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Strong-Electric-Field Eigenvalue Asymptotics for the Perturbed Magnetic Schrödinger Operator

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Abstract. We consider the Schrödinger operator with constant full-rank magnetic field, perturbed by an electric potential which decays at infinity, and has a constant sign. We study the asymptotic behaviour for large values of the electric-field coupling constant of the eigenvalues situated in the gaps of the essential spectrum of the unperturbed operator.

0. Introduction

On $C_0^\infty(\mathbb{R}^m)$ define the Schrödinger operator

$$H_g^{\pm} := (i\nabla + A)^2 \mp gV .$$

Here $A : \mathbb{R}^m \to \mathbb{R}^m$ is the magnetic potential, $V : \mathbb{R}^m \to \mathbb{R}_+$ is the electric potential, and $g > 0$ is the electric-field coupling constant. Our further assumptions about A and V will imply, in particular, the essential selfadjointness of the operator $H_g^{\pm}$, so that in the sequel $H_g^{\pm}$ will denote the operator selfadjoint in $L^2(\mathbb{R}^m)$.
We assume that the entries

$$B_{ij} = \partial_{X_i} A_j - \partial_{X_j} A_i , \quad i, j = 1, \ldots, m ,$$

of the magnetic-field tensor $B = \{B_{ij}\}_{i,j=1}^m$ are constant in X. Moreover, we assume

$$\operatorname{rank} B = m . \tag{0.1}$$

Note that the condition (0.1) may hold only if the dimension m is even, i.e. $m = 2d$, $d \in \mathbb{Z}$, $d \geq 1$. Let $b_1 \geq \cdots \geq b_d > 0$ be such numbers that the eigenvalues of the skew-symmetric matrix B are equal together with the multiplicities to the imaginary

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numbers $-ib_j$ and ib_j, $j = 1, \dots, d$. Let $\{\Lambda_q\}$, $q \geq 1$, be the nondecreasing sequence consisting of positive numbers of the form

$$\Lambda_q = \sum_{j=1}^{d} (2n_j - 1) b_j \,, \tag{0.2}$$

where n_j, $j = 1, \dots, d$, are positive integers. The numbers Λ_q are known in the physical literature as Landau levels. If $\varkappa_s$ different sets $\{n_1, \dots, n_d\}$ yield one and the same level Λ_s according to (0.2), then Λ_s is repeated $\varkappa_s$ times in the sequence $\{\Lambda_q\}_{q \geq 1}$. It is well-known (see e.g. [Av. Her. Si]) that we have

$$\sigma(H_0) = \sigma_{\text{ess}}(H_0) = \bigcup_{q=1}^{\infty} \{\Lambda_q\} \,,$$

where $\sigma(T)$ (resp. $\sigma_{\text{ess}}(T)$) denotes the spectrum (resp. the essential spectrum) of a selfadjoint operator T.

Further, we assume that the electric potential V decays at infinity. Therefore, the multiplier by V is relatively compact with respect to H_0. Hence, we have

$$\sigma_{\text{ess}}(H_0) = \sigma_{\text{ess}}(H_g^{\pm}) \,, \quad \forall g > 0 \,.$$

Fix a real number $\lambda \in \rho(H_0) = \mathbb{R} \setminus \sigma(H_0)$ and denote by $\mathcal{N}_g^{\pm}(\lambda)$ the number of the eigenvalues of the operator $H_t^{\pm}$ crossing λ as the parameter t grows from 0 to the value $g > 0$. The paper is devoted to the study of the asymptotic behaviour of the functions $\mathcal{N}_g^{\pm}(\lambda)$ as $g \to \infty$, the value $\lambda = \bar{\lambda} \in \rho(H_0)$ being fixed.

The paper is organized as follows. In Sect. 1 we state our main results and comment briefly on them. Sect. 2 contains some necessary auxiliary results. In Sect. 3 we prove Theorems 1.1–1.2, while the proof of Theorem 1.3 can be found in Sect. 4.

1. Statement of Main Results

1.1. We shall say that the potential V belongs to the class $\mathscr{D}_\alpha$, $\alpha > 0$, if and only if $V \in C^\infty(\mathbb{R}^m)$, and the estimates

$$|D^\beta V(X)| \leq C_\beta \langle X \rangle^{-\alpha - |\beta|} \,, \quad \langle X \rangle := (1 + |X|^2)^{1/2} \,,$$

hold for each $X \in \mathbb{R}^m$ and each multiindex β for some constants C_β.

For $s > 0$ set

$$\Phi_V(s) = \text{vol}\{X \in \mathbb{R}^m : V(X) > s\} \,.$$

We shall say that the potential V belongs to the class $\mathscr{D}_\alpha^+$, $\alpha > 0$, if and only if:

i) $V \in \mathscr{D}_\alpha$;

ii) the estimate

$$C_0 \langle X \rangle^{-\alpha} \leq V(X) \,, \quad |X| > R \,,$$

holds for some constants $C_0 > 0$ and $R > 0$;

iii) the function $\Phi_V(s)$ is differentiable for $s \in (0, s_0]$, $s_0 > 0$, and the estimate

$$-s \Phi_V'(s) \leq C \Phi_V(s) \,, \quad s \in (0, s_0] \,,$$

holds for some constant C.

Remark. Various sufficient conditions which guarantee the validity of condition iii) can be found in [Shu], Subsect. 28.7, or in [Dau. Rob], Subsect. 2.B. In particular, condition iii) holds if the estimate

$$C_1 V(X) \le |X \cdot \nabla V(X)|$$

is fulfilled for some positive C_1 and sufficiently large $|X|$.

Let $F_j(t)$, $j = 1, 2$, $t > 0$, be two nondecreasing (resp., nonincreasing) positive functions. We shall write

$$F_1(t) \asymp F_2(t), \quad t \to \infty,$$

(respectively,

$$F_1(t) \asymp F_2(t), \quad t \downarrow 0)$$

if and only if there exists a constant $C \ge 1$ such that we have

$$C^{-1} F_1(t) \le F_2(t) \le C F_1(t)$$

for sufficiently large (resp. small) t.

