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Classification of the Indecomposable Bounded Admissible Modules over the Virasoro Lie Algebra with Weightspaces of Dimension not Exceeding Two

Christiane Martin and Alain Piard

Physique Mathematique, U.A. CNRS 1102, University of Bourgogne, B.P. 138, F-21004 Dijon Cedex, France

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Abstract. In view of [1, 2] any bounded admissible module $\mathscr{A}$ over the Virasoro Lie algebra $\mathscr{V}$ is a finite length extension of irreducible modules with one-dimensional weightspaces. To each extension of finite length n are associated $n + 1$ invariants $(a_1, \Lambda_1, \ldots, \Lambda_n)$. We prove that we have $\Lambda_i - \Lambda_j \in \{0, 1, \ldots 6(n-1)\}$ for all (i, j) with $1 \leqq i \leqq j \leqq n$. In the case $n = 2$ this result allows us to construct all the indecomposable bounded admissible $\mathscr{V}$ modules, where the dimensions of the weightspaces are less than or equal to two. In particular we obtain all the extensions of two irreducible bounded $\mathscr{V}$-modules.

I. Introduction

The Virasoro algebra $\mathscr{V}$ is the complex Lie algebra with basis $\{C, x_n, n \in \mathbb{Z}\}$ and commutation relations:

$$[x_i, x_j] = (j - i)x_{i+j} + \delta_{i,-j}\frac{j^3 - j}{12}C \quad \forall i, \forall j \in Z ,$$

$$[C, x_i] = 0 .$$

We set also $Q_1 = -x_1 x_{-1} + x_0^2 - x_0$.

A $\mathscr{V}$-module is said to be admissible if it satisfies the two conditions:

a) x_0 acts semi-simply.

b) The eigenspaces of x_0 (also called weight-spaces) are finite-dimensional.

Recently, the classification of irreducible admissible $\mathscr{V}$-modules has been achieved in [1, 2]. Besides the highest or lowest weight $\mathscr{V}$-modules, it furnishes a second class of $\mathscr{V}$-modules where the weightspaces are one-dimensional. These latter are the following:

– The $\mathscr{V}$-modules of Feigin–Fuchs $A(a, \Lambda)$ with $(a, \Lambda) \in \mathbb{C}^2$ and $0 \leqq \operatorname{Re} a < 1$ $(a = 0 \Rightarrow \Lambda \neq 0, 1)$, whose action is given on a basis $\{v_n, n \in \mathbb{Z}\}$ by:

$$x_i v_n = (a + n + i\Lambda)v_{n+i} \quad Cv_n = 0 \quad \forall n, \forall i . \tag{I.1}$$

– The trivial $\mathscr{V}$-module, called $D(0)$.

– The maximal proper $\mathscr{V}$-submodule of $A(0, 1)$, called $\tilde{A}$ ($A(0, 1)/\tilde{A} \simeq D(0)$ and $A(0, 0)/D(0) \simeq \tilde{A}$) whose action is given on a basis $\{v_n, n \in \mathbb{Z}^*\}$ by:

$$x_i v_n = (n + i) v_{n+i} \quad C v_n = 0 \quad \forall n, \forall i \,. \tag{I.2}$$

Similarly to the irreducible case and as it is proved in [3], two classes of indecomposable admissible $\mathscr{V}$-modules emerge which are sufficient to describe all other ones:

a) the bounded $\mathscr{V}$-modules (the weightspace dimensions are bounded),
b) the $\mathscr{V}$-modules where the weights set is upper or lower bounded.

In this paper we are interested in the indecomposable admissible $\mathscr{V}$-modules of the class a) which appear as finite-length extensions of the irreducible $\mathscr{V}$-modules of type $A(a, \Lambda)$, $\tilde{A}$ or $D(0)$. Our aim is to prove that many such $\mathscr{V}$-modules do exist and to describe them by giving necessary conditions on the possible irreducible components of the finite-length extensions.

The main results of this paper are the following:

1. In any indecomposable bounded admissible $\mathscr{V}$-module, n-length extension of irreducible $\mathscr{V}$-modules, the invariants $\{\Lambda_i \; i = 1 \ldots p, p \leqq n\}$ must verify:

$$|\Lambda_i - \Lambda_j| \in \{0, 1, \ldots, 6(n-1)\} \,.$$

In the case $n = 2$, we obtain a complete precise result.

2. a) There exists, up to equivalence, a unique admissible extension of $A(a, \Lambda_1)$ by $A(a, \Lambda_2)$ if and only if (Λ_1, Λ_2) verifies:

$$\begin{aligned}
&\Lambda_1 - \Lambda_2 = 0 \quad (\Lambda_1, \Lambda_2) \neq (0, 0) \text{ and } (1, 1)\,,\\
&\Lambda_1 - \Lambda_2 = 2, 3, 4\,,\\
&\Lambda_1 - \Lambda_2 = 5 \quad \text{with } (\Lambda_1, \Lambda_2) = (1, -4) \text{ or } (5, 0)\,,\\
&\Lambda_1 - \Lambda_2 = 6 \quad \text{with } (\Lambda_1, \Lambda_2) = \frac{7 + \varepsilon\sqrt{19}}{2}, \frac{-5 + \varepsilon\sqrt{19}}{2}\,.
\end{aligned}$$

b) There exists, up to equivalence, two admissible extensions of $A(a, \Lambda)$ by $A(a, \Lambda)$ if $\Lambda = 0$ or 1, for all a, of $A(0, 0)$ by $A(0, 1)$ and three admissible extensions of $A(0, 1)$ by $A(0, 0)$.

c) There exists, up to equivalence, a unique admissible extension of $\tilde{A}$ by $A(a, \Lambda)$ and of $A(a, 1 - \Lambda)$ by $\tilde{A}$ if and only if

$$a = 0, \quad \Lambda = 0, -2, -3, -4$$

d) Besides the extensions of $\tilde{A}$ and $D(0)$ given in [4], we obtain a unique admissible extension of $A(0, \Lambda)$ by $D(0)$ and of $D(0)$ by $A(0, 1 - \Lambda)$ if and only if $\Lambda = 0, 1, 2$.

For each of these extensions we calculate explicitly the action of the Lie generators of $\mathscr{V}$.

The result 1 generalizes and improves Proposition IV.5 of [2], and its proof together with a careful study of the case $n = 2$ are given in Sect. II. The result 2 gives all the admissible extensions of two $\mathscr{V}$-modules among $\{\tilde{A}, D(0), A(a, \Lambda), (a, \Lambda) \in \mathbb{C}^2\}$. Consequently, besides all the admissible extensions of two irreducible bounded $\mathscr{V}$-modules, we also get extensions of length three or four (for example, the extensions of $\tilde{A}$ or $A(0,0)$ by $A(0,0)$). Finally, we give a complete classification of all bounded $\mathscr{V}$-modules with weightspace dimensions less than or equal to two. In particular, we have all the admissible extensions of two $\mathscr{V}$-modules given in [4].

Sections III to V are devoted to this classification as follows:

– In Sect. III. we obtain the result 2 a).

– In Sect. IV, we obtain all the admissible extensions of an irreducible $\mathscr{V}$-module $A(a, \Lambda)$ by $\tilde{A}$, $D(0)$ or any indecomposable $\mathscr{V}$-module given in [4] (which are extensions of $D(0)$ and $\tilde{A}$).

– In Sect. V, we obtain all the admissible extensions of two $\mathscr{V}$-modules among $\tilde{A}$, $D(0)$ or any indecomposable $\mathscr{V}$-module of [4]. The results 2 b) are given in Sect. V, Proposition (V.4.1). The results 2 c) and d) are given in Sects. IV and V but summarized in Sect. V (Propositions (V.1.1) and (V.3.2)).

Adding the $\mathscr{V}$-modules of [4], we conclude in part VI that we have all the indecomposable admissible $\mathscr{V}$-modules where the weightspace dimensions are less than or equal to two. We also remark that we obtain some results of [6].

Now, recall, for the following, the classification of the admissible $\mathscr{V}$-modules with one-dimensional weightspaces given in [4]. Besides the $\mathscr{V}$-modules $A(a, \Lambda)$, $\tilde{A}$, defined by (I.1) (I.2), appear two series A_α and B_β, ($\alpha, \beta \in \mathbb{C}$) which are respectively extensions of $\tilde{A}$ by $D(0)$ and $D(0)$ by $\tilde{A}$. On a basis $\{v_n, n \in \mathbb{Z}\}$ they are given by:

$$A_\alpha : \begin{cases} x_i v_n = (i+n) v_{i+n} & \forall n \neq 0 \\ x_i v_0 = i(\alpha + i) v_i \end{cases} ; \; C = 0 ,$$

$$B_\beta : \begin{cases} x_i v_0 = 0 \quad \forall i \\ x_i v_n = (i+n) v_{n+i}, n+i \neq 0, n \neq 0; C = 0 . \\ x_i v_{-i} = (\beta + i) v_0 \end{cases} \tag{I.3}$$

Remarks I.4. Let us notice that the above parametrization A_α, B_β is slightly different from the parametrization $A(\alpha')$, $B(\beta')$ in [4]. The correspondence is the following:

$$A_\alpha \sim A(\alpha') \quad \text{if } 1 + 2\alpha' = \frac{\alpha+1}{\alpha-1} ,$$

$$B_\beta \sim B(\beta') \quad \text{if } 1 + 2\beta' = \frac{\beta+1}{\beta-1} .$$

The advantage is that the $\mathscr{V}$-modules A_1 and B_1 are not obtained in [4].

II. Extensions of Irreducible Bounded Admissible $\mathscr{V}$-Modules: First Results and Consequences for Indecomposable Bounded Admissible $\mathscr{V}$-Modules

In this section we denote by $\mathscr{A} = \bigoplus_{n \in Z} \mathscr{A}_{a+n}$ an indecomposable bounded admissible $\mathscr{V}$-module, where $\mathscr{A}_{a+n}$ is the weightspace relative to the weight $a+n$, and

$\{\dim \mathscr{A}_{a+n}, n \in \mathbb{Z}\}$ is bounded. We also denote $\mathscr{A}^*$ the contragredient $\mathscr{V}$-module of $\mathscr{A}$:

$$\mathscr{A}^* = \bigoplus_{n \in Z} (\mathscr{A}_{a+n})^*. \text{ Then } \mathscr{A}^* = \bigoplus_{n \in Z} (\mathscr{A}^*)_{-a+n} \text{ with } (\mathscr{A}^*)_{-a+n} = (\mathscr{A}_{a-n})^*.$$

Recall the simple following properties on $\mathscr{A}^*$:

Property II.1. If $A(a, \Lambda)$, $\tilde{A}$, A_α, B_β are defined as in (I.1), (I.2) and (I.3), we have:

a) $[A(a, \Lambda)]^* = A(1 - a, 1 - \Lambda)$; $(\tilde{A})^* = \tilde{A}$; $D(0)^* = D(0)$; $A_\alpha^* = B_\alpha$.
b) Suppose $\dim \mathscr{A}_{a+n} = p$, $\forall n \in Z$. Then, we have:
x_{-1} (respectively x_1) is annihilated in $\mathscr{A}_{a+n} \Leftrightarrow x_{-1}$ (respectively x_1) is annihilated in $(\mathscr{A}^*)_{-a+1-n}$ (respectively $(\mathscr{A}^*)_{-a-1-n}$).

From [1] and [2] we know that any indecomposable bounded admissible $\mathscr{V}$-module $\mathscr{A}$ is a finite length extension of irreducible $\mathscr{V}$-modules of type $A(a, \Lambda)$ ($\Lambda \neq 0, 1$, if $a = 0$), $\tilde{A}$ or $D(0)$. Recall that for any $\mathscr{V}$-module $\mathscr{A}'$ and $\mathscr{A}''$, the first cohomology space $H^1(\mathscr{V}; \mathrm{Hom}_{\mathbb{C}}(\mathscr{A}'', \mathscr{A}'))$ classifies the short exact sequences: $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$, also called the extension of $\mathscr{A}'$ by $\mathscr{A}''$.

We are only interested in the admissible extensions and they are classified by a group of relative cohomology $H^1(\mathscr{V}, x_0, \mathrm{Hom}_{\mathbb{C}}(\mathscr{A}'', \mathscr{A}'))$. Actually, we prove in the following that this cohomology vanishes on the center C if $\mathscr{A}'$ and $\mathscr{A}''$ are irreducible bounded admissible $\mathscr{V}$-modules, except if $\mathscr{A}'$ or $\mathscr{A}'' = D(0)$. From now on, $\mathscr{A}'$ (respectively $\mathscr{A}''$) is identified with a submodule of $\mathscr{A}$ (respectively a factor of $\mathscr{A}$).

We prove now the following proposition.

Proposition II.2. *Let $\mathscr{A}$ be a non-trivial admissible extension of two irreducible $\mathscr{V}$-modules $\mathscr{A}'$ and $\mathscr{A}''$ of type $A(a, \Lambda)$ or $\tilde{A}$: $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ (a has necessarily the same value in $\mathscr{A}'$ and $\mathscr{A}''$). Then:*

1. *The center C is trivial in $\mathscr{A}$.*
2. *If $\mathscr{A} \cap \mathrm{Ker}\, x_{-1} \neq \{0\}$, setting $m_0 = \sup\{n/\mathrm{Ker}\, x_{-1} \cap \mathscr{A}_{a+n} \neq \{0\}\}$. Then*

$$\mathrm{Ker}\, x_{-1} \cap \mathscr{A}_{a+m_0} = \mathscr{A}'_{a+m_0}\,.$$

3. $\mathscr{A}' \cap \mathrm{Ker}\, x_{-1} \neq \{0\} \Leftrightarrow \mathscr{A}'' \cap \mathrm{Ker}\, x_{-1} \neq \{0\}$.
4. *If $\mathscr{A} \cap \mathrm{Ker}\, x_{-1} \neq \{0\}$ and m_0 as in 2, then*

$$\mathrm{Sup}\{n/\mathrm{Ker}\, x_{-1} \cap \mathscr{A}''_{a+n} \neq \{0\}\} \leqq m_0\,.$$

Proof.
1. From Theorem (II.7) of [2], C has the only eigenvalue 0 and if C is not zero, the trivial $\mathscr{V}$-module appears as a factor of $\mathscr{A}$ and we have then a proper $\mathscr{V}$-submodule $\mathscr{A}_3$ of $\mathscr{A}$ such that $\mathscr{A}/\mathscr{A}_3 = D(0)$. We obtain a contradiction with the irreducibility of $\mathscr{A}'$ and $\mathscr{A}''$.
2. To prove the second assertion, we use Proposition III.1 of [2] which can be written as follows:

Proposition. *Let $\mathscr{A}$ be an indecomposable bounded admissible $\mathscr{V}$-module with $\mathrm{Ker}\, x_{-1} \neq \{0\}$. Let m_0 defined as above. Let v be a vector of $\mathscr{A}_{a+m_0} \cap \mathrm{Ker}\, x_{-1}$. Suppose that v verifies one of the following properties:*

a) $x_1^n v \neq \{0\} \quad \forall n \in \mathbb{N}$,
b) $\exists m_1 \in \mathbb{N}$ *such that* $x_1^{m_1+1} v = 0, x_1^{m_1} v \neq 0$ *and there exists*

$$v' \in \mathscr{A}_{a+m_1+1} \text{ with } x_{-1} v' = x_1^{m_1} v .$$

Then v belongs to a $\mathscr{V}$*-submodule of* $\mathscr{A}$*, all of whose weightspaces are one dimensional, except, maybe, the weightspace relative to the weight* 0.

