_q(N)$ implies that every finite-dimensional representation of $\mathscr{U}_h(\mathfrak{sl}(N,\mathbb{C}))$ can be turned into a $*$-representation [19]. The invariant scalar product on $\mathscr{M}_\sigma$ can be introduced as follows. The vacuum-value functional ε_0 on the algebra $\mathscr{C}$ corresponds to the normal ordering when the elements z^*_{st} stand to the left and the elements z_{uv} stand to the right, $\varepsilon_0(1) = 1$. Set

$$\langle c \mid d\rangle = \varepsilon_0(c^* d)\,. \tag{59}$$

The number $\langle c\mid d\rangle$ depends only on the images of c and d in the fact space $\mathscr{C}/\mathscr{T}$ and it holds

$$\langle c \mid fd\rangle = \langle f^* c \mid d\rangle\,.$$

We can proceed to the general case and describe the module $\mathscr{M}_{\sigma(1),\ldots,\sigma(N-1)}$ with the lowest weight $(\sigma_1,\ldots,\sigma_{N-1}) \in \mathbb{N}_0^{N-1}$. In the notation below the superscript (m) refers to the type of the Grassmann manifold $\mathbb{G}_m$. Denote by

$$\Delta_k : \mathscr{A}_q(AN) \to \mathscr{A}_q(AN)\otimes\ldots\otimes\mathscr{A}_q(AN) \quad (k \textit{ copies})$$

the iterated comultiplication in $\mathscr{A}_q(AN)$. The morphism $\psi|(\psi^{(1)}\otimes\ldots\otimes\psi^{(N-1)})\circ\Delta_{N-1}$ enables one to define a tensor product of modules $\mathscr{M}^{(1)}_{\sigma(1)}\otimes\ldots\otimes\mathscr{M}^{(N-1)}_{\sigma(N-1)}$. The cyclic submodule $\mathscr{M}_{\sigma(1),\ldots,\sigma(N-1)}$ generated by the lowest weight vector $v = 1\otimes\ldots\otimes 1$ is irreducible,

$$\exp(hH_j)\cdot v = q^{\sigma(j)} v\,, \qquad j = 1,2,\ldots,N-1\,. \tag{60}$$

Acknowledgements. The author is indebted to the reviewer for his comments.

References

1. Drinfeld, V.G.: Quantum groups. In: Proc. ICM Berkeley 1986, Providence, RI: AMS 1987, p. 798
2. Jimbo, M.: A q-difference analogue of $U(g)$ and the Yang-Baxter equation. Lett. Math. Phys. **10**, 63 (1985)
3. Rosso, M.: Finite dimensional representations fo the quantum analog of the enveloping algebra of a complex simple Lie algebra. Commun. Math. Phys. **117**, 581 (1988)
4. Lusztig, G.: Quantum deformations of certain simple modules over enveloping algebras. Adv. Math. **70**, 237 (1988)
5. Parshall, B., Wang, J.: Quantum linear groups. Providence, RI: AMS 1991
6. Lakshmibai, V., Reshetikhin, N.: Quantum deformations of flag and Schubert schemes. C.R. Acad. Sci. Paris **313**, Série I, 121 (1991)
7. Levendorskii, S., Soibelman, Ya.: Algebras of functions on cmpact quantum groups, Schubert cells and quantum tori. Commun. Math. Phys. **139**, 141 (1991)
8. Semenov-Tian-Shansky, M.A.: Dressing transformation and Poisson Lie group actions. Publ. RIMS, Kyoto University **21**, 1237 (1985)
9. Babelon, O., Bernard, D.: Dressing transformations and the origin of the quantum group symmetries. Preprint SPhT-91-016 (1991)
10. Lu, J.H., Weinstein, A.: Poisson Lie groups, dressing transformations and Bruhat decompositions. J. Diff. Geom. **31**, 501 (1990)
11. Reshetikhin, N.Yu., Semenov-Tian-Shansky, M.A.: Quantum R-matrices and factorization problems. J. Geom. Phys. **5**, 533 (1987)
12. Podles, P., Woronowicz, S.L.: Quantum deformation of Lorentz group. Commun. Math. Phys. **130**, 381 (1990)
13. Reshetikhin, N.Yu., Takhtajan, L.A., Faddeev, L.D.: Quantization of Lie groups and Lie algebras. Algebra i analiz **1**, 178 (1989) (in Russian)
14. Jurčo, B., Šťovíček, P.: Quantum dressing orbits on compact groups. Commun. Math. Phys. **152**, 97 (1993)
15. Podles, P.: Quantum spheres. Lett. Math. Phys. **14**, 193 (1987)
16. Soibelman, Y.: On quantum flag manifolds. RIMS-780 (1991)
17. Taft, E., Towber, J.: Quantum deformation of flag schemes and Grassmann schemes I. A q-deformation of the shape-algebra for $GL(n)$. J. Algebra **142**, No. 1 (1991)
18. Šťovíček, P.: Quantum line bundles on S^2 and the method of orbits for $SU_q(2)$. J. Math. Phys. **34**, 1606 (1993)
19. Woronowicz, S.L.: Compact matrix pseudogroups. Commun. Math. Phys. **111**, 613 (1987)

Communicated by N. Yu. Reshetikhin