Note that if $V \in \mathscr{D}_\alpha^+$, $\alpha > 0$, we have

$$\Phi_V(s) \asymp s^{-m/\alpha}, \quad s \downarrow 0. \tag{1.1}$$

Let $V \in \mathscr{D}_\alpha^+$, $\alpha > 0$, I be an arbitrary lower-bounded interval. For $\lambda \in \rho(H_0)$ introduce the function

$$\Psi_g(I) \equiv \Psi(I;\lambda) := \frac{b_1 \cdots b_d}{(2\pi)^{m/2}} \sum_{\substack{q \ge 1: \\ \Lambda_q \in I}} \Phi_V(g^{-1}|\Lambda_q - \lambda|). \tag{1.2}$$

Since V is a bounded function, the sum at the right-hand side of (1.2) may contain just a finite number of non-zero terms for any fixed $g > 0$. If the interval I is bounded, then the estimate (1.1) implies

$$\Psi_g(I) \asymp g^{m/\alpha}, \quad g \to \infty, \tag{1.3}$$

for all $\alpha > 0$. If I is unbounded, then the asymptotic behaviour of $\Psi_g(I)$ is essentially different for $\alpha \in (0, 2)$, $\alpha = 2$ and $\alpha > 2$. Namely, if $\alpha \in (0, 2)$, then the estimate (1.3) holds, if $\alpha = 2$, then we have

$$\Psi_g(I) \asymp g^{m/2} \log g, \quad g \to \infty, \tag{1.4}$$

and if $\alpha > 2$ we have

$$\Psi_g(I) = g^{m/2} \frac{(4\pi)^{-m/2}}{\Gamma(1 + m/2)} \int_{\mathbb{R}^m} V(X)^{m/2} dX (1 + o(1)), \quad g \to \infty. \tag{1.5}$$

Note that in the case where I is unbounded and $\alpha \ge 2$, the estimates (1.4)–(1.5) together with (1.1) imply that the main asymptotic term of $\Psi_g(I)$ is independent of I. More precisely, we have

$$\lim_{g \to \infty} \Psi_g(I_1)/\Psi_g(I_2) = 1 \tag{1.6}$$

for $\alpha \ge 2$ and any pair of unbounded intervals I_1 and I_2.

Assume that $V \in \mathscr{D}_\alpha^+$, $\alpha > 0$, and, moreover, V obeys the asymptotics

$$V(X) = v(\hat{X})|X|^{-\alpha}(1 + o(1)), \quad \hat{X} := X/|X|, \quad |X| \to \infty.$$

Then we can replace (1.3)–(1.4) by more precise asymptotic relations. Namely, we have

$$\lim_{g\to\infty} g^{-m/\alpha}\Psi_g(I) = \frac{b_1\cdots b_d}{m(2\pi)^{m/2}} \int_{\mathbb{S}^{m-1}} v(\omega)^{m/\alpha} d\sigma(\omega) \sum_{\substack{q\geq 1:\\ \Lambda_q\in I}} |\Lambda_q - \lambda|^{-m/\alpha}$$

if $\alpha > 0$ and I is bounded, or if $\alpha \in (0,2)$ and I is arbitrary, and

$$\lim_{g\to\infty} g^{-m/2}(\log g)^{-1}\Psi_g(I) = \frac{(4\pi)^{-m/2}}{2\Gamma(1+m/2)} \int_{\mathbb{S}^{m-1}} v(\omega)^{m/2} d\sigma(\omega)$$

if $\alpha = 2$ and I is unbounded (cf. [Rai 2], p. 46).

1.3 **Theorem 1.1.** *Let $V \in \mathscr{D}_\alpha^+$, $\alpha > 0$. Then we have*

$$\mathscr{N}_g^{-}(\lambda) = \Psi_g([\Lambda_1,\lambda))(1+o(1)), \quad g\to\infty, \tag{1.7}$$

for any $\lambda = \bar{\lambda} \in \rho(H_0)$, $\lambda > \Lambda_1$.

Remark. If $\lambda < \Lambda_1$, then $\mathscr{N}_g^{-}(\lambda) = 0$.

Theorem 1.2. *Let $V \in \mathscr{D}_\alpha^+$, $\alpha \in (0,2)$. Then we have*

$$\mathscr{N}_g^{+}(\lambda) = \Psi_g((\lambda,\infty))(1+o(1)), \quad g\to\infty, \tag{1.8}$$

for any $\lambda = \bar{\lambda} \in \rho(H_0)$.

Remark. We shall prove Theorem 1.2 for $\lambda > \Lambda_1 = \inf\sigma(H_0) \equiv \inf\sigma_{\mathrm{ess}}(H_0)$. The case $\lambda < \Lambda_1$ is included as a special case in [Rai 2], Theorem 2.1; in this case the asymptotics (1.8) are valid as well.

Theorem 1.3. *Let $V \in \mathscr{D}_2^+$. Then we have*

$$\mathscr{N}_g^{+}(\lambda) = \Psi_g((\lambda,\infty))(1+o(1)) = \Psi_g([\Lambda_1,\infty))(1+o(1)), \quad g\to\infty, \tag{1.9}$$

for any $\lambda = \bar{\lambda} \in \rho(H_0)$.

Remark. The conditions i)–iii) imposed on the potential V may seem rather restrictive and, as a matter of fact, they really are. As it will be seen from the sequel, these assumptions enable us to use some known results on the spectral asymptotics for pseudodifferential operators with Weyl symbols. Since we investigate only the main asymptotic term of $\mathscr{N}_g^{\pm}(\lambda)$, our results could be considerably extended applying the variational technique developed by M. Sh. Birman and M. Z. Solomyak (see e.g. [Bir. Sol]). Here we would mention just one possible generalization. Let $V_1 \in \mathscr{D}_\alpha^+$, $\alpha \in (0,2]$. Assume that $V_2 \geq 0$ and $V_2 \in L^{m/2}(\mathbb{R}^m)$ if $m > 2$, $V_2 \in L^p(\mathbb{R}^2)$, $p > 1$ and $\operatorname{supp} V_2$ is compact if $m = 2$. Then the asymptotic formulae (1.8)–(1.9) remain valid for $V = V_1 + V_2$. Note that in this case the main asymptotic term of $\Psi_g(I)$ as $g\to\infty$ depends only on V_1 but not on V_2. In particular, we find that our results are valid for the Coulomb potential $V(X) = 1/|X|$.

Note that our results do not contain the asymptotics of $\mathscr{N}_g^{+}(\lambda)$ for $\alpha > 2$. These asymptotics are included as a special case in the general result of [Bir. Rai], Theorem 1.1, where a much more general class of "regular" potentials (A, V) has been studied. Bearing in mind (1.5) and (1.6), we find that the asymptotic formula in Theorem 1.1 of [Bir. Rai] can be written in the form of (1.9) if B is constant, $\operatorname{rank} B = m$, and $V \in \mathscr{D}_\alpha^+$, $\alpha > 2$.