Here, any vector v of $\operatorname{Ker} x_{-1} \cap \mathscr{A}_{a+m_0}$ satisfies the hypotheses of the preceding proposition. Indeed, if it is not true, in view of Theorem (III.8) of [2], we have $a + m_0 = 0$ and thus $x_1 v = 0$. We deduce, from $[x_{-1}, x_2] v = 0$, that v generates the trivial submodule $D(0)$ of $\mathscr{V}$. We obtain a contradiction with the hypothesis of irreducibility of $\mathscr{A}'$ and $\mathscr{A}''$. Thus, we can apply the preceding proposition: v belongs to a $\mathscr{V}$-submodule $\mathscr{A}_3$ with one-dimensional weightspaces except maybe the weightspace relative to 0. The irreducibility of $\mathscr{A}'$ implies:

$$\mathscr{A}' \cap \mathscr{A}_3 = \{0\} \quad \text{or} \quad \mathscr{A}' \cap \mathscr{A}_3 = \mathscr{A}' .$$

If $\mathscr{A}' \cap \mathscr{A}_3 = \{0\}$, $\mathscr{A}_3$ is a submodule of $\mathscr{A}/\mathscr{A}' = \mathscr{A}''$, and thus $\mathscr{A}'' = \mathscr{A}_3$. We obtain a contradiction with the indecomposability of $\mathscr{A}$. Necessarily, we have $\mathscr{A}' \cap \mathscr{A}_3 = \mathscr{A}'$ and from the irreducibility of $\mathscr{A}''$, we deduce: $\mathscr{A}_3 = \mathscr{A}'$ and thus $\operatorname{Ker} x_{-1} \cap \mathscr{A}_{a+m_0} = \mathscr{A}'_{a+m_0}$.
3. Suppose $\mathscr{A}' \cap \operatorname{Ker} x_{-1} \neq \{0\}$. Then x_{-1} is annihilated in $\mathscr{A}$ and consequently in $\mathscr{A}^*$ and $\mathscr{A}'^*$ (Property II.1.b). We can look at $\mathscr{A}^*$ as the following extension:

$$0 \to \mathscr{A}''^* \to \mathscr{A}^* \to \mathscr{A}'^* \to 0 .$$

In view of II.1.a $\mathscr{A}^*$ satisfies the hypotheses of Proposition II.2, Part 2 and thus, we have:

$$\operatorname{Ker} x_{-1} \cap (\mathscr{A}^*)_{a+m_0^*} = (\mathscr{A}''^*)_{a+m_0^*} ,$$

where $m_0^* = \sup\{n \in \mathbb{Z} / \operatorname{Ker} x_{-1} \cap (\mathscr{A}^*)_{a+n} \neq \{0\}\}$. x_{-1} vanishes in $\mathscr{A}''^*$ and consequently in $\mathscr{A}''$. Applying the result to $\mathscr{A}^*$, we obtain the third assertion of Proposition II.2.
4. From parts 1 and 2 of Proposition (II.2), we deduce that $\operatorname{Ker} x_{-1}$ is not trivial in $\mathscr{A}''$. Set:

$$m_1 = \sup\{n \in \mathbb{Z} / \operatorname{Ker} x_{-1} \cap \mathscr{A}''_{a+n} \neq \{0\}\} .$$

Thus, there exists in $\mathscr{A}_{a+m_1}$ a vector v in a supplementary subspace of $\mathscr{A}'_{a+m_1}$, with $x_{-1} v \in \mathscr{A}'_{a+m_1-1}$. Necessarily we have: $\operatorname{Ker} x_{-1} \cap \mathscr{A}_{a+m_1} \neq \{0\}$ and thus $m_1 \leq m_0$.

Remark. We obtain an analogous proposition with the condition $\operatorname{Ker} x_1 \neq \{0\}$.

Proposition II.3. *Let* $\mathscr{A}$ *be a nontrivial admissible extension of two irreducible* $\mathscr{V}$*-modules* $\mathscr{A}'$ *and* $\mathscr{A}''$*. Suppose:* $\mathscr{A}'' = D(0)$ *and* $\mathscr{A}'$ *of type* $A(0, \Lambda)$ $(\Lambda \neq 0, 1)$ *or* $\tilde{A}$*, or the contragredient hypothesis. Then,* Q_1 *has either the unique eigenvalue* 0 *or two eigenvalues* 0 *and* 2*, and the center* C *is zero. In the second case,* $\mathscr{A}$ *is either the unique extension* $\mathscr{F}$ *of* $A(0, 2)$ *by* $D(0)$ *or the contragredient extension* $\mathscr{F}^*$ *of* $D(0)$ *by* $A(0, -1)$.

Proof. We suppose: $\mathscr{A}'' = D(0)$

- If $\mathscr{A}' = \tilde{A}$, we have $Q_1 = 0$ and $C = 0$, [4, 5].
- If $\mathscr{A}' = A(0, \Lambda)$ $\Lambda \neq 0, 1$, Q_1 has the eigenvalue $\Lambda(\Lambda - 1) \neq 0$ in $\mathscr{A}'$. We write $\mathscr{A} = \bigoplus_{n \in Z} \mathscr{A}_n$ with $\dim \mathscr{A}_n = 1 \ \forall n \in \mathbb{Z}^*$ and $\dim \mathscr{A}_0 = 2$. There

exists $v'_0 \in \mathscr{A}_0 (v'_0 \notin \mathscr{A}')$ such that $x_1 v'_0 = 0$. Thus $x_1 x_{-1} v'_0 = 0$. As $Q_1 x_{-1} v'_0 = \Lambda(\Lambda - 1) x_{-1} v'_0 = x_{-1} Q_1 v'_0 = 0$, we deduce $x_{-1} v'_0 = 0$. In view of the indecomposability of $\mathscr{A}$, $x_2 v'_0$ is different from zero. Thus $[x_{-1} x_2] v'_0 = 0$ implies $x_{-1}(x_2 v'_0) = 0$ and $Q_1(x_2 v'_0) = 2x_2 v'_0 = \Lambda(\Lambda - 1) x_2 v'_0$. We get $\Lambda = 2$. C is trivial: indeed if $Cv'_0 \neq 0$, Cv'_0 is in $\mathscr{A}'_0$, $Q_1 Cv'_0 = 2Cv'_0 = 0$, and we obtain a contradiction.

So, there exists a unique extension of $A(0, \Lambda)$ by $D(0)$ for $\Lambda = 2$. It is denoted by $\mathscr{F}$. Up to equivalence, we can choose a basis of $\mathscr{F}$, $\{v_n, n \in \mathbb{Z}, v'_0\}$ such that:

$$x_i v_n = (n + 2i) v_{n+1}, \quad \forall n, \forall i \in \mathbb{Z}; \qquad x_0 v'_0 = x_1 v'_0 = x_{-1} v'_0 = 0\,,$$

$$x_2 v'_0 = v_2, \qquad x_{-2} v'_0 = -v_{-2}\,,$$

$$Cv_n = Cv'_0 = 0 \qquad \forall n \in \mathbb{Z}\,.$$

All other cases are the contragredient cases of the previous ones. In particular, there exists a unique extension of $D(0)$ by $A(0, \Lambda)$ for $\Lambda = -1$ which is the contragredient extension $\mathscr{F}^*$ of $\mathscr{F}$. Up to equivalence, we can choose a basis of $\mathscr{F}^* \{v_0, v'_n \in \mathbb{Z}\}$ such that:

$$x_i v_0 = 0, \quad \forall i \in \mathbb{Z}\,,$$

$$x_1 v'_n = (n - 1) v'_{n+1}, \quad x_2 v'_n = (n - 2) v'_{n+2} + \delta_{n,-2} v_0\,,$$

$$x_{-1} v'_n = (n + 1) v'_{n-1}, \quad x_{-2} v'_n = (n + 2) v'_{n-2} - \delta_{n,2} v_0\,,$$

$$Cv_0 = Cv_n = 0, \forall n \in \mathbb{Z}\,.$$

Corollary 11.4. *Let $\mathscr{A}$ be a nontrivial admissible extension of $\mathscr{A}'$ by $\mathscr{A}''$, where $\mathscr{A}'$ and $\mathscr{A}''$ are of type $A(a, \Lambda)(\Lambda \neq 0, 1,$ if $a = 0)$, $\tilde{A}$ or $D(0)$.*
1. *If $\mathscr{A} \cap \operatorname{Ker} x_{-1} \neq 0$ or $\mathscr{A} \cap \operatorname{Ker} x_1 \neq 0$, then Q_1 has, at most, two eigenvalues $\Lambda_1(\Lambda_1 - 1), \Lambda_2(\Lambda_2 - 1)$ with $\Lambda_1 - \Lambda_2 \in \mathbb{Z}$.*
2. *If $\operatorname{Ker} x_{-1} = \operatorname{Ker} x_1 = 0$, then Q_1 has at most two eigenvalues $\Lambda_1(\Lambda_1 - 1)$, $\Lambda_2(\Lambda_2 - 1)$* with $\Lambda_1 \pm \Lambda_2 \in \mathbb{Z}$.

The first assertion results from Proposition II.2 and Proposition II.3. The second assertion was proved in [2] (§IV.2). In this case the condition $\Lambda_1 + \Lambda_2 \in \mathbb{Z}$ cannot be a priori rejected if we choose in $\mathscr{A}'$ and $\mathscr{A}''$ a basis of Feigin–Fuchs type (I.1) (I.2) (a condition which was not imposed in [2]).

We can generalize the results (II.3) and (II.4) as follows:

Theorem II.5. *Let $\mathscr{A}$ be an indecomposable bounded admissible $\mathscr{V}$-module. Then the eigenvalues $\{\Lambda_1(\Lambda_i - 1), i = 1, \ldots, p\}$ of Q_1 verify $\Lambda_i - \Lambda_j \in \mathbb{Z}$ or $\Lambda_i + \Lambda_j \in \mathbb{Z}$, $\forall i, \forall j$.*

Proof. We look at $\mathscr{A}$ as a finite length extension of irreducible bounded admissible $\mathscr{V}$-modules and we prove the result by induction over the length n of the extension.

For $n = 1$ the result is obvious. For $n = 2$ it is given by Corollary II.4. Then, the result is easily proved by induction over n.

Now we want to improve Corollary II.4 and Theorem II.5. Let $\mathscr{A} = \bigoplus_{n \in Z} \mathscr{A}_{a+n}$ be an indecomposable bounded admissible $\mathscr{V}$-module with an asymptotic dimension 2. From [1, 2] (Theorems (III.9) and (IV.13)), $\mathscr{A}$ contains a submodule $\mathscr{A}'$ with an asymptotic dimension 1 and $\mathscr{A}'' = \mathscr{A}/\mathscr{A}'$ has also an asymptotic dimension 1. Thus there exists an integer $n_0 \in \mathbb{Z}$ and a basis

$\{v_n, v'_n, n \geqq n_0\}$ of $\bigoplus_{n \leqq n_0} \mathscr{A}_{a+n}$ such that:

$$\begin{cases} x_i v_n = (a+n+i\Lambda_1)v_{n+i} \\ x_i \bar{v}'_n = (a+n+i\Lambda_2)\bar{v}'_{n+i} \end{cases} \quad \forall n, \forall i \text{ with } n+i \geqq n_0, n \geqq n_0\,, \tag{II.6}$$

where $\{v_n, n \geqq n_0\}$ (respectively $\{\bar{v}'_n, n \geqq n_0\}$) is a basis of $\bigoplus_{n \geqq n_0} \mathscr{A}'_{a+n}$ (respectively $\bigoplus_{n \geqq n_0} \mathscr{A}''_{a+n}$).

Remark II.7. $\mathscr{A}'$ is necessarily of type $A(a, \Lambda)$, $\tilde{A}$, A_α, B_α or an extension of $D(0)$ by one of these $\mathscr{V}$-modules. The choice of the parametrization of these $\mathscr{V}$-modules given by (I.1), (I.2), (I.3) implies that Λ_1 is unique except for $\mathscr{A}' = A(a, 0)$ or $\mathscr{A}' = (A(a, 1)$ and $a \neq 0$. We have the same conclusion for the choice of Λ_2 in $\mathscr{A}''$.

In such a $\mathscr{V}$-module $\mathscr{A}$, as x_1 and x_{-1} are one-to-one from $\mathscr{A}_{a+n}$ to $\mathscr{A}_{a+n+1}$ or $\mathscr{A}_{a+n-1}$ for enough large n, we have only the two following possibilities for Q_1:

- either Q_1 is diagonalisable on $\mathscr{A}_{a+n}$ for $n \geqq N_0$
- or Q_1 is not diagonalisable on $\mathscr{A}_{a+n}$ for $n \geqq N_0$.

Definition II.8. *Such a $\mathscr{V}$-module $\mathscr{A}$ is asymptotically Q_1-diagonalisasble (respectively asymptotically non-Q_1-diagonalisable) if there exists $N_0 \in \mathbb{N}$ such that Q_1 is diagonalisable (respectively non-diagonalisable) on $\mathscr{A}_{a+n}$ $\forall n \geqq N_0$.*

Theorem II.9. *Let $\mathscr{A}$ be an indecomposable admissible bounded $\mathscr{V}$-module, asymptotic extension of $\mathscr{A}'$ by $\mathscr{A}''$. Λ_1 and Λ_2 are their invariants defined by* (II.6) *and Remark* (II.7).

A. *If Q_1 is asymptotically diagonalisable, we have necessarily*:

1. $a = 0$: $\Lambda_1 = 0$ $\Lambda_2 = 0$, or $\Lambda_1 = 1$ $\Lambda_2 = 0$, or $\Lambda_1 = 0$ $\Lambda_2 = 1$, or $\Lambda_1 = \Lambda_2 = 1$.
2. $a = 0, \Lambda_1 = \Lambda_2 = 2$.
3. $a \neq \frac{1}{2}, \Lambda_1 = \Lambda_2 = \frac{1}{2}$.
4. $a \neq 0, \Lambda_1 = \Lambda_2 = 0$.
5. $\Lambda_1 = 2, \Lambda_2 = 1$ or $\Lambda_1 = 0, \Lambda_2 = -1$.
6. $\Lambda_1 - \Lambda_2 = 2, \Lambda_1 \neq \frac{3}{2}$.
7. $\Lambda_1 - \Lambda_2 = 3, \Lambda_1 \neq 2$.
8. $\Lambda_1 - \Lambda_2 = 4, \Lambda_1 \neq \frac{5}{2}$.
9. $\Lambda_1 = 1, \Lambda_2 = -4$ or $\Lambda_1 = 5, \Lambda_2 = 0$.
10. $\Lambda_1 = \dfrac{7+\sqrt{19}}{2}, \Lambda_2 = \dfrac{-5+\sqrt{19}}{2}$ or $\Lambda_1 = \dfrac{7-\sqrt{19}}{2}, \Lambda_2 = \dfrac{-5-\sqrt{19}}{2}$.

B. *If Q_1 is asymptotically non-diagonalisable we have necessarily*:

1. $\Lambda_1 = \Lambda_2$.
2. $\Lambda_2 = 1 - \Lambda_1$, *with* $\Lambda_1 = 0, 1, \frac{3}{2}, 2, \frac{5}{2}$.

Proof.
A. Q_1 is asymptotically diagonalisable.