1.4. For $t \in \mathbb{R}$ set

$$k(t) := \frac{b_1 \cdots b_d}{(2\pi)^{m/2}} \#\{q : \Lambda_q < t\}.$$

Then the asymptotics (1.7) can be written in the form

$$\mathcal{N}_g^-(\lambda) = \int_{-\infty}^{\lambda} \Phi_V(g^{-1}(\lambda - t))dk(t)\,(1 + o(1)), \quad g \to \infty,$$

and the asymptotics (1.8)–(1.9) — in the form

$$\mathcal{N}_g^+(\lambda) = \int_{\lambda}^{\infty} \Phi_V(g^{-1}(t - \lambda))dk(t)\,(1 + o(1)), \quad g \to \infty.$$

Comparing these formulae with the analogous asymptotics obtained in [Al. De. Hem], [Hem] and [Sob] for the case where the unperturbed operator has the form $-\Delta + V_0$, V_0 being a periodic function over $\mathbb{R}^m$, we find that the role of the integrated density of states for the operator considered in the present paper is played by the function $k(t)$ introduced by Y. Colin de Verdière in [CdV].

2. Auxiliary Results

2.1. It is well known that on $\mathbb{R}^m \equiv \mathbb{R}^{2d}$ there exist rectangular coordinates (x, y) with $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ such that the operator $H_g^\pm$ can be written in the form

$$H_g^\pm = \sum_{j=1}^{d} \left\{ \left(-i\frac{\partial}{\partial x_j} - b_j y_j/2 \right)^2 + \left(-i\frac{\partial}{\partial y_j} + b_j x_j/2 \right)^2 \right\} \mp gV(x, y).$$

Introduce the selfadjoint operator

$$h = \sum_{j=1}^{d} b_j \left(-\frac{\partial^2}{\partial x_j^2} + x_j^2 \right)$$

defined originally on $C_0^\infty(\mathbb{R}^d)$ and then closed in $L^2(\mathbb{R}^d)$. The spectrum of h is purely discrete and the eigenvalues of h together with the multiplicities coincide with the Landau levels Λ_q, $q \geq 1$. Let $\{f_q\}_{q \geq 1}$ be the orthonormal in $L^2(\mathbb{R}^d)$ eigenfunctions of h such that

$$hf_q = \Lambda_q f_q, \quad q \geq 1.$$

The eigenfunctions f_q can be written in the form

$$f_q(x) = \exp(-|x|^2/2)\mathcal{P}_q(x), \quad q \geq 1,$$

where $\mathcal{P}_q$, $q \geq 1$, are some polynomials with real coefficients.

Introduce the operator

$$\mathcal{H}_0 = \int_{\mathbb{R}^d}^{\oplus} dy\, h.$$

which is selfadjoint in $L^2(\mathbb{R}^{2d})$.

Further, set

$$V_b(x,y) = V(b_1^{-1/2}x_1,\dots,b_d^{-1/2}x_d,b_1^{-1/2}y_1,\dots,b_d^{-1/2}y_d)$$

and define the operator $\mathscr{V}$, selfadjoint and bounded on $L^2(\mathbb{R}^{2d})$, as a pseudodifferential operator (ΨDO) with Weyl symbol

$$S(x,y,\xi,\eta) := V_b(x-\eta,y-\xi)\,,\quad (x,y;\xi,\eta)\in T^*\mathbb{R}^{2d}\,,$$

(see [Shu], Chapter 4, or [Hö]). Thus, for $u\in L^2(\mathbb{R}^m)$ we have

$$(\mathscr{V}u)(x,y) = \frac{1}{(2\pi)^{2d}}\int\limits_{\mathbb{R}^{4d}} \exp\{i[\xi(x-x')+\eta(y-y')]\}$$
$$\times V_b\left(\frac{1}{2}(x+x')-\eta,\frac{1}{2}(y+y')-\xi\right)u(x',y')dx'\,dy'\,d\xi\,d\eta\,.$$

For $g\geq 0$ set

$$\mathscr{H}_g^{\pm} := \mathscr{H}_0 \mp g\mathscr{V}\,.$$

The operators $H_g^{\pm}$ and $\mathscr{H}_g^{\pm}$ are unitarily equivalent. To see this, first of all change the variables $x_j\to b_j^{-1/2}x_j$, $y_j\to b_j^{-1/2}y_j$, $j=1,\dots,d$, in $\mathbb{R}_{x,y}$. Let (ξ,η) be the variables dual to (x,y). In $T^*\mathbb{R}^{2d}=\mathbb{R}^{4d}$ define the linear symplectic transformation

$$x\to x-\eta\,,\quad y\to y-\xi\,,\quad \xi\to(y+\xi)/2\,,\quad \eta\to(x+\eta)/2\,.$$

The superposition of these two transformations maps the symbol of the operator $H_g^{\pm}$ into the symbol of $\mathscr{H}_g^{\pm}$. Taking into account Theorem 4.3 in [Hö], we establish the unitary equivalence of $H_g^{\pm}$ and $\mathscr{H}_g^{\pm}$.

2.2. Let T be a linear compact operator in a Hilbert space. Then $\nu(\mu;T)$, $\mu>0$, is the number of the singular values of T (i.e. the eigenvalues of $(T^*T)^{1/2}$) greater than μ. Assume in addition that T is selfadjoint. Then $n_{\pm}(\mu;T)$, $\mu>0$, is the number of the eigenvalues of the operator $\pm T$ greater than μ. Note that we have

$$\nu(\mu;T)=\nu(\mu;T^*)=n_+(\mu^2;T^*T)=n_+(\mu^2;TT^*)\,,\quad \mu>0\,,\tag{2.1}$$

for any compact operator T. Moreover, recall the well-known Weyl inequalities

$$n_+(\mu;T_1+T_2)$$
$$\leq n_+(\mu(1-\tau);T_1)+n_+(\mu\tau;T_2)\,,\quad \forall\mu>0\,,\forall\tau\in(0,1),\tag*{$(2.2)_+$}$$
$$n_+(\mu;T_1+T_2)$$
$$\geq n_+(\mu(1+\tau);T_1)-n_-(\mu\tau;T_2)\,,\quad \forall\mu>0\,,\forall\tau>0\,,\tag*{$(2.2)_-$}$$

which are valid for any couple of compact operators $T_j=T_j^*$, $j=1,2$.