We can thus choose the basis defined by the formulas (II.6) as follows $(n \geqq \sup(n_0, N_0) = N_1)$:

$$\left\{\begin{array}{ll} \begin{cases} x_1 v_n = (a+n+\Lambda_1)v_{n+1} \\ x_1 v'_n = (a+n+\Lambda_2)v'_{n+1} \end{cases} & \begin{cases} x_{-1} v_n = (a+n-\Lambda_1)v_{n-1} \\ x_{-1} v'_n = (a+n-\Lambda_2)v'_{n-1} \end{cases} \\ \begin{cases} x_2 v_n = (a+n+2\Lambda_1)v_{n+2} \\ x_2 v'_n = (a+n+2\Lambda_2)v'_{n+2} + \alpha_n v_{n+2} \end{cases} & \begin{cases} x_{-2} v_n = (a+n-2\Lambda_1)v_{n-2} \\ x_{-2} v'_n = (a+n-2\Lambda_2)v'_{n-2} + \beta_n v_{n-2} \end{cases} \end{array}\right. \tag{II.10}$$

From the relations $[x_{-1}x_2]v'_n = 3x_1 v'_n$ and $[x_{-2}x_1]v'_n = 3x_{-1}v'_n$ we get:

$$(a+n+2-\Lambda_1)\alpha_n - (a+n-\Lambda_2)\alpha_{n-1} = 0\,,$$

$$(a+n+\Lambda_2)\beta_{n+1} - (a+n-2+\Lambda_1)\beta_n = 0\,.$$

We deduce the existence of two constants α_+ and β_+ such that:

$$\begin{cases} \alpha_n = \dfrac{\Gamma(a+n+1-\Lambda_2)}{\Gamma(a+n+3-\Lambda_1)}\alpha_+ & \forall n \geqq N_1 \\ \beta_n = \dfrac{\Gamma(a+n-2+\Lambda_1)}{\Gamma(a+n+\Lambda_2)}\beta_+ & \forall n \geqq N_1+2 \end{cases}. \tag{II.11}$$

Recall that the center C is zero on $\mathscr{A}_{a+n}$, $n \geqq N_1$ ([2], Theorem (II.7)). Then, the relation $[x_2, x_{-2}]v'_n = 4x_0 v'_n$ together with the formulas (II.11) gives:

$$\begin{aligned} &\alpha_+ \frac{\Gamma(a+n-1-\Lambda_2)}{\Gamma(a+n+1-\Lambda_1)}\Bigg[-2+\frac{(\Lambda_1-\Lambda_2-1)(\Lambda_1-\Lambda_2-2)(1-\Lambda_1)}{a+n+1-\Lambda_1} \\ &\quad + \frac{\Lambda_1(\Lambda_1-\Lambda_2-2)(\Lambda_1-\Lambda_2-3)}{a+n+2-\Lambda_1}\Bigg] \\ &= -\beta_+ \frac{\Gamma(a+n-2+\Lambda_1)}{\Gamma(a+n+\Lambda_2)}\Bigg[-2+\frac{(\Lambda_1-\Lambda_2-1)(\Lambda_1-\Lambda_2-2)\Lambda_2}{a+n+\Lambda_2} \\ &\quad + \frac{(1-\Lambda_2)(\Lambda_1-\Lambda_2-2)(\Lambda_1-\Lambda_2-3)}{a+n+1+\Lambda_2}\Bigg]. \end{aligned} \tag{II.12}$$

From Theorem (II.5) we know that $\Lambda_1 \pm \Lambda_2 = p \in \mathbb{Z}$. Let us discuss the solutions of (II.12):

– Either $\alpha^+ = \beta^+ = 0$. Then the two $\mathscr{V}$-submodules generated by v_{N_1} and v'_{N_1} have both an asymptotic dimension 1. To get an indecomposable $\mathscr{V}$-module $\mathscr{A}$, these two submodules must have an intersection which is necessarily either the submodule $D(0)$ or $D(0) \oplus D(0)$. Using Proposition II.3, we only have the following possibilities:

$a=0, \Lambda_1 = 0, \Lambda_2 = 0$ (case 1 of Theorem II.9),
$a=0, \Lambda_1 = \Lambda_2 = 2$ (case 2 of Theorem II.9)
$a=0, \Lambda_1 = 2, \Lambda_2 = 0$ (case 6 of Theorem II.9)

– Or $\alpha^+ \cdot \beta^+ \neq 0$.

1st case. $\Lambda_1 + \Lambda_2 = p$. From (II.12), we immediately get that $p = 1$. We want to prove that necessarily: $\Lambda_1 = 0$ or 1 or $\Lambda_1 = \frac{1}{2}$ and $a \neq \frac{1}{2}$. We use Theorems II.10 and III.2 of [2]. They claim that the $\mathscr{V}$-submodule generated by an eigenvector of Q_1, $v \in \mathscr{A}_{a+n}$ $(n \geqq N_1)$ such that $x_2 v = \lambda x_1^2 v$, has an asymptotic dimension equal to 1. Setting $v''_{N_1} = v'_{N_1} + k v_{N_1}$, the equation $x_2 v''_{N_1} = \lambda x_1^2 v''_{N_1}$, $\lambda \in \mathbb{C}$, together with II.11, imply:

$$-2k\Lambda_1(\Lambda_1-1)(2\Lambda_1-1) = \alpha_+ \frac{\Gamma(a+N_1+\Lambda_1)}{\Gamma(a+N_1-\Lambda_1)}\,.$$

If $\Lambda_1 \neq 0, 1, \frac{1}{2}$, there exists v''_{N_1} which generates a submodule $\mathscr{A}_1$ with an asymptotic dimension 1. Necessarily, we have $\mathscr{A}' \cap \mathscr{A}_1 = D(0)$ or $D(0) \oplus D(0)$ and we are again in the preceding case $\alpha^+ = \beta^+ = 0$.

If $\Lambda_1 = 0$ or 1, we are either in case 1 of the theorem, or in case 4 ($A(a, 1) \sim A(a, 0)$ if $a \neq 0$).

If $\Lambda_1 = \Lambda_2 = \frac{1}{2}$ the diagonalisability of Q_1 implies $\operatorname{Ker} x_{-1} \cap \mathscr{A}_{1/2} = \mathscr{A}_{1/2}$. Then, using $[x_{-1} x_2] = 3x_1$ and the injectivity of x_{-1} on $\mathscr{A}_{1/2+n}$ for $n \in \mathbb{N}^*$, the two vectors v_0, v'_0 of $\mathscr{A}_{1/2}$ verify the condition $x_2 v = \lambda x_1^2 v$. Consequently, each of them generates a $\mathscr{V}$-module with an asymptotic dimension 1, and $\mathscr{A}$ is decomposable. Thus, we have necessarily $a \neq \frac{1}{2}$ if $\Lambda_1 = \Lambda_2 = \frac{1}{2}$ (case 3 of the theorem).

2nd case. $\Lambda_1 - \Lambda_2 = p \in \mathbb{Z}$. Setting $x = a + n$ in (II.12), we obtain a polynomial identity. We first deduce in all cases $\beta_+ = -\alpha_+$. Then, we look at the zeros of the right and left members. We have to discuss according to the hypotheses $p < 4$, $p = 4$, $p > 4$, and we get the necessary condition $0 \leqq p \leqq 6$. For $p = 2, 3, 4$ Λ_1 is arbitrary (cases 6, 7, 8). For $p = 5, 6$ we have only two values for Λ_1 (cases 9 and 10). For $p = 0, 1$ all solutions are listed in the cases 1 to 5.

B. Q_1 is asymptotically non-diagonalisable:

As Q_1 has a unique eigenvalue $\Lambda(\Lambda - 1)$, we only have the two following possibilities: $\Lambda_1 = \Lambda_2$ or $\Lambda_2 = 1 - \Lambda_1$. Suppose $\Lambda_2 = 1 - \Lambda_1$. We can choose the basis defined by formulas (II.6) for all $n \geqq \sup(n_0, N_0) = N_1$:

$$\begin{cases} \begin{cases} x_1 v_n = (a+n+\Lambda_1) v_{n+1} \\ x_1 v'_n = (a+n+1-\Lambda_1) v'_{n+1} \\ \qquad + \delta_n v_{n+1} \end{cases} & \begin{cases} x_{-1} v_n = (a+n-\Lambda_1) v_{n-1} \\ x_{-1} v'_n = (a+n-1+\Lambda_1) v'_{n-1} \\ \qquad + \gamma_n v_{n-1} \end{cases} \\ \begin{cases} x_2 v_n = (a+n+2\Lambda_1) v_{n+2} \\ x_2 v'_n = (a+n+2-2\Lambda_1) v'_{n+2} \\ \qquad + \alpha_n v_{n+2} \end{cases} & \begin{cases} x_{-2} v_n = (a+n-2\Lambda_1) v_{n-2} \\ x_{-2} v'_n = (a+n-2+2\Lambda_1) v'_{n-2} \\ \qquad + \beta_n v_{n-2} \end{cases} \end{cases} \tag{II.13}$$

From the relation $[x_{-1}, x_1] v_n = 2x_0 v_n$, we get:

$$(a+n+1-\Lambda_1)(\gamma_{n+1} + \delta_n) - (a+n+\Lambda_1 - 1)(\gamma_n + \delta_{n-1}) = 0 \quad \forall n \geqq N_1 + 1 .$$

As Q_1 is not diagonalisable on $\mathscr{A}_{a+n}$ ($\forall n \geqq N_1 + 1$), $\gamma_n + \delta_{n-1} \neq 0$ and we obtain

$$\delta_n + \gamma_{n+1} = \varepsilon \frac{\Gamma(a+n+\Lambda_1)}{\Gamma(a+n+2-\Lambda_1)} \quad \forall n \geqq N_1 + 1, \varepsilon \neq 0 . \tag{II.14}$$

From the relations $[x_{-2}, x_1] = 3x_{-1}$ and $[x_{-1}, x_2] = 3x_1$ applied on v'_n, follows the relation:

$$\begin{aligned} &(a+n+2-\Lambda_1)(a+n+1-\Lambda_1)(\alpha_n + \beta_{n+2}) \\ &\quad - (a+n-2+\Lambda_1)(a+n-1+\Lambda_1)(\alpha_{n-2} + \beta_n) \\ &= \frac{\Gamma(a+n+\Lambda_1)}{\Gamma(a+n+1-\Lambda_1)} F(n), \quad \forall n \geqq N_1 , \end{aligned}$$

where

$$F(n) = 8 + 2\Lambda_1(\Lambda_1 - 1)\left[\frac{1}{a+n+2-\Lambda_1} - \frac{1}{a+n-1+\Lambda_1} + \frac{1}{a+n+1-\Lambda_1} - \frac{1}{a+n-2+\Lambda_1}\right].$$

From $[x_{-2}, x_2]v'_n = 4x_0 v'_n$ we have:

$$(a+n+2-2\Lambda_1)(\alpha_n + \beta_{n+2}) - (a+n-2+2\Lambda_1)(\alpha_{n-2} + \beta_n) = 0 \quad \forall n \geqq N_1 .$$

These two inducing relations lead to the following necessary compatibility condition:

$$D(n+2)(a+n+2-\Lambda_1)(a+n+1-\Lambda_1)(a+n-2+2\Lambda_1)F(n)$$
$$= D(n)(a+n+4-2\Lambda_1)(a+n+1+\Lambda_1)(a+n+\Lambda_1)F(n+2),$$

where $D(n) = 2(a+n)^2 + 4(\Lambda_1 - 1)^2(\Lambda_1 - 2)$. A careful study of the poles of this last equation shows that it is generally impossible except for the particular values $\Lambda_1 = 0, \Lambda_1 = 1, \Lambda_1 = \frac{3}{2}, \Lambda_1 = 2, \Lambda_1 = \frac{5}{2}$. The proof of Theorem II.9 is achieved.

We can deduce the following corollary:

Corollary II.15. *Let $\mathscr{A}'$ and $\mathscr{A}''$ be two irreducible $\mathscr{V}$-modules of type $A(a, \Lambda)$ (if $a = 0, \Lambda \neq 0, 1$), $\tilde{A}$ $(a = 0, \Lambda = 1)$ or $D(0)$. We denote by $H^1(\mathscr{V}, x_0, \mathrm{Hom}_{\mathbb{C}}(\mathscr{A}'', \mathscr{A}'))$ the first group of relative cohomology of $\mathscr{V}$ with values in $\mathrm{Hom}_{\mathbb{C}}(\mathscr{A}'', \mathscr{A}')$. Then:*

1. *If $\mathscr{A}' = A(a, \Lambda_1)$ or $\tilde{A}(\Lambda_1 = 1)$, and $\mathscr{A}'' = A(a, \Lambda_2)$ or $\tilde{A}(\Lambda_2 = 1)$: $H^1(\mathscr{V}, x_0, \mathrm{Hom}_{\mathbb{C}}(\mathscr{A}'', \mathscr{A}')) \neq \{0\} \Rightarrow \Lambda_1 - \Lambda_2 \in \{0, 1, 2, 3, 4, 5, 6\}$.*

2. *If $\mathscr{A}' = D(0)$, $\mathscr{A}'' = A(a, \Lambda_2)$ or $\tilde{A}$, $H^1(\mathscr{V}, x_0, \mathrm{Hom}_{\mathbb{C}}(\mathscr{A}'', \mathscr{A}')) \neq \{0\} \Rightarrow \mathscr{A}'' = A(0, -1)$ or $\mathscr{A}'' = \tilde{A}$.*

3. *If $\mathscr{A}' = A(a, \Lambda_1)$ or $\tilde{A}$, $\mathscr{A}'' = D(0)$, $H^1(\mathscr{V}, x_0, \mathrm{Hom}_{\mathbb{C}}(\mathscr{A}'', \mathscr{A}')) \neq \{0\} \Rightarrow \mathscr{A}' = A(0, 2)$ or $\mathscr{A}' = \tilde{A}$.*

Proof. The first assertion results from Theorem II.9. Indeed, in Theorem II.9 we have always $\Lambda_1 - \Lambda_2 \in \mathbb{Z}$ with $0 \leqq \Lambda_1 - \Lambda_2 \leqq 6$ except in the cases A1 and B2, for $\Lambda_1 = 0, \Lambda_2 = 1$. For these values of Λ_1 and Λ_2, the irreducibility of $\mathscr{A}'$ and $\mathscr{A}''$ implies $a \neq 0$. Thus, the hypothesis $\Lambda_1 = 0, \Lambda_2 = 1$ is equivalent to $\Lambda_1 = \Lambda_2 = 0$. The second and third assertions result from Proposition II.3.

Now we can improve Theorem II.5 as follows:

Theorem II.16. *Let $\mathscr{A}$ be an indecomposable bounded admissible $\mathscr{V}$-module.*

1. *Then the eigenvalues $\{\Lambda_i(\Lambda_i - 1)\}$ of Q_1 verify $\Lambda_i - \Lambda_j \in \mathbb{Z}$, $\forall i, \forall j$.*
2. *Moreover if $\mathscr{A}$ is a n-length extension of irreducible bounded admissible $\mathscr{V}$-modules ($n \geqq 2$),* the *eigenvalues $\{\Lambda_i(\Lambda_i - 1)\}$ of Q_1 verify:*

$$0 \leqq |\Lambda_i - \Lambda_j| \leqq 6(n-1) \text{ with } \Lambda_i - \Lambda_j \in \mathbb{Z} .$$

The proof is the same as in Theorem II.5, substituting the induction hypothesis $\Lambda_i \pm \Lambda_j \in \mathbb{Z}$ by $\Lambda_i - \Lambda_j \in \mathbb{Z}$ with $|\Lambda_i - \Lambda_j| \leqq 6(n-1)$.

III. Non Trivial Admissible Extensions of Two Irreducible $\mathscr{V}$-Modules, $A(a, \Lambda_1)$ by $A(a,\Lambda_2)$ $(a = 0 \Rightarrow \Lambda_i \neq 0, 1)$

Let $\mathscr{A} = \bigoplus_{n \in \mathbb{Z}} \mathscr{A}_{a+n}$ be such a $\mathscr{V}$-module. Then, $\dim \mathscr{A}_{a+n} = 2$, $\forall n \in \mathbb{Z}$. In view of Theorem II.9, we distinguish the following cases:

- $\Lambda_1 = \Lambda_2$ and Q_1 is asymptotically non-diagonalisable except:

$$\Lambda_1 = \Lambda_2 = 0 \text{ and } a \neq 0 ,$$

$$\Lambda_1 = \Lambda_2 = \frac{1}{2} \text{ and } a \neq \frac{1}{2} ,$$

$$\Lambda_1 = \Lambda_2 = 2 \text{ and } a = 0 ,$$

where we can have, a priori, the two possibilities for Q_1.