Set $\mathscr{W}:=\mathscr{V}^{1/2}$. Obviously, under the assumptions of Theorems 1.1–1.2 the operator $\mathscr{W}\mathscr{H}_0^{-1/2}$ is compact. Bearing in mind the unitary equivalence of the operators $H_g^{\pm}$ and $\mathscr{H}_g^{\pm}$, and applying a suitable version of the Birman-Schwinger principle (see [Al. De. Hem], Theorem 1.3, or [Bir], Proposition 1.6), we obtain the following

Lemma 2.1. *Let $\lambda=\bar\lambda\in\varrho(H_0)$. Then we have*

$$\mathscr{N}_g^{\pm}(\lambda)=n_{\pm}(g^{-1};\mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W})\,,\quad \forall g>0\,.\tag{2.3}$$

2.3. Introduce the operator $\chi(\lambda)$ acting in

$$\ell^2(\mathbb{N}_+;\, \mathbb{R}^d_y) := \left\{ \mathbf{u} = \{u_q(y)\}_{q=1}^{\infty} : \sum_{q=1}^{\infty} \int_{\mathbb{R}^d} |u_q(y)|^2\, dy < \infty \right\}$$

according to the formula

$$(\chi(\lambda)\mathbf{u})_q(y) = |\Lambda_q - \lambda|^{-1/2} u_q(y)\,, \quad q \geq 1\,,\ y \in \mathbb{R}^d\,,\ \lambda = \bar{\lambda} \in \varrho(H_0)\,.$$

The same notation $\chi(\lambda)$ will be also used for the restrictions of the operator $\chi(\lambda)$ onto the subspaces $L^2(\mathbb{R}^d_y)^{\varkappa}$, $1 \leq \varkappa < \infty$, of $\ell^2(\mathbb{N}_+;\, \mathbb{R}^d_y)$.

Further, we shall say that an operator acting in $L^2(\mathbb{R}^d_y)$ belongs to the class $\mathscr{S}_\alpha$ (resp. $\mathscr{S}_\alpha^+$) if and only if it is a ΨDO with Weyl symbol $s \in \mathscr{D}_\alpha$ (resp. $s \in \mathscr{D}_\alpha^+$), $\alpha > 0$. It is well-known that if $T \in \mathscr{S}_\alpha$, then $T^* \in \mathscr{S}_\alpha$, if $T_j \in \mathscr{S}_\alpha$, $j = 1, 2$, then $T_1 + T_2 \in \mathscr{S}_\alpha$, and if $T_1 \in \mathscr{S}_\alpha$, $T_2 \in \mathscr{S}_\beta$, then $T_1 T_2 \in \mathscr{S}_{\alpha+\beta}$ (see e.g. [Shu], Chapter 4).

Fix $\lambda > \Lambda_1$, $\lambda \in \varrho(H_0)$, and $\Lambda > \lambda$. Denote by P_-, $P_+ = P_+(\Lambda)$ and $P_\infty = P_\infty(\Lambda)$ the spectral projections of the operator $\mathscr{H}_0$ corresponding respectively to the intervals $I_- = [\Lambda_1, \lambda)$, $I_+ = (\lambda, \Lambda)$ and $I_\infty = [\Lambda, \infty)$. Obviously, the projections P_-, P_+ and P_∞ are pairwise orthogonal. Since $\Lambda_1 = \inf \sigma(H_0)$ and $\lambda \in \varrho(H_0)$, we have $P_- + P_+ + P_\infty = \mathrm{Id}$ as well. We shall use also the notation $Q = P_+ + P_\infty$.

Lemma 2.2. *For any* $\alpha > 0$, $\lambda > \Lambda_1$, $\lambda \in \varrho(H_0)$. *we have*

$$n_\pm(\mu;\, \mathscr{W} P_\pm (\mathscr{H}_0 - \lambda)^{-1} P_\pm \mathscr{W}) = \Psi_{1/\mu}(I_\pm)(1 + o(1))\,,\ \mu \downarrow 0\,, \qquad (2.4)_\pm$$

$$n_\pm(\mu;\, P_\pm \mathscr{W} P_\pm (\mathscr{H}_0 - \lambda)^{-1} P_\pm \mathscr{W} P_\pm) = \Psi_{1/\mu}(I_\pm)(1 + o(1))\,,\ \mu \downarrow 0\,. \qquad (2.5)_\pm$$

Proof. For definiteness we shall prove the relations $(2.4)_-$–$(2.5)_-$. The proofs of $(2.4)_+$–$(2.5)_+$ are quite similar.

Recalling (2.1), we obtain the identity

$$\begin{aligned} &n_-(\mu;\, \mathscr{W} P_- (\mathscr{H}_0 - \lambda)^{-1} P_- \mathscr{W}) \\ &\quad = n_+(\mu;\, |\mathscr{H}_0 - \lambda|^{-1/2} P_- \mathscr{V} P_- |\mathscr{H}_0 - \lambda|^{-1/2})\,,\ \mu > 0\,. \end{aligned} \qquad (2.6)$$

Set $\varkappa_- = \#\{q : \Lambda_q \in I_-\}$. Introduce the operator $\chi(\lambda) T \chi(\lambda)$: $L^2(\mathbb{R}^d_y)^{\varkappa_-} \to L^2(\mathbb{R}^d_y)^{\varkappa_-}$, where T is a ΨDO with matrix valued Weyl symbol

$$\begin{aligned} s(y,\, \eta) &= \{s_{rs}(y,\, \eta)\}_{r,\, s=1}^{\varkappa_-}\,, \\ s_{rs}(y,\, \eta) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{3d}} f_r(x) f_s(x') e^{i\xi(x-x')} \\ &\qquad V_b\left(\frac{1}{2}(x + x') - \eta,\, y - \xi\right) dx\, dx'\, d\xi\,, \quad r,\, s = 1, \ldots, \varkappa_-\,. \end{aligned} \qquad (2.7)$$

The nonzero eigenvalues of the operators $|\mathscr{H}_0 - \lambda|^{-1/2} P_- \mathscr{V} P_- |\mathscr{H}_0 - \lambda|^{-1/2}$ and $\chi(\lambda) T \chi(\lambda)$ coincide. Therefore, we obtain

$$n_+(\mu;\, |\mathscr{H}_0 - \lambda|^{-1/2} P_- \mathscr{V} P_- |\mathscr{H}_0 - \lambda|^{-1/2}) = n_+(\mu;\, \chi(\lambda) T \chi(\lambda))\,, \quad \mu > 0\,. \quad (2.8)$$

Further, we have $T = T_1 + T_2$, where T_1 is a ΨDO with matrix-valued Weyl symbol $\{\delta_{rs} V_b(-\eta, y)\}_{r,s=1}^{\varkappa_-}$ and T_2 is a ΨDO with matrix-valued Weyl symbol with entries

$$\tilde{s}_{rs}(y, \eta) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \mathfrak{P}_{rs}(x, \xi) \exp\{-(|x|^2 + |\xi|^2)\}$$

$$\int_0^1 \{x \cdot \nabla_x V_b(\tau x - \eta, y - \tau\xi) - \xi \cdot \nabla_y V_b(\tau x - \eta, y - \tau\xi)\}\, d\tau\, dx d\xi\,, \quad (2.9)$$

where $\mathfrak{P}_{rs}(x, \xi)$ are some polynomials, $r, s = 1, \dots, \varkappa_-$, (see [Rai 1], Sect. 5). Applying $(2.2)_\pm$, we get

$$\pm n_+(\mu; \chi(\lambda) T \chi(\lambda)) \le \pm\, n_+(\mu(1 \mp \tau); \chi(\lambda) T_1 \chi(\lambda)) + n_\pm(\mu\tau; \chi(\lambda) T_2 \chi(\lambda))\,, \quad \forall \mu > 0\,, \ \forall \tau \in (0, 1)\,. \quad (2.10)_\pm$$

It is not difficult to check that the entries of the symbol of T_2 (see (2.9)) belong to the class $\mathscr{D}_{\alpha+1}$. Since $\varkappa_- < \infty$, we have

$$n_\pm(s; \chi(\lambda) T_2 \chi(\lambda)) = O(s^{-2d/(\alpha+1)}) = o(\Psi_{1/s}(I_-))\,, \quad s \downarrow 0\,, \quad (2.11)_\pm$$

(see [Dau. Rob]). On the other hand, we have

$$T_1 = \sum_{q=1}^{\varkappa_-} \oplus t\,, \quad (2.12)$$

where t is a ΨDO with (scalar) symbol $V_b(-\eta, y)$. Thus we get

$$n_+(s; \chi(\lambda) T_1 \chi(\lambda)) = \sum_{q=1}^{\varkappa_-} n_+(s(\Lambda_q - \lambda); t)\,. \quad (2.13)$$

Applying the standard asymptotic formulae describing the eigenvalue distribution for elliptic ΨDOs of negative order, we find that

$$n_+(s(\Lambda_q - \lambda); t) = (2\pi)^{-d} \operatorname{vol}\{(y, \eta) \in T^*\mathbb{R}^d : V_b(-\eta, y) > s(\Lambda_q - \lambda)\}(1 + o(1))\,, \quad s \downarrow 0\,, \quad (2.14)$$

(see [Dau. Rob]). Note that $d = m/2$, and the volume at the right-hand side of (2.14) coincides with $b_1 \dots b_d \Phi_V(s(\Lambda_q - \lambda))$.

Combining (2.6), (2.8), $(2.10)_\pm$, $(2.11)_\pm$, (2.13) and (2.14), and taking account of the limiting relation

$$\lim_{\tau \downarrow 0} \limsup_{\mu \downarrow 0} \pm \Psi_{1/\mu(1\mp\tau)}(I_-)/\Psi_{1/\mu}(I_-) \le \pm 1\,,$$

we come to $(2.4)_-$.

In order to prove $(2.5)_-$, note the identity

$$n_-(\mu; P_- \mathscr{W} P_- (\mathscr{H}_0 - \lambda)^{-1} P_- \mathscr{W} P_-) = n_+(\mu; \chi(\lambda) \tilde{T}^2 \chi(\lambda))\,, \quad \mu > 0\,,$$

where $\tilde{T}$ is a matrix-valued ΨDO whose symbol is defined by analogy with (2.7) but V_b is replaced by $V_b^{1/2}$. Arguing as in the case of T, we can show that $\tilde{T} = \tilde{T}_1 + \tilde{T}_2$, where $\tilde{T}_1$ is a ΨDO with matrix-valued symbol $\{\delta_{rs} V_b^{1/2}(-\eta, y)\}_{r,s=1}^{\varkappa_-}$, and the entries of the symbol of the operator $\tilde{T}_2$ belong to $\mathscr{D}_{(\alpha+2)/2}$. Thus we find that $\tilde{T}^2 = T_1 + \hat{T}_2$, where T_1 is the operator described in (2.12), and the entries of the symbol of the operator $\hat{T}_2$ belong to $\mathscr{D}_{\alpha+1}$. Further the proof of $(2.5)_-$ is analogous to the one of $(2.4)_-$. $\square$

Lemma 2.3. *For any $\alpha > 0$, $\lambda > \Lambda_1$, $\lambda \in \varrho(H_0)$, we have*

$$\nu(\mu^{1/2};\, Q\mathscr{W}P_-|\mathscr{H}_0 - \lambda|^{-1/2}) = O(\mu^{-2d/(\alpha+1)}), \quad \mu \downarrow 0, \tag{2.15}$$

Proof. Obviously we have

$$\begin{aligned}\nu(\mu^{1/2};\, &Q\mathscr{W}P_-|\mathscr{H}_0 - \lambda|^{-1/2})\\ &= n_+(\mu;\, |\mathscr{H}_0 - \lambda|^{-1/2}P_-\mathscr{W}(\mathrm{Id} - P_-)\mathscr{W}P_-|\mathscr{H}_0 - \lambda|^{-1/2})\\ &= n_+(\mu;\, \chi(\lambda)(T - \tilde{T}^2)\chi(\lambda)) = n_+(\mu;\, \chi(\lambda)(T_2 - \hat{T}_2)\chi(\lambda)),\end{aligned}$$

where the operators T, $\tilde{T}$, T_2 and $\hat{T}_2$ have been introduced in the proof of the previous lemma. Since the entries of the symbols of the operators T_2 and $\hat{T}_2$ belong to the class $\mathscr{D}_{\alpha+1}$, we obtain the estimate

$$n_+(\mu;\, \chi(\lambda)(T_2 - \hat{T}_2)\chi(\lambda)) = O(\mu^{-2d/(\alpha+1)}), \quad \mu \downarrow 0,$$

which entails (2.15). $\square$

Lemma 2.4. *For any $\alpha > 0$, $\lambda > \Lambda_1$, $\lambda \in \varrho(H_0)$, we have*

$$\nu(\mu^{1/2};\, P_-\mathscr{W}P_+|\mathscr{H}_0 - \lambda|^{-1/2}) = O(\mu^{-2d/(\alpha+2)}), \quad \mu \downarrow 0, \tag{2.16}$$