- $\Lambda_2 = 1 - \Lambda_1$ with $\Lambda_1 = \frac{3}{2}$, $\Lambda_1 = 2$ or $\Lambda_1 = \frac{5}{2}$ and Q_1 is asymptotically non-diagonalisable.

- $\Lambda_1 = 2, \Lambda_2 = 1$; $\Lambda_1 = 0, \Lambda_2 = -1$ $(a \neq 0)$.
- $\Lambda_1 - \Lambda_2 = 2, 3, 4, \Lambda_1 + \Lambda_2 \neq 1$.
- $\Lambda_1 = 1, \Lambda_2 = -4$; $\Lambda_1 = 5, \Lambda_2 = 0$ $(a \neq 0)$.
- $\Lambda_1 = \dfrac{7 + \varepsilon\sqrt{19}}{2}, \Lambda_2 = \dfrac{-5 + \varepsilon\sqrt{19}}{2}, \varepsilon = \pm 1$.

In the four latter cases, Q_1 is asymptotically diagonalisable.

Remark. If $a \neq 0$, the cases $\Lambda_1 = 2$ $\Lambda_2 = 1$ and $\Lambda_1 = 0$ $\Lambda_2 = -1$ are respectively equivalent to the cases $\Lambda_1 = 2, \Lambda_2 = 0$ and $\Lambda_1 = 1, \Lambda_2 = 1$ and are included in the case $\Lambda_1 - \Lambda_2 = 2$.

III.1 Extensions of $A(a\ \Lambda)$ by $A(a, \Lambda)$ $(a = 0 \Rightarrow \Lambda \neq 0, 1)$.

A) Q_1 is asymptotically non-diagonalisable:

Then, Q_1 is non-diagonalisable on $\mathscr{A}_{a+n}$ for all n in $\mathbb{Z}$. Thus we can choose the basis defined by (II.6) for all n in $\mathbb{Z}$ as follows:

$$\left\{\begin{array}{ll} \begin{cases} x_1 v_n = (a+n+\Lambda_1) v_{n+1} \\ x_1 v'_n = (a+n+\Lambda_1) v'_{n+1} + \delta_n v_{n+1} \end{cases} & \begin{cases} x_2 v_n = (a+n+2\Lambda_1) v_{n+2} \\ x_2 v'_n = (a+n+2\Lambda_1) v'_{n+2} + \alpha_n v_{n+2} \end{cases} \\ \begin{cases} x_{-1} v_n = (a+n-\Lambda_1) v_{n-1} \\ x_{-1} v'_n = (a+n-\Lambda_1) v'_{n-1} + \gamma_n v_{n-1} \end{cases} & \begin{cases} x_{-2} v_n = (a+n-2\Lambda_1) v_{n-2} \\ x_{-2} v'_n = (a+n-2\Lambda_1) v'_{n-2} + \beta_n v_{n-2} \end{cases} \end{array}\right. . \tag{III.1.1}$$

From $[x_{-1} x_1](v'_n) = 2x_1 v'_n$ we deduce:

$$(a+n+\Lambda_1)\gamma_{n+1} + (a+n+1-\Lambda_1)\delta_n$$

$$= (a+n-1+\Lambda_1)\gamma_n + (a+n-\Lambda_1)\delta_{n-1}, \quad \forall n \in \mathbb{Z} ,$$

and we also have:

$$Q_1 v'_n = \Lambda_1(\Lambda_1 - 1)v'_n - [(a+n-1+\Lambda_1)\gamma_n + (a+n-\Lambda_1)\delta_{n-1}]v_n \quad \forall n .$$

The non-diagonalisability of Q_1 on $\mathscr{A}_{a+n}$ implies:

$$(a+n-1+\Lambda_1)\gamma_n + (a+n-\Lambda_1)\delta_{n-1} \neq 0, \quad \forall n .$$

– If $\Lambda_1 = \frac{1}{2}$, this condition together with the relations $[x_{-1}x_2](v'_n) = 3x_1(v'_n)$, $[x_{-2}x_1](v'_n) = 3x_{-1}(v'_n)$, $[x_2 x_{-2}](v'_n) = 4x_0(v'_n)$ ($c = 0$, Proposition II.2) leads to a contradiction.

– If $\Lambda_1 \neq \frac{1}{2}$, the basis of $\mathscr{A}$ defined by (III.1.1) can be chosen so that $\{v_n, v'_n\}$ is a Jordan basis of Q_1 on $\mathscr{A}_{a+n}$ ($\forall n \in \mathbb{Z}$) and:

$$\delta_n = \frac{1}{2\Lambda_1 - 1}; \quad \gamma_n = -\frac{1}{2\Lambda_1 - 1} \quad \forall n .$$

Writing $\alpha_n = \frac{2}{2\Lambda_1 - 1} + \alpha'_n$, $\beta_n = -\frac{2}{2\Lambda_1 - 1} + \beta'_n$, the relations $[x_{-1}x_2]\, v'_n = 3x_1 v'_n$, $[x_{-2}x_1]v'_n = 3x_{-1}v'_n$, $[x_{-2}x_2]\, v'_n = 4x_0 v'_n$ imply:

$$\alpha'_n(a+n+2-\Lambda_1) - \alpha'_{n-1}(a+n-\Lambda_1) = 0 ,$$

$$\beta'_{n+1}(a+n+\Lambda_1) - \beta'_n(a+n-2+\Lambda_1) = 0 ,$$

$$\alpha'_n(a+n+2-2\Lambda_1) + \beta'_{n+2}(a+n+2\Lambda_1) - \beta'_n(a+n-2+2\Lambda_1) - \alpha'_{n-2}(a+n-2\Lambda_1) = 0 .$$

By a straightforward calculation, we prove that this system only admits the trivial solution $\alpha'_n = \beta'_n = 0$, $\forall n$, except in the particular cases $a = 0, \Lambda_1 = 0$ and $a = 0, \Lambda_1 = 1$. But, these latter are not considered in this section.

Thus, if Q_1 is non-diagonalisable and $\Lambda_1 \neq \frac{1}{2}$ we get a unique non-trivial admissible extension $\mathscr{A}$ of $A(a, \Lambda)$ by $A(a, \Lambda)$ ($a = 0 \Rightarrow \Lambda \neq 0, 1$) defined by the formulas (III.1.1) with

$$\delta_n = -\gamma_n = \frac{1}{2\Lambda_1 - 1}, \quad \alpha_n = -\beta_n = \frac{2}{2\Lambda_1 - 1}, \quad \forall n \in \mathbb{Z} . \tag{III.1.2}$$

B. Q_1 is asymptotically diagonalisable:

As either x_{-1} or x_1 is one-to-one from $\mathscr{A}_{a+n}$ to $\mathscr{A}_{a+n-1}$ or $\mathscr{A}_{a+n+1}$, Q_1 is diagonalisable on $\mathscr{A}_{a+n}$, $\forall n \in \mathbb{Z}$. The basis given by (II.6) and (II.10), (II.11) can be defined for all n in $\mathbb{Z}$. Equation (II.12) gives us $\alpha^+ + \beta^+ = 0$.

a) $\Lambda_1 = \Lambda_2 = 2, a = 0$. We have $\alpha^+ = \beta^+ = 0$ and $\mathscr{A}$ is decomposable.
b) $\Lambda_1 = \Lambda_2 = \frac{1}{2}, a \neq \frac{1}{2}$. Up to equivalence, we get a unique non-trivial admissible extension of $A(a, \frac{1}{2})$ by $A(a, \frac{1}{2})$ defined by the formulas (II.10), (II.11) for all n in $\mathbb{Z}$ with:

$$\alpha_n = \frac{1}{(a+n+\frac{3}{2})(a+n+\frac{1}{2})}, \quad \beta_n = -\frac{1}{(a+n-\frac{1}{2})(a+n-\frac{3}{2})} . \tag{III.1.3}$$

c) $\Lambda_1 = \Lambda_2 = 0, a \neq 0$. Up to equivalence, we get a unique non-trivial admissible extension of $A(a, 0)$ by $A(a, 0)$ defined by the formulas (II.10) (II.11) with:

$$\alpha_n = \frac{1}{(a+n+2)(a+n+1)}, \quad \beta_n = -\frac{1}{(a+n-2)(a+n-1)}. \quad \text{(III.1.4)}$$

We can thus claim the following theorem.

Theorem III.1.5. *$A(a, \Lambda)$ is an irreducible $\mathscr{V}$-module of Feigin–Fuchs (defined by* I.1) *($a = 0$ implies $\Lambda \neq 0, 1$). We have:*

1. *If $\Lambda \neq 0, \frac{1}{2}$ $\forall a$, or $\Lambda = \frac{1}{2}, a \neq \frac{1}{2}$:*

$$\dim \mathscr{H}^1[\mathscr{V}, x_0, \mathrm{Hom}_{\mathbb{C}}(A(a, \Lambda), A(a, \Lambda))] = 1 ,$$

and the cocycle is defined on $x_1, x_{-1}, x_2, -x_2$ either by (III.1.1) and (III.1.2) if Λ is different than $\frac{1}{2}$ or by (II.10) and (III.1.3) if $\Lambda = \frac{1}{2}, a \neq \frac{1}{2}$.

2. *If $\Lambda = 0$ $(a \neq 0)$:*

$$\dim \mathscr{H}^1[\mathscr{V}, x_0, \mathrm{Hom}_{\mathbb{C}}(A(a, 0), A(a, 0))] = 2 .$$

We have a basis of two independent cocycles, one defined by (III.1.1) and (III.1.2) for $\Lambda_1 = 0$ and one defined by (II.10) and (III.1.4).

III.2. Extensions of $A(a, \Lambda)$ by $A(a, \Lambda - p)$ $p = 2, 3, 4$.

Although $A(0, \Lambda)$ (respectively $A(0, \Lambda - p)$) is not irreducible when $\Lambda = 0, 1$ (respectively $\Lambda = p, p + 1$), we also consider here these cases which are not different from the general case.

1st case. $p = 2$.

A) Q_1 is asymptotically diagonalisable: necessarily, from Theorem (II.9), we have $(\Lambda, \Lambda - 2) \neq (\frac{3}{2}, \frac{1}{2})$. As either x_{-1} or x_1 is one-to-one from $\mathscr{A}_{a+n}$ on $\mathscr{A}_{a+n-1}$ or $\mathscr{A}_{a+n+1}$, for all n in $\mathbb{Z}$, Q_1 is diagonalisable on $\mathscr{A}_{a+n}$, for all n in $\mathbb{Z}$.

Then we can choose, up to equivalence, a basis of $\mathscr{A}$ where x_1, x_{-1}, x_2, x_{-2} are defined by the formulas (II.10), (II.11) for all n in $\mathbb{Z}$ with:

$$\alpha_n = -\beta_n = 1 \quad \forall n \in \mathbb{Z} . \quad \text{(III.2.1)}$$

B) Q_1 is asymptotically non-diagonalisable: $(\Lambda, \Lambda - 2) = (\frac{3}{2}, \frac{1}{2})$. For the same reasons as in A) Q_1 is non-diagonalisable on $\mathscr{A}_{a+n}$, for all n in $\mathbb{Z}$.

Thus we can choose a basis $\{v_n, v'_n, n \in \mathbb{Z}\}$ of $\mathscr{A}$ so that the formulas (II.13) are true for all n in $\mathbb{Z}$.

From the relation (II.14) we get:

$$\delta_n + \gamma_{n+1} = \varepsilon\left(a + n + \frac{1}{2}\right), \varepsilon \neq 0 .$$

Using $[x_{-1}x_2]v'_n = 3x_1 v'_n$ and $[x_{-2}x_1]v'_n = 3x_{-1}v'_n$, we obtain:

$$\left(a + n + \frac{1}{2}\right)[(\alpha_n + \beta_{n+2}) - (\alpha_{n-1} + \beta_{n+1})] = 4\varepsilon\left(a + n + \frac{1}{2}\right).$$

From $[x_2, x_{-2}]v'_n = -4x_0 v'_n$, we deduce:

$$(a + n - 1)(\alpha_n + \beta_{n+2}) - (a + n + 1)(\alpha_{n-2} + \beta_n) = 0 .$$

For all values of a, these two equations admit a unique solution:

$$\alpha_n + \beta_{n+2} = 4\varepsilon(a + n + 1) .$$

In other respects, it can be proved that, on a given reference level n, δ_n and α_n can be chosen independently (by taking a suitable basis). Therefore we can fix $\varepsilon = 1$. We get:

$$\delta_n = \frac{1}{2}\left(a + n + \frac{1}{2}\right); \quad \alpha_n = 2(a + n + 1) ,$$

$$\gamma_n = \frac{1}{2}\left(a + n - \frac{1}{2}\right); \quad \beta_n = 2(a + n - 1) . \tag{III.2.2}$$

The formulas (II.13) for all n with $\Lambda_1 = \frac{3}{2}$, together with (III.2.2), define a unique non-trivial admissible extension of $A(a, \frac{3}{2})$ by $A(a, \frac{1}{2})$.

2nd case. $p = 3$.

A) Q_1 is asymptotically diagonalisable: Necessarily from Theorem II.9 we have $(\Lambda, \Lambda - 3) \neq (2, -1)$. As in the preceding case, Q_1 is diagonalisable on $\mathscr{A}_{a+n}$, $\forall n \in \mathbb{Z}$. Then we can choose, up to equivalence, a basis of $\mathscr{A}$, $\{v_n, v'_n, n \in \mathbb{Z}\}$, where x_1, x_{-1}, x_2, x_{-2} are defined by (II.10), (II.11) for all n in $\mathbb{Z}$ with:

$$\alpha_n = (a + n - \Lambda + 3) \quad \beta_n = -(a + n + \Lambda - 3) \quad \forall n \in \mathbb{Z} , \tag{III.2.3}$$

and we obtain a unique non-trivial admissible exstension $\mathscr{A}$ of $A(a, \Lambda)$ by $A(a, \Lambda - 3)$.

B) Q_1 is asymptotically non-diagonalisable: $(\Lambda, \Lambda - 3) = (2, -1)$. Q_1 is non diagonalisable on $\mathscr{A}_{a+n}$, $\forall n \in \mathbb{Z}$. We can choose a basis $\{v_n, v'_n, n \in \mathbb{Z}\}$ of $\mathscr{A}$ so that the formulas (II.13) are verified for all $n \in \mathbb{Z}$.

The arguments used in case 1 B) ($\Lambda = \frac{3}{2}$) lead to the following result:

$$\begin{cases} \delta_n = \frac{1}{2}(a + n)(a + n + 1), & \alpha_n = 2(a + n)(a + n + 2), \\ \gamma_n = \frac{1}{2}(a + n - 1)(a + n), & \beta_n = 2(a + n - 2)(a + n). \end{cases} \tag{III.2.4}$$

We get a unique non-trivial admissible extension $\mathscr{A}$ of $A(a, 2)$ by $A(a, -1)$, $\forall a$.

3rd case. $p = 4$.