Proof. Set $\varkappa_+ = \#\{q : \Lambda_q \in I_+\}$. Let $\mathscr{T} : L^2(\mathbb{R}^d_y)^{\varkappa_+} \to L^2(\mathbb{R}^d_y)^{\varkappa_-}$ be a ΨDO with matrix-valued Weyl symbol with entries

$$\begin{aligned}\hat{s}_{rs}(y,\eta) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{3d}} f_r(x)f_s(x')e^{i\xi(x-x')}V_b^{1/2}\Big(\frac{1}{2}(x+x') - \eta,\, y - \xi\Big)\,dx\,dx'\,d\xi\,,\\ r = 1,\dots,\varkappa_-\,,\quad s = \kappa_- + 1,\,\dots,\,\kappa_- + \kappa_+\,.\end{aligned} \tag{2.17}$$

Then we have

$$\nu(\mu^{1/2};\, P_-\mathscr{W}P_+|\mathscr{H}_0 - \lambda|^{-1/2}) = n_+(\mu;\, \chi(\lambda)\mathscr{T}^*\mathscr{T}\chi(\lambda)), \quad \mu > 0.$$

Since the functions f_r and f_s appearing in (2.17) are pairwise orthogonal, we find that the entries of the symbol of the operator $\mathscr{T}$ belong to $\mathscr{D}_{(\alpha+2)/2}$, and hence the entries of the symbol of the operator $\mathscr{T}^*\mathscr{T}$ belong to $\mathscr{D}_{\alpha+2}$. Thus we come to the estimate

$$n_+(\mu;\, \chi(\lambda)\mathscr{T}^*\mathscr{T}\chi(\lambda)) = (\mu^{-2d/(\alpha+2)}), \quad \mu \downarrow 0,$$

entailing (2.16). $\square$

Lemma 2.5. *For any $\alpha > 0$, $\lambda > \Lambda_1$, $\lambda \in \varrho(H_0)$, we have*

$$\lim_{\Lambda\to\infty}\limsup_{\mu\downarrow 0}\mu^{2d/\alpha}\nu(\mu^{1/2};\, P_-\mathscr{W}P_\infty(\Lambda)|\mathscr{H}_0 - \lambda|^{-1/2}) = 0\,. \tag{2.18}$$

Proof. Obviously we have

$$\nu(\mu^{1/2};\, P_-\mathscr{W}P_\infty(\Lambda)|\mathscr{H}_0-\lambda|^{-1/2}) \le \nu(\mu^{1/2}(\Lambda-\lambda)^{1/2};\, P_-\mathscr{W}) \le n_+(\mu\Lambda;\, P_-\mathscr{V}P_-) = n_+(\mu\Lambda;\, T)\,, \quad \mu>0\,,$$

where the operator T has been introduced in the proof of Lemma 2.2. Since the quantity $\limsup_{s\downarrow 0} s^{2d/\alpha} n_+(s;\, T)$ is bounded, the relation (2.18) holds. □

Lemma 2.6. *Let $\alpha\in(0,\,2)$. Then for any $\lambda=\bar\lambda\in\varrho(H_0)$, we have*

$$\lim_{\Lambda\to\infty}\limsup_{\mu\downarrow 0}\mu^{2d/\alpha}\nu(\mu^{1/2};\, \mathscr{W}P_\infty(\Lambda)|\mathscr{H}_0-\lambda|^{-1/2})=0\,. \tag{2.19}$$

Proof. For sufficiently large Λ and some $c\in(0,\,1)$ we have

$$\begin{aligned}\nu(\mu^{1/2};\, \mathscr{W}P_\infty(\Lambda)|\mathscr{H}_0-\lambda|^{-1/2}) &\le \nu(c\mu^{1/2};\, \mathscr{W}(\mathscr{H}_0+\Lambda)^{-1/2})\\ &= n_+(c^2\mu;\, (\mathscr{H}_0+\Lambda)^{-1/2}\mathscr{V}(\mathscr{H}_0+\Lambda)^{-1/2})\,.\end{aligned} \tag{2.20}$$

Applying the Birman-Schwinger principle, we find that the rightmost quantity in (2.20) coincides with the number of the eigenvalues of the operator $H_0-(c^2\mu)^{-1}V$ smaller than $-\Lambda$. Employing Theorem 2.1 in [Rai 2], we find that the estimate

$$\begin{aligned}&\limsup_{s\downarrow 0} s^{2d/\alpha}n_+(s;\, (\mathscr{H}_0+\Lambda)^{-1/2}\mathscr{V}(\mathscr{H}_0+\Lambda)^{-1/2})\\ &\quad\le C\sum_{q\ge 1:}(\Lambda_q+\Lambda)^{-2d/\alpha}\end{aligned} \tag{2.21}$$

holds with some constant C independent of Λ. Since $\alpha\in(0,\,2)$, the series at the right-hand side of (2.21) is absolutely convergent. Letting $\Lambda\to\infty$, we conclude that the relation (2.19) is valid. □

3. Proof of Theorems 1.1–1.2

3.1. In this subsection we prove Theorem 1.1.

Replacing the operator $(\mathscr{H}_0-\lambda)^{-1}$ by its negative part and using the minimax principle, we get

$$\begin{aligned}n_-(\mu;\, \mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}) &\le n_-(\mu;\, \mathscr{W}P_-(\mathscr{H}_0-\lambda)^{-1}P_-\mathscr{W}\\ &= n_+(\mu;\, \mathscr{W}P_-|\mathscr{H}_0-\lambda|^{-1}P_-\mathscr{W})\,, \quad \mu>0\,.\end{aligned}$$

The rightmost quantity can be estimated directly according to $(2.4)_-$. Thus we get

$$\limsup_{\mu\downarrow 0}\Psi_{1/\mu}([\Lambda_1,\,\lambda))^{-1}n_-(\mu;\, \mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W})\le 1\,. \tag{3.1}$$

Restricting the operator $\mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}$ onto the range of P_-, we get

$$n_-(\mu;\, \mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W})\ge n_-(\mu;\, P_-\mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}P_-)\,. \tag{3.2}$$

Write the operator identity

$$\begin{aligned} P_-\mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}P_- = {} & P_-\mathscr{W}P_-(\mathscr{H}_0-\lambda)^{-1}P_-\mathscr{W}P_- \\ & + P_-\mathscr{W}P_+(\mathscr{H}_0-\lambda)^{-1}P_+\mathscr{W}P_- \\ & + P_-\mathscr{W}P_\infty(\mathscr{H}_0-\lambda)^{-1}P_\infty\mathscr{W}P_- . \end{aligned}$$

Using the relations (2.1)–$(2.2)_-$, we get

$$\begin{aligned} n_-(\mu;\, P_-\mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}P_-) \geq {} & n_-(\mu(1+2\tau);\, P_-\mathscr{W}P_-(\mathscr{H}_0-\lambda)^{-1}P_-\mathscr{W}P_-) \\ & - \nu((\tau\mu)^{1/2};\, P_-\mathscr{W}P_+|\mathscr{H}_0-\lambda|^{-1/2}) \\ & - \nu((\tau\mu)^{1/2};\, P_-\mathscr{W}P_\infty|\mathscr{H}_0-\lambda|^{-1/2}), \\ & \mu>0,\ \tau>0 . \end{aligned}$$

The first term at the right-hand side is estimated according to $(2.5)_-$; this term is responsible for the main term of the asymptotics. The second term is estimated according to Lemma 2.4, the third term is estimated according to Lemma 2.5. Thus we obtain the inequality

$$\liminf_{\mu\downarrow 0} \Psi_{1/\mu}([\Lambda_1,\, \lambda))^{-1} n_-(\mu;\, P_-\mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}P_-) \geq 1 ,$$

which combined with (3.2), and then with (3.1), entails (1.7).

3.2. In this subsection we prove Theorem 1.2.

Replacing the operator $(\mathscr{H}_0-\lambda)^{-1}$ by its positive part and using the minimax principle, we get

$$n_+(\mu;\, \mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}) \leq n_+(\mu;\, \mathscr{W}Q(\mathscr{H}_0-\lambda)^{-1}Q\mathscr{W}), \quad \mu>0 .$$

Fix $\Lambda>\lambda$ and apply $(2.2)_+$. Thus we get

$$\begin{aligned} n_+(\mu;\, \mathscr{W}Q(\mathscr{H}_0-\lambda)^{-1}Q\mathscr{W}) \leq {} & n_+(\mu(1-\tau);\, \mathscr{W}P_+(\mathscr{H}_0-\lambda)^{-1}P_+\mathscr{W}) \\ & + n_+(\mu\tau;\, \mathscr{W}P_\infty(\mathscr{H}_0-\lambda)^{-1}P_\infty\mathscr{W}), \\ & \forall\mu>0,\ \forall\tau\in(0,\,1) . \end{aligned} \tag{3.3}$$

Applying the asymptotics $(2.4)_+$, and taking into account (2.1), (2.19) and the limiting relations

$$\lim_{\Lambda\to\infty} \limsup_{s\downarrow 0} \Psi_{1/s}((\lambda,\, \Lambda))/\Psi_{1/s}((\lambda,\, \infty)) = 1 ,$$

$$\lim_{\tau\downarrow 0} \limsup_{\mu\downarrow 0} \Psi_{1/\mu(1-\tau)}((\lambda,\, \infty))/\Psi_{1/\mu}((\lambda,\, \infty)) = 1 ,$$

we get

$$\limsup_{\mu\downarrow 0} \Psi_{1/\mu}((\lambda,\, \infty))^{-1} n_+(\mu;\, \mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}) \leq 1 . \tag{3.4}$$

Now, restricting the operator $\mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}$ onto the range of Q, we obtain the estimate

$$n_+(\mu;\, \mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}) \geq n_+(\mu;\, Q\mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}Q), \quad \mu>0 . \tag{3.5}$$

Write the operator inequalities

$$\begin{aligned} Q\mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}Q &\geq \mathscr{W}Q(\mathscr{H}_0-\lambda)^{-1}Q\mathscr{W} - 2\operatorname{Re} P_-\mathscr{W}Q(\mathscr{H}_0-\lambda)^{-1}Q\mathscr{W} \\ &\quad + Q\mathscr{W}P_-(\mathscr{H}_0-\lambda)^{-1}P_-\mathscr{W}Q \\ &\geq (1-\varepsilon)\mathscr{W}Q(\mathscr{H}_0-\lambda)^{-1}Q\mathscr{W} \\ &\quad - \varepsilon^{-1}P_-\mathscr{W}P_+(\mathscr{H}_0-\lambda)^{-1}P_+\mathscr{W}P_- \\ &\quad - \varepsilon^{-1}P_-\mathscr{W}P_\infty(\mathscr{H}_0-\lambda)^{-1}P_\infty\mathscr{W}P_- \\ &\quad - Q\mathscr{W}P_-|\mathscr{H}_0-\lambda|^{-1}P_-\mathscr{W}Q\,, \ \forall\varepsilon\in(0\,,\,1)\,. \end{aligned}$$

Using the relations (2.1) and $(2.2)_+$, we get

$$\begin{aligned} n_+(\mu;\, Q\mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}Q) &\geq n_+(\mu(1-\varepsilon)^{-1}(1+3\tau)\,;\, \mathscr{W}Q(\mathscr{H}_0-\lambda)^{-1}Q\mathscr{W}) \\ &\quad - \nu((\tau\varepsilon\mu)^{1/2};\, P_-\mathscr{W}P_+|\mathscr{H}_0-\lambda|^{-1/2}) \\ &\quad - \nu((\tau\varepsilon\mu)^{1/2};\, P_-\mathscr{W}P_\infty|\mathscr{H}_0-\lambda|^{-1/2}) \\ &\quad - \nu((\tau\mu)^{1/2};\, Q\mathscr{W}P_-|\mathscr{H}_0-\lambda|^{-1/2})\,, \end{aligned}$$

for each sufficiently small $\varepsilon > 0$ and each $\tau > 0$. The first term at the right-hand-side is handled in the same way as the estimate (3.3) was derived; this term yields the main term of the asymptotics. The second term is estimated according to Lemma 2.4, the third term is estimated according to Lemma 2.5, and the fourth term is estimated according to Lemma 2.3. Thus we come to the estimate

$$\liminf_{\mu\downarrow 0} \Psi_{1/\mu}((\lambda,\, \infty))^{-1} n_+(\mu;\, Q\mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}Q) \geq 1\,,$$

which combined with (3.5), and then with (3.4), entails (1.8).

4. Proof of Theorem 1.3

In the proof of Theorem 1.3 we employ the scheme developed in [Bir].