A) Q_1 is asymptotically diagonalisable: necessarily, from Theorem (II.9), we have $(\Lambda, \Lambda - 4) \neq (\frac{5}{2}, -\frac{3}{2})$. As in the preceding cases, Q_1 is diagonalisable on $\mathscr{A}_{a+n}$, $\forall n \in \mathbb{Z}$. Then, up to equivalence, we can choose a basis of $\mathscr{A}$: $\{v_n, v'_n, n \in \mathbb{Z}\}$, where x_1, x_{-1}, x_2, x_{-2} are defined by (II.10), (II.11) for all n in $\mathbb{Z}$ with:

$$\alpha_n = (a + n + 3 - \Lambda)(a + n + 4 - \Lambda),$$

$$\beta_n = -(a + n - 3 + \Lambda)(a + n - 4 + \Lambda) \quad \forall n \in \mathbb{Z} \tag{III.2.5}$$

B) Q_1 is asymptotically non-diagonalisable: $(\Lambda, \Lambda - 4) = (\frac{5}{2}, -\frac{3}{2})$. We always get that Q_1 is non-diagonalisable on $\mathscr{A}_{a+n}$, $\forall n \in \mathbb{Z}$. We can choose a basis $\{v_n, v'_n, n \in \mathbb{Z}\}$ so that formulas (II.13) are verified for all n in $\mathbb{Z}$. The same arguments and a similar calculation as in case 1 B) and 2 B) lead to choose up to equivalence:

$$\begin{cases} \delta_n = \frac{1}{2}\left(a+n+\frac{3}{2}\right)\left(a+n+\frac{1}{2}\right)\left(a+n-\frac{1}{2}\right) \\ \gamma_n = \frac{1}{2}\left(a+n+\frac{1}{2}\right)\left(a+n-\frac{1}{2}\right)\left(a+n-\frac{3}{2}\right) \end{cases},$$

$$\begin{cases} \alpha_n = 2(a+n+3)(a+n+1)(a+n-1) \\ \beta_n = 2(a+n+1)(a+n-1)(a+n-3) \end{cases}. \tag{III.2.6}$$

We get a unique non-trivial admissible extension of $A(a, \frac{5}{2})$ by $A(a, -\frac{3}{2})$ defined by (II.13) and (III.2.6).

We can summarize the results of this paragraph as follows:

Theorem (III.2.7). *Let $A(a, \Lambda)$ and $A(a, \Lambda - p)$ $(p = 2, 3, 4)$ be two $\mathscr{V}$-modules of Feigin–Fuchs defined by* (I.1). *We have:*

1) *For $p = 2$, $\dim \mathscr{H}^1[\mathscr{V}, x_0, \mathrm{Hom}_{\mathbb{C}}(A(a, \Lambda - 2), A(a, \Lambda))] = 1$ $\forall \Lambda$, $\forall a$ and the cocycle is defined on x_1, x_{-1}, x_2, x_{-2} either by* (II.10) *for all n and* (III.2.1) *if $\Lambda \neq \frac{3}{2}$, or by* (II.13), (III.2.2) *if $\Lambda = \frac{3}{2}$.*
2) *For $p = 3$, $\dim \mathscr{H}^1[\mathscr{V}, x_0, \mathrm{Hom}_{\mathbb{C}}(A(a, \Lambda - 3), A(a, \Lambda))] = 1$ $\forall \Lambda$, $\forall a$ and the cocycle is defined on x_1, x_{-1}, x_2, x_{-2} either by* (II.10) *and* (III.2.3) *if $\Lambda \neq 2$, or by* (II.13), (III.2.4) *if $\Lambda = 2$.*
3) *$p = 4$, $\dim \mathscr{H}^1[\mathscr{V}, x_0, \mathrm{Hom}_{\mathbb{C}}(A(a, \Lambda - 4), A(a, \Lambda))] = 1$ $\forall \Lambda$, $\forall a$ and the cocycle is defined either by* (II.10) *and* (III.2.5) *if $\Lambda \neq \frac{5}{2}$ or by* (II.13), (III.2.6) *if $\Lambda = \frac{5}{2}$.*

III.3. Extensions of $A(a, 1)$ by $A(a, -4)$ and $A(a, 5)$ by $A(a, 0)$ $(a \neq 0)$.

Having two different values, Q_1 is diagonalisable on each $\mathscr{A}_{a+n}$, $\forall n \in \mathbb{Z}$, in these two cases. As $A(a, 1)$ and $A(a, 0)$ are equivalent $(a \neq 0)$, these two contragredient extensions are respectively equivalent to the extension of $A(a, 0)$ by $A(a, -4)$ and to the extension of $A(a, 5)$ by $A(a, 1)$. They are included in III.2, case 3. The case $a = 0$ is studied in Sect. IV.

III.4. Extension of $A\left(a, \frac{7+\varepsilon\sqrt{19}}{2}\right)$ by $A\left(a, \frac{-5+\varepsilon\sqrt{19}}{2}\right)$ $(\varepsilon = \pm 1)$.

Q_1 is always diagonalisable on each $\mathscr{A}_{a+n}$, $\forall n \in \mathbb{Z}$. The relations (II.10), (II.11) are defined for all n in $\mathbb{Z}$ with

$$\begin{cases} \alpha_n = \alpha_+\left(a+n+\frac{5-\varepsilon\sqrt{19}}{2}\right)\left(a+n+\frac{3-\varepsilon\sqrt{19}}{2}\right)\left(a+n+\frac{1-\varepsilon\sqrt{19}}{2}\right) \\ \qquad \times\left(a+n-\frac{1+\varepsilon\sqrt{19}}{2}\right) \\ \beta_n = -\alpha_+\left(a+n+\frac{1+\varepsilon\sqrt{19}}{2}\right)\left(a+n-\frac{1-\varepsilon\sqrt{19}}{2}\right)\left(a+n-\frac{3-\varepsilon\sqrt{19}}{2}\right) \\ \qquad \times\left(a+n-\frac{5-\varepsilon\sqrt{19}}{2}\right) \end{cases}.$$

Up to equivalence we can fix $\alpha_+ = 1$ and we have a unique non-trivial admissible extension $\mathscr{A}$ of $\mathscr{A}\left(a, \frac{7+\varepsilon\sqrt{19}}{2}\right)$ by $\mathscr{A}\left(a, \frac{-5+\varepsilon\sqrt{19}}{2}\right)$, $(\varepsilon = \pm 1)$, for each a.

IV. Non-Trivial Admissible Extensions $\mathscr{A}$ of an Irreducible $\mathscr{V}$-Module $A(0, \Lambda)$ ($\Lambda \neq 0, 1$) by $\mathscr{A}'$ (Where $\mathscr{A}' = \tilde{A}, \tilde{A} \oplus D(0), A_\alpha, B_\beta, A(0,1), A(0,0), D(0)$) and Their Contragredient $\mathscr{V}$-Modules

IV.1. Extensions of $A(0, \Lambda)$ ($\Lambda \neq 0, 1$) by $\mathscr{A}'$.

In the following, we suppose $\mathscr{A}'$ of type $\tilde{A}$ or $\tilde{A} \oplus D(0)$ or A_α or B_α, or $A(0,1)$ or $A(0,0)$ or $D(0)$. They are all the $\mathscr{V}$-modules with one-dimensional weightspaces, where $Q_1 = 0$.

In view of Proposition II.3 and Theorem II.9 we have the only following possibilities: $\Lambda = 2$ or $\Lambda = 3$ or $\Lambda = 4$ or $\Lambda = 5$. Thus Q_1 is diagonalisable on $\mathscr{A}_n$, $\forall n \in \mathbb{Z}$.

Case 1. Extensions of $A(0, \Lambda)$ ($\Lambda = 2, 3, 4, 5$) by $\tilde{A}$. In all cases, we can define a basis of $\mathscr{A}$, according to (II.10) and (II.11) by:

$$\begin{cases} x_1 v_n = (n+\Lambda) v_{n+1}, & \forall n, \\ x_{-1} v_n = (n-\Lambda) v_{n-1}, & \forall n, \\ x_2 v_n = (n+2\Lambda) v_{n+2}, & \forall n, \\ x_{-2} v_n = (n-2\Lambda) v_{n-2}, & \forall n, \end{cases} \qquad \begin{cases} x_1 v'_n = (n+1) v'_{n+1}, & \forall n \neq 0, \\ x_{-1} v'_n = (n-1) v'_{n-1}, & \forall n \neq 0, \\ x_2 v'_n = (n+2) v'_{n+2} + \alpha_n v_{n+2}, & \forall n \neq 0, -2, \\ x_{-2} v'_n = (n-2) v'_{n-2} + \beta_n v_{n-2}, & \forall n \neq 0, 2, \end{cases} \tag{IV.1.1}$$

where α_n (respectively β_n) is given by (II.11) for $n \geqq 1$ (respectively $n \geqq 3$) and by analogous formulas for $n \leqq -3$ (respectively $n \leqq -1$), with another constant α_- (respectively β_-).

- If $\Lambda = 2$, $\mathscr{A}$ is the direct sum $A(0,2) \oplus \tilde{A}$.
- If $\Lambda = 3, 4, 5$, let us set:

$$x_2 v'_{-2} = \alpha_{-2} v_0, \; x_{-2} v'_2 = \beta_2 v_0 . \tag{IV.1.2}$$

Writing the commutators $[x_1 x_{-2}]$, $[x_{-1} x_2]$ and $[x_{-2} x_2]$, we obtain: $\alpha_+ = \alpha_-$.

Up to equivalence, we can write (IV.1.1) and (IV.1.2) with:

- if $\Lambda = 3$ $\quad \alpha_n = -\beta_n = 1 \quad \forall n \neq 0$
- if $\Lambda = 4$ $\quad \alpha_n = n-1 \quad \beta_n = -(n+1) \quad \forall n \neq 0$
- if $\Lambda = 5$ $\quad \alpha_n = (n-2)(n-1) \quad \beta_n = -(n+2)(n+1) \quad \forall n \neq 0$. (IV.1.3)

We obtain a unique non-trivial admissible extension of $A(0, \Lambda)$ by $\tilde{A}$ for Λ: 3, 4, 5.

Case 2. Extensions of $A(0, \Lambda)$ ($\Lambda = 2, 3, 4, 5$) by $\tilde{A} \oplus D(0)$. All these extensions are reducible.

Case 3. Extensions of $A(0, \Lambda)$ ($\Lambda = 2, 3, 4, 5$) by A_α. We can use the results of case 1. If $\Lambda = 3, 4, 5$, we can choose a basis of $\mathscr{A}\{v_n, v'_n, n \in \mathbb{Z}\}$ such that the formulas

(IV.1.1), (IV.1.2) and (IV.1.3) are verified. Now, we must add the following relations:

$$\begin{cases} x_1 v'_0 = (1+\alpha)v'_1 \\ x_{-1}v'_0 = (1-\alpha)v'_{-1} \end{cases} \quad \begin{cases} x_2 v'_0 = 2(2+\alpha)v'_2 + \alpha_0 v_2 \\ x_{-2}v'_0 = 2(2-\alpha)v'_{-2} + \beta_0 v_{-2} \end{cases} \tag{IV.1.4}$$

We apply the commutators $[x_{-1}, x_2]$, $[x_1, x_{-2}]$, $[x_2, x_{-2}]$ on v'_0. For $\Lambda = 5$, we only get a reducible $\mathscr{V}$-module. For $\Lambda = 3, 4$ we get:

$$\alpha_0 = \alpha - 1, \quad \beta_0 = -(\alpha + 1). \tag{IV.1.5}$$

Thus for $\Lambda = 3, 4$ we have, up to equivalence, a unique non-trivial admissible extension $\mathscr{A}$ of $A(0, \Lambda)$ by A_α defined by the formulas ((IV.1.1) → (IV.1.5)).

For $\Lambda = 2$ from case 1 and Proposition (II.3), we can also look at $\mathscr{A}$ as an extension of $\tilde{A}$ by the affine $\mathscr{V}$-module $\mathscr{F}$. Up to equivalence, this extension is defined on a basis $\{v_n, v'_n, n \in \mathbb{Z}\}$ of $\mathscr{A}$ as follows:

$$\begin{cases} x_i v_n = (n+2i)v_{n+i} \quad \forall n, \forall i \\ x_i v'_n = (n+i)v'_{n+i} \quad \forall i, \forall n \text{ with } n+i \neq 0; n \neq 0 \\ x_1 v'_0 = (1+\alpha)v'_1, x_{-1}v'_0 = (1-\alpha)v'_{-1} \\ x_2 v'_0 = 2(2+\alpha)v'_2 + 2v_2; x_{-2}v'_0 = 2(2-\alpha)v'_{-2} - 2v_{-2}. \end{cases} \tag{IV.1.6}$$

Case 4. Extensions of $A(0, \Lambda)$ $(\Lambda = 2, 3, 4, 5)$ by $A(0, 1)$. For $\Lambda = 3, 4, 5$, this case is included in III.2 for $\Lambda - p = 1$ and $p = 2, 3, 4$. If $\Lambda = 2$ we obtain, as in the previous case, an extension of $\tilde{A}$ by the affine $\mathscr{V}$-module $\mathscr{F}$. Up to equivalence, we can define a basis of this extension $\mathscr{A}$ by the formulas (IV.1.5) except:

$$\begin{cases} x_1 v'_0 = v'_1 \\ x_2 v'_0 = 2v'_2 + v_2 \end{cases} \quad \begin{cases} x_{-1}v'_0 = -v'_{-1} \\ x_{-2}v'_0 = -2v'_{-2} - v_{-2} \end{cases}.$$

Case 5. Extensions of $A(0, \Lambda)$ $(\Lambda \neq 0, 1)$ by B_β. If $\Lambda = 3, 4, 5$, Proposition (II.3) implies that $A(0, \Lambda) \oplus D(0)$ is a $\mathscr{V}$-submodule of $\mathscr{A}$. From case 1, for each of these values of Λ and each β, we have a unique, non-trivial, admissible extension of $A(0, \Lambda)$ by B_β. It is defined on a basis $\{v_n, v'_n, n \in \mathbb{Z}\}$ by the formulas (IV.1.1) and (IV.1.3) except $x_2, v'_2, x_{-2}v'_2, x_1 v'_1, x_{-1}v'_1$ given by:

$$x_1 v'_{-1} = (\beta+1)v'_0, \quad x_2 v'_{-2} = (\beta+2)v'_0 + \alpha_{-2}v_0,$$

$$x_{-1}v'_1 = (\beta-1)v'_0, \quad x_{-2}v'_2 = (\beta-2)v'_0 + \beta_2 v_0,$$

$$x_i v'_0 = 0,$$

where α_{-2} and β_2 also satisfy (IV.1.3). If $\Lambda = 2$, we only get the direct sum $A(02) \oplus B_\beta$.

Case 6. Extension of $A(0, \Lambda)$ $(\Lambda \neq 0, 1)$ by $A(0, 0)$. If $\Lambda = 2, 3, 4$ this case is included in (III.2) for $\Lambda - p = 0$ and $p = 2, 3, 4$. If $\Lambda = 5$, Proposition (II.3) implies the existence of the submodule $A(0, 5) \oplus D(0)$ in $\mathscr{A}$. Thus $\mathscr{A}$ i's an extension of $A(0, 5) \oplus D(0)$ by $\tilde{A}$. From case 1, we obtain a unique extension $\mathscr{A}$, which is defined by:

$$\begin{cases} x_i v_n = (n+5i)v_{n+i} \quad \forall n, \forall i \in \mathbb{Z} & \\ x_1 v'_n = n v'_{n+1} & x_2 v'_n = n v'_{n+2} + \alpha_n v_{n+2} \quad \forall n \\ x_{-1} v'_n = n v'_{n-1} & x_{-2} v'_n = n v'_{n-2} + \beta_n v_{n-2} \quad \forall n \end{cases}$$

with $\alpha_n = -\beta_n = n(n-1)(n-2)\ \forall n$.

Case 7. Extensions of $A(0,\Lambda), (\Lambda = 2, 3, 4, 5)$ by $D(0)$. Recall that there exists a unique extension of $A(0,2)$ by $D(0)$ denoted by $\mathscr{F}$, given by Proposition (II.3).