Theorem 2.1 in [Rai 2] implies, in particular, the validity of the asymptotic formula

$$\mathscr{N}_g^+(0) = \Psi_g([\Lambda_1,\, \infty))(1+o(1))\,, \quad g\to\infty\,, \tag{4.1}$$

under the hypotheses of Theorem 1.3. On the other hand, the resolvent identity

$$(\mathscr{H}_0-\lambda)^{-1} - \mathscr{H}_0^{-1} = \lambda\mathscr{H}_0^{-1}(\mathscr{H}_0-\lambda)^{-1}\,, \quad \lambda=\bar\lambda\in\varrho(H_0)\,,$$

and the Weyl inequalities $(2.2)_\pm$ entail

$$\begin{aligned} \pm n_+(s;\, \mathscr{W}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}) &\leq \pm n_+(s(1\mp\tau)\,;\, \mathscr{W}\mathscr{H}_0^{-1}\mathscr{W}) \\ &\quad + n_\pm(s\tau;\, \lambda\mathscr{W}\mathscr{H}_0^{-1}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W})\,, \\ &\quad \forall s>0\,,\ \forall\tau\in(0\,,\,1)\,. \end{aligned} \tag{4.2$_\pm$}$$

Combining (1.4), (1.6), (2.3), (4.1) and $(4.2)_\pm$, we find that it suffices to prove the asymptotic estimates

$$n_\pm(s;\, \mathscr{W}\mathscr{H}_0^{-1}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}) = o(s^{-d}|\log s|)\,, \quad s\downarrow 0\,, \tag{4.3$_\pm$}$$

in order to derive (1.9). Denote by $\widetilde{\mathscr{W}}$ the operator defined by analogy with $\mathscr{W}$ but $V_b(X)$ is replaced by $\langle X\rangle^{-2}$. Applying the minimax principle, we get

$$\begin{aligned} n_{\pm}(s;\, \mathscr{W}\mathscr{H}_0^{-1}(\mathscr{H}_0-\lambda)^{-1}\mathscr{W}) &\le n_+(sc;\, \widetilde{\mathscr{W}}\mathscr{H}_0^{-2}\widetilde{\mathscr{W}}) \\ &= \nu((sc)^{1/2};\, \mathscr{H}_0^{-1}\widetilde{\mathscr{W}}), \quad \forall s>0, \end{aligned}$$

for some constant c. Hence, it suffices to prove the estimate

$$\nu(\mu;\, \mathscr{H}_0^{-1}\widetilde{\mathscr{W}}) = o(\mu^{-2d}|\log\mu|), \quad \mu\downarrow 0, \tag{4.4}$$

in order to derive $(4.3)_{\pm}$.

By (2.1) and $(2.2)_+$ we get

$$\begin{aligned} \nu(\mu;\, \mathscr{H}_0^{-1}\widetilde{\mathscr{W}}) &= \nu(\mu;\, \widetilde{\mathscr{W}}\mathscr{H}_0^{-1}) \\ &\le \nu(\mu/2;\, \mathscr{H}_0^{-1/2}\widetilde{\mathscr{W}}\mathscr{H}_0^{-1/2}) \\ &\quad + \nu(\mu/2;\, (\widetilde{\mathscr{W}}\mathscr{H}_0^{-1/2} - \mathscr{H}_0^{-1/2}\widetilde{\mathscr{W}})\mathscr{H}_0^{-1/2}), \\ &\quad \forall\mu>0. \end{aligned} \tag{4.5}$$

Theorem 2.1 in [Rai 2] with $\lambda=0$ and $\alpha=1$ entails

$$\nu(\mu/2;\, \mathscr{H}_0^{-1/2}\widetilde{\mathscr{W}}\mathscr{H}_0^{-1/2}) = n_+(\mu/2;\, \mathscr{H}_0^{-1/2}\widetilde{\mathscr{W}}\mathscr{H}_0^{-1/2}) = O(\mu^{-2d}), \quad \mu\downarrow 0. \tag{4.6}$$

Set $\mathscr{K} = i(\widetilde{\mathscr{W}}\mathscr{H}_0^{-1/2} - \mathscr{H}_0^{-1/2}\widetilde{\mathscr{W}})$; note that the operator $\mathscr{K}$ is selfadjoint. Applying the minimax principle, we obtain

$$\begin{aligned} \nu(\mu/2;\, (\widetilde{\mathscr{W}}\mathscr{H}_0^{-1/2} - \mathscr{H}_0^{-1/2}\widetilde{\mathscr{W}})\mathscr{H}_0^{-1/2}) &\le \nu(\mu\Lambda_1^{1/2}/2;\, (\widetilde{\mathscr{W}}\mathscr{H}_0^{-1/2} - \mathscr{H}_0^{-1/2}\widetilde{\mathscr{W}})) \\ &= \nu(\mu\Lambda_1^{1/2}/2;\, \mathscr{K}) \\ &= n_+(\mu\Lambda_1^{1/2}/2;\, \mathscr{K}) + n_-(\mu\Lambda_1^{1/2}/2;\, \mathscr{K}), \\ &\quad \forall\mu>0. \end{aligned} \tag{4.7}$$

The Weyl symbol of the operator $\mathscr{K}$ can be written in the form (see [Hö, p. 374])

$$\begin{aligned} &-2\pi^{-2d}\int_{\mathbb{R}^{4d}} \sin\{2[(\tau,z)-(t,\zeta)]\}(1+|x-\eta+t|^2+|y-\xi-\tau|^2)^{-1/2} \\ &\qquad \mathscr{R}(x+z,\,\xi+\zeta)\,dz\,d\zeta\,dt\,d\tau, \quad (x,y,\xi,\eta)\in\mathscr{T}^*\mathbb{R}^{2d}, \end{aligned}$$

where

$$\begin{aligned} &\mathscr{R}(x,\xi) := \\ &\pi^{-1/2}\int_0^\infty t^{-1/2}\prod_{j=1}^d(\operatorname{ch} b_jt)^{-1}\exp\{-(x_j^2+\xi_j^2)\operatorname{th} b_jt\}dt, \quad (x;\xi)\in\mathscr{T}^*\mathbb{R}^d, \end{aligned}$$

is the Weyl symbol of the operator $h^{-1/2}$. Applying the general results in [Dau. Rob], we find that

$$n_{\pm}(\mu;\, \mathscr{K}) = O(\mu^{-d}|\log\mu|), \ \mu\downarrow 0, \tag{4.8}$$

Putting together (4.5)–(4.8), we come to (4.4). Thus the proof of Theorem 1.3 is complete.

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