IV.2. Extensions of $\mathscr{A}'$ by $A(0,\Lambda)(\Lambda \neq 0, 1)$.

$\mathscr{A}'$ is always either $\tilde{A}$, or $\tilde{A} \oplus D(0)$, or A_α, or B_β or $A(0,1)$ or $A(0,0)$ or $D(0)$. In view of Property (II.1), these extensions are necessarily exactly all the contragredient $\mathscr{V}$-modules of the preceding ones (Sect. IV.1).

Proposition II.3 and Theorem II.9 imply the only following possibilities for Λ:

$$\Lambda = -1, \quad \Lambda = -2, \quad \Lambda = -3, \quad \Lambda = -4 .$$

Case 1. Extensions of $\tilde{A}$ by $A(0,\Lambda)(\Lambda = -1, -2, -3, -4)$. For $\Lambda = -2$ or -3 or -4, we have unique non-trivial admissible extensions $\mathscr{A}$, contragredient of those defined in IV.1, case 1, for $\Lambda = 3$ or 4 or 5. Up to equivalence, $\mathscr{A}$ is defined on a basis $\{v_n, v'_n, n \in \mathbb{Z}\}$ by:

$$x_i v_n = (n+i)v_{n+i} \quad \text{if } n+i \neq 0$$

$$\begin{cases} x_1 v'_n = (n+\Lambda)v'_{n+1} \\ x_{-1} v'_n = (n-\Lambda)v'_{n-1} \end{cases} \begin{cases} x_2 v'_n = (n+2\Lambda)v'_{n+2} + \alpha_n v_{n+2} \\ x_{-2} v'_n = (n-2\Lambda)v'_{n-2} + \beta_n v_{n-2} \end{cases}, \tag{IV.2.1}$$

where

$$\begin{cases} \bullet \text{ if } \Lambda = -2: \alpha_n = n+2, \beta_n = -(n-2) \quad \forall n \\ \bullet \text{ if } \Lambda = -3: \alpha_n = (n+2)(n+3), \beta_n = -(n-2)(n-3) \quad \forall n \\ \bullet \text{ if } \Lambda = -4: \alpha_n = (n+4)(n+3)(n+2), \beta_n = -(n-4)(n-3)(n-2) \quad \forall n . \end{cases} \tag{IV.2.2}$$

Case 2. Extensions of $\tilde{A} \oplus D(0)$ by $A(0,\Lambda)$ $(\Lambda = -1, -2, -3, -4)$. In view of (IV.1), case 2, there is no indecomposasble admissible $\mathscr{V}$-module $\mathscr{A}$, extension of $\tilde{A} \oplus D(0)$ by $A(0,\Lambda)(\Lambda \neq 0, 1)$.

Case 3. Extensions $\mathscr{A}$ of B_β by $A(0,\Lambda)(\Lambda = -1, -2, -3, -4)$. In view of (IV.1) case 3, $\mathscr{A}$ is indecomposable if and only if $\Lambda = -1$ or -2, or -3. Up to equivalence, we can choose a basis $\{v_n, v'_n\}$ of $\mathscr{A}$ such that

$$\begin{cases} x_1 v_n = (n+1)v_{n+1}, n \neq 0, -1 \\ x_1 v_{-1} = (\beta+1)v_0 \\ x_1 v_0 = 0 \end{cases} \quad \begin{cases} x_{-1} v_n = (n-1)v_{n-1}, n \neq 0, 1 \\ x_{-1} v_1 = (\beta-1)v_0 \\ x_{-1} v_0 = 0 \end{cases}$$

$$\begin{cases} x_2 v_n = (n+2) v_{n+2}, \ n \neq 0, -2 \\ x_2 v_{-2} = (\beta + 2) v_0 \\ x_2 v_0 = 0 \end{cases} \qquad \begin{cases} x_{-2} v_n = (n-2) v_{n-2}, \ n \neq 0, 2 \\ x_{-2} v_2 = (\beta - 2) v_0 \\ x_{-2} v_0 = 0 \end{cases}$$

$$\begin{cases} x_1 v'_n = (n + \Lambda) v'_{n+1} \\ x_{-1} v'_n = (n - \Lambda) v'_{n-1} \end{cases} \qquad \begin{cases} x_2 v'_n = (n + 2\Lambda) v'_{n+2} + \alpha_n v_{n+2} \\ x_{-2} v'_n = (n - 2\Lambda) v'_{n-2} + \beta_n v_{n-2} \end{cases},$$

where

- if $\Lambda = -1$ $\quad \alpha_n = 0 \quad \forall n \neq -2, \alpha_{-2} = 1$,

  $\beta_n = 0 \quad \forall n \neq 2, \beta_2 = -1$,
- if $\Lambda = -2$ $\quad \alpha_n = (n+2) \quad \forall n \neq -2, \alpha_{-2} = \beta - 1$,

  $\beta_n = -(n-2) \quad \forall n \neq 2, \beta_2 = -(\beta + 1)$,
- if $\Lambda = -3$ $\quad \alpha_n = (n+2)(n+3) \ n \neq -2, \alpha_{-2} = \beta - 1$

  $\beta_n = -(n-2)(n-3) \ n \neq 2, \beta_2 = -(\beta + 1)$.

Remark. We can also consider the case $\Lambda = -1$ as an extension of the affine $\mathscr{V}$-module $\mathscr{F}^*$ (Prop. II.3) by the $\mathscr{V}$-module $\tilde{A}$.

Case 4. Extensions $\mathscr{A}$ of $A(0,0)$ by $A(0,\Lambda)$ ($\Lambda = -1, -2, -3, -4$). The cases $\Lambda = -2, \Lambda = -3, \Lambda = -4$ are included in III.2. If $\Lambda = -1$ we obtain as in the previous case another extension of the affine $\mathscr{V}$-module $\mathscr{F}^*$ by $\tilde{A}$, defined up to equivalence by:

$$x_i v_n = n v_{n+i}, \quad \forall n, \forall i$$

$$\begin{cases} x_1 v'_n = (n-1) v'_{n+1} & \forall n \\ x_{-1} v'_n = (n+1) v'_{n-1} & \forall n \end{cases} \qquad \begin{cases} x_2 v'_n = (n-2) v'_{n+2} & n \neq -2 \\ x_{-2} v'_n = (n+2) v'_{n-2} & n \neq 2 \\ x_2 v'_{-2} = -4 v'_0 + v_0 & \\ x_{-2} v'_2 = 4 v'_0 + v_0 & \end{cases}$$

Case 5. Extensions $\mathscr{A}$ of A_α by $A(0,\Lambda)$ ($\Lambda = -1, -2, -3, -4$). In view of (IV.1) case 5 and Proposition II.3, if $\Lambda = -2$ or -3, or -4 we have an extension of $\tilde{A}$ by $D(0) \oplus A(0,\Lambda)$. For each value of Λ and each α we get a unique indecomposable admissible $\mathscr{V}$-module $\mathscr{A}$ defined on a basis $\{v_n, v'_n, n \in \mathbb{Z}\}$ by the formulas (IV.2.1) and (IV.2.2) and:

$$x_1 v_0 = (1+\alpha) v_1, \qquad x_2 v_0 = 2(2+\alpha) v_2,$$

$$x_{-1} v_0 = (1-\alpha) v_{-1}, \quad x_{-2} v_0 = 2(2-\alpha) v_{-2}.$$

If $\Lambda = -1$ in view of (IV.1) case 5, $\mathscr{A}$ is necessarily the direct sum $A_\alpha \oplus A(0, -1)$.

Case 6. Extensions $\mathscr{A}$ of $A(0,1)$ by $A(0,\Lambda)$ ($\Lambda = -1, -2, -3, -4$). The cases $\Lambda = -1$ or $\Lambda = -2$ or $\Lambda = -3$ are included in III.2. For $\Lambda = -4$, $\mathscr{A}$ can be looked as an extension of $\tilde{A}$ by $D(0) \oplus A(0, -4)$ (Prop. III.3). From (IV.2) case 1, we obtain a unique non-trivial admissible extension $\mathscr{A}$, which is defined by the formulas (IV.2.1) and (IV.2.2) and:

$$x_1 v_0 = v_1 \qquad x_2 v_0 = 2v_2$$

$$x_{-1} v_0 = -v_{-1} \qquad x_{-2} v_0 = -2v_{-2}$$

Case 7. Extensions of $D(0)$ by $A(0, \Lambda)$ ($\Lambda = -1, -2, -3, -4$).
Recall that, from Proposition (II.3), there exists a unique extension of $D(0)$ by $A(0, -1)$ which is the contragredient $\mathscr{V}$-module $\mathscr{F}^*$ of $\mathscr{F}$ (case 7 of IV.1).

Now we can summarize the results of Sect. IV:

Theorem IV.3. *Set* $\mathscr{A}' = D(0), \tilde{A}, \tilde{A} \oplus D(0), A_\alpha, A(0,1), B_\beta, A(0,0)$.

a) *The only non-trivial admissible extensions of* $A(0, \Lambda)$ ($\Lambda \neq 0, 1$) *by* $\mathscr{A}'$ *are the unique following ones*:

- $\mathscr{A}' = D(0)$ *and* $\Lambda = 2$
- $\mathscr{A}' = \tilde{A}$ *and* $\Lambda = 3, 4, 5$
- $\mathscr{A}' = A_\alpha$ *and* $\Lambda = 2, 3, 4$
- $\mathscr{A}' = B_\beta$ *and* $\Lambda = 3, 4, 5$
- $\mathscr{A}' = A(0, 1)$ *or* $A(0, 0)$ *and* $\Lambda = 2, 3, 4, 5$.

b) *The only non-trivial admissible extensions of* $\mathscr{A}'$ *by* $A(0, \Lambda)$ *are the contragredient extensions of the previous ones.*

V. Indecomposable Admissible $\mathscr{V}$-Modules $\mathscr{A} = \bigoplus_{n \in \mathbb{Z}} \mathscr{A}_n$ such that $\dim \mathscr{A}_n \leqq 2\ \forall n$, $Sp(x_0) = \mathbb{Z}$ and $Q_1^2 = 0$

A $\mathscr{V}$-module $\mathscr{A} = \bigoplus_{n \in \mathbb{Z}} \mathscr{A}_n$ with $\dim \mathscr{A}_n = 1\ \forall n$ and $Q_1 = 0$ may be $D(0) \oplus \tilde{A}$, $A(0, 1)$, A_α, $A(0, 0)$, B_β. If $\mathscr{A}$ contains a trivial $\mathscr{V}$-submodule $D(0)$, it is $D(0) \oplus \tilde{A}$, $A(0, 0)$ or B_β. In other cases, namely $D(0) \oplus \tilde{A}$, $A(0, 1)$, A_α, $\mathscr{A}$ contains an irreducible $\mathscr{V}$-module $\tilde{\mathscr{A}}$. In order to be able to discuss at once the three first cases or the three other ones, we use the following notations:

$$1.\ \begin{cases} x_1 v_0 = 0 \\ x_{-1} v_0 = 0 \end{cases} \begin{cases} x_2 v_0 = 0 \\ x_{-2} v_0 = 0 \end{cases} \begin{cases} x_1 v_{-1} = \delta_{-1} v_0 \\ x_{-1} v_1 = \gamma_1 v_0 \end{cases} \begin{cases} x_2 v_{-2} = \frac{1}{2}(3\delta_{-1} - \gamma_1) v_0 \\ x_{-2} v_2 = \frac{1}{2}(-\delta_{-1} + 3\gamma_1) v_0 \end{cases}$$

with $\delta_{-1} = \gamma_1 = 0$ for $D(0) \oplus \tilde{A}$,

$\delta_{-1} = \gamma_1 = 1$ for $A(0, 0)$,

$\delta_{-1} = \beta + 1$ $\quad \gamma_1 = \beta - 1$ for B_β.

$$2.\ \begin{cases} x_1 v_0 = \delta_0 v_1 \\ x_{-1} v_0 = \gamma_0 v_{-1} \end{cases} \begin{cases} x_2 v_0 = (3\delta_0 + \gamma_0) v_2 \\ x_{-2} v_0 = (\delta_0 + 3\gamma_0) v_{-2} \end{cases} \begin{cases} x_1 v_{-1} = 0 \\ x_{-1} v_1 = 0 \end{cases} \begin{cases} x_2 v_{-2} = 0 \\ x_{-2} v_2 = 0 \end{cases}$$

with $\delta_0 = \gamma_0 = 0$ for $D(0) \oplus \tilde{A}$,

$\delta_0 = -\gamma_0 = 1$ for $A(0, 1)$,

$\delta_0 = 1 + \alpha$ $\quad \gamma_0 = 1 - \alpha$ for A_α.

V.1. Indecomposable admissible $\mathscr{V}$*-modules* $\mathscr{A} = \bigoplus_{n \in \mathbb{Z}} \mathscr{A}_n$ *such that* $\dim \mathscr{A}_n = 1$ $\forall n \neq 0$. We are interested here in affine $\mathscr{V}$-modules: $\dim \mathscr{A}_n = 1$ $\forall n \neq 0$,

$\dim \mathscr{A}_0 = 2$. For all $n \neq 0$, $\{v_n\}$ will be a basis of $\mathscr{A}_n$ and $\{v_0, v'_0\}$ a basis of $\mathscr{A}_0$. Let us first recall that we already got in part II (Proposition II.3) two inequivalent affine $\mathscr{V}$-modules with $Sp(x_0) = \mathbb{Z}$ and $Q_1(Q_1 - 2) = 0$. They are the extension $\mathscr{F}$ of $A(0, 2)$ by $D(0)$ and its contragredient $\mathscr{V}$-module $\mathscr{F}^*$.

From Proposition (II.3), we deduce that all other affine $\mathscr{V}$-modules verify $Sp(x_0) = \mathbb{Z}$ and $Q_1^2 = 0$. Thus we shall get the complete classification of affine $\mathscr{V}$-modules after the following discussion according to the three assumptions:

(a) $x_1 v_{-1}$ and $x_{-1} v_1$ are independent vectors,
(b) $x_1 v_{-1} = x_{-1} v_1 = 0$,
(c) $x_1 v_{-1}$ and $x_{-1} v_1$ are dependent vectors which are not both equal to zero.

(a) $x_1 v_{-1}$ and $x_{-1} v_1$ are independent vectors. We get an indecomposable affine $\mathscr{V}$-module defined by the relations:

$$\begin{cases} x_i v_j = (i + j) v_{i+j} & \forall j \neq 0 \quad \text{and} \quad i + j \neq 0\,, \\ x_i v_0 = 0 & \forall i\,, \\ x_i v'_0 = 0 & \forall i\,, \\ x_i v_{-i} = (1 + i) v_0 + (1 - i) v'_0 & \forall i \neq 0\,, \end{cases}$$

where we have $c v'_0 = 0$.

(b) $x_1 v_{-1} = x_{-1} v_1 = 0$. We get an indecomposable affine $\mathscr{V}$-modules defined by the relations:

$$\begin{cases} x_i v_j = (i + j) v_{i+j} & \forall j \neq 0\,, \\ x_i v_0 = i(i + 1) v_i & \forall i\,, \\ x_i v'_0 = i(i - 1) v_i & \forall i\,, \end{cases}$$

where we have $c v'_0 = 0$.

(c) $x_1 v_{-1}$ and $x_{-1} v_1$ are dependent vectors which are not both equal to zero. It appears that three cases may occur:

– The $\mathscr{V}$-submodule generated by v_1 is B_β and the quotient $\mathscr{V}$-module $\mathscr{A}/\{v_0\}$ is $A_{1/\beta}$, $\beta \neq 0$. For each $\beta \neq 0$ we get a unique indecomposable affine $\mathscr{V}$-module defined by the relations:

$$\begin{cases} x_i v_j = (i + j) v_{i+j} & \forall j \neq 0 \quad \text{and} \quad i + j \neq 0\,, \\ x_i v_{-i} = (\beta + i) v_0 & \forall i\,, \\ x_i v_0 = 0 & \forall i\,, \\ x_i v'_0 = i\left(\dfrac{1}{\beta} + i\right) v_i & \forall i\,, \end{cases}$$

where we have $c v'_0 = -24 v_0$.

– The $\mathscr{V}$-submodule generated by v_1 is B_0 and the quotient $\mathscr{V}$-module $\mathscr{A}/\{v_0\}$ is $A(0, 1)$. We get a unique indecomposable affine $\mathscr{V}$-module defined by the relations:

$$\begin{cases} x_i v_j = (i+j) v_{i+j} & \forall j \neq 0 \quad \text{and } i+j \neq 0 \\ x_i v_{-i} = i v_0 & \forall i\,, \\ x_i v_0 = 0 & \forall i\,, \\ x_i v'_0 = i v_i & \forall i\,, \end{cases}$$

where we have $cv'_0 = 0$.

– The $\mathscr{V}$-submodule generated by v_1 is $A(0,0)$ and the quotient $\mathscr{V}$-module $\mathscr{A}/\{v_0\}$ is A_0. We get a unique indecomposable affine $\mathscr{V}$-module defined by the relations:

$$\begin{cases} x_i v_j = (i+j) v_{i+j} & \forall j \neq 0 \quad \text{and } i+j \neq 0\,, \\ x_i v_{-i} = v_0 & \forall i\,, \\ x_i v'_0 = i^2 v_i & \forall i\,, \\ x_i v_0 = 0 & \forall i\,, \end{cases}$$

where we have $cv'_0 = 0$.

Proposition V.1.1. *Any affine* $\mathscr{V}$*-module is one of the following*:

1) *the* $\mathscr{V}$*-module* $\mathscr{F}$ *or* $\mathscr{F}^*$;

2) *the unique extension of* $D(0) \oplus D(0)$ *by* $\tilde{A}$ *which can be looked at as the extension of* $D(0)$ *by* $A(0,0)$ *or by* B_0 *or its contragredient* (*case* V.1 (a) *and* (b));

3) *the unique extension of* $A(0,0)$ *by* $D(0)$ *which can be also looked at as the extension of* $D(0)$ *by* A_0 (*third subcase of case* V.1.(c)) *or its contragredient* (*second subcase of case* V.1.(c));

4) *the unique extension of* B_β *by* $D(0)$ ($\beta \neq 0$) *which can be also looked at as the extension of* $D(0)$ *by* $A_{1/\beta}$ (*first subcase of case* V.1.(c)).

We have $c = 0$ *in case* 1), 2), 3) *and* $c \neq 0$ (*but* $c^2 = 0$) *in case* 4).

V.2. Asymptotic relations for all $\mathscr{V}$*-modules* $\mathscr{A} = \bigoplus_{n \in \mathbb{Z}} \mathscr{A}_n$ *such that* $Sp(x_0) = \mathbb{Z}$, $Q_1^2 = 0$ *and* $\dim \mathscr{A}_n = 2\ \forall n \neq 0$. In all cases, there exists a $\mathscr{V}$-submodule with an asymptotic dimension one which may be $\tilde{A}$, $\tilde{A} \oplus D(0)$, $A(0,1)$, A_α, $A(0,0)$, B_β or an affine $\mathscr{V}$-module containing $D(0)$ (V.1) and the corresponding factor $\mathscr{V}$-module is also one of these $\mathscr{V}$-modules. Thus, from Remarks (I.8.c)) and Sect. (V.1), we can choose a basis $\{v_n, v'_n\}$ of $\mathscr{A}_n$, $\forall n \in \mathbb{Z}$, such that:

$$\begin{cases} x_1 v_n = (n+1) v_{n+1} & \forall n \neq -1, 0 \\ x_1 v'_n = (n+1) v'_{n+1} + \delta_n v_{n+1} & \forall n \neq -1, 0 \\ x_{-1} v_n = (n-1) v_{n-1} & \forall n \neq 0, 1 \\ x_{-1} v'_n = (n-1) v'_{n-1} + \gamma_n v_{n-1} & \forall n \neq 0, 1 \end{cases}$$

$$\begin{cases} x_2 v_n = (n+2) v_{n+2} & \forall n \neq -2, 0 \\ x_2 v'_n = (n+2) v'_{n+2} + \alpha_n v_{n+2} & \forall n \neq -2, 0 \\ x_{-2} v_n = (n-2) v_{n-2} & \forall n \neq 0, 2 \\ x_{-2} v'_n = (n-2) v'_{n-2} + \beta_n v_{n-2} & \forall n \neq 0, 2\,. \end{cases} \tag{V.2.1}$$

From the relation $[x_{-1}x_1](v'_n) = 2x_0(v'_0)$, we deduce that there exist two constants ε_+ and ε_- such that:

$$n\delta_n + (n+1)\gamma_{n+1} = \varepsilon_+ \quad \forall n \geqq 1\,,$$

$$n\delta_n + (n+1)\gamma_{n+1} = \varepsilon_- \quad \forall n \leqq -2\,.$$

For fixed vectors v'_1 and v'_{-1}, we can choose v'_n $\forall n \neq 0$ such that: $\delta_n = \varepsilon_+$, $\gamma_n = -\varepsilon_+$ $\forall n > 0$ and $\delta_n = \varepsilon_- \gamma_n = -\varepsilon_-$ $\forall n < -1$. From the relations $[x_{-1}x_2](v'_n) = 3x_1(v'_n)$ and $[x_{-2}x_1]\,(v'_n) = 3x_{-1}(v'_n)$ we deduce the existence of a constant α_+ such that:

$$\alpha_n = 2\varepsilon_+ + \frac{\alpha_+}{n(n+1)} \quad \forall n \geqq 1, \qquad \beta_n = -2\varepsilon_+ - \frac{\alpha_+}{n(n-1)} \quad \forall n \geqq 3\,.$$

A similar calculation gives a constant α_- such that:

$$\alpha_n = 2\varepsilon_+ + \frac{\alpha_-}{n(n+1)} \quad \forall n \leqq -3, \quad \beta_n = -2\varepsilon_+ - \frac{\alpha_-}{n(n-1)} \quad \forall n \leqq -1\,.$$

Writing now the relations: $[x_{-2}x_2]\,(v'_n) = 4x_0(v'_n) + \frac{1}{2}c(v'_n)$ $\forall n \neq -2, 0, 2$ as we know from Theorem (I.2) that $cv'_n = 0$, we conclude that necessarily $\varepsilon_+ = \varepsilon_- = \varepsilon$.

As $Q_1 v'_n = \varepsilon v_n$ $\forall n \neq 0$ we see here that in all cases Q_1 is simultaneously diagonalisable or non-diagonalisable on all $\mathscr{A}_n$, $n \neq 0$. Up to equivalence we can suppose $\varepsilon = 0$ or $\varepsilon = 1$.

V.3. Indecomposable admissible $\mathscr{V}$-modules $\mathscr{A} = \oplus_{n\in\mathbb{Z}} \mathscr{A}_n$ such that $\dim \mathscr{A}_n = 2$ $\forall n \neq 0$, *and* $\dim \mathscr{A}_0 = 1$ *and* $Q_1^2 = 0$. Let us first recall that we already got in part (IV) six indecomposable $\mathscr{V}$-modules satisfying $\dim \mathscr{A}_n = 2\,\forall n \neq 0$ and $\dim \mathscr{A}_0 = 1$. They verify the equations $Q_1(Q_1 - 6) = 0$, $Q_1(Q_1 - 12) = 0$ and $Q_1(Q_1 - 20) = 0$.

All other indecomposable $\mathscr{V}$-modules such that $\dim \mathscr{A}_n = 2$ $\forall n \neq 0$ and $\dim \mathscr{A}_0 = 1$ satisfy $Q_1^2 = 0$. We construct them as follows.

Let $\{v_0\}$ be a basis of $\mathscr{A}_0$ and let us discuss according to the following assumptions:

(a) $\begin{cases} x_1 v_0 \neq 0 \\ x_{-1} v_0 \neq 0 \\ x_{-2}(x_1 v_0) = \lambda x_{-1} v_0 \quad \lambda \in \mathbb{C} \end{cases}$ (b) $\begin{cases} x_1 v_0 \neq 0 \\ x_{-1} v_0 \neq 0 \\ x_{-2}(x_1 v_0) \text{ and } x_{-1} v_0 \text{ are independent vectors} \end{cases}$

(c) $\begin{cases} x_1 v_0 = 0 \\ x_{-1} v_0 \neq 0 \end{cases}$ (d) $\begin{cases} x_1 v_0 \neq 0 \\ x_{-1} v_0 = 0 \end{cases}$ (e) $\begin{cases} x_1 v_0 = 0 \\ x_{-1} v_0 = 0 \end{cases}$.

Obviously, these different assumptions will furnish a complete classification of such $\mathscr{V}$-modules and each one leads to $\mathscr{V}$-modules which cannot be isomorphic to the others.

(a) The $\mathscr{V}$-submodule generated by v_0 may be A_α ($\alpha \neq \pm 1$) or $A(0, 1)$. We must add to the relations (V.2.1) the following relations:

$$\begin{cases} x_1 v_0 = \delta_0 v_1 \\ x_{-1} v_0 = \gamma_0 v_{-1} \end{cases} \begin{cases} x_2 v_0 = (3\delta_0 + \gamma_0) v_2 \\ x_{-2} v_0 = (\delta_0 + 3\gamma_0) v_{-2} \end{cases} \begin{cases} x_1 v'_{-1} = \delta_{-1} v_0 \\ x_{-1} v'_1 = \gamma_1 v_0 \end{cases} \begin{cases} x_2 v'_{-2} = \alpha_{-2} v_0 \\ x_{-2} v'_2 = \beta_0 v_0 \end{cases} . \tag{V.3.1}$$

Writing the commutators which were not calculated in the previous asymptotic discussion, it appears that Q_1 must be asymptotically non-diagonalisable: $\varepsilon = 1$. We get two indecomposable $\mathscr{V}$-modules:

(i) the extension of $\tilde{A}$ by $A(00)$: $\delta_0 = \gamma_0 = 1$,

$$\gamma_1 = \delta_{-1} = \alpha_{-2} = \beta_2 = -1, \quad \alpha_+ = \alpha_- = -2, \quad \alpha_{-1} = -\beta_1 = 2 .$$

(ii) the extension of $A(01)$ by $\tilde{A}$: $\delta_0 = -\gamma_0 = 1$,

$$\gamma_1 = -\delta_{-1} = -1, \quad \alpha_{-2} = -\beta_2 = 2, \quad \alpha_+ = \alpha_- = 0, \quad \alpha_{-1} = -\beta_1 = 2 .$$

(b) $x_{-1}(x_1 v_0) = 0$ and $x_1(x_{-1} v_0) = 0$. We get a unique indecomposable $\mathscr{V}$-module, extension of $\tilde{A} \oplus \tilde{A}$ by $D(0)$ (or $\tilde{A}$ by A_1 or $\tilde{A}$ by A_{-1}) which is defined by the relations:

$$\begin{cases} x_i v_j = (i+j) v_{i+j} & \forall j \neq 0 , \\ x_i v'_j = (i+j) v'_{i+j} & \forall j \neq 0 , \\ x_i v_0 = i(i+1) v'_i + i(i-1) v_i & \forall i , \end{cases}$$

and we have $cv_i = 0 \ \forall i$, $cv'_i = 0 \ \forall i$.

(c) and (d): These two cases lead to reducible $\mathscr{V}$-modules.

(e) There exists v_1 and v_{-1} such that $x_1 v_{-1} = x_{-1} v_1 = 0$. We get a unique indecomposable $\mathscr{V}$-module extension of $D(0)$ by $\tilde{A} \oplus \tilde{A}$ (or B_1 by $\tilde{A}$ or B_{-1} by $\tilde{A}$) which is defined by the relations:

$$\begin{cases} x_i v_j = (i+j) v_{i+j} & \forall i+j \neq 0 , \\ x_i v'_j = (i+j) v'_{i+j} & \forall i+j \neq 0 , \\ x_i v_0 = 0 & \forall i , \\ x_i v_{-1} = (i+1) v_0 & \forall i , \\ x_i v'_{-i} = (i-1) v_0 & \forall i , \end{cases}$$

and we have $cv_i = cv'_i = 0 \ \forall i$.

Proposition (V.3.2). *Any indecomposable admissible* $\mathscr{V}$*-module* $\mathscr{A} = \bigoplus \mathscr{A}_n$ *such that* $\dim \mathscr{A}_n = 2$, $\forall n \in \mathbb{Z}^*$ *and* $\dim \mathscr{A}_0 = 1$, *is one of the following*:

1) *The unique extension of* $A(0, \Lambda)$ (*for* $\Lambda = 3$ *or* $\Lambda = 4$ *or* $\Lambda = 5$) *by* $\tilde{A}$ *or its contragredient.*

2) *The unique extension of* $\tilde{A}$ *by* $A(0, 0)$ *which can also be looked at as the extension of* A_0 *by* $\tilde{A}$ (*case* V.3.(a) (i)) *or its contragredient* (*case* V.3.(a)$'$(ii)).

3) *The unique extension of* $\tilde{A}$ *by* A_1 *which can also be looked at as the extension of* $\tilde{A}$ *by* A_{-1} (*case* V.3.(b)) *or its contragredient* (*case* V.3.(e)).

V.4. Indecomposable admissible $\mathscr{V}$*-modules* $\mathscr{A} = \bigoplus_{n\in\mathbb{Z}} \mathscr{A}_n$ *such that* $\dim \mathscr{A}_n = 2$ $\forall n \in \mathbb{Z}$. This case will be discussed according to the following properties of the $\mathscr{V}$-submodule $\mathscr{A}' = \bigoplus_{n\in\mathbb{Z}^*} \mathscr{A}_n \oplus \mathscr{A}'_0$ generated by $\mathscr{A}_1$:

a) $\dim \mathscr{A}'_0 = 0$,
b) $\dim \mathscr{A}'_0 = 2$ and $\mathscr{A}'_0$ does not contain a trivial $\mathscr{V}$-submodule $D(0)$,
c) $\dim \mathscr{A}'_0 = 2$ and $\mathscr{A}'_0$ contains exactly one trivial $\mathscr{V}$-submodule $D(0)$,
d) $\dim \mathscr{A}'_0 = 2$ and $\mathscr{A}'_0$ is a direct sum of two trivial $\mathscr{V}$-submodules,
e) $\dim \mathscr{A}'_0 = 1$ and $\mathscr{A}'_0$ does not contain any trivial $\mathscr{V}$-submodule,
f) $\dim \mathscr{A}'_0 = 1$ and $\mathscr{A}'_0$ is a trivial $\mathscr{V}$-submodule.

Evidently, these different assumptions furnish a complete classification of such $\mathscr{V}$-modules and each one leads to indecomposable $\mathscr{V}$-modules which are not isomorphic to the others.

(a) Dim $\mathscr{A}'_0 = 0$: the $\mathscr{V}$-module $\mathscr{A}'$ is $\tilde{A} \oplus \tilde{A}$

– Suppose first that any vector of $\mathscr{A}_0$ is such that $x_{-1}v_0$ and $x_{-2}(x_1 v_0)$ are dependent vectors. Then the $\mathscr{V}$-module $\mathscr{A}$ is reducible.
– Suppose now that there exists $v_0 \in \mathscr{A}_0$ such that $x_{-1}v_0$ and $x_{-2}(x_1 v_0)$ are independent vectors. The $\mathscr{V}$-submodule generated by v_0 is the indecomposable $\mathscr{V}$-module which we got in (V.3.b). The corresponding factor $\mathscr{V}$-module is $D(0)$. Let $\{v_0, v'_0\}$ be a basis of $\mathscr{A}_0$ and set:

$$x_1 v'_0 = \delta_0 v_1 + \delta'_0 v'_1 \,,$$

$$x_{-1} v'_0 = \gamma_0 v_{-1} + \gamma'_0 v'_{-1} \,.$$

We can choose v'_0 such that $\delta'_0 = 0$.

A necessary condition to get an indecomposable $\mathscr{V}$-module is: $\gamma_0^2 + 4\delta_0\gamma'_0 = 0$. If $\delta_0\gamma'_0 \neq 0$, we obtain the unique extension of A_α by A_α, $\alpha \neq \pm 1$, and the unique extension of $A(0, 1)$ by $A(0, 1)$ such that Q_1 is asymptotically diagonalisable.

For $\delta_0 = 0$ or $\gamma'_0 = 0$, we get the unique extensions of A_{-1} by A_{-1} and A_1 by A_1.

(b) $\dim \mathscr{A}'_0 = 2$ and $\mathscr{A}'_0$ does not contain a trivial $\mathscr{V}$-submodule $D(0)$. Then it appears that it must contain an indecomposable $\mathscr{V}$-submodule of type (V.1.b). We add the relations:

$$x_1 v'_{-1} = \delta_{-1} v_0 + \delta'_{-1} v'_0 \,, \quad x_2 v'_{-2} = \alpha_{-2} v_0 + \alpha'_{-2} v'_0 \,,$$
$$x_{-1} v'_1 = \gamma_1 v_0 + \gamma'_1 v'_0 \,, \qquad x_{-2} v'_2 = \beta_2 v_0 + \beta'_2 v'_0 \,.$$

Writing the commutators which were not calculated in the asymptotic discussion, we get a system which, up to equivalence, admits the unique solution:

$$\varepsilon = 0, \quad \gamma'_1 = \delta_{-1} = 2, \quad \gamma_1 = \delta'_{-1} = 0, \quad \alpha'_{-2} = \beta_2 = -1, \quad \alpha_{-2} = \beta'_2 = 3,$$

$$\alpha_+ = \alpha_- = 2 \,.$$

We can suppose $\alpha_{-1} = \beta_1 = 0$ and we get a unique indecomposable $\mathscr{V}$-module, extension of $A(0, 1)$ by $A(0, 0)$.

(c) $\dim \mathscr{A}'_0 = 2$ and $\mathscr{A}'_0$ contains exactly one trivial submodule $D(0)$. The corresponding factor $\mathscr{V}$-module is necessarily one of the two indecomposable $\mathscr{V}$-modules which we constructed in (V.3.a). In both cases, we have the relations (V.2.1) with $\delta_n = 1$ and $\gamma_n = -1$.

– First case: we use the formulas defining (V.3.a.i) and we set:

$$\begin{cases} x_1 v'_0 = v_1 \\ x_{-1} v'_0 = v_{-1} \end{cases} \begin{cases} x_1 v'_{-1} = -v'_0 + \delta'_{-1} v_0 \\ x_{-1} v'_1 = -v'_0 + \gamma'_1 v_0 \end{cases} \begin{cases} x_{-1} v_1 = \gamma_1 v_0 \\ x_1 v_{-1} = \delta_{-1} v_0 \end{cases} \quad x_i v_0 = 0 \quad \forall i\,.$$

We can choose v'_0 so that $\gamma'_1 = 0$ and we get $\gamma_1 = \delta_{-1}$.

- If $\gamma_1 = \delta_{-1} = 0$, v_0 can be chosen so that $\delta'_{-1} = 1$ and we get a unique indecomposable $\mathscr{V}$-module, extension of A_0 by $A(0, 0)$ (or any B_β) where we have $cv'_0 = 0$.
- If $\gamma_1 = \delta_{-1} = 1$, we get a unique indecomposable $\mathscr{V}$-module, extension of $A(0, 0)$ by $A(0, 0)$ such that Q_1 is asymptotically diagonalisable. It satisfies $cv'_0 = 0$.

– Second case: A similar discussion as in the preceding case gives:

- a unique indecomposable $\mathscr{V}$-module, extension of $A(0, 1)$ by $A(0, 0)$ (or any B_β) where we have $cv'_0 = 0$.
- the unique extension of B_0 by B_0 such that Q_1 is asymptotically diagonalisable. It satisfies $cv'_0 = 0$.

(d) $\dim \mathscr{A}'_0 = 2$ and $\mathscr{A}'_0$ is a direct sum of two trivial $\mathscr{V}$-submodules $D(0)$.

– Suppose first that there exists a trivial $\mathscr{V}$-submodule $\{v_0\}$ such that the corresponding factor $\mathscr{V}$-module is indecomposable.

A similar discussion as in the case (V.4.a) gives:

- the unique extension of B_β by B_β for each β.
- the unique extension of $A(0, 0)$ by $A(0, 0)$ such that Q_1 is asymptotically diagonalisable.

– Suppose now that for all trivial $\mathscr{V}$-submodules of $\mathscr{A}'_0$, the corresponding factor $\mathscr{V}$-module is reducible. Then the $\mathscr{V}$-module is reducible.

(e) $\dim \mathscr{A}'_0 = 1$ and $\mathscr{A}'_0$ does not contain a trivial $\mathscr{V}$-submodule. Here, we have a trivial quotient module $\mathscr{A}/\mathscr{A}' = D(0)$. The $\mathscr{V}$-submodule $\mathscr{A}'$ generated by $\{v_1, v'_1\}$ may be one among the two indecomposable $\mathscr{V}$-modules of type (V.3.a). We discuss separately the two cases in the same way as in (V.4.c):

– First case: We find here

- the unique extension of $A(0, 1)$ (or any A_α) by $A(0, 0)$ and
- the unique extension of A_0 by A_0, such that Q_1 is not asymptotically diagonalisable.

– Second case: We get

- the unique extension of $A(0, 1)$ (or any A_α) by B_0 and
- the unique extension of $\mathscr{A}(0, 1)$ by $A(0, 1)$, such that Q_1 is not asymptotically diagonalisable.

(f) $\dim \mathscr{A}'_0 = 1$ and $\mathscr{A}'_0$ is a trivial $\mathscr{V}$-submodule. Thus $\mathscr{A}'$ is either an indecomposable $\mathscr{V}$-module of type (V.3.e) or a reducible $\mathscr{V}$-module $B_\beta \oplus \tilde{A}$ or $A(0,0) \oplus \tilde{A}$ and $\mathscr{A}/\mathscr{A}'$ is $D(0)$.

– If $\mathscr{A}'$ is an indecomposable $\mathscr{V}$-module of type (V.3.e), we set:

$$\begin{cases} x_1 v'_0 = \delta'_0 v'_1 + \delta_0 v_1 \\ x_{-1} v'_0 = \gamma'_0 v'_{-1} + \gamma_0 v_{-1} \end{cases}.$$

Writing the commutator $[x_{-1} x_1](v'_0) = 2x_0 v'_0$, we get $\delta'_0 = -\gamma_0$. Thus we get the two following possible solutions:

(i) $\delta'_0 = -\gamma_0 = 1, \gamma'_0 = -\delta_0 = -1$. This gives an extension of $A(0,0)$ (or any B_β) by $A(0,1)$.
(ii) $\delta'_0 = -\gamma_0 = 1 + \alpha$, $\gamma'_0 = -\delta_0 = 1 - \alpha$: for each α we define a unique indecomposable $\mathscr{V}$-module, extension of $A(0,0)$ (or any B_β) by A_α. In both cases the commutator $[x_{-2} x_2](v'_0)$ gives $cv'_0 = 0$.

– If $\mathscr{A}'$ is a reducible $\mathscr{V}$-module. We have the relations

$$\begin{cases} x_1 v_{-1} = \delta_{-1} v_0 \\ x_{-1} v_1 = \gamma_1 v_0 \end{cases} \quad \begin{cases} x_1 v'_{-1} = 0 \\ x_{-1} v'_1 = 0 \end{cases} \quad \begin{cases} x_1 v'_0 = \delta_0 v_1 + \delta'_0 v'_1 \\ x_{-1} v'_0 = \gamma_0 v_{-1} + \gamma'_0 v'_{-1} \end{cases}.$$

Considering the $\mathscr{V}$-submodule $\mathscr{A}'' \simeq \tilde{A}$ generated by $\{v'_{-1}, v'_1\}$, the quotient $\mathscr{V}$-module $\mathscr{A}/\mathscr{A}''$ is either reducible or affine indecomposable. If this quotient module is reducible, the $\mathscr{V}$-module $\mathscr{A}$ is itself reducible. Therefore we have only to consider the case where $\mathscr{A}/\mathscr{A}''$ is an affine indecomposable $\mathscr{V}$-module. From the relation $[x_{-1} x_1](v'_0) = 2x_0(v'_0)$ we deduce $\delta_0 \gamma_1 = \gamma_0 \delta_{-1}$. The assumptions $\delta_0 = \gamma_0 = 0$ or $\delta'_0 = \gamma'_0 = 0$ leads to reducible $\mathscr{V}$-modules. Thus we get the following solutions:

(i) $\gamma_1 = 1$ $\delta_{-1} = 1$: it defines an extension of $A(0,0)$ by $A(0,1)$ (or any A_α) such that Q_1 is asymptotically diagonalisable.
(ii) $\gamma_1 = \beta - 1$ $\delta_1 = \beta + 1$: we get an extension of B_β by $A(0,1)$ (or any A_α).

Proposition V.4.1. *Any indecomposable admissible $\mathscr{V}$-module $\mathscr{A} = \bigoplus_{n \in \mathbb{Z}} \mathscr{A}_n$ such that $\dim \mathscr{A}_n = 2$, $\forall n \in \mathbb{Z}$ and $Q_1^2 = 0$, is one of the following extensions of length four:*

1) *The unique extensions of A_α, B_β, $A(0,1)$, $A(0,0)$ by themselves, and of $A(0,1)$ by $A(0,0)$ such that Q_1 is diagonalisable on $\mathscr{A}_n$ $\forall n$.*
2) *The unique extensions of $A(0,0)$, $A(0,1)$, A_0, B_0 by themselves, such that Q_1 is non-diagonalisable on $\mathscr{A}_n$ $\forall n$.*
3) *The unique extension of A_0 by $A(0,0)$, of $A(0,1)$ by B_0 and two extensions of $A(0,1)$ by $A(0,0)$ such that Q_1 is non-diagonalisable on all $\mathscr{A}_n$ except on $\mathscr{A}_0$.*
4) *The unique extension of $A(0,0)$ by A_α (for each α), of B_β by $A(0,1)$ (for each β) and two extensions of $A(0,0)$ by $A(0,1)$ such that Q_1 is diagonalisable on all $\mathscr{A}_n$ except on $\mathscr{A}_0$.*

VI. Conclusion

Now we can conclude with the following Theorem:

Theorem VI.1. *Any indecomposable admissible $\mathscr{V}$-module $\mathscr{A}$ where the weightspace dimensions are less than or equal to two is such that:*

– *either, all weightspaces are one-dimensional and $\mathscr{A}$ belongs to the classification given in* [4].
– *or one weightspace, at least, has a dimension two and $\mathscr{A}$ is one of the $\mathscr{V}$-modules classified in the Sects. (III), (IV), (V).*

Proof. Let us suppose that $\mathscr{A}$ has at least a two-dimensional weightspace.

First case. The asymptotic dimension of $\mathscr{A}$ is one. From Theorem (III.8) of [2], only the zero-weightspace is two-dimensional. Then, $D(0)$ is either a submodule of $\mathscr{A}$ or a factor module of $\mathscr{A}$, and $\mathscr{A}$ is an affine $\mathscr{V}$-module. Using Proposition (II.3), $\mathscr{A}$ appears either in (IV.1) (case 7) or (IV.2) (case 7) or in (V.1).

Second case. The asymptotic dimension of $\mathscr{A}$ is two. From [1, 2], we know that $\mathscr{A}$ contains an irreducible $\mathscr{V}$-module $A(a, \Lambda)$ ($a = 0 \Rightarrow \Lambda \neq 0, 1$) or $\tilde{A}$ or $D(0)$ and hence, in all cases, a $\mathscr{V}$-submodule $\mathscr{A}'$ with an asymptotic dimension equal to one. $\mathscr{A}'$ can be $A(a, \Lambda)$, $\tilde{A}$, A_α, B_β, $\tilde{A} \oplus D(0)$ or an affine $\mathscr{V}$-module containing the trivial $\mathscr{V}$-module. If $\mathscr{A}'$ and $\mathscr{A}/\mathscr{A}'$ is of type $A(a, \Lambda)$ or $\tilde{A}$ or A_α, or B_β or $\tilde{A} \oplus D(0)$, then $\mathscr{A}$ occurs in (III) or (IV) or (V). In the other cases, either $\mathscr{A}'$ is an affine $\mathscr{V}$-module containing the trivial $\mathscr{V}$-module, or $\mathscr{A}/\mathscr{A}'$ is an affine $\mathscr{V}$-module which does not contain the trivial $\mathscr{V}$-module. These two cases are contragredient, and it is sufficient to prove Theorem (VI.1) for one of them. If $\mathscr{A}'$ is an affine $\mathscr{V}$-module containing the trivial $\mathscr{V}$-module, there exists two cases (Proposition II.3):

– either in $\mathscr{A}'$, $Q_1^2 = 0$ and $a = 0$. Then $\mathscr{A}/\mathscr{A}'$ is $\tilde{A}$. Necessarily we have $\mathscr{A}$ such that $Q_1^2 = 0$, $a = 0$ and $\mathscr{A}$ appears in (V).
– or $\mathscr{A}' = \mathscr{F}^*$. Then $\mathscr{A}/\mathscr{A}' = \tilde{A}$ and $\mathscr{A}/D(0)$ is an extension of $A(0, -1)$ by $\tilde{A}$ which is trivial (IV.1 case 1). Thus, we can look at $\mathscr{A}$ as an extension of B_β or $A(0, 0)$ by $A(0, -1)$ and $\mathscr{A}$ occurs in (IV.2), case 3 or 4.

Finally, let us notice a last remark:

Consider the subalgebra W_1 of $\mathscr{V}$, whose a basis is $\{x_i, i \geqq -1\}$. Each $\mathscr{V}$-module $A(a, \Lambda)$ verifying $\Lambda - a \in \mathbb{Z}$, when restricted to the subalgebra W_1, contains a W_1 submodule $F_{-\Lambda}$. $F_{-\Lambda}$ is generated by the weightspaces $\mathscr{A}_{a+n}$ verifying $a + n \geqq \Lambda_1$. All the extensions of F_μ by F_λ have been obtained by Feigin–Fuchs in [7]. Then, consider an admissible extension of two $\mathscr{V}$-modules $A(a, \Lambda_1)$ and $A(a, \Lambda_2)$ such that $a - \Lambda_i \in \mathbb{Z}$ ($i = 1, 2$), and restrict it to the subalgebra W_1. A natural question is to ask whether it contains an extension of $F_{-\Lambda_1}$ by $F_{-\Lambda_2}$. It appears that all extensions obtained in (III.2) for $a - \Lambda \in \mathbb{Z}$, or (III.4) for $a = 0$, the extension of $A(0, 5)$ by $A(0, 0)$ and its contragredient ((IV.1) case 6 and (IV.2) case 6) and the extension of $\tilde{A}$ by $A(0, 0)$ ((V.3.a). (i)) are convenient. Moreover, we obtain like this, all admissible extensions of two W_1-modules, F_λ by F_μ of [7].